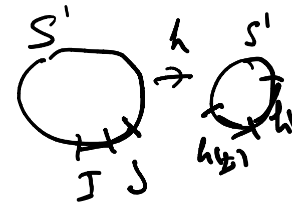
Bishop's Class 11/3/20

SHEARS ON THE FAREY TESSELATION AND CIRCLE MAPS

DRAGOMIR ŠARIĆ



1. Introduction

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ unit disk equipped with the hyperbolic metric $\rho(z) = \frac{2|dz|}{1-|z|^2}$; ideal boundary at infinity is $S^1 = \{|z| = 1\}$

Theorem 1.1. *A quasiconformal map $f : \mathbb{D} \to \mathbb{D}$ extends by continuity to a quasisymmetric map $h : S^1 \to S^1$, and every quasisymmetric map $h : S^1 \to S^1$ has many quasiconformal extensions $f : \mathbb{D} \to \mathbb{D}$.*

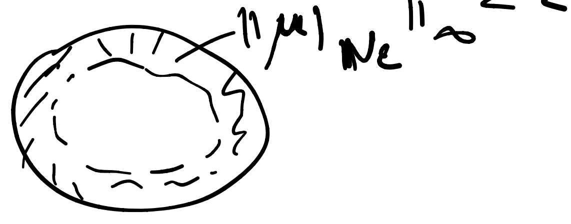
Definition 1.2. The *universal Teichmüller space* $T(\mathbb{D})$ is the space of all quasiconformal maps $f : \mathbb{D} \to \mathbb{D}$ up to an equivalence: $f_1 \sim f_2$ if there exists $\gamma \in$Möb($\mathbb{D}$) such that $\gamma \circ f_1$ is homotopic to f_2 modulo the ideal boundary S^1 (or bounded homotopy).

By Theorem 1.1,

$$T(\mathbb{D}) = \{h : S^1 \to S^1 : h \text{ quasisymmetric and fixes } 1, i, -1\}$$

Facts about $T(\mathbb{D})$:

- complete metric space in the Teichmüller metric
$$d_T([f],[g]) = \inf_{f_1 \in [f], g_1 \in [g]} \frac{1}{2} \log K(g_1^{-1} \circ f_1)$$
- infinite-dimensional non-separable complex Banach manifold (Bers embedding)
- contains multiple copies of Teichmüller space of any Riemann surface as closed complex Banach submanifolds
- contains the Weil-Petersson qc maps and the closure in d_T of the set of WP maps is the submanifold $T_0(\mathbb{D})$ of asymptotically conformal qc maps or symmetric maps

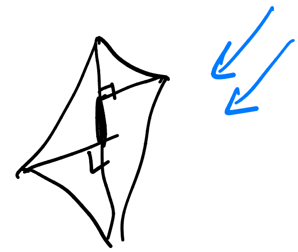


Date: November 3, 2020.

When S is a closed surface of genus $g \geq 2$, the Teichmüller space

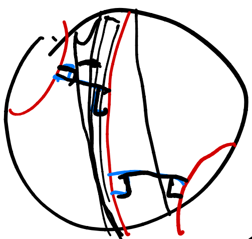
$$T(S) := \{f : S \to S_1 : \text{ quasiconformal } \}/\sim$$

Choose a pants decomposition of S with cuffs $\{\gamma\}_{j=1}^{3g-3}$, the *lengths* $\ell(\gamma_j)$ of the cuffs of pairs of pants and the *twists* $t(\gamma)$ about the cuffs uniquely determine a marked hyperbolic surfaces-i.e., point in $T(S)$

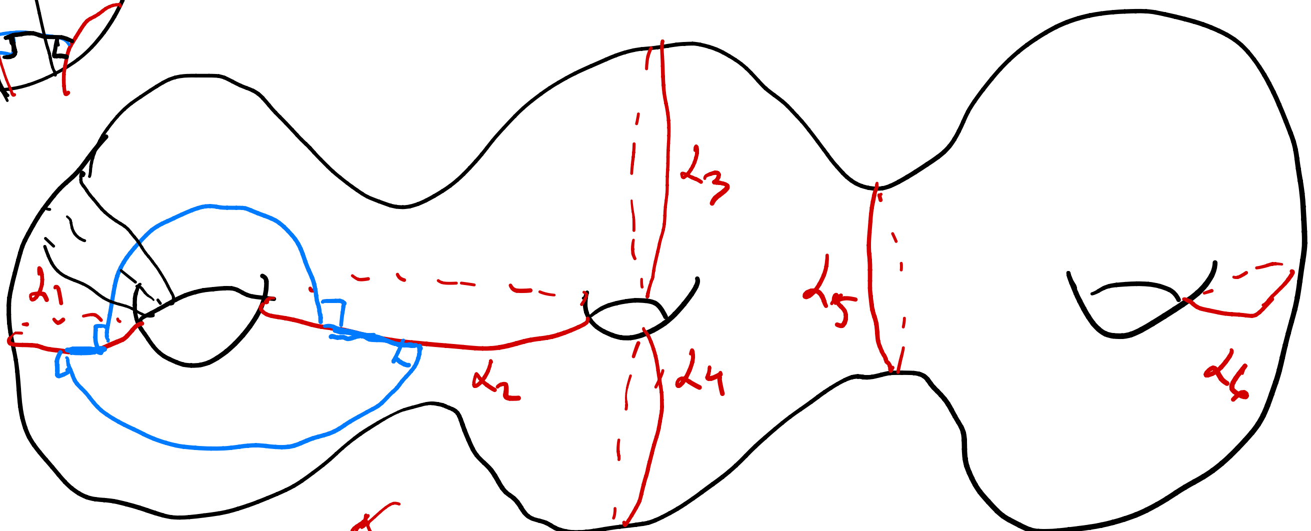


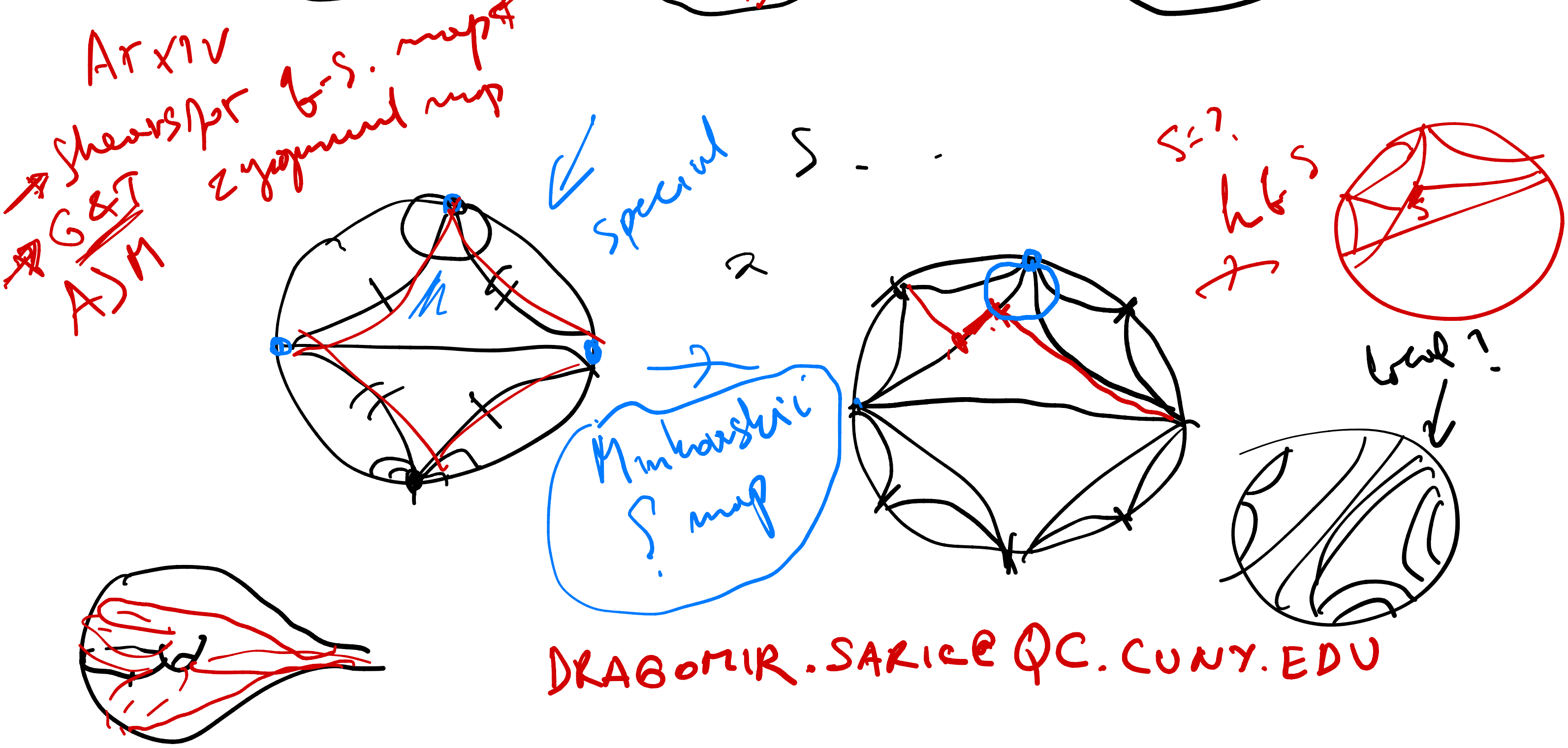
The *Fenchel-Nielsen parametrization* of $T(S)$ is given by:

$$[f, S_1] \in T(S) \mapsto \{(\ell_{S_1}(f(\gamma_j)), t_{S_1}(f(\gamma_j)))\}_{j=1}^{3g-3}$$



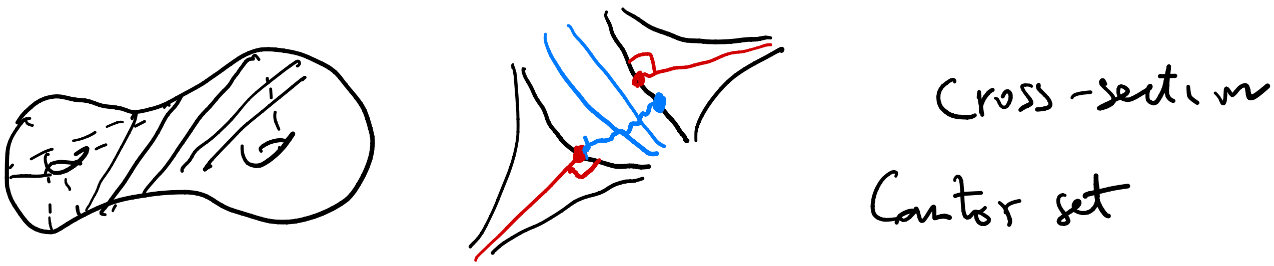
Theorem 1.3. *$T(S)$ is homeomorphic to* $(\mathbb{R}^+)^{3g-3} \times \mathbb{R}^{3g-3}$.



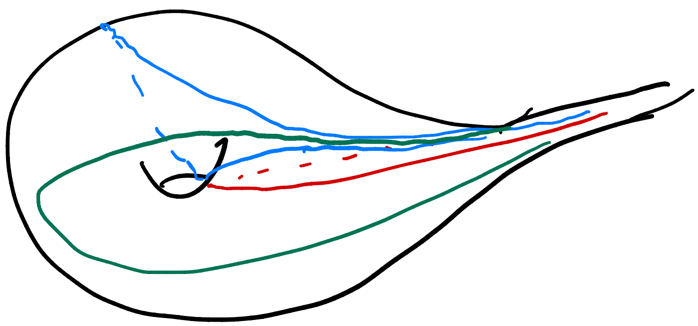


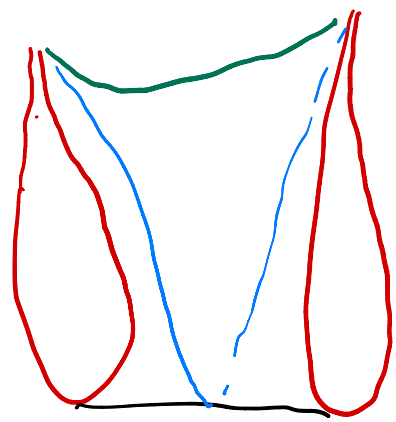
If we do not want to use lengths for parametrization of $T(S)$, Thurston introduced cataclysm coordinates and its equivalent formulation is called *shear coordinates* (Bonahon and Penner)

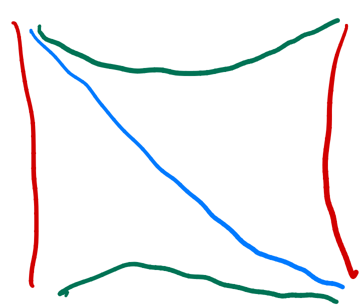
Fix a *maximal geodesic lamination* λ on S (complementary regions are ideal hyperbolic triangles) and measure the relative position of the ideal hyperbolic triangles (shears along the geodesic lamination λ)



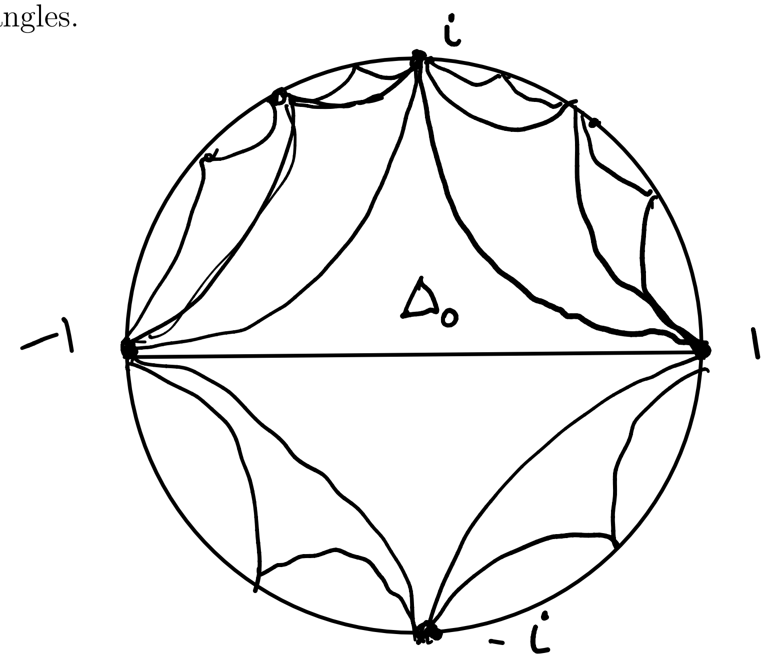
When S has punctures and λ consists of finitely many infinite geodesics Penner gives parametrization of $T(S)$; when S is closed and λ arbitrary (cross-sections Cantor sets) Bonahon gives parametrization and further extensive study of analytic properties of the shear coordinates



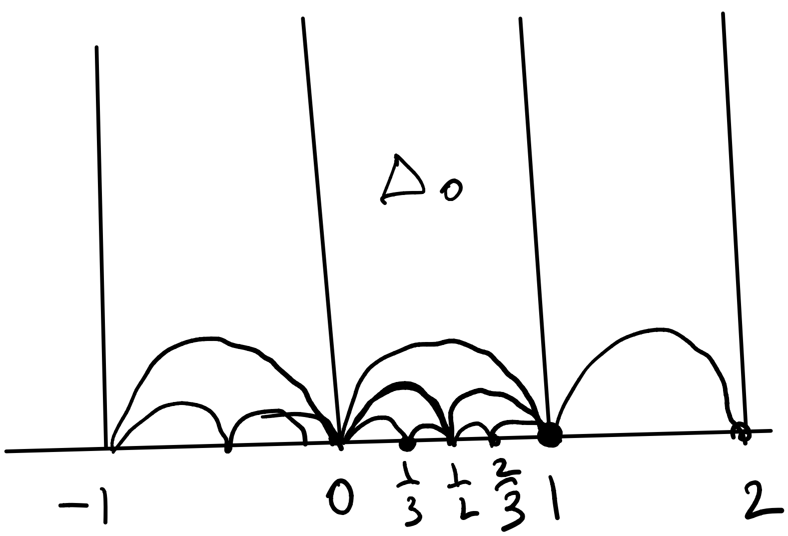




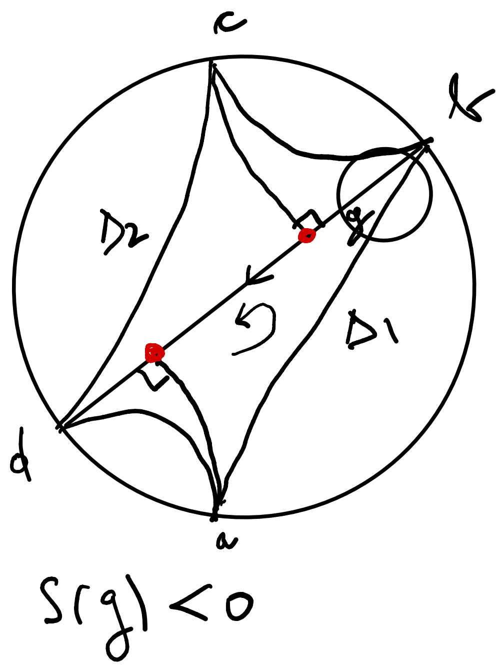
Definition 1.4. The *Farey tesselation* $\mathcal{F}$ of $\mathbb{D}$ is a triangulation of $\mathbb{D}$ by ideal hyperbolic triangles: A triangle Δ_0 with vertices 1, i and -1 is a generation 0 triangle of $\mathcal{F}$. The generation 1 triangles are obtained by hyperbolic reflections of Δ_0 in its sides. The generation $n+1$ triangles are obtained by reflecting generation n triangles in the sides not shared with generation $n-1$. The tesselation $\mathcal{F}$ is the collection of all these triangles.

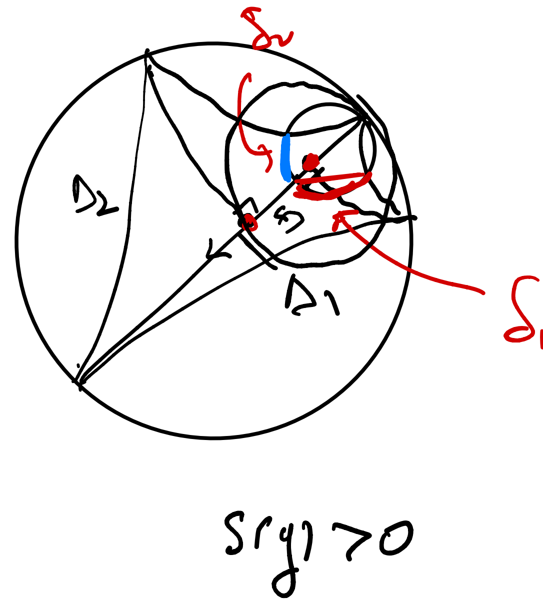


Definition 1.5. The *Farey tesselation* on the upper half plane $\mathbb{H}$ is obtained by inverting the hyperbolic triangle with vertices 0, 1 and ∞. It is invariant under the action of $PSL_2(\mathbb{Z})$ and the action is simply transitive on the oriented edges of $\mathcal{F}$.



Definition 1.6. Given two ideal hyperbolic triangles Δ_1 and Δ_2 sharing a common boundary geodesic g, the *shear* $s(g)$ is the signed distance along g between the foots of the orthogonals from the third vertices of Δ_1 and Δ_2.





$s(g)$ is the signed translation length of a hyperbolic translation with axis g that moves the third vertex of the reflected triangle Δ_1' to the third vertex of Δ_2

fix a horocycle C based at an endpoint of g such that the triangle Δ_1 comes before Δ_2 for the natural orientation on C

$$s(g) = \log \frac{\delta_2}{\delta_1},$$

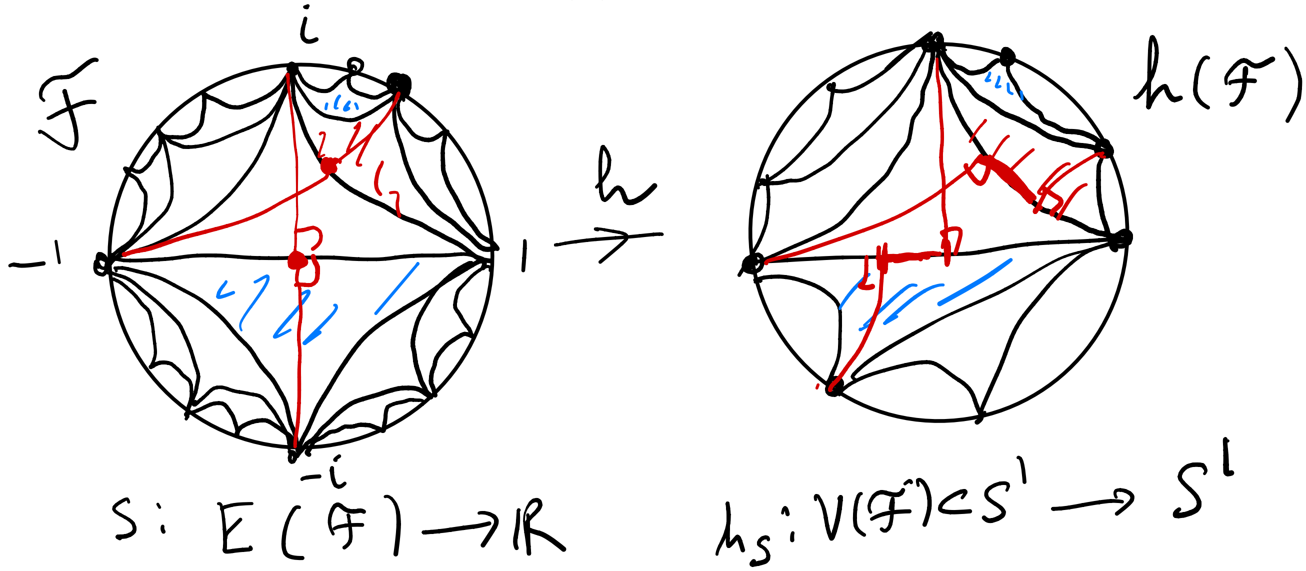
where δ_i is the length of $C \cap \Delta_i$

if a, b, d are vertices of Δ_1, b, c, d are vertices of Δ_2 and g the geodesic with endpoints b, d, then

$$s(g) = \log \frac{(a-d)(b-c)}{(a-b)(c-d)} = \log \frac{TM}{LR}.$$

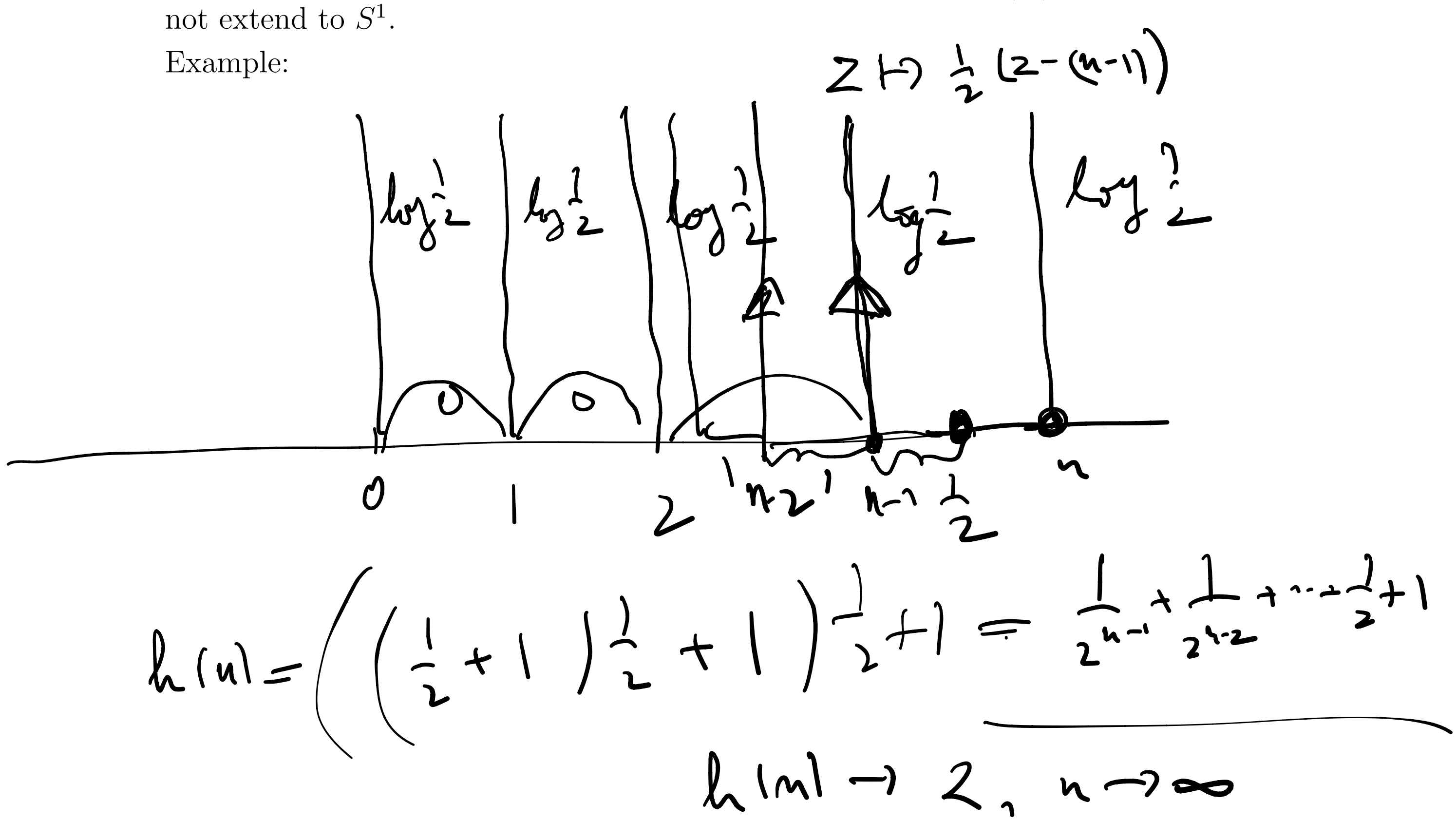
To parametrize $T(\mathbb{D})$, start with Farey tesselation $\mathcal{F}$ of the unit disk; shears on adjacent triangles are equal to 0

Given a homeomorphism $h : S^1 \to S^1$ we get a new tesselation $h(\mathcal{F})$ with new shears $s_h : E(\mathcal{F}) \to \mathbb{R}$. The homeomorphism h (up to post-composition by Mobius maps) is uniquely determined by the shears s_h because there is a developing map corresponding to any $s : E(\mathcal{F}) \to \mathbb{R}$.



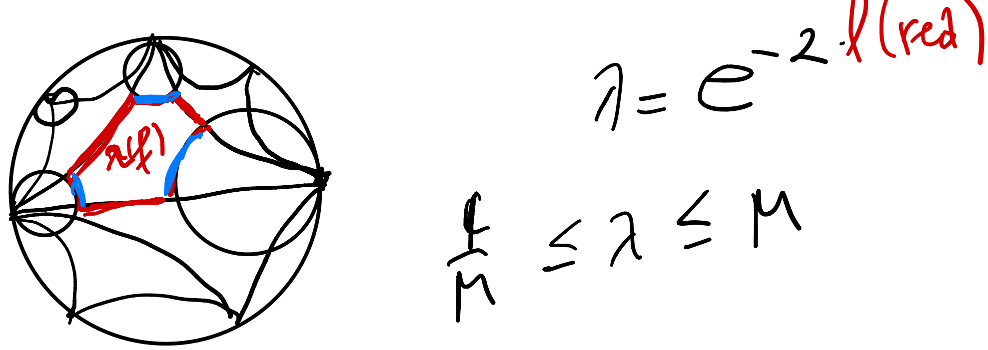
Given a map $s : E(\mathcal{F}) \to \mathbb{R}$ is there a homeomorphism $h : S^1 \to S^1$ which induces s? Not always! There is s such that the developing map $h_s : V(\mathcal{F}) \subsetneq S^1 \to S^1$ does not extend to S^1.

Example:



Problem (Penner): Find necessary and sufficient conditions on the shears such that the developing map extends to a homeomorphism, quasisymmetric map, symmetric map, $C^{k+\alpha}$-map, WP map, ...

Definition 1.7. A *decoration* on a tesselation is a choice of a horocycle at each vertex of the tesselation. A *lambda length* of an edge of a decorated tesselation is the distance between along the geodesic between the intersections with the horocycles.



An assignment of lambda lengths $\lambda : E(\mathcal{F}) \to \mathbb{R}$ develops into a map from $V(\mathcal{F})$ into S^1.

Theorem 1.8 (Penner-Sullivan)**.** *A shear function $s : E(\mathcal{F}) \to \mathbb{R}$ induces a quasisymmetric map of S^1 if there is a choice of a horocycle at each vertex of $h_s(\mathcal{F})$ such that the lambda lengths are pinched between two positive constants.*

Remark 1.9. Sufficient condition which is not explicit in terms of the shear function.

Remark 1.10. Not every quasisymmetric map has such a choice of horocycles.

2. Main results

Definition 2.1. A fan of edges $\mathcal{F}^p$ with the tip p consists of all edges of $\mathcal{F}$ with one endpoint p.

Theorem 2.2. *A function $s:\mathcal{F}\to\mathbb{R}$ is induced by shears of $h(\mathcal{F})$ of a quasisymmetric map of $h:S^1\to S^1$ if and only if there exists a constant $M\geq 1$ such that for each fan of geodesics $\mathcal{F}^p$ of $\mathcal{F}$ and for all $m\in\mathbb{Z}$ and $k\in\mathbb{N}\cup\{0\}$, we have*

$$\frac{1}{M}\leq e^{s(f^p_m)}\frac{1+e^{s(f^p_{m+1})}+\cdots+e^{s(f^p_{m+1})+s(f^p_{m+2})+\cdots+s(f^p_{m+k})}}{1+e^{-s(f^p_{m-1})}+\cdots+e^{-s(f^p_{m-1})-s(f^p_{m-2})-\cdots-s(f^p_{m-k})}}\leq M. \tag{1}$$

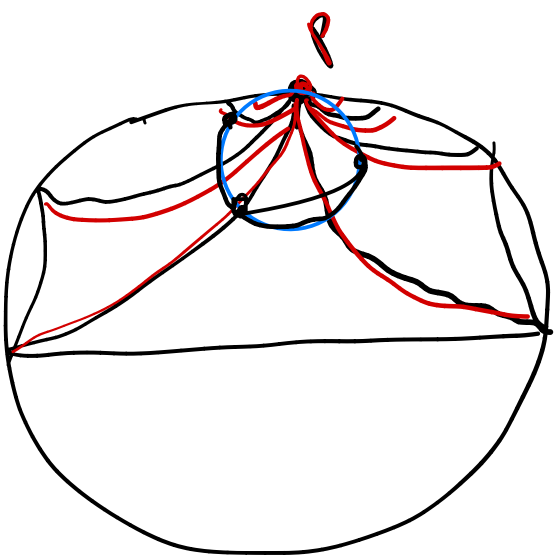
where δ^p_n is the length of the arc of C_p between f^p_n and f^p_{n+1}.

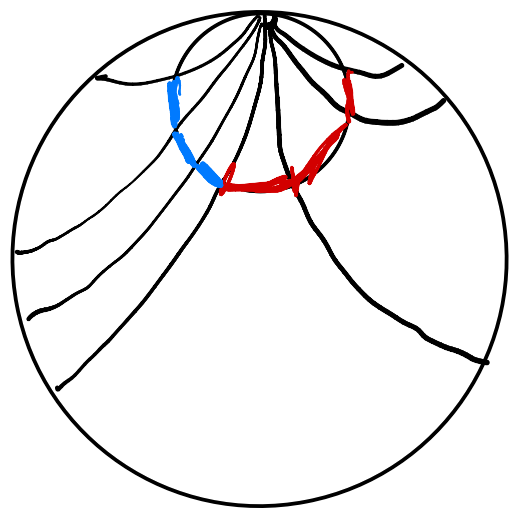
Geometric interpretation: choose a horocycle C_p based at the tip p; δ^p_n is the length of the arc of C_p between f^p_n and f^p_{n+1}

Then (1) is equivalent to

$$\frac{1}{M}\leq\frac{\delta^p_m+\delta^p_{m+1}+\cdots+\delta^p_{m+k}}{\delta^p_{m-1}+\delta^p_{m-2}+\cdots+\delta^p_{m-k-1}}\leq M.$$

The condition is only in the fans!





Definition 2.3. A homeomorphism $h : S^1 \to S^1$ is *symmetric* if

$$|h(I)/h(J)| \to 1$$

when $|I| \to 0$ for all $I, J \subset S^1$ adjacent arcs with $|I| = |J|$.

Fact: A homeomorphism $h : S^1 \to S^1$ is symmetric if and only if it extends to asymptotically conformal quasiconformal map of the unit disk $\mathbb{D}$.

Definition 2.4. A *generation* of an edge f of $\mathcal{F}$ is the number of edges between f and closests edge of the initial ideal hyperbolic triangle plus one.

Thus large generation of an edge implies that the endpoints on S^1 are close to each other.

Theorem 2.5. *A function $s : E(\mathcal{F}) \to \mathbb{R}$ corresponds to a symmetric homeomorphism of S^1 if and only if*

$$\frac{\delta^p_m + \delta^p_{m+1} + \cdots + \delta^p_{m+k}}{\delta^p_{m-1} + \delta^p_{m-2} + \cdots + \delta^p_{m-k-1}} \rightrightarrows 1$$

as the generations of f_{m+k} and f_{m-k} converge to infinity.

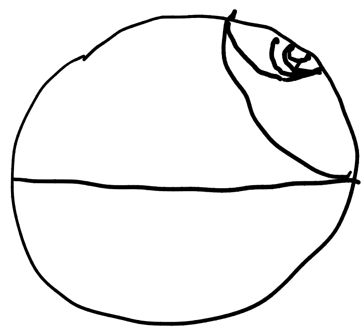
Definition 2.6. A *chain of geodesics* of $\mathcal{F}$ is a sequence $\{e_n\}_n$ in $E(\mathcal{F})$ such that e_n and e_{n+1} are adjacent and the geodesics are not repeating.

Theorem 2.7. *A function $s : E(\mathcal{F}) \to \mathbb{R}$ is induced by a homeomorphism of S^1 if and only if*

$$\sum_{n=1}^{\infty} e^{s_1^n + s_2^n + \cdots + s_n^n} = \infty$$

where $s_i^n = \pm s(f_i)$.

The choice of the sign is combinatorial: $f < f'$ depends on horocycle C



$s_i^n = s(f_i)$ if $f_i < f_{i+1}$ and the number of times we change the fans going from f_i to f_n is even, or if $f_i > f_{i+1}$ and the number of times we change the fans going from f_i to f_n is odd;

otherwise $s_i^n = -s(f_i)$

consider a differentiable path $t \mapsto h_t$, for $t \in (-\epsilon, \epsilon)$, of quasisymmetric maps with $h_0 = id$

the tangent vector field $V := \frac{d}{dt} h_t|_{t=0}$ on S^1 is identified with a real valued function $V(e^{i\theta})$ which is *Zygmund*:

$$\sup_t \frac{\|V(e^{i(x+t)} + V(e^{i(x-t)}) - 2V(e^{ix})\|_\infty}{t} < \infty$$

Fact: V is a Zygmund function iff there exists a differentiable path of qs maps h_t with $h_0 = id$ and $\frac{d}{dt} h_t|_{t=0} = V$

$s_t(f) = \log \frac{(h_t(a)-h_t(c))(h_t(b)-h_t(d))}{(h_t(a)-h_t(d))(h_t(b)-h_t(c))}$ is the shear at $f \in E(\mathcal{F})$ of h_t

Definition 2.8. The shear of V at $f \in E(\mathcal{F})$ is

$$\dot{s}(f) := \frac{d}{dt} s_t(f)|_{t=0} = \frac{V(a)-V(c)}{a-c} - \frac{V(a)-V(d)}{a-d} + \frac{V(b)-V(d)}{b-d} - \frac{V(b)-V(c)}{b-c}.$$

Given $\dot{s} : E(\mathcal{F}) \to \mathbb{R}$, there is a developing vector field defined on $V(\mathcal{F}) \subset S^1$.

Question: Find a necessary and sufficient condition on $\dot{s}$ to be induced by a Zygmund function.

Theorem 2.9. *A function $\dot{s}: E(\mathcal{F}) \to \mathbb{R}$ induces a Zygmund function on S^1 if and only if there is a constant $C > 0$ such that for every fan of geodesics $\{f_n\}_{n\in\mathbb{Z}}$ and for every $n \in \mathbb{Z}$ and $k \geq 0$ we have*

$$|\dot{s}(f_n) + \frac{k}{k+1}[\dot{s}(f_{n+1}) + \dot{s}(f_{n-1})] + \frac{k-1}{k+1}[\dot{s}(f_{n+2}) + \dot{s}(f_{n-2})] + \cdots + \frac{1}{k+1}[\dot{s}(f_{n+k}) + \dot{s}(f_{n-k})]| \leq C.$$

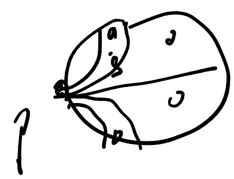
Remark 2.10. There is no explicit necessary and sufficient condition on the Fourier coefficients to guarantee that the function is Zygmund. The above theorem gives explicit condition on the shears.

If $\mathcal{F}^p$ is the fan of geodesic with tip p, denote by V_p the Zygmund function whose shears $\dot{s}_p$ are zero everywhere except at $\mathcal{F}_p$ and equal to (one-half of) $\dot{s}$ on $\mathcal{F}^p$

Denote by $H(V)$ the Hilbert transform of the Zygmund function V (it gives the almost complex structure at the identity in $T(\mathbb{D})$)

Theorem 2.11. *Let V be Zygmund and $\dot{s}: E(\mathcal{F}) \to \mathbb{R}$ its shear function. Then*

$$V = \sum_{p\in V(\mathcal{F})} V_p$$



where the convergence is uniform and absolute on S^1. In addition,

$$H(V) = \sum_{p\in V(\mathcal{F})} H(V_p)$$

and

$$H(V_p)(z) = \frac{1}{2}\sum_{f_n=(a_n,b_n)\in E(\mathcal{F}^p)} \dot{s}(f_n)\frac{|z-a_n|}{|z-b_n|}\log\frac{|z-a_n|}{|z-b_n|}.$$

3. Proof for quasisymmetric maps

Theorem 3.1. *A function $s : \mathcal{F} \to \mathbb{R}$ is induced by shears of $h(\mathcal{F})$ of a quasisymmetric map of $h : S^1 \to S^1$ if and only if there exists a constant $M \geq 1$ such that for each fan of geodesics $\mathcal{F}^p$ of $\mathcal{F}$ and for all $m \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$, we have*

$$\frac{1}{M} \leq e^{s(f^p_m)} \frac{1 + e^{s(f^p_{m+1})} + \cdots + e^{s(f^p_{m+1})+s(f^p_{m+2})+\cdots+s(f^p_{m+k})}}{1 + e^{-s(f^p_{m-1})} + \cdots + e^{-s(f^p_{m-1})-s(f^p_{m-2})-\cdots-s(f^p_{m-k})}} \leq M. \tag{2}$$

where δ^p_n is the length of the arc of C_p between f^p_n and f^p_{n+1}.

Proof. ($\Longrightarrow$) Change coordinates such that $\mathbb{D}$ is the upper half-plane $\mathbb{H}$ and the tip $p = \infty$. Then the condition (2) is simply the definition of qs maps.

($\Longleftarrow$) (Step 1.) The developing map h of s is a homeomorphism of S^1. If not then $h(V(\mathcal{F}))$ is not dense in S^1 which is prevented by (2) as illustrated by the figure below.

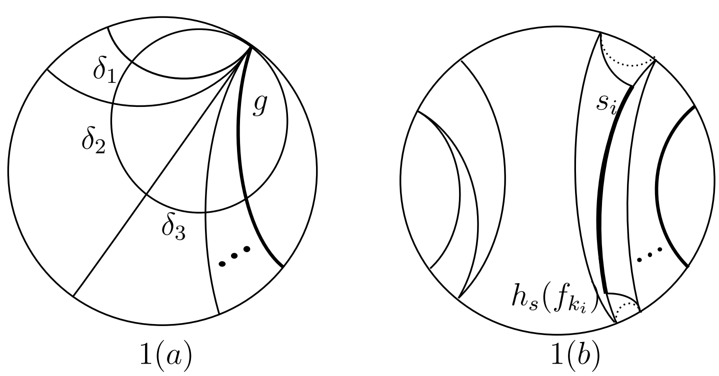


Figure 1. The continuity of a developing map.

(Step 2.) The developing map h is quasisymmetric.

Lemma 3.2. *Let $s : \mathcal{F} \to \mathbb{R}$ be a shear function that is equal to zero everywhere except on a single fan $\mathcal{F}^p = \{f_n^p\}_{n\in\mathbb{Z}}$ with tip p. If there exists $M \geq 1$ such that for all $m \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$,*

$$\frac{1}{M} \leq \frac{\delta_m^p + \delta_{m+1}^p + \cdots + \delta_{m+k}^p}{\delta_{m-1}^p + \delta_{m-2}^p + \cdots + \delta_{m-k-1}^p} \leq M$$

then s is induced by an M'-quasisymmetric map $h_s : S^1 \to S^1$, where M' depends only on M.

Proof of lemma. Consider $\mathbb{H}$ and tip $p = \infty$. For symmetric triples $x - t, x, x + t$ the qs condition holds uniformly when t is bounded because the relative slopes are bounded over $\mathbb{R}$ independent of x. When t is large, we approximate the intervals $[x-t, x]$ and $[x, x+t]$ by the intervals of with integer endpoints and the qs condition is satisfied there. So it is satisfied for $[x-t, x]$ and $[x, x+t]$. *End of proof of lemma.*

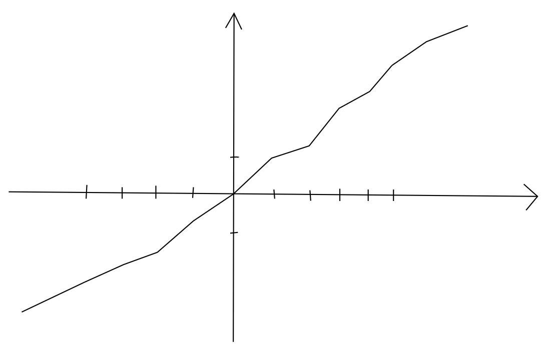


FIGURE 2. The graph of the developing map h_1.

Let $ex(h) : \mathbb{D} \to \mathbb{D}$ be the barycentric extension of h introduced by Douady and Earle. The extension satisfies:

- $ex(h)$ is real analytic
- conformally natural: if $\alpha, \beta \in \mathrm{Mob}(\mathbb{D})$ then $\alpha \circ ex(h) \circ \beta = ex(\alpha \circ h \circ \beta)$
- if h qs then $ex(h)$ qc
- if $h_n \to h$ pointwise then $ex(h_n) \to ex(h)$ in C^∞ topology uniformly on compact subsets of $\mathbb{D}$

Assume that h satisfies the condition (2) but is not quasisymmetric. Then there exists $z_n \in \mathbb{D}$ such that

$$D(ex(h))(z_n) := \frac{|ex(h)_z| + |ex(h)_{\bar{z}}|}{|ex(h)_z| - |ex(h)_{\bar{z}}|}(z_n) \to \infty. \tag{3}$$

Then $|z_n| \to 1$ as $n \to \infty$ and wee seek a contradiction.

Let $z_n \in \Delta_n$ and $\alpha_n, \beta_n \in \text{Mob}(\mathbb{D})$ such that $\alpha_n(\Delta_0) = \Delta_n$ and $\beta_n \circ ex(h) \circ \alpha_n$ fixes 1, i and -1. Let $\Delta_0 \ni z_n' := \alpha_n^{-1}(z_n)$.

Two cases: z_n' stays in a compact subset of Δ_0 or it leaves towards one vertex on S^1

Case 1. After taking a subsequence, $z_n' \to z^*$ because it stays in a compact subset. We have that the discrete triangulation $\alpha_n^{-1}(\mathcal{F})$ converge to a discrete triangulation $\mathcal{F}^*$ and the shears converge as well. Then $\beta_n \circ h \circ \alpha_n$ converges to a homeomorphism h^* corresponding to $\mathcal{F}^*$ and the limiting shears. This contradicts (3) by the conformal naturallity:

$$D(ex(h))(z_n) = D(ex(\beta_n \circ h \circ \alpha_n))(z_n') \to D(ex(h^*))(z^*) < \infty$$

which gives a contradiction.

Case 2. $z'_n \to S^1$. Let $\alpha_n \in \text{Mob}(\mathbb{D})$ be such that $\alpha'_n(0) = z'_n$.

Then $\mathcal{F}'_n := (\alpha_n \circ \alpha'_n)^{-1}(\mathcal{F})$ is a tesselation that converges to a foliation of $\mathbb{D}$ but it satisfies the condition (2) with the same constant in each fan and for each n. Let h'_n be the map with shears in a single fan $\mathcal{F}'_{p_n}$ of $\mathcal{F}'_n$ that accumulates to the foliation, then the above Lemma guaranties the convergence of

$$\beta'_n \circ h_n \to h_\infty$$

where h_∞ is a quasisymmetric map and $\beta'_n \circ h_n$ fixes 1, i and -1.

Since the endpoints of $\mathcal{F}{p_n}'$ are dividing the unit circle S^1 onto smaller and smaller intervals, we have that the map $\beta'_n \circ h \circ \alpha_n \circ \alpha'_n$ converges to the same quasisymmetric map h_∞. Then the conformal naturality of the barycentric extension implies that

$$D(ex(\beta_n \circ h \circ \alpha_n))(z'_n) = D(ex(\beta'_n \circ h \circ \alpha_n \circ \alpha'_n))(0) \to D(ex(h_\infty))(z_n) < \infty$$

which is again a contradiction

□