

THE ISOPERIMETRIC INEQUALITY

BY ROBERT OSSERMAN¹

The circle is uniquely characterized by the property that among all simple closed plane curves of given length L , the circle of circumference L encloses maximum area. This property is most succinctly expressed in the isoperimetric inequality

$$L^2 > 4\pi A, \quad (1)$$

where A is the area enclosed by a curve C of length L , and where equality holds if and only if C is a circle.

The purpose of this paper is to recount some of the most interesting of the many sharpened forms, generalizations, and applications of this inequality, with emphasis on recent contributions. Earlier work is summarized in the book of Hadwiger [1]. Other general references, varying from very elementary to quite technical are Kazarinoff [1], Pólya [2, Chapter X], Porter [1], and the books of Blaschke listed in the bibliography. Most books on convexity also contain a discussion of the isoperimetric inequality from that perspective. One aspect of the subject is given by Burago [1]. Others may be found in a recent paper of the author [4] on Bonnesen inequalities and in the book of Santaló [4] on integral geometry and geometric probability.

An important note: we shall *not* go into the area of so-called “isoperimetric problems”. Those are simply variational problems with constraints, whose name derives from the fact that inequality (1) corresponds to the first example of such a problem: maximize the area of a domain under the constraint that the length of its boundary be fixed. There are also the “isoperimetric inequalities” of mathematical physics. They are special cases of isoperimetric problems in which typically some physical quantity, usually represented by the eigenvalues of a differential equation, is shown to be extremal for a circular or spherical domain. Extensive discussions of such problems can be found in the book of Pólya and Szegő [1] and the review article by Payne [1]. We shall discuss them here only insofar as they relate to the main subject of this paper.

What we shall concentrate on here is “the” isoperimetric inequality (1) and other geometric versions and generalizations of it. We shall also consider

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various analytic inequalities closely connected, and in some cases equivalent to the isoperimetric inequalities, such as Wirtinger's, Poincaré's and Sobolev's inequalities.

This paper is divided into six sections whose contents in brief are as follows:

1. The classical case: refinements of (1) for curves in the plane; Wirtinger's inequality.
2. Extensions of (1) to domains in $\mathbf{R}^n$: variational approach and constant mean curvature; Minkowski's theory; integral geometry.
3. Analytic inequalities: Sobolev inequality, Rayleigh quotient, eigenvalues of the Laplacian, Poincaré inequality.
4. Analogs of (1) for domains on surfaces: Bonnesen inequalities; minimal surfaces; inequalities depending on Gauss curvature.
5. Variants of (1): other inequalities between L and A ; submanifolds of $\mathbf{R}^n$ and Riemannian manifolds; Rayleigh quotient on compact manifolds.
6. Applications: physical problems; conformal and quasi-conformal mappings; symmetrization; geometry and analysis.

Before starting, a historical note is worth inserting. Many mathematicians have been attracted to the isoperimetric inequality, either by its intrinsic geometric interest or with a view to applications, and they have approached it from a variety of directions. In some cases this has led to parallel lines of development by different groups of mathematicians, each happily oblivious to the existence of the others. Although no claim to completeness is made here, one of our goals is to tie together various threads of this development in an effort to convey a more comprehensive picture of the current state of the subject.

This paper has benefited from conversations and correspondence with a large number of mathematicians. I should like to express my appreciation to them, as well as to the Guggenheim Foundation for financial support during its preparation, and to the University of Warwick and Imperial College, London, who generously provided the use of their facilities.

1. The classical case: curves in the plane. To begin, consider how one might prove the classical isoperimetric inequality. If C is a simple closed smooth curve given parametrically, then its arc length L can be expressed as

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \quad (1.1)$$

The area A enclosed by C can also be expressed as a line integral:

$$A = - \int_C y dx = - \int_a^b y \frac{dx}{dt} dt, \quad (1.2)$$

where the orientation determined by C may be assumed to be the positive one with respect to its interior. A little experimentation reveals that the usual integral inequalities go the wrong way, giving an upper bound on L^2 , and one is forced to the simple artifice of introducing a special parameter in order to eliminate the square root in the integral (1.1). Any multiple of the parameter s of arc length will do. The most convenient is $t = (2\pi/L)s$. Then

$$\int_0^{2\pi} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] dt = \int_0^{2\pi} \left(\frac{ds}{dt} \right)^2 dt = \frac{L^2}{2\pi},$$

and

$$\begin{aligned} L^2 - 4\pi A &= 2\pi \int_0^{2\pi} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + 2y \frac{dx}{dt} \right] dt \\ &= 2\pi \int_0^{2\pi} \left(\frac{dx}{dt} + y \right)^2 dt + 2\pi \int_0^{2\pi} \left[\left(\frac{dy}{dt} \right)^2 - y^2 \right] dt. \quad (1.3) \end{aligned}$$

The first term on the right is obviously nonnegative, and the result will be proved if we can show that

$$\int_0^{2\pi} \left(\frac{dy}{dt} \right)^2 dt \geq \int_0^{2\pi} y^2 dt. \quad (1.4)$$

The inequality (1.4) cannot hold for an arbitrary function $y(t)$, since it fails when $y(t)$ is a nonzero constant. However, one has the following classical result. (For a history of this result, and a discussion of its usual attribution, "Wirtinger's inequality," see Mitrinović [1].)

LEMMA 1.1. *If $y(t)$ is a smooth function with period 2π , and if $\int_0^{2\pi} y(t) dt = 0$, then (1.4) holds, with equality if and only if $y = a \cos t + b \sin t$.*

The easiest proof of this lemma is by using a Fourier expansion of $y(t)$. The hypothesis guarantees that the constant term is zero, and (1.4) follows immediately. For another proof, and a reference to this whole discussion, see the book of Hardy, Littlewood, and Pólya [1, p. 185]. A somewhat different earlier version is in Lewy [1, p. 41].

In order to apply Wirtinger's lemma to our case, we need only observe that the hypothesis $\int_0^{2\pi} y(t) dt = 0$ can always be satisfied by suitable choice of coordinates. Specifically, choose the x -axis to pass through the center of gravity of the curve C . Then both terms on the right of (1.3) are nonnegative, giving

$$L^2 \geq 4\pi A. \quad (1.5)$$

Equality holds in (1.5) only if both terms on the right of (1.3) vanish, and using Lemma 1.1, one sees immediately that C must be a circle.

It is interesting to note that this proof does not use anywhere the assumption that C is simple. The inequality (1.5) holds for an arbitrary smooth closed curve, where L and A are defined by (1.1) and (1.2). In fact, one has

LEMMA 1.2. *Wirtinger's inequality (Lemma 1.1) is equivalent to the statement that the isoperimetric inequality (1.5) holds for every smooth closed curve, with equality only for a circle, where L and A in (1.5) are defined by (1.1) and (1.2).*

PROOF. Let $y(t)$ be a smooth function with period 2π , satisfying $\int_0^{2\pi} y(t) dt = 0$. Let $x(t) = -\int_0^t y(\tau) d\tau$. Then $x(t + 2\pi) - x(t) = -\int_t^{t+2\pi} y(\tau) d\tau = 0$, so that $x(t)$ has period 2π , and the pair $(x(t), y(t))$ defines a smooth closed

curve. Its length L , by Schwarz' inequality, satisfies

$$\begin{aligned} L^2 &= \left[\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \right]^2 \\ &\leq 2\pi \int_0^{2\pi} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right] dt. \end{aligned} \quad (1.6)$$

Thus, using (1.2) and (1.5) gives

$$0 \leq L^2 - 4\pi A \leq 2\pi \int_0^{2\pi} \left(\frac{dx}{dt} + y\right)^2 dt + 2\pi \int_0^{2\pi} \left[\left(\frac{dy}{dt}\right)^2 - y^2\right] dt. \quad (1.7)$$

The first term on the right vanishes by the definition of $x(t)$, and thus (1.7) reduces to Wirtinger's inequality (1.4). For equality to hold it must hold in (1.6) which can only happen if

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \equiv c^2, \quad \text{a constant.}$$

By (1.1), it follows that $L = 2\pi c$, and hence $ds/dt = L/2\pi$. Since the curve must be a circle, it now follows easily, using $\int_0^{2\pi} y(t) dt = 0$, that $y(t)$ is of the form $y = a \cos t + b \sin t$. This proves the lemma.

The obvious question is whether the quantity A in (1.2) has any geometric meaning in the case of a curve C with self-intersections. In fact it does. The complement of C consists of a number of components D_k , and with respect to each domain D_k , C has a well-defined winding number n_k . Then the expression (1.2) has the interpretation $A = \sum n_k A_k$, where A_k is the area (in the usual sense) of the domain D_k . (See Radó [1], or [3, III. 3.88].) Thus, the proof given above for (1.5) shows that for an arbitrary closed curve one has

$$L^2 \geq 4\pi \sum_k n_k A_k, \quad (1.8)$$

and Lemma 1.2 says that this inequality for all curves is equivalent to Wirtinger's inequality. An application of (1.8) to a physical problem will be given in §3.

Actually a much stronger result is true. Namely

$$L^2 \geq 4\pi \sum_k |n_k| A_k. \quad (1.9)$$

Note that for a lemniscate, for example, the right-hand side of (1.8) is zero, whereas (1.9) gives the sum of the areas inside each loop.

Inequality (1.9) was proved by Radó [2, §4.6]. It also appears as a special case of the isoperimetric inequality given by Federer and Fleming in their basic paper on normal and integral currents [1, Corollary 6.5 and Remark 6.6 on p. 487]. In fact, the expression $T = \sum n_k D_k$ denotes an integral current, defined as a linear functional on 2-forms ω by

$$T\omega = \sum n_k \int_{D_k} \omega.$$

The boundary of T is the current ∂T acting on 1-forms α by

$$(\partial T)\alpha = T(d\alpha) = \sum n_k \int_{D_k} d\alpha = \int_C \alpha.$$

The mass of T and of ∂T correspond to

$$M(T) = \sum_k |n_k| A_k, \quad M(\partial T) = L,$$

and (1.9) is just the isoperimetric inequality for currents

$$[M(\partial T)]^2 \geq 4\pi M(T).$$

(See also Federer [1, Theorem 4.5.9(31), p. 486].)

Surprisingly, an even stronger inequality was obtained recently by Banchoff and Pohl [1], using quite different methods. They showed that

$$L^2 \geq 4\pi \sum_k n_k^2 A_k, \quad (1.10)$$

with equality if and only if C is a circle traversed a finite number of times in a given direction. This result is a special case of a general inequality concerning curves in higher dimensional spaces that we shall discuss below in §5.

As a curious note, we remark that inequality (1.10) is contained implicitly in the work of Federer and Fleming [1, p. 487], but is never explicitly mentioned, presumably because the right-hand side is not a quantity that arises naturally in the context of currents. That the *square* of the winding number should enter in seems a bit mysterious at first. It will appear more natural after the discussion of analytic inequalities in §3.

Finally we mention that at the end of §6 we give an application of (1.10) to an apparently quite distant part of mathematics.

2. Domains in $\mathbf{R}^n$. The isoperimetric problem in $\mathbf{R}^n$ is to minimize the surface area among all domains having given volume, or equivalently, maximize the volume among all domains whose boundary surfaces has fixed $((n-1)$ -dimensional) area. The solution in both cases is that the unique extremal is the domain bounded by a sphere. However, for $n > 2$ there are no proofs approaching the simplicity of the one given above for plane domains. the domain bounded by a sphere. However, for $n > 2$ there are no proofs approaching the simplicity of the one given above for plane domains.

Perhaps the most direct approach, assuming one works with smooth boundaries, is to try to use the methods of the calculus of variations. Consider, for example, a domain D in $\mathbf{R}^3$ bounded by a smooth surface S . Let $h: S \rightarrow \mathbf{R}$ be a smooth real-valued function on S , and let S_t denote the surface obtained by displacing each point of S by the vector thN , where N is the unit exterior normal field to S . If $A(t)$ is the area of S_t and $V(t)$ is the volume enclosed by S_t , then the formulae for the first variation are

$$A'(0) = - \int_S h H \, dA, \quad (2.1)$$

$$V'(0) = \int_S h \, dA, \quad (2.2)$$

where H is the mean curvature of S with respect to the normal N . Now it is intuitively clear, and not hard to verify, that if there exists a function h for which $V'(0) = 0$ and $A'(0) \neq 0$, then applying a similarity transformation with factor $(V/V(t))^{1/3}$ transforms the surface S_t into a surface $\hat{S}_t$ bounding a volume V and having surface area $\hat{A}(t)$ which for all small values of t will be either strictly greater than or strictly less than A , depending on the sign of t . Thus, in order for the original surface S to have minimum area among all surfaces bounding the same volume V , it must be true that whenever $\int_S h \, dA = 0$, also $\int_S hH \, dA = 0$. It follows then that H must be constant on S . Namely, if H had different values at two points then one could choose h to be zero except on small neighborhoods of each point, and to have opposite signs on these neighborhoods in such a manner that $\int_S h \, dA = 0$, but $\int_S hH \, dA > 0$. The corresponding variation would have the effect geometrically of "rounding out" the surface S , in the sense that it would pull in the surface at the point where the mean curvature was larger, and push it out a roughly equal amount at the point where it was smaller. This would preserve the volume, but decrease the surface area.

The conclusion of this argument is therefore:

LEMMA 2.1. *If a surface S has minimum area among all surfaces bounding the same volume, then the mean curvature of S must be constant.*

One is thus led to the question, "is a surface of constant mean curvature necessarily a sphere?" This question has a long and interesting history. It has a physical counterpart in the question "can a soap bubble have any other shape than a sphere?" The physical properties of soap films have as a consequence that the mean curvature of the film at each point is proportional to the difference in air pressure on the two sides. Thus a soap bubble must have a constant mean curvature determined by the difference in pressure on the inside and outside.

The first result obtained was due to Liebmann in 1900 [1]. He showed that if a compact, strictly convex surface in $\mathbf{R}^3$ has constant mean curvature, then it must be a sphere.

It may be worth remarking here that one of the differences between the isoperimetric problem in two dimensions and in higher dimensions is that in two dimensions the result for convex domains immediately implies the general result. Namely, given a nonconvex domain in the plane, its convex hull has greater area than the original domain and shorter boundary length. On the other hand, for certain nonconvex domains in $\mathbf{R}^3$, such as those with a sharp exterior spike, the convex hull has both greater volume and greater surface area.

Returning to the problem of surfaces of constant mean curvature, Heinz Hopf in 1951 [1] proved a much stronger version of Liebmann's theorem in which no convexity assumptions were needed, and in fact the surface could even be allowed to have self-intersections. The only requirement was that the surface be defined by a regular map of a 2-sphere into $\mathbf{R}^3$. A. D. Aleksandrov in 1958 [3] generalized Liebmann's theorem in a different direction. Using an ingenious geometric argument, he showed that any surface of constant mean curvature, with no assumptions on its topological type, must be a sphere. On

the other hand, the surface was not allowed to have self-intersections. In a later paper [4], Aleksandrov generalized his result in various ways, including the admission of certain surfaces with self-intersections. However, the general question of whether there may exist surfaces of higher topological type (say, like the torus) with self-intersections, and with constant mean curvature, remains unanswered.

Just in the past year a purely analytic proof of Aleksandrov's theorem has been obtained by Reilly [1], [2]. Unlike Hopf's method of proof, which uses complex variables arguments and is valid only for two dimensional surfaces, both Aleksandrov's and Reilly's proofs hold for hypersurfaces of constant mean curvature in $\mathbf{R}^n$ for all $n \geq 3$. Since the variational arguments given above extend immediately to arbitrary dimension, and since the surfaces occurring there are boundaries of domains and hence free of self-intersections, one arrives at the following result: *Suppose a hypersurface S in $\mathbf{R}^n$ has minimum $(n - 1)$ -dimensional area among all surfaces enclosing the same volume. Then it must have constant mean curvature, and hence be a sphere.*

At first glance this may seem to settle the isoperimetric problem in higher dimensions. However, on closer inspection it turns out to be essentially a strong *uniqueness* theorem. No surface *other* than a sphere can have minimum area with respect to all those enclosing the same volume. What is missing is an *existence* theorem asserting that there does indeed exist *some* surface of least area. Exactly the same objection applies to the many ingenious geometric arguments of Steiner for showing the isoperimetric property of the circle and the sphere. (For an excellent discussion of this question, see Chapter X of the book of Pólya [2].) The lack of an existence theorem was explicitly pointed out by H. A. Schwarz [1, Vol. II, p. 327], who went on to give the first complete proof of the isoperimetric inequality in $\mathbf{R}^3$. We shall not, however, pursue this line any further, but rather, we turn to an entirely different approach to the problem.

First, a general remark. If we start with a relatively smooth boundary, adding "wiggles" to it will have very little effect on the volume enclosed, but will greatly increase the surface area. Thus, one has the somewhat ironic situation that the more irregular the boundary, the stronger will be the isoperimetric inequality, but the harder it is to prove. The fact is, the isoperimetric inequality holds in the greatest generality imaginable, but one needs suitable definitions even to state it.

In the two-dimensional case, there is no problem. If the boundary curve is not rectifiable, then we may set $L = \infty$, and the inequality holds in a trivial sense. If on the other hand it is rectifiable, then by the very definition of rectifiability, its length is the limit of the lengths of approximating polygons, and one can easily derive the isoperimetric inequality in the most general case from the special case of polygons.

In higher dimensions, complications of an entirely different order arise. There are many different definitions of surface area, various ones being more suited for various purposes, and although they all agree for sufficiently smooth surfaces, they may well give different values in less standard circumstances. This problem is particularly critical in the calculus of variations, since in order to obtain a solution, one wants typically to assume the least

possible regularity to begin with, and then show that smoothness is a consequence of some extremal property. It is only by examining the consequences of a given definition of surface area that one can decide upon its appropriateness or "correctness". The validity of the isoperimetric inequality is, in fact, one criterion that has been used. See, in particular, the highly interesting papers of Besicovitch [1], [3] and Radó [2]; also Radó [3, p. 560].

The notion that is most suited for our purposes is that of the *Minkowski content*. In order to define it, we must assume that we have a well-defined notion of volume for open sets in $\mathbf{R}^n$, the obvious one being n -dimensional Lebesgue measure. We shall use the following notation:

$$\begin{aligned} V(A) &= \text{volume of the set } A, \\ B_r^n(a) &= \{x \in \mathbf{R}^n: |x - a| < r\}, & S_r^{n-1}(a) &= \{x \in \mathbf{R}^n: |x - a| = r\}, \\ B_r^n &= B_r^n(0), & S_r^n &= S_r^n(0), \\ \omega_n &= V(B_1^n). \end{aligned}$$

Further, given an arbitrary set E , define

$$E_r = \{x \in \mathbf{R}^n: \exists y \in E \ni |x - y| < r\}.$$

Thus E_r is an open set consisting of all points within distance r of E : a "thickening" or "tube-domain" about E .

For any integer k , $1 \leq k \leq n - 1$, set

$$M_k(E) = \lim_{r \rightarrow 0} \frac{V(E_r)}{\omega_{n-k} r^{n-k}}. \quad (2.3)$$

$M_k(E)$ is called the k -dimensional Minkowski content of E . More properly, it should be called the "lower outer k -dimensional Minkowski content," but we will opt for informality. For a complete discussion of its properties, and a proof that in favorable circumstances its value coincides with that obtained from a whole array of other definitions, we refer to the book of Federer [1, 3.2.37, 3.2.39, and 3.2.26].

From elementary properties of volume, it follows from $V(B_1^n) = \omega_n$, that

$$V(B_r^n) = \omega_n r^n. \quad (2.4)$$

Thus the denominator in the definition of $M_k(E)$ is the volume of a ball of radius r in $\mathbf{R}^{n-k}$. If one thinks of E as being a k -dimensional manifold, then one may think of this ball as lying in the $(n - k)$ -dimensional affine space perpendicular to the k -dimensional tangent plane to E at each point, and the definition of $M_k(E)$ is based on the idea that for small r , the volume of the tube domain E_r is approximately equal to the measure of E times the "cross-sectional area" perpendicular to E .

Let us test out the definition to compute the surface area of the sphere S_r^{n-1} . If $E = S_r^{n-1}$, then for $0 < \rho < r$, $E_\rho = B_{r+\rho}^n - B_{r-\rho}^n$, and

$$V(E_\rho) = \omega_n(r + \rho)^n - \omega_n(r - \rho)^n = \omega_n \left[2nr^{n-1}\rho + 2\binom{n}{3}r^{n-3}\rho^3 + \cdots \right].$$

Thus

$$M_{n-1}(S_r^{n-1}) = \lim_{\rho \rightarrow 0} \frac{V(E_\rho)}{\omega_1 \rho} = n\omega_n r^{n-1}, \quad (2.5)$$

since $\omega_1 = 2$.

From this we conclude that if D is the ball B_r^n , then by (2.4) and (2.5), its volume V and surface area A are given by

$$V = \omega_n r^n, \quad A = n\omega_n r^{n-1},$$

so that

$$A^n = n^n \omega_n V^{n-1}. \quad (2.6)$$

The isoperimetric inequality for domains in $\mathbf{R}^n$ then states that if D is an arbitrary domain in $\mathbf{R}^n$, its volume V and surface area A are related by

$$A^n \geq n^n \omega_n V^{n-1}, \quad (2.7)$$

with equality if and only if $D = B_r^n(a)$ for some r and a .

In accordance with our discussion above, the quantity A in (2.7) is understood to be $M_{n-1}(S)$, where S is the boundary of D . Note that no regularity assumptions whatever are made concerning S .

A very short proof of the isoperimetric inequality can be given by using the Brunn-Minkowski inequality:

$$[V(A+B)]^{1/n} \geq [V(A)]^{1/n} + [V(B)]^{1/n}, \quad (2.8)$$

for two sets A, B in $\mathbf{R}^n$, where the sum of two sets is defined by

$$A+B = \{x+y: x \in A, y \in B\}.$$

For example, the set E_r occurring in the definition of Minkowski content can be written as

$$E_r = E + B_r^n.$$

Consider now an arbitrary domain D in $\mathbf{R}^n$. Let E be its boundary and let $D_r = D + B_r^n$. Then by (2.8) and (2.4), setting $V(D) = V$,

$$\begin{aligned} V(D_r) &\geq ([V(D)]^{1/n} + [V(B_r^n)]^{1/n})^n \\ &= (V^{1/n} + [\omega_n r^n]^{1/n})^n \geq V + nV^{(n-1)/n} \omega_n^{1/n} r, \end{aligned}$$

and

$$(V(D_r) - V(D))/r \geq n\omega_n^{1/n} V^{(n-1)/n}.$$

But the numerator on the left-hand side corresponds to "half" of the set E_r ; namely, it is the part of E_r lying outside the domain D . An analogous argument leads to a similar inequality for the part of E_r lying inside D . Combining the two, and letting r tend to zero, yields

$$M_{n-1}(E) \geq n\omega_n^{1/n} V^{(n-1)/n},$$

which is the isoperimetric inequality (2.7).

For complete details of this argument, as well as a proof of the Brunn-Minkowski inequality, we refer to the book of Federer [1, 3.2.41, 3.2.43].

We conclude with several remarks. First, we note that the generalized forms of the isoperimetric inequality for self-intersecting curves also extend to self-intersecting surfaces. If S is such a surface in $\mathbf{R}^n$, its complement will be a number of domains D_k with respect to which S has a well-defined index n_k . If V_k is the volume of D_k and A the area of S , then (1.9) generalizes to

$$A^n \geq n^n \omega_n \left[\sum |n_k| V_k \right]^{n-1} \quad (2.9)$$

while (1.10) becomes

$$A^n \geq n^n \omega_n \left[\sum |n_k|^{n/(n-1)} V_k \right]^{n-1}. \quad (2.10)$$

For $n = 3$, (2.9) is proved by Radó [2, §4.7], while for arbitrary n , (2.9) and (2.10) follow from Federer and Fleming [1, p. 487].

Next we note the importance of an additional quantity:

$$M = \int_S H, \quad (2.11)$$

which arises as the first variation of area for a family of parallel surfaces to S , as one sees by setting $h \equiv -1$ in (2.1). Note that for the sphere S_r^2 , $H = 1/r$ and $M = 4\pi r$. The relevance of this quantity to the isoperimetric inequality was pointed out by Minkowski, who derived two inequalities for convex domains in $\mathbf{R}^3$:

$$A^2 \geq 3MV \quad (2.12)$$

and

$$M^2 \geq 4\pi A. \quad (2.13)$$

Combining these two, one obtains

$$A^3 \geq 36\pi V^2, \quad (2.14)$$

which is precisely the case $n = 3$ of the isoperimetric inequality (2.7), since $\omega_3 = 4\pi/3$.

Finally, we note that Pólya [1] in 1917 gave an interesting interpretation of Minkowski's inequality (2.13) using the notions of geometric probability introduced by Crofton. Given two sets E, E' in space, one can ask what is the relative probability that a random line (or a random plane) will intersect E or E' . In order to answer the question, one must define a measure on the set of all lines (or planes) and compare the measures of the subsets whose members intersect E and E' respectively. This was done by Crofton for lines in the plane, and he showed that if E and E' were domains bounded by closed convex curves C, C' (or equivalently, if E and E' were the curves themselves) then the relative probability of a random line intersecting E or E' was proportional to the relative lengths L, L' of C and C' . In his paper, Pólya [1] showed that there was a unique measure, up to a constant factor, on the set of lines in the plane, if one placed the obvious requirements on the measure, such as invariance under Euclidean motion, and that this measure is precisely the one used by Crofton. Thus by suitable normalization, the value of the measure for the set of lines intersecting a closed convex curve would equal

the length of the curve. Similarly, there is a unique measure on the set of lines in space, and suitably normalized, its value on the set of lines intersecting a closed convex surface is precisely the area of that surface. Finally, the appropriately normalized measure on the set of all planes in space gives as the measure of the subset of planes intersecting a given closed convex surface S , precisely the quantity M defined in (2.11). The interpretation of Minkowski's inequality (2.13), in view of the fact that equality holds for a sphere, is that "among all closed convex surfaces that are equally likely to be hit by an arbitrary plane, the sphere is the most likely to be hit by an arbitrary line." This property in fact *characterizes* the sphere, since equality holds in (2.13) only for the sphere.

The theory of geometric probability was later taken up by Blaschke [3] who rechristened the subject "integral geometry" and initiated a period of renewed activity. One of the main contributors was Santaló, whose results included a number of new isoperimetric inequalities, as well as new proofs of known inequalities. For details on this, as well as an overview of the entire field, see the recent book of Santaló [4].

3. Analytic inequalities. In §1 we saw that the isoperimetric inequality in the plane was equivalent to the purely analytic inequality of Wirtinger. Wirtinger's inequality holds for functions of period 2π , which may be considered as functions on the unit circle S_1^1 . There is a *generalized Wirtinger's inequality*

$$\int_{S_r^n} f = 0 \Rightarrow \int_{S_r^n} |\nabla f|^2 \geq \frac{n}{r^2} \int_{S_r^n} f^2, \quad (3.1)$$

for functions defined on an n -sphere. An equivalent form of (3.1) is obtained by considering an arbitrary function g on S_r^n and subtracting off its average value $\bar{g} = \int_{S_r^n} g / \int_{S_r^n} 1$:

$$\int_{S_r^n} |\nabla g|^2 \geq \frac{n}{r^2} \int_{S_r^n} (g - \bar{g})^2. \quad (3.1)'$$

It does not seem that these higher dimensional Wirtinger inequalities (also called in some places, *Poincaré inequalities*) can be used to derive the higher dimensional isoperimetric inequality. There are, however, interesting relations discovered recently by Chavel and Reilly. We shall return to this question at the end of §5. We turn now to another important analytic inequality with even closer ties to the isoperimetric inequality.

Let us consider first the case of a plane domain D . Then the *Sobolev inequality* states

$$f \text{ has compact support in } D \Rightarrow \left(\int_D |\nabla f| \right)^2 \geq 4\pi \int_D f^2. \quad (3.2)$$

THEOREM 3.1.² *The Sobolev inequality (3.2) is equivalent to the isoperimetric inequality*

$$L^2 \geq 4\pi A. \quad (3.3)$$

²The relation between isoperimetric inequalities and Sobolev inequalities was apparently first pointed out independently by Federer and Fleming [1, p. 487] and by V. G. Maz'ya [1, pp. 884–885].

More specifically, the validity of the isoperimetric inequality for all domains with smooth boundary implies the Sobolev inequality for arbitrary f and D , and conversely.

In one direction, the implication is almost immediate. Let D be a domain with smooth boundary C , and for small $\varepsilon > 0$ define

$$f_\varepsilon(p) = \begin{cases} 1, & \text{if } d(p, C) > \varepsilon, \\ d(p, C)/\varepsilon, & \text{if } d(p, C) < \varepsilon, \end{cases}$$

where $d(p, C)$ is the distance from p to C . Then f_ε can be approximated by smooth functions with compact support so that Sobolev's inequality will also hold for f_ε . As $\varepsilon \rightarrow 0$, $f_\varepsilon \rightarrow \chi_D$, the characteristic function of D , and

$$\int_D f_\varepsilon^2 \rightarrow \int_D 1 = A.$$

On the other hand, if $C_\varepsilon = \{p: d(p, C) < \varepsilon\}$, then

$$|\nabla f_\varepsilon| = \begin{cases} 1/\varepsilon & \text{in } D \cap C_\varepsilon, \\ 0 & \text{in } D \setminus D_\varepsilon. \end{cases}$$

Thus

$$\int_D |\nabla f_\varepsilon| = \frac{\text{Area}(D \cap C_\varepsilon)}{\varepsilon}.$$

As $\varepsilon \rightarrow 0$, the limit on the right-hand side is essentially the quantity defined in (2.3) as the Minkowski content $M_1(C)$, and for a smooth curve C it coincides with the length of C . Thus (3.2) $\Rightarrow$ (3.3).

To obtain the reverse inequality, the idea is to carry out the integration first along the level curves of f , and then with respect to the parameter defining the level curves. We introduce the following notation:

Let

$$D(t) = \{(x, y) \in D: |f(x, y)| > t\}, \quad A(t) = \text{Area}(D(t)),$$

$$C(t) = \{(x, y) \in D: |f(x, y)| = t\}, \quad L(t) = \text{length}(C(t)),$$

s = parameter of arc length along $C(t)$,

σ = parameter of arc length along an orthogonal trajectory

to the family $\{C(t)\}$, increasing with t .

Note that by Sard's Theorem, the set of singular values t for which $\nabla f = 0$ somewhere on $C(t)$ has measure zero. For all other values of t , $C(t)$ is a regular level set consisting of a finite number of smooth curves that together bound the domain $D(t)$. Note that

$$|\nabla f| = |\partial f / \partial \sigma| = dt / d\sigma,$$

so that the area element in D has the form

$$dx \, dy = ds \, d\sigma = |\nabla f|^{-1} ds \, dt$$

in a neighborhood of each point on a regular curve $C(t)$. It follows that for

an arbitrary function $h(x, y)$,

$$\iint_D h(x, y) |\nabla f| \, dx \, dy = \int_0^\infty \left[\int_{C(t)} h \, ds \right] dt. \quad (3.4)$$

This is sometimes called the *coarea formula*.

In the case of interest to us, $h \equiv 1$, and

$$\iint_D |\nabla f| \, dx \, dy = \int_0^\infty L(t) \, dt. \quad (3.5)$$

Thus the left-hand side of the Sobolev inequality represents precisely an integrated length of level curves of f . Applying the isoperimetric inequality to each of the domains $D(t)$, yields

$$\iint_D |\nabla f| \, dx \, dy \geq 2\sqrt{\pi} \int_0^\infty \sqrt{A(t)} \, dt. \quad (3.6)$$

We next express the right-hand side of the Sobolev inequality in a similar manner. If one considers the domain defined by $0 < t < |f(x, y)|$ in 3-dimensional x, y, t -space, and integrates two different ways over this domain, one finds

$$\begin{aligned} \iint_D f^2 \, dx \, dy &= \iint_D \left[\int_0^{|f(x, y)|} 2t \, dt \right] dx \, dy \\ &= \int_0^\infty 2t \left[\iint_{t < |f(x, y)|} 1 \, dx \, dy \right] dt = \int_0^\infty 2t A(t) \, dt. \end{aligned} \quad (3.7)$$

It remains to compare the right-hand sides of (3.6) and (3.7). But since $A(t)$ is a decreasing function of t , one has

$$\begin{aligned} t\sqrt{A(t)} &\leq \int_0^t \sqrt{A(\tau)} \, d\tau, \\ tA(t) &\leq \sqrt{A(t)} \int_0^t \sqrt{A(\tau)} \, d\tau = \frac{1}{2} \frac{d}{dt} \left[\int_0^t \sqrt{A(\tau)} \, d\tau \right]^2, \\ \int_0^\infty 2tA(t) \, dt &\leq \left[\int_0^\infty \sqrt{A(t)} \, dt \right]^2. \end{aligned} \quad (3.8)$$

Combining (3.6), (3.7) and (3.8) gives Sobolev's inequality (3.2).

COROLLARY.

$$\inf_{f \in \mathcal{F}_0} \frac{(\int_D |\nabla f|)^2}{\int_D f^2} = 4\pi, \quad (3.9)$$

where $\mathcal{F}_0$ is the family of smooth functions with compact support in D .

Our proof of the equivalence of (3.2) and (3.3) showed that whatever constant worked in one inequality would also hold for the other. Thus, the left-hand side of (3.9) could not be smaller than 4π , since that would imply the isoperimetric inequality with a constant less than 4π , which is false.

We are now in a position to explain why the *squares* of the winding

numbers occur in the Banchoff-Pohl inequality (1.10). Given a closed plane curve C , let f be the function whose value at each point in the complement of C is the winding number of C with respect to the point. Then the right-hand side of the Sobolev inequality coincides with the right-hand side of (1.10). Since crossing a simply-traversed arc of the curve C has the effect of changing the value of f by one, an approximation argument analogous to that used in Theorem 3.1 yields again L for the limiting value of $\int |\nabla f_k|$. Thus (3.2) implies (1.10). The coefficients n_k^2 in (1.10) occur because it is the L -two norm of f that is bounded above (by the L_1 norm of $|\nabla f|$) in the Sobolev inequality (3.2).

It is worth noting that the value 4π that enters in (3.9) is totally independent of the domain D . The situation is radically different if one considers a slightly different quotient, by taking the square in the numerator inside the integral sign.

THEOREM 3.2. *If*

$$\inf_{f \in \mathfrak{F}_1} \frac{\int_D |\nabla f|^2}{\int_D f^2} = \lambda_1, \quad (3.10)$$

where $\mathfrak{F}_1$ is the set of piecewise smooth functions in D vanishing on the boundary, then λ_1 is the smallest eigenvalue of the equation

$$\Delta f + \lambda f = 0 \quad (3.11)$$

for solutions having zero boundary values.

This is a well-known result of partial differential equations, and can be found for instance, in the book of Garabedian [1, Chapter 11].

The quotient on the left of (3.10) is called the *Rayleigh quotient*. Unlike the left-hand side of (3.9), it is not dimensionally invariant. If one applies a similarity transformation to the domain D , multiplying distances by a factor h , then the left-hand side of (3.10) is divided by h^2 . Thus, to understand the dependence of λ_1 on the domain D , it is sufficient to normalize by fixing for example the area of D . One then has the result

THEOREM 3.3. *Among all domains D having fixed area, the left side of (3.10) attains a minimum if and only if D is a circular disk.*

The interest in this theorem derives from the physical interpretation of the quantity λ_1 . Equation (3.11) arises from separating space and time variables in the wave equation. If a homogeneous stretched membrane has the shape of the domain D , and is attached at the boundary, then solutions of (3.11) with zero boundary values represent the amplitude of vibrations of the membrane with frequency $\sqrt{\lambda}$. The eigenvalues λ_n are thus the squares of the frequencies of free vibration of the membrane, and the quantity λ_1 given by (3.10) corresponds to the lowest frequency, or the "fundamental tone" of the membrane.

The first statement of Theorem 3.3 is due to Rayleigh in his fundamental treatise *The theory of sound* [1, §210]. He writes, "If the area of a membrane be given, there must evidently be some form of boundary for which the pitch

(or the principal tone) is the gravest possible, and this form can be no other than the circle." By way of evidence, he offers a variational argument showing that the first variation of λ_1 is positive under variations that start with a circular domain and vary it keeping the area constant. He also lists a number of special domains for which λ_1 can be computed, such as various rectangles, triangles, and circular sectors, and it seems apparent that the more the domains deviate from circularity, the higher the value of λ_1 .

The first actual proofs of Theorem 3.3 were given by Faber [1] and Krahn [1], and the theorem itself is generally referred to as the Faber-Krahn theorem. Its proof makes use of the standard isoperimetric inequality $L^2 \geq 4\pi A$, in a manner analogous to the proof of the Sobolev inequality given above, together with the technique of symmetrization. (See, for example, Garabedian [1, p. 413].)

Rayleigh also has some discussion of the case of nonhomogeneous membranes, corresponding to a drum made of material of variable density. The Faber-Krahn theorem was generalized by Nehari [1] to membranes of variable density $p(x, y)$ provided that $\log p(x, y)$ is a subharmonic function. Specifically, among all such membranes with the same total mass, the minimum of λ_1 is attained for a circular membrane with constant density.

Nehari's proof follows along similar lines to that of Faber and Krahn, but then reduces to showing that for simply connected subdomains G of D with boundary curve C ,

$$\left(\int_C \sqrt{p} \, ds \right)^2 \geq 4\pi \int_G p \, dx \, dy.$$

(For p constant, this is the familiar $L^2 \geq 4\pi A$.) By taking the least harmonic majorant of $\log p$, and completing to an analytic function $f(z)$, the above inequality follows from

$$\left(\int_C |f(z)| \, ds \right)^2 \geq 4\pi \int_G |f(z)|^2 \, dx \, dy,$$

where $f(z)$ is analytic in G and different from zero in $G \cup C$. (We shall meet this same inequality in a different context in (4.10) below.) Nehari concludes the proof by setting $f(z) = F'(z)$, and observing that this inequality then reduces to the generalized isoperimetric inequality (1.8) for self-intersecting curves, applied to the image curve $F(C)$ and the multiply-covered image domain $F(G)$.

There are many other results of a similar nature, referred to as isoperimetric inequalities of mathematical physics, where extrema are sought for various quantities of physical significance. We shall go into somewhat more detail about this type of problem in §6 below.

There are several remarks worth making about the particular extremal problem (3.10).

First, in the one-dimensional case, where D is an interval, $a < x < b$, equation (3.11) becomes $f'' + \lambda f = 0$, and the solutions with zero boundary values are $\sin n\pi(x - a)/(b - a)$, for all integers n . The corresponding eigenvalues are $\lambda_n = [n\pi/(b - a)]^2$. Thus (3.10) becomes

$$\inf_{f(a)=f(b)=0} \frac{\int_a^b (f')^2 dx}{\int_a^b f^2 dx} = \lambda_1 = \left(\frac{\pi}{b-a} \right)^2. \quad (3.12)$$

This is clearly a close relative of (1.4), and it is also referred to as Wirtinger's inequality. It may also be used in essentially the same way that we used (1.4) to give another proof of the isoperimetric inequality in the plane. (See Lewy [1, p. 41].)

In the one-dimensional case, the physical model is the vibrating string. By the remarks above, the fundamental frequency is π/l , where l is the length of the string, and all the other frequencies are integer multiples of that one.

In the two-dimensional case, the set of frequencies obeys no such simple law, and in fact, depends in a complicated way on the shape of the domain D . A problem much studied in recent years is the "isospectral problem". Can two different domains (i.e.—not congruent) have the same set of λ_n , or equivalently, the same set of frequencies? This problem has also been given the picturesque name: "Can you hear the shape of a drum?" One approach has been to try to find expressions for geometric properties of D in terms of the λ_n . Among those that have been so expressed are the connectivity of D , the length of its boundary, and its area. It follows, for example, that a multiply-connected domain can never have the same set of frequencies as a simply-connected one. It follows further that from the frequencies one can immediately determine if the drum is circular: evaluate L , evaluate A and check whether $L^2 = 4\pi A$!

Incidentally, this is the *only* case so far in which the isospectral problem for plane domains has been settled. For more details on this question we refer to the papers of Kac [1], Berger [1], [3].

Let us conclude by noting the following extensions from two dimensions to higher dimensional euclidean spaces. Let D be a domain in $\mathbf{R}^n$.

1. The Sobolev inequality takes the form

$$\inf_{f \in \mathfrak{G}_0} \frac{(\int_D |\nabla f|)^n}{(\int_D |f|^{n/(n-1)})^{n-1}} = n^n \omega_n; \quad (3.13)$$

the equivalence to the isoperimetric inequality in $\mathbf{R}^n$ follows along similar lines to the 2-dimensional proof. (See for example, Federer and Fleming [1, p. 487], or Bombieri [1, p. 17].) For further aspects of the relationship between isoperimetric and Sobolev inequalities see Aubin [1], [3], Maz'ya [1], [2], and Talenti [1].

2. The Rayleigh quotient

$$\inf_{f \in \mathfrak{G}_1} \frac{\int_D |\nabla f|^2}{\int_D f^2} = \lambda_1 \quad (3.14)$$

again gives the lowest eigenvalue of the problem $\Delta f + \lambda f = 0$ in D , $f = 0$ on the boundary. Krahn [2] showed by means of the n -dimensional isoperimetric inequality (2.7), that among all domains D with given volume, the sphere provides the smallest value of λ_1 .

3. Wirtinger's inequality, Lemma 1.1, has the following analog in n -dimensions. Let D be a convex domain in $\mathbf{R}^n$, and let d be the diameter of D .

Then

$$\int_D f = 0 \Rightarrow \int_D |\nabla f|^2 > \left(\frac{\pi}{d}\right)^2 \int_D f^2. \quad (3.15)$$

The constant $(\pi/d)^2$ cannot be replaced by any smaller one that is valid for all convex domains.

This result is due to Payne and Weinberger [1]. (See also Chavel and Feldman [1].) To place it in the context of our earlier inequalities, let us denote by $\mathfrak{F}_2$ the family of functions in D satisfying $\int_D f = 0$, and let

$$\inf_{f \in \mathfrak{F}_2} \frac{\int_D |\nabla f|^2}{\int_D f^2} = \nu_1. \quad (3.16)$$

Then ν_1 is the first nontrivial eigenvalue of the "free membrane problem":

$$\Delta f + \nu f = 0 \quad \text{in } D, \quad \partial f / \partial n = 0 \quad \text{on } \partial D. \quad (3.17)$$

The first eigenvalue is $\nu_0 = 0$, corresponding to the nonzero constant functions. The family $\mathfrak{F}_2$ consists of those functions orthogonal to the first eigenspace of constant functions. The fact that $\nu_1 > 0$ guarantees that for any domain D one has a "Poincaré inequality":

$$\int_D f = 0 \Rightarrow \int_D |\nabla f|^2 > \nu_1 \int_D f^2, \quad (3.18)$$

where ν_1 depends only on D . When D is convex, the Payne-Weinberger result (3.15) asserts that $\nu_1 > (\pi/d)^2$.

4. The isoperimetric inequality on surfaces. Most histories of the isoperimetric problem begin with its legendary origins in the "Problem of Queen Dido". Her problem (or at least one of them) was to enclose an optimal portion of land using a leather thong fashioned from oxhide. I only mention it here to point out that if Dido's was the true original isoperimetric problem, then all that we have said so far is irrelevant, since what is wanted is a solution not in the plane, but on a curved surface.

An interesting solution to the isoperimetric problem for curves on the sphere was given by F. Bernstein in 1905 [1]. Before stating his result, let us see what form one should expect the isoperimetric inequality to take on a sphere. Given a spherical cap, we want to find a relation between its area and the length of the boundary circle. Concerning the area, it is convenient to remember that on a given sphere the area of a zone cut out by a pair of parallel planes depends only on the distance between the planes. It follows that the area is proportional to the distance h between the planes; $A = ch$. If the sphere has radius R , then when $h = 2R$ one gets the full area of the sphere: $A = 4\pi R^2$. Hence $c = 2\pi R$, and the area of any zone is $A = 2\pi Rh$. Applying this to the case where one plane is tangent to the sphere, we obtain a circular cap whose boundary circle lies in a plane and has a radius there equal to the mean proportional between h and $2R - h$. Thus its length is $L = 2\pi\sqrt{h(2R - h)}$, and we have the relation

$$L^2 = 4\pi A - A^2/R^2 \quad (4.1)$$

for a circular cap on a sphere of radius R .

Thus, $L^2 - 4\pi A$ is *not* nonnegative on a sphere. However, if the circle is to

have smallest length among all curves bounding a fixed area, then the isoperimetric inequality on the sphere should take the form

$$L^2 \geq 4\pi A - A^2/R^2 \quad (4.2)$$

with equality only for a circle.

What Bernstein proved was the inequality

$$L^2 - 4\pi A + A^2/R^2 \geq (2Rg(R))^2(2\pi + g(R)^2), \quad (4.3)$$

for convex curves on a sphere of radius R , where

$$g(R) = \sin[d/4(1 + 2\pi)R],$$

and d is the minimum width of circular annuli on the sphere containing the given curve. Since the right-hand side of (4.3) is nonnegative, (4.2) is an immediate consequence.

An unexpected additional feature of Bernstein's result is that it yields a brand new relation for convex curves in the plane by simply using the inequality (4.3) and letting the radius R of the sphere tend to infinity:

$$L^2 - 4\pi A \geq cd^2 \quad (4.4)$$

where c is a positive constant, and d is the width of the narrowest circular annulus containing the curve.

The left-hand side of (4.4) is called the *isoperimetric deficit* of the curve. It provides a measure of how far the curve deviates from a circle. Inequality (4.4) accomplished three things simultaneously:

- (i) it shows $L^2 \geq 4\pi A$ for every curve;
- (ii) it shows that $L^2 = 4\pi A$ only when $d = 0$ and the curve is a circle;
- (iii) it gives a quantitative estimate of the isoperimetric deficit for any given curve.

This is in contrast to most of the nineteenth-century proofs, which, as we mentioned earlier, needed two separate arguments: one, such as Steiner's or the calculus of variations argument, to show that *only* the circle could be an extremum, and a separate one to show that an extremum exists.

Starting in 1921, Bonnesen wrote a series of papers proving inequalities like (4.4). He showed that (4.4) holds with $c = 4\pi$, and that this is the best possible constant. Other inequalities of this type obtained by Bonnesen are

$$L^2 - 4\pi A \geq (L - 2\pi\rho)^2, \quad (4.5)$$

$$L^2 - 4\pi A \geq (L - 2\pi R)^2, \quad (4.6)$$

and, as a consequence of these two,

$$L^2 - 4\pi A \geq \pi^2(R - \rho)^2. \quad (4.7)$$

The quantities ρ and R represent the radii of inscribed and circumscribed circles, respectively, for the curve C . Bonnesen proved inequalities (4.5)–(4.7) for convex curves only, but they are in fact true for arbitrary rectifiable Jordan curves. For these, and many related results, see the book of Bonnesen [2] and the recent paper of Osserman [4].

To return to domains on the sphere, note that inequality (4.2) can also be

written as

$$L^2 > A\bar{A}/R^2, \quad (4.8)$$

where $\bar{A} = 4\pi R^2 - A$ is the area of the complementary domain on the sphere. In fact, if one starts with a simple closed rather than a domain, then to speak of "the area bounded by the curve" is not meaningful, since it divides the sphere in two parts, with areas A and $\bar{A} = 4\pi R^2 - A$. The meaning of (4.8) is that if $L < 2\pi R$, then the smaller of A and $\bar{A}$ is maximized when the curve is a circle of length L . (If $L > 2\pi R$, then (4.8) holds trivially, since the right-hand side is never more than $(2\pi R)^2$. In that case (4.8) does not imply any bounds on A and $\bar{A}$, which is as it should be, since when $L > 2\pi R$, one may choose any positive numbers $A, \bar{A}$, whose sum is $4\pi R^2$, and there will exist infinitely many distinct curves of length L dividing the sphere into two parts with areas A and $\bar{A}$.)

Since a closed curve on the surface of the sphere always bounds at least two domains, it seems even more natural than in the plane to allow the curve to have self-intersections and ask how its length is related to the areas of the various domains into which it divides the sphere. This has been done recently by Weiner [1]. Using the techniques of Banchoff and Pohl, he obtains an inequality analogous to (1.10) involving the squares of winding numbers, such that when the curve is free of self-intersections, the inequality reduces to (4.8).

The consideration of the isoperimetric problem on curved surfaces goes quite a way back, at least to an 1842 paper of Steiner [1]. If one assumes that a smooth closed curve on a surface bounds a domain of maximum area among all curves of the same length, then a calculus of variations argument, analogous to the one we have given in §2, shows that the curve must have constant geodesic curvature. This fact is mentioned in Steiner's paper [1, p. 150], and a proof was given in 1878 by Minding [1]. A detailed discussion is given in §18 of an extraordinary paper of Erhard Schmidt [4]. This paper provides an extended analysis of the isoperimetric problem on surfaces. In particular, in §17 Schmidt gives conditions under which there do not exist any simple closed curves of constant geodesic curvature. It follows from the above variational argument that for such surfaces, the isoperimetric problem does not have a solution, in the sense that there is no domain that has maximal area for fixed boundary length. On the other hand, one may still be able to find an isoperimetric inequality giving an upper bound for the area of a domain in terms of the length of its boundary. The remainder of this section is devoted to that formulation of the problem. It turns out that for many applications it is precisely that sort of a bound that is needed, rather than the existence of an extremal domain.

It is convenient to consider separately two parts of the general problem.

Part I. Find those classes of surfaces and domains on them for which the classical isoperimetric inequality

$$L^2 - 4\pi A > 0 \quad (4.9)$$

remains valid.

Part II. In those cases where (4.9) does not hold, find appropriate analogs, such as (4.2) for the sphere.

We begin with the first of these questions.

Part I. The first result of this nature is due to Carleman [2], who showed in 1921 that (4.9) holds for a simply-connected domain on a minimal surface. His proof uses complex function theory. Note that if $F(z)$ maps the unit disk conformally onto a domain D , then the area A of D and its boundary length L are given by

$$L = \int_0^{2\pi} |f(e^{i\theta})| d\theta, \quad A = \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2 r dr d\theta,$$

where $f(z) = F'(z)$. Thus the isoperimetric inequality (4.9) implies

$$\left[\int_0^{2\pi} |f(e^{i\theta})| d\theta \right]^2 \geq 4\pi \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2 r dr d\theta, \quad (4.10)$$

for functions f of the type considered. Carleman turned this around, and gave a direct proof of (4.10) for arbitrary analytic functions $f(z)$.

A simply-connected minimal surface in $\mathbf{R}^n$ is given by a map of the unit disk into $\mathbf{R}^n$ for which each coordinate function x_k is harmonic and has the further property that the functions

$$\varphi_k(z) = \partial x_k / \partial x - i \partial x_k / \partial y, \quad z = x + iy, \quad (4.11)$$

are analytic and satisfy

$$\sum_{k=1}^n [\varphi_k(z)]^2 \equiv 0. \quad (4.12)$$

The area A of the image and length L of its boundary are given by

$$L = \int_0^{2\pi} \sqrt{\frac{1}{2} \sum_{k=1}^n |\varphi_k(e^{i\theta})|^2} d\theta \quad (4.13)$$

$$A = \int_0^1 \int_0^{2\pi} \frac{1}{2} \sum_{k=1}^n |\varphi_k(re^{i\theta})|^2 r dr d\theta. \quad (4.14)$$

Applying (4.10) to each φ_k , summing over k , and then applying the Minkowski inequality (see, for example, Hardy, Littlewood and Pólya [1, p. 146])

$$\sum \left(\int g_k^{1/2} \right)^2 \leq \left(\int \left[\sum g_k \right]^{1/2} \right)^2,$$

with $g_k = |\varphi_k|^2$, gives (4.9).

A quite different argument, due to Blaschke [2, p. 247] gives some insight into why one might expect the classical isoperimetric inequality to hold also on minimal surfaces. Consider a surface S in $\mathbf{R}^3$ that has least area among all surfaces with the same boundary curve C . Let its area be A and its boundary length L . Let S' be the cone over C with vertex at some point of C . Then S' has the same boundary as S , and hence its area A' is not less than A . But since it is a cone, S' can be developed onto a plane domain D , preserving its area and the length of its boundary. By the classical isoperimetric inequality in the plane, $L^2 \geq 4\pi A' \geq 4\pi A$. This argument is certainly a pretty one, but

should be regarded as more heuristic than rigorous. For one thing, there is no reason to assume that S' will develop onto a simply covered plane domain, so at the least one would have to invoke a generalized inequality for self-interesting curves in the plane, such as (1.9). In any case, it is valid only for area-minimizing surfaces, while Carleman's holds for arbitrary minimal surfaces.

In 1933 Beckenbach and Radó [1] generalized Carleman's inequality (4.10), showing that it was valid if in place of $|f|$, one used any non-negative function whose logarithm is subharmonic. As a consequence they showed that the isoperimetric inequality (4.9) holds for simply-connected domains on arbitrary surfaces of nonpositive Gauss curvature. But the converse of this statement was already known: if $L^2 - 4\pi A > 0$ for all simply-connected domains on a surface, then the Gauss curvature K of the surface can never be positive. Namely, given any point p of the surface, let $L(r)$ be the length of the geodesic circle of radius r centered at p , and let $A(r)$ be the area of the corresponding disk. One has the asymptotic formulas

$$\begin{aligned} L(r) &= 2\pi r - \frac{\pi}{3} K(p)r^3 + O(r^4), \\ A(r) &= \pi r^2 - \frac{\pi}{12} K(p)r^4 + O(r^5), \end{aligned}$$

(see, for example, doCarmo [1, p. 292] or Blaschke and Leichtweiss [1, p. 204]) from which it follows that

$$K(p) = -\lim_{r \rightarrow 0} \frac{L(r)^2 - 4\pi A(r)}{\pi^2 r^4}. \quad (4.15)$$

We thus conclude

THEOREM 4.1 (BECKENBACH AND RADÓ). *Let S be a surface with Gauss curvature K . Necessary and sufficient that $L^2 - 4\pi A > 0$ for all simply-connected domains on S is that $K \leq 0$ everywhere on S .*

It would seem at first glance that this provides a complete solution to Part I of our program, but that is not the case for several reasons. First of all, this theorem applies to regular surfaces, and it is often important to allow surfaces with various kinds of singularities. Shiffman [1] has shown that the inequality $L^2 > 4\pi A$ continues to hold on two important classes of surfaces with singularities: polyhedral surfaces whose behavior at vertices generalizes the condition $K \leq 0$, and harmonic surfaces.

More important still, the restriction to simply-connected domains is not a natural one. For example, a simple closed curve in $\mathbb{R}^3$ may bound a minimal surface of the type of a Möbius strip, or a genus-one domain on a torus. One would like to know if $L^2 > 4\pi A$ also in those cases. Another important case arises when a portion of surface is cut off by a sphere and one wants area bounds for the part of the surface lying inside the sphere. That part may have arbitrary topological type and be bounded by one or more separate curves lying on the sphere. Area bounds of this sort can be obtained by means of isoperimetric inequalities. (See Alexander, Hoffman, and Osserman [1, p. 452] and the discussion in §6B below.)

The first point worth noting when one drops simple connectivity is that the inequality $L^2 > 4\pi A$ does *not* hold for general domains on surfaces satisfying $K < 0$. For example, on a cylinder (where $K \equiv 0$), the length of each boundary circle is $2\pi r$ and the area is $2\pi rh$. Thus the area can be made arbitrarily large by increasing h , while the total boundary length $L = 4\pi r$ remains fixed. Also, one can construct an example with a single boundary curve using a flat torus obtained by identifying opposite pairs of sides on a rectangle. The complement of a small disk is a domain of genus one on the torus bounded by a single curve whose length L can be made as small as one wishes.

In view of these examples it is something of a surprise that similar examples cannot be constructed within the class of minimal surfaces. One has the following results:

THEOREM 4.2. *The isoperimetric inequality $L^2 \geq 4\pi A$ holds for domains D lying on minimal surfaces in $\mathbf{R}^n$ in the following cases:*

- (i) *the boundary of D consists of a single rectifiable Jordan curve;*
- (ii) *D is doubly connected and is bounded by two rectifiable curves;*
- (iii) *D is bounded by a finite number of rectifiable curves lying on a sphere centered at a point of D ;*
- (iv) *D is bounded by a finite number of rectifiable Jordan curves, and minimizes area among all surfaces with the same boundary.*

A number of different methods are needed for proving the various parts of this theorem, but there is one basic formula that serves as a common starting point. In the case $n = 3$, the formula goes back at least to Schwarz [1, Vol. I, p. 329]. It is

$$2A = -2 \iint_D (x - c) \cdot H dA + \int_C (x - c) \cdot \nu ds, \quad (4.16)$$

where D is a domain on an oriented surface in $\mathbf{R}^n$, A is the area of D , C its boundary, ν the unit exterior normal to D , H the mean curvature vector³ of the surface, and c is an arbitrary point in $\mathbf{R}^n$. Note that C may consist of one or more curves. For minimal surfaces, $H \equiv 0$ and the first term on the right vanishes, so that the area is expressed by a boundary integral. When C consists of a single curve, then choosing $c \in C$ and making an adept application of Wirtinger's inequality (3.12), one can show that the absolute value of the boundary integral is at most equal to $L^2/2\pi$, with equality only if C is a circle and ν the normal vector field to the circle in the plane. This proves part (i) in the case of an oriented surface. This proof is due to Reid [1] for $n = 3$, and was extended by Hsiung [1] to arbitrary n . (See also Osserman [1] for further details of this proof, and other uses of formula (4.16).)

It has just recently been observed by Chavel [1] that one can arrive at the same result much more easily by using the other form of Wirtinger's

³For a discussion of the mean curvature vector and its geometric significance in conjunction with variation of area see, for example, §1 of Osserman, *Minimal varieties*, Bull. Amer. Math. Soc. 75 (1969), 1092–1120. See also Alexander, Hoffman and Osserman [1, §3], for a proof of (4.16) on submanifolds of arbitrary dimension.

inequality, Lemma 1.1. (See also, a closely related paper of Reilly [3].)* In fact, following the method used in §1, we may choose the parameter $t = 2\pi s/L$ on the boundary curve C , and choose the center of gravity of C as the origin of coordinates. This means that

$$\int_0^{2\pi} x_k(t) dt = \frac{2\pi}{L} \int_C x_k ds = 0, \quad k = 1, \dots, n,$$

and by Lemma 1.1,

$$\int_0^{2\pi} [x'_k(t)]^2 dt \geq \int_0^{2\pi} [x_k(t)]^2 dt, \quad k = 1, \dots, n.$$

Summing over k gives

$$\frac{L^2}{2\pi} = \int_0^{2\pi} \left(\frac{ds}{dt} \right)^2 dt \geq \int_0^{2\pi} |x|^2 dt = \frac{2\pi}{L} \int_0^L |x|^2 ds.$$

Using this inequality together with Schwarz' inequality and (4.16) (with $c = 0$), we find

$$2A = \int_0^L x \cdot \nu ds \leq \int_0^L |x| ds \leq \sqrt{L \int_0^L |x|^2 ds} \leq \frac{L^2}{2\pi}.$$

This proves (i) when the surface is oriented. If D is nonorientable, one may apply the same argument to the two sheeted oriented covering surface of D , whose boundary consists of the original boundary curve of D described twice: once in each direction.

The proof of part (ii) depends on a more detailed study of doubly-connected minimal surfaces. Let such a surface be given by a map F of an annulus

$$r_1 < |z| < r_2$$

into $\mathbf{R}^n$, where F is assumed to extend continuously to the boundary circles and to map them onto Jordan curves C_1, C_2 . On the interior F has the behavior described in (4.11) and (4.12). Let $L(r)$ be the length of the image of the circle $|z| = r$, for $r_1 < r < r_2$. The only case of interest is that where C_1 and C_2 are rectifiable, of length L_1, L_2 say. In that case, $\lim_{r \rightarrow r_j} L(r) = L_j$, $j = 1, 2$. (See Nitsche [3, p. 517] and Feinberg [2].) The key lemma for the case $n = 3$ is the following:

THEOREM 4.3. *The function $L(r)$ satisfies*

$$d^2L/d(\log r)^2 \geq L \tag{4.17}$$

with equality possible in only two cases: if F is a conformal map onto a plane annulus, or if the image of F is a catenoid bounded by a pair of coaxial circles in parallel planes.

Using this inequality together with (4.16) and the specific expressions for $L(r)$ on a catenoid, one finds that not only $L^2 > 4\pi A$, but a stronger

*See also a paper by Chakerian [1] that has just appeared. Using a variant of the same argument, Chakerian proves the inequality

$$L^2 - 4\pi A \geq \frac{2\pi^2}{L} \int_C \left| x - \frac{L}{2\pi} \nu \right|^2 ds.$$

inequality:

$$L^2 - 4\pi A > 2L_1L_2(1 - \log 2). \quad (4.18)$$

(See Osserman and Schiffer [1, p. 297].)

The proof of (4.17) depends on the representation theorem for minimal surfaces in $\mathbb{R}^3$, and does not go through for $n > 3$. The proof that $L^2 > 4\pi A$ holds for all n was carried out by Feinberg [1] as follows. He first notes that a weaker form of (4.17) will yield a weaker form of (4.18). Specifically,

Step 1. The inequality

$$d^2L/d(\log r)^2 > \alpha^2 L \quad (4.19)$$

for $r_1 < r < r_2$ implies

$$L^2 - 4\pi A > 4\pi^2 [\cosh \alpha t_1 \cosh \alpha t_2 - (t_2 - t_1)], \quad t_j = \log r_j. \quad (4.20)$$

Step 2. If we denote the right-hand side of (4.20) by $4\pi^2 F(t_1, t_2)$, then

$$\min_{t_1, t_2} F(t_1, t_2) = \frac{1}{\alpha} \left(\alpha + \sqrt{\alpha^2 + 1} - \sinh^{-1} \frac{1}{\alpha} \right). \quad (4.21)$$

Step 3. The right-hand sides of (4.21) is positive if and only if $\alpha^2 > \beta^2$, where β is the unique solution of

$$\sinh(\beta + \sqrt{\beta^2 + 1}) = 1/\beta.$$

Step 4. The numerical value of β^2 in Step 3 is

$$\beta^2 \sim 0.20047,$$

whereas (4.19) can be shown to hold with the value

$$\alpha^2 = 2/\pi^2 \sim 0.20264.$$

Combining these four steps yields the inequality $L^2 > 4\pi A$ for all doubly-connected minimal surfaces in $\mathbb{R}^n$.

The key step is thus the proof of (4.19) with the value $\alpha^2 = 2/\pi^2$. Feinberg does this by proving an analog of the Wirtinger inequality, Lemma 1.1, but "without the squares". Namely

$$\int_0^{2\pi} y(t) dt = 0 \Rightarrow \int_0^{2\pi} |y'(t)| dt > \frac{2}{\pi} \int_0^{2\pi} |y(t)| dt. \quad (4.22)$$

Note that unlike (1.4), the inequality here is a strict one. However, the constant $2/\pi$ is best possible, an observation of some interest in view of the remarkably little room for maneuver in Step 4 of Feinberg's argument.

For proofs of parts (iii) and (iv) of Theorem 4.2, see Alexander, Hoffman, and Osserman [1, p. 453], or Osserman [1, pp. 213–214].

That is the current state of progress in the problem we have considered in Part I of this section. Before going on to Part II, I would like to mention some questions still to be settled.

1. Does the inequality $L^2 \geq 4\pi A$ hold for all domains on minimal surfaces in $\mathbb{R}^n$, regardless of topological type?

2. Does $L^2 \geq 4\pi A$ hold for surfaces in $\mathbf{R}^3$ with $K \leq 0$ and bounded by a single curve?⁴

3. More generally, does $L^2 \geq 4\pi A$ hold for surfaces in $\mathbf{R}^n$ satisfying the convex hull property⁵ and bounded by a single curve?

4. Does inequality (4.17) hold for doubly-connected minimal surfaces in $\mathbf{R}^n$ for all n , and if not, what is the best value of α in (4.19)?

Part II. Generalized inequalities on arbitrary surfaces. To see what one might expect in general to replace the inequality $L^2 \geq 4\pi A$, let us return to the case of the sphere, with which we started the section. Since the Gauss curvature of a sphere of radius R is given by $K = 1/R^2$, we may write the isoperimetric inequality (4.2) for the sphere in the form

$$L^2 \geq 4\pi A - KA^2. \quad (4.23)$$

In fact, in this form it is valid both for the sphere and for the plane, where $K \equiv 0$. One might guess that it would hold equally for the hyperbolic plane, where $K \equiv -1$, and this turns out to be the case. A proof was given in 1940 by Schmidt [2, p. 209].

Thus, (4.23) is the precise form of the isoperimetric inequality for all surfaces of constant curvature, with equality in each case only for geodesic circles. As a consequence, one sees that the effect of the Gauss curvature is to decrease the value of $L^2 - 4\pi A$ as K increases. For surfaces of variable curvature this effect has been expressed in two different forms:

THEOREM 4.3. *For a simply-connected domain D on a surface, one has the two inequalities*

$$L^2 \geq 4\pi A - 2 \left[\int \int_D K^+ \right] A, \quad (4.24)$$

$$L^2 \geq 4\pi A - \left[\sup_D K \right] A^2, \quad (4.25)$$

where $K^+(p) = \max\{K(p), 0\}$.

Inequalities (4.24) and (4.25) have been proved, using a variety of different methods, by Fiala [1], Bol [1], Schmidt [4], Aleksandroff [1], [2], Huber [1], Toponogov [1], Karcher [2], Bandle [4], Aubin [1], [3], and Chavel-Feldman [2]. (See also Barbosa-doCarmo [3].)

Fiala [1] appears to have been the first to prove a general isoperimetric inequality for surfaces of variable Gauss curvature. He proved (4.24) in the real analytic case, when $K \geq 0$, using an argument based on parallel curves. Bol [1], also using parallel curves, gives a proof of both (4.24) and (4.25), without the restriction that $K \geq 0$ in (4.24). Schmidt [4, p. 618], proves (4.25) for rotationally symmetric metrics using methods of the calculus of variations. Aleksandrov [1] proves (4.25), and later [2, p. 509], states that (4.24)

⁴The answer to this question turns out to be "no". E. Calabi and M. Gage have provided counterexamples. In particular, Gage gives examples of surfaces of Euler characteristic χ such that L^2/A tends to zero as $|\chi|$ tends to infinity. That leads to a refinement of question 2: "Is there a lower bound for L^2/A depending on χ ?" The same modification should be made in question 3.

⁵For a definition and geometric characterization of the convex hull property, see Osserman, *The convex hull property of immersed manifolds*, J. Differential Geometry 6 (1971), 267-270.

can be proved for very general surfaces using his methods of polyhedral approximation. A paper by Aleksandrov and Strel'cov [1], [2] gives a complete proof of (4.24) together with characterization of the case where equality holds which we discuss below. (Note that for surfaces of constant curvature equality can hold in (4.25), but not in (4.24).) Huber [1] proves (4.24) using methods of potential theory. He derives an analytic inequality that contains as special cases the inequality (4.10) of Carleman, and its generalization to subharmonic functions by Beckenbach and Radó. Toponogov [1] removes certain restrictions in Aleksandrov's proof of (4.25) in the case $\sup_D K > 0$. Karcher [2] proves (4.25) for convex curves, using Aleksandrov angle comparison theorems. Both Bandle [4] and Chavel-Feldman [2] use variants of the parallel-curve argument. Aubin [3] applies a symmetrization argument to reduce the proof of (4.25) to the known case of constant curvature surfaces.

A few words on the two inequalities themselves may be helpful.

First, note that the right-hand side of (4.25) is exactly equal to the square of the length of a geodesic circle enclosing an area A on a surface of constant curvature equal to $\sup_D K$. (For constant positive curvature, that fact is just equation (4.1).) Thus, (4.25) is a concise form of the following statement: *the length L of the boundary of a simply-connected domain D of area A on a surface S is greater than or equal to the length of the geodesic circle bounding a disk of the same area A on a surface of constant curvature equal to the supremum over D of the Gauss curvature of S .* It is in fact in this form that Schmidt, Aleksandrov, and Aubin state the result.

Concerning (4.24), let us rewrite it in the form

$$L^2 \geq 2A \left(2\pi - \iint_D K^+ \right).$$

This inequality may be viewed as providing an upper bound for the area A in terms of L , provided $\iint_D K^+ < 2\pi$. Otherwise it gives no information. But this proviso is just as it should be, since if $\iint_D K^+ \geq 2\pi$, then no bound for A is possible in terms of L . For example, if S consists of a semi-infinite cylinder of radius r capped by a hemisphere, then a simply-connected domain D bounded by a cross-section of the cylinder has $\iint_D K^+ = 2\pi$ and $L = 2\pi r$, but it may have arbitrarily large area.

For smooth surfaces, equality can never hold in (4.24) unless $K \equiv 0$ and the boundary curve is a geodesic circle (Huber [1, p. 245]). However, among the more general surfaces with singularities allowed by Aleksandrov, equality can occur, and it does so if and only if D is a right circular cone having the given values of L and A . (See Aleksandrov and Strel'cov [2]). By rounding off the vertex of the cone, one sees that the right side of (4.24) is best possible within the category of smooth surfaces.

If one goes back to Fiala's paper and examines it more closely, one finds [1, p. 336] that he proves in fact an inequality that contains both (4.24) and Bonnesen's inequality (4.5). His proof is carried out under the restrictive conditions that $K > 0$, the metric on the surface is analytic, and the domain is bounded by an analytic curve. A proof without those restrictions was later

given by Burago and Zalgaller [1]. The result is

$$\rho L \geq A + \left(\pi - \frac{1}{2}\omega^+\right)\rho^2, \quad (4.26)$$

where

$$\omega^+ = \int \int_D K^+ \quad (4.27)$$

and ρ is the greatest distance of a point in D to the boundary. (If D is a plane domain, ρ coincides with the *inradius* of D : the maximum radius of a circular disk lying in D .)

If $\omega^+ < 2\pi$, (4.26) is equivalent to

$$L^2 - 4\pi A + 2\omega^+ A \geq (L - (2\pi - \omega^+)\rho)^2, \quad (4.28)$$

which is a stronger form of (4.24). (If $\omega^+ \geq 2\pi$, then (4.24) holds trivially, with strict inequality.) For a plane domain, $\omega^+ = 0$ and (4.28) reduces to Bonnesen's inequality (4.5).

Let us conclude this section by citing one more result, generalizing (4.24) in a quite different way than (4.26). (See Ionin [1] and Burago [2].) For any real number λ , let

$$E_\lambda = \{p \in D: K(p) \geq \lambda\}, \quad \omega_\lambda^+ = \int \int_{E_\lambda} (K - \lambda).$$

Thus, when $\lambda = 0$, $\omega_\lambda^+ = \int \int_D K^+$.

THEOREM 4.4. *Let D be a domain of area A , Euler characteristic χ , and length of boundary L . Then for any real number λ ,*

$$L^2 \geq 2(2\pi\chi - \omega_\lambda^+)A - \lambda A^2. \quad (4.29)$$

Note that if D is simply-connected, then $\chi = 1$, and choosing $\lambda = 0$, (4.29) reduces to (4.24). In general, when $\lambda = 0$, (4.29) is of no interest unless D is simply-connected, since otherwise $\chi \leq 0$ and the right-hand side of (4.29) is nonpositive. However, when λ is negative the situation is different. For example if D is a domain on a surface where $K \leq -1$, and if we choose $\lambda = -1$, then $\omega_{-1}^+ = 0$ and (4.29) becomes

$$L^2 \geq 4\pi\chi A + A^2. \quad (4.30)$$

For simply-connected domains, this is just (4.25). However, for doubly-connected domains, $\chi = 0$, and we have the

COROLLARY. *For any doubly-connected domain with $K \leq -1$, one has the inequality*

$$L \geq A. \quad (4.31)$$

To place this result in context, let us note that the inequality (4.31) follows immediately from (4.25) for simply-connected domains with $K \leq -1$, whereas for triply-connected domains it is not necessarily true, as one sees by taking the metric of constant curvature $K \equiv -1$ on the sphere punctured at three points. On the other hand, it is worth noting that the inequality

$$L^2 \geq 4\pi A + \alpha^2 A^2 \quad (4.32)$$

and its consequence

$$L \geq \alpha A \quad (4.33)$$

hold for *arbitrary* domains D on a *simply-connected* surface S , provided $K < -\alpha^2$, $\alpha > 0$, on S . Namely the outer boundary of D bounds a simply-connected domain D' on S for which

$$(L')^2 \geq 4\pi A' + \alpha^2 (A')^2$$

by (4.25). Since $L \geq L'$ and $A' \geq A$, (4.32) follows immediately.

An application of the above remark is given in the proof of Theorem 6.1 below.

Inequality (4.33) is an example of a different kind of inequality between length and area, and as such, it leads us to the subject of the following section.

5. Isoperimetric-type inequalities. The inequalities we have considered so far have been modeled closely on the initial classical one $L^2 \geq 4\pi A$. However, there are many cases in which other inequalities between length and area (or their higher dimensional equivalents) arise naturally. Some of these are stronger inequalities for restricted cases, some are weaker inequalities for more general cases, and some are simply different inequalities. We shall see examples of each of these.

A. Stronger inequalities: polygons and polyhedra. A theorem dating back to Lhuillier in 1782 states that for a convex polygon P in the plane one has the inequality

$$L^2 \geq 4\alpha A \quad (5.1)$$

where α is the area enclosed by the polygon circumscribed about the unit circle with sides parallel to the sides of P . Clearly $\alpha > \pi$. Furthermore, equality holds in (5.1) if and only if P is itself circumscribed about a circle. In fact, if P is circumscribed about a circle of radius r , then $\alpha = A/r^2$, and $4\alpha A = (2A/r)^2$. But it is easily verified that

$$rL = 2A \quad (5.2)$$

for a polygon circumscribed about a circle of radius r . We may note parenthetically that the proof of the isoperimetric inequality for regular polygons given by Galileo [1, p. 62] actually uses only the fact that (5.2) holds both for the polygon and the circle, and the proof is therefore valid for any polygon circumscribed about a circle.

Returning to (5.1), if one fixes the number of sides of the polygon, say n , then the minimum of α is attained for a regular polygon. Substituting the corresponding value of α , one finds

THEOREM 5.1. *For any n -sided polygon in the plane,*

$$\frac{L^2}{A} \geq \frac{4}{n} \tan \frac{\pi}{n} > 4\pi. \quad (5.3)$$

The above argument actually applies only to convex polygons, but then (5.3) holds for arbitrary polygons by passing to the convex hull.

Given a rectifiable plane curve, one can use inscribed polygons, apply (5.3), and pass to the limit to deduce $L^2 \geq 4\pi A$. (For details of this argument, see Blaschke [1].) The shortcoming of this proof is that it does not allow a characterization of the case in which equality holds. However, a modified version will do it. Namely, with the same notation as in (5.1), if ρ is the largest radius of a circle inscribed in the polygon, then

$$\rho L \geq A + \alpha \rho^2 > A + \pi \rho^2. \quad (5.4)$$

If C is convex and is approximated by convex polygons, this gives $\rho L \geq A + \pi \rho^2$ for the curve C , and that is equivalent to Bonnesen's inequality (4.5): $L^2 - 4\pi A \geq (L - 2\pi\rho)^2$. This shows that for convex curves $L^2/A \geq 4\pi$, with equality only for a circle. But L^2/A is always reduced when passing from a nonconvex domain to its convex hull, so that equality is impossible for a nonconvex domain.

For a proof of (5.4), see Fejes Tóth [1, p. 10].

The step from two to three dimensions involves difficulties of a whole new order of magnitude. Despite important contributions by Steiner, Lindelöf, Minkowski, Steinitz, and many others, some of the most basic questions remain open. For example, Steiner's conjecture that a regular polyhedron minimizes the quantity S^3/V^2 among all polyhedra of the same type is still not solved for the icosahedron. However, one has the following 3-dimensional analog of Theorem 5.1.

THEOREM 5.2. *Let S be the surface area of an n -sided convex polyhedron and V the volume enclosed. Then*

$$\int_M |\nabla f| dA \geq \int_M \frac{f(x)}{|x - y|} dA_x$$

and

$$\int_M \frac{|(\nabla f)(x)|}{|x - y|} dA_x \geq 2\pi f(y),$$

where ∇f refers to the intrinsic gradient of f on the surface and $|x - y|$ is the distance in $\mathbf{R}^3$ between the points x and y . Combining these two gives the Sobolev inequality

$$\left(\int_M |\nabla f| dA \right)^2 \geq 2\pi \int_M f^2 dA \quad (5.7)$$

from which (5.6) follows as before.

We should mention that the slightly weaker inequality $L^2 > 6A$ had been obtained earlier by Nitsche [3, §554] for doubly-connected minimal surfaces, but for that case the sharp inequality $L^2 > 4\pi A$ is now known, as we indicated in §4 above.

Isoperimetric inequalities have been demonstrated for “minimal objects” of varying descriptions and arbitrary dimensions in euclidean space, usually in the context of Sobolev inequalities. Among the objects that have been considered are integral currents (Federer and Fleming [1, §6.2], Federer [1, §§4.2.10, 4.5.14]), varifolds (Almgren [1], [2], Allard [1]), minimal graphs (Miranda [1], Bombieri, deGiorgi, and Miranda [1]), weak solutions of the minimal surface equation (Michael and Simon [1]) and minimal submanifolds (Allard [1] and Michael and Simon [1]). The conclusion of all of these is that if M is an m -dimensional minimal object (in a suitable sense) in $\mathbf{R}^n$, then

$$S^m \geq cV^{m-1}, \quad (5.8)$$

where V is the m -dimensional measure of M , S is the $(m - 1)$ -dimensional measure of its boundary, and c is an absolute positive constant; that is, c is independent of M , depending at most on the dimensions m and n . The conjecture is that (5.8) holds with $c = m^m \omega_m$.

The method of Michael and Simon has been generalized by Hoffman and Spruck [1] to minimal submanifolds of an arbitrary Riemannian manifold. We shall discuss their results in more detail below.

C. Other inequalities: submanifolds of $\mathbf{R}^n$ and Riemannian manifolds. There are a number of inequalities, not of the form (5.8), that relate the volume of a domain with the measure of its boundary, sometimes involving other quantities as well.

These other inequalities arise generally in the context of arbitrary curved surfaces and Riemannian manifolds, where the particular exponents in (5.8) are of less significance because of the lack of homogeneity. For example, the results cited in Part II of §4 are of this form, where an additional term involving the Gauss curvature of the surface is included.

There is another variant that has proved valuable in recent investigations. That is an isoperimetric inequality on submanifolds, where the extra term involves the mean curvature of the submanifolds. When the mean curvature

is zero, the extra term vanishes, and one obtains the isoperimetric inequalities for minimal surfaces that we have discussed above.

One approach to these results is by means of formula (4.16) and its higher dimensional analogs. The formula, once more, is

$$A = - \iint_D (x - c) \cdot H \, dA + \frac{1}{2} \int_C (x - c) \cdot \nu \, ds. \quad (5.9)$$

In applying this formula to minimal surfaces, not much attention need be paid to the exact definition of the mean curvature vector H . However, when working with arbitrary surfaces, it is important to bear in mind that there are two conflicting traditions regarding the mean curvature vector; in one, it is the trace of the second fundamental form, and in the other it is the trace divided by the dimension of the manifold. For a hypersurface, this amounts to the distinction between the sum of the principal curvatures and their average. Here we shall use their average, so that the mean curvature of a sphere of radius r is $1/r$.

The methods of Reid [1], Hsiung [1], and Chavel [1], for proving Theorem 4.2(i) consist of estimating the line integral in (5.9), and they therefore yield an inequality valid for arbitrary domains D bounded by a single curve C :

$$L^2 \geq 4\pi \left(A + \iint_D (x - c) \cdot H \, dA \right), \quad (5.10)$$

where c may be an arbitrary point of C (using the method of Reid and Hsiung) or else the center of gravity of C (by the argument of Chavel; see the proof of Theorem 4.2(i) above).

The cases where an inequality such as (5.10) is most useful are those where the geometry of the situation guarantees that the right-hand side is positive. For example, if D lies in the unit ball, and there is a uniform bound, $|H(x)| \leq h$ at all points of D , then one has $|x - c| \leq 2$, and consequently

$$L^2 \geq 4\pi(1 - 2h)A. \quad (5.11)$$

For simply-connected surfaces lying in the unit sphere, Heinz and Hildebrandt [1] have proved the inequality

$$L^2 \geq 8(1 - h)A, \quad (5.12)$$

extending an earlier result of Heinz [1] for surfaces of constant mean curvature. This result is weaker than (5.11) when h is small, and it is restricted to simply-connected surfaces. On the other hand, it is proved under weaker regularity conditions than (5.11) (note that (5.9) needs enough smoothness up to the boundary to apply Green's Theorem) and most importantly, it gives a positive result in the range $\frac{1}{2} < h < 1$. Heinz and Hildebrandt use (5.12) to give an upper bound on the number of possible branch points of the surface.

Kaul [1], [2], [3] has generalized (5.10) to surfaces lying in a Riemannian manifold, and has given a corresponding extension of the Heinz-Hildebrandt bound on branch points. He also obtains inequalities in the case of more than one boundary curve.

An attempt to extend the method of Reid and Hsiung to higher dimensions was made by Hanes [1]. However, Hanes has noted that his results are not

correct as presented, since the inequality in his Theorem 4 requires the condition that $\int f = 0$, whereas it is applied on p. 531 of his paper to the function $|X|$.

A more naive approach does give some kind of a bound that can be obtained in any number of dimensions. Suppose that D lies in a ball of radius R . We may apply (5.9), using for c the center of the ball. Since ν is a unit vector, one immediately obtains the bound

$$A \leq R \left(\frac{1}{2} L + \iint_D |H| dA \right), \quad (5.13)$$

with the special case

$$RL \geq 2A \quad \text{for minimal surfaces.} \quad (5.14)$$

Note that equality holds in (5.14) if D is a disk of radius R .

For certain applications, an inequality such as (5.13) or (5.14) is all one needs. For example, (5.14) gives a uniform bound on the area of all minimal surfaces lying in a fixed ball and having boundary lengths uniformly bounded (totally independent, incidentally, of topological type). For an application of such a bound, see for example Courant [1, p. 131] or Nitsche [2, §327].

The higher-dimensional version of (5.9) for an m -dimensional submanifold M of $\mathbf{R}^n$ is

$$V = - \int_M (x - c) \cdot H dV + \frac{1}{m} \int_{\partial M} (x - c) \cdot \nu dS \quad (5.15)$$

where V is the m -dimensional measure of M , x is the position vector of M , c is an arbitrary constant vector, $H = H(x)$ is the mean curvature vector of M at x , dV is m -dimensional measure on M , dS is $(m - 1)$ -dimensional measure on ∂M , and ν is the unit exterior normal to M ; that is, ν is a unit vector lying in the tangent space to M , orthogonal to ∂M and directed outward from M .

If M lies in a ball of radius R , then choosing c to be the center of the ball gives

$$V \leq R \left(\frac{1}{m} S + \int_M |H| dV \right) \quad (5.16)$$

where S is the $(m - 1)$ -dimensional measure of ∂M . In particular, if $\sup_M |H| = h < 1/R$, then (5.16) implies

$$S \geq m(1/R - h)V \quad (5.17)$$

and for minimal submanifolds, where $H \equiv 0$,

$$S \geq mV/R. \quad (5.18)$$

Again, equality holds in (5.18) when M is an m -ball of radius R in an affine $(m + 1)$ -dimensional subspace of $\mathbf{R}^n$.

The approach via Sobolev inequalities yields inequalities analogous to (5.16), but with the correct exponents, and independent of R .

The Sobolev inequality on an arbitrary m -dimensional submanifold M in $\mathbf{R}^n$ is of the form

$$\int_M (|\nabla f| + mf|H|) \geq c \left[\int_M f^{m/(m-1)} \right]^{(m-1)/m}, \quad (5.19)$$

for all functions f with compact support in M . (See Michael and Simon [1], where the constant c is $\omega_m^{1/m}/4^{m+1}$, and where H is m times our H .)

If M has a smooth boundary, then by the same method that was used in Theorem 3.1, one can deduce from (5.19) the isoperimetric inequality

$$S + m \int_M |H| \geq cV^{(m-1)/m}. \quad (5.20)$$

(See also Allard [1, p. 461].)

Almgren ([1], [2, Corollary 8.9]) has noted that a standard isoperimetric inequality of the form (5.8) holds on a submanifold M , provided $V \leq (c'/h)^m$, where $|H| \leq h$ on M , and c' is a constant depending only on m . In fact, (following a suggestion of D. Hoffman) we may deduce from (5.20) that

$$cV^{(m-1)/m} \leq S + mhV \leq S + mc'V^{(m-1)/m}$$

or

$$S \geq (c - mc')V^{(m-1)/m}.$$

Choosing for c' any value less than c/m gives an inequality of the form (5.8).

Note that the role of the mean curvature in the various inequalities (5.10)–(5.20) is analogous to that of the Gauss curvature in (4.24) and (4.25). One has the intuitive notion that a surface can be enlarged by producing a bulge in it without increasing the size of the boundary. The additional term involving curvature is a quantitative measure of the size of the bulge.

The culmination of this approach is the recent work of Hoffman and Spruck [1] who generalized the method of Michael and Simon to submanifolds of a Riemannian manifold. They arrive at the following result. (See also Otsuki [1] who gives a corrected version of their work.)

THEOREM 5.4. *The isoperimetric inequality (5.20) holds for submanifolds M of a Riemannian manifold $\bar{M}$, under the following conditions. Denote by $\bar{K}$ the sectional curvature of $\bar{M}$, and by $\bar{R}(M)$ the injectivity radius of $\bar{M}$ restricted to M (i.e. the minimum distance to the cut locus in $\bar{M}$ for all points of M). Suppose $\bar{K} \leq b^2$ on M , where b may be real or imaginary. Let α be a free parameter, $0 < \alpha < 1$, and denote*

$$\hat{V} = [V / (1 - \alpha)\omega_m]^{1/m},$$

where $m = \dim M$, $\omega_m = \text{volume of unit ball in } \mathbb{R}^m$. Then under the assumptions

$$b^2 \hat{V}^2 \leq 1 \quad (5.21)$$

and

$$\bar{R}(M) \geq 2\rho_0 = \begin{cases} \frac{2}{b} \sin^{-1}(b\hat{V}), & b \text{ real,} \\ 2\hat{V}, & b \text{ imaginary,} \end{cases} \quad (5.22)$$

(5.20) holds with the constant

$$c = \frac{2\alpha(m-1)^2[(1-\alpha)\omega_m]^{1/m}}{\pi m[(m-\alpha)2^{m-1} - (1-\alpha)]}. \quad (5.23)$$

Note that if $\bar{K} < 0$, then (5.21) is trivially satisfied. If in addition $\bar{M}$ is simply connected, then by the Hadamard-Cartan theorem, $\bar{R}(M) = \infty$, and (5.22) is automatic. In that case the inequality holds with no restriction on M .

Two important special cases of Theorem 5.4 are where $\bar{M}$ is a minimal submanifold of $\bar{M}$, and where M is an open subdomain of $\bar{M}$. In both cases, the term involving $|H|$ vanishes, and (5.20) reduces to the standard form

$$S \geq cV^{(m-1)/m}, \quad (5.24)$$

with the constant c given by (5.23).⁶

We note next two results of Yau for domains on negatively curved manifolds.

THEOREM 5.5 (YAU [1, p. 491]). *Let D be a doubly-connected domain on a surface of nonpositive curvature. Let C_1, C_2 be the boundary curves of D ; L_1, L_2 their lengths; d the distance between them. Then if $L = L_1 + L_2$,*

$$L(L/2 + d) \geq A. \quad (5.25)$$

THEOREM 5.6 (YAU [1, p. 498]). *Let D be a domain with smooth boundary in an n -dimensional complete simply-connected Riemannian manifold whose sectional curvature $\bar{K}$ satisfies $\bar{K} \leq -\alpha^2$, $\alpha > 0$. Then the n -dimensional measure V of D and the $(n-1)$ -dimensional measure S of its boundary satisfy*

$$S \geq (n-1)\alpha V. \quad (5.26)$$

We conclude this section with two theorems that are the furthest from the standard isoperimetric inequality. In the first of these, due to Banchoff and Pohl [1], there is no mention of area, and indeed there is no surface. It is concerned with a closed curve C in $\mathbf{R}^n$, and compares the length L of C with a certain integral. Specifically, given any line in $\mathbf{R}^n$ that does not intersect C , there is a well-defined *linking number*, $\lambda(l)$, of l with the curve C . The *Banchoff-Pohl inequality* is

$$L^2 \geq 4 \int_G \lambda^2 dG, \quad (5.27)$$

where G is the space of all lines in $\mathbf{R}^n$, and dG is the invariant measure on G , suitably normalized. Equality holds only if C is a circle traversed one or more times.

In the case that C is a simple plane curve, (5.27) is just the classical isoperimetric inequality $L^2 \geq 4\pi A$. If C is a self-intersecting plane curve, then the right-hand side of (5.27) may still be interpreted in terms of the areas bounded by C , and indeed, (5.27) reduces to the inequality (1.10). In the general case, it would be interesting to know whether the right-hand side has an interpretation as the area of some naturally defined surface bounded by C .

⁶R. Schoen [1, §IV. 3], has given a short proof that an isoperimetric inequality (5.24) holds for domains on an arbitrary compact Riemannian manifold with boundary. His proof, however, is indirect, and gives no value for the constant c .

For a discussion of the properties of the integral in (5.27), as well as generalizations from curves to higher-dimensional submanifolds, we refer to the original paper of Banchoff and Pohl.

Finally, we have a recent result obtained independently by Chavel [1] and Reilly [3]. By way of background, we note that if M is a compact Riemannian manifold, then the Laplace operator on M has a spectrum consisting of eigenvalues

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots \quad (5.28)$$

for which the equation

$$\Delta \varphi + \mu \varphi = 0 \quad (5.29)$$

has a solution $\varphi_k \not\equiv 0$. (The study of this spectrum is the subject of the book by Berger, Gauduchon, and Mazet [1].) The eigenfunctions corresponding to $\mu_0 = 0$ are the nonzero constant functions. The first nontrivial eigenvalue μ_1 may be represented in a manner analogous to the Rayleigh quotient (3.10) as

$$\mu_1(M) = \inf_{f \in \mathcal{F}_1} \frac{\int_M |\nabla f|^2}{\int_M f^2} \quad (5.30)$$

where $\mathcal{F}_1$ is the family of smooth functions f on M defined by

$$f \in \mathcal{F}_1 \quad \text{if } f \not\equiv 0, \int_M f = 0. \quad (5.31)$$

Thus $\mathcal{F}_1$ consists of those nonzero functions on M orthogonal to the eigenfunctions corresponding to μ_0 , i.e. the constant functions.

In the special case of an n -sphere of radius r : $M = S_r^n$, the solutions of (5.29) are the spherical harmonics, and in particular, one may choose as the first nontrivial eigenfunction φ_1 , the restriction to S_r^n of any linear function in $\mathbb{R}^{n+1}$. It follows that

$$\mu_1(S_r^n) = n/r^2. \quad (5.32)$$

The generalized Wirtinger inequality (3.1) follows immediately from (5.30)–(5.32).

For a complete discussion of this subject, see the book of Berger, Gauduchon and Mazet [1]. For specific relations between $\mu_1(M)$ and geometric isoperimetric constants associated with M , we refer to papers of Cheeger [1] and Yau [1]. (See also Buser [1], [3], [5], [6].) Before stating Chavel and Reilly's theorem we cite a related result of Hersch [1] concerning the two-sphere S_r^2 . We note that the eigenspace corresponding to $\mu_1(S_r^n)$ is $(n+1)$ -dimensional, since it consists of the restrictions to S_r^n of the linear functions in $\mathbb{R}^{n+1}$. Thus

$$\mu_1 = \mu_2 = \dots = \mu_{n+1} = n/r^2 \quad \text{on } S_r^n. \quad (5.33)$$

THEOREM 5.7 (HERSCH [1]). *Let M be a two-dimensional Riemannian manifold homeomorphic to the two-sphere. Let A denote the area of M , and μ_1, μ_2, μ_3 the first three nontrivial eigenvalues. Then*

$$\frac{1}{A} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} \right) \geq \frac{3}{8\pi}. \quad (5.34)$$

Equality holds on the standard sphere S_r^n .

For further discussion of this theorem, and related results, see Berger [2].

THEOREM 5.8 (CHAVEL [1], REILLY [3]). *Let D be a domain in $\mathbf{R}^{n+1}$ bounded by a smooth compact manifold M . Let V be the volume of D , and S the n -dimensional measure of M . Then*

$$\mu_1(M) \leq \left(\frac{S}{V} \right)^2 \frac{n}{(n+1)^2}. \quad (5.35)$$

Equality holds if and only if $M = S_r^n$.

PROOF. By a translation of coordinates we may assume that each of the coordinate functions x_k in $\mathbf{R}^{n+1}$ satisfies

$$\int_M x_k = 0.$$

Applying (5.30), gives

$$\mu_1(M) \leq \frac{\int_M |\nabla x_k|^2}{\int_M x_k^2},$$

and therefore

$$\mu_1(M) \int_M |x|^2 \leq \int_M \sum_{k=1}^{n+1} |\nabla x_k|^2 = nS.$$

By the divergence theorem over the domain D , and Schwarz' inequality,

$$(n+1)V = \int_M x \cdot \nu \leq \int_M |x| \leq \sqrt{\int_M |x|^2 \cdot S} \leq \sqrt{nS^2/\mu_1(M)},$$

which is (5.35). For equality to hold in Schwarz' inequality, $|x|$ must be constant on M , hence M is a sphere.

REMARKS. 1. Theorem 5.8 is in fact a special case of the results of Chavel and Reilly. Chavel shows that the same result holds for domains on a minimal submanifold of a complete simply-connected Riemannian manifold with nonpositive curvature, while Reilly obtains a whole series of related inequalities in euclidean space (inspired by earlier work of Bleecker and Weiner [1]).

2. Reilly has pointed out that in the case $n = 2$ and M homeomorphic to the two-sphere, Hersch's inequality (5.34) is stronger than (5.35). In fact, since $\mu_3 \geq \mu_2 \geq \mu_1$, (5.34) implies

$$\mu_1 \leq 8\pi/A, \quad (5.36)$$

and applying the isoperimetric inequality (2.14) for domains in $\mathbf{R}^3$ yields (5.35) with $n = 2$.

3. The proof of Theorem 5.8 may be viewed as a direct extension of the proof that we gave in §1 of the isoperimetric inequality in the plane. In fact, if M is one-dimensional, then $\mu_1(M) = (2\pi/L)^2$, where L is the length of M , and (5.30) is equivalent to Wirtinger's inequality, Lemma 1.1. In the case that M is a simple closed plane curve, (5.35) reduces to the isoperimetric

inequality $L^2 \geq 4\pi A$, and we find that our excursion through analytic inequalities and eigenvalues has brought us full circle to our starting point in §1.

6. Applications. The isoperimetric property of the circle and the sphere has been the subject of speculation and fascination for literally thousands of years. The search for proofs and generalizations has proceeded without any thought that outside justification seemed necessary. However, it is also true that increasingly in recent years, isoperimetric inequalities have proved useful in a number of problems in geometry, analysis, and physics. We have given several examples already, but in this section, we shall try to present in a more orderly fashion some of the applications that have been made.

A. Mathematical physics. Perhaps the most obvious isoperimetric question in physics is "Why is the earth (the planets, the stars) roughly a sphere?" The notion that the physical shape of the universe may be related to the geometric isoperimetric inequality goes back to Ptolemy [1, Book I, Chapter 3]. However, the first attempt at a precise demonstration appears to be due to Poincaré ([1]; see also [2, pp. 15–24], and [3, pp. 143–150]). He addresses himself to the basic problem of showing that for a nonrotating homogeneous fluid mass, acted upon only by the internal forces of gravitation, the only figure of stable equilibrium is a sphere. His approach is to show that any such equilibrium figure must have the property that it has minimum electrostatic capacity among all figures of equal volume. He then uses the isoperimetric inequality in three dimensions together with a variational argument to conclude that the sphere minimizes electrostatic capacity among all figures of equal volume. A complete proof of this fact, again using the isoperimetric inequality, was given in 1930 by Szegő [1].

Another physical quantity (whose mathematical expression we shall give shortly) is *torsional rigidity*. In 1856 Saint-Venant conjectured that of all cross-sections with a given area, the circle has maximum torsional rigidity. It was almost 100 years before Pólya gave the first proof of this conjecture in 1948. His proof, like Szegő's for capacity, used the method of symmetrization first invented by Steiner to prove the original geometric isoperimetric inequality.

The book of Pólya and Szegő [1] treats these results, as well as the Faber-Krahn proof of Rayleigh's statement on the fundamental tone of a drum, discussed above in §3. For a more modern discussion of symmetrization, together with a quite different application to the motion of an ideal fluid see the paper of Fraenkel and Berger [1, especially Appendix I and Corollary 3B on p. 27]. (See also the discussion following Theorem 6.6 below.)

In order to put some of the more recent results into perspective, let us carry a little further our discussion of analytic inequalities in §3.

Let D be a domain on a two-dimensional Riemannian manifold, let $\mathcal{F}$ be the family of smooth real-valued functions with compact support in D , and consider the four ratios

$$R_{ij} = \inf_{f \in \mathcal{F}} \frac{B_i}{C_j}, \quad i, j = 1, 2, \quad (6.1)$$

where

$$\begin{aligned} B_1 &= \iint_D |\nabla f|^2, & B_2 &= \left[\iint_D |\nabla f| \right]^2, \\ C_1 &= \iint_D f^2, & C_2 &= \left[\iint_D f \right]^2. \end{aligned} \quad (6.2)$$

The two ratios involving B_1 have physical interpretations, whereas the two involving B_2 turn out to have purely geometric interpretations. Namely, in the case where D is a domain with smooth boundary,

$$R_{11} = \lambda_1, \quad R_{12} = 4/P, \quad (6.3)$$

where λ_1 is the smallest eigenvalue of the equation $\Delta f + \lambda f = 0$ in D , with zero boundary values, and P (in the case of a plane domain D) is the *torsional rigidity* of D . On the other hand, one has

$$R_{21} = \inf_{D'} \frac{L^2}{A}, \quad R_{22} = \inf_{D'} \left(\frac{L}{A} \right)^2 \quad (6.4)$$

where D' is a relatively compact subdomain of D , A its area, and L the length of its boundary.

The proof of (6.4) follows the lines in Theorem 3.1 above. The first equation in (6.4) is simply the equivalence of the Sobolev inequality and the isoperimetric inequality. The second equation in (6.4) was first noted as recently as 1975 by Yau [1], following the fundamental result of Cheeger [1] in 1970, that

$$\lambda_1 \geq \frac{1}{4} h^2, \quad (6.5)$$

where

$$h = \inf_{D'} \frac{L}{A}. \quad (6.6)$$

Thus, in our notation, (6.5) becomes $R_{11} \geq R_{22}/4$.

Other inequalities between the R_{ij} are known, such as $R_{11}/R_{12} < A$, or

$$P\lambda_1 < 4A,$$

where A is the area of D . (See Pólya-Szegő [1, p. 91].)

Pólya and Szegő [1, p. 18, top] conjectured that the expression $P\lambda_1^2$ is minimized for a circular disk. This was just recently proved by Kohler-Jobin [1]. Analytically this fact is expressed by the inequality

$$P\lambda_1^2 \geq \frac{\pi}{2} j^4$$

with equality only in the case of a circular disk. Here j is the first positive zero of the Bessel function J_0 .

The dependence of each of the separate quantities λ_1 and P on the geometry of the domain D is strikingly illustrated by the inequalities

$$\frac{\pi j^2}{A} < \lambda_1 < \frac{\pi j^2}{A} [1 + (2.8)\delta], \quad (6.7a)$$

$$\frac{A^2}{2\pi} \left[1 - 2\delta \left(1 + \delta \log \frac{\delta}{1 + \delta} \right) \right] \leq P \leq \frac{A^2}{2\pi}, \quad (6.7b)$$

where

$$\delta = L^2/4\pi A - 1$$

measures the deviation of the domain from circularity, with $\delta = 0$ if and only if D is a circular disk.

The left-hand inequality in (6.7a) is simply the Faber-Krahn inequality (Theorem 3.3), since $\pi j^2/A$ is the value of λ_1 for a circular disk of area A . Similarly, the right-hand inequality in (6.7b) is the result of Pólya cited above, confirming the conjecture of Saint-Venant. The other two inequalities are due to Payne and Weinberger [1]. In both cases the expressions in brackets tend to 1 as $\delta \rightarrow 0$, thus effectively encircling the quantities λ_1 and P .

Generalizations of the Faber-Krahn and Pólya theorems have been obtained for domains on surfaces. For example, Peetre [1] showed that the Faber-Krahn inequality (the left-hand inequality in (6.7a)) remains valid if D is a domain of area A on a simply-connected surface with Gauss curvature $K \leq 0$. More recently, Bandle [1], [3] and Chavel-Feldman [2] have obtained generalizations to arbitrarily-curved surfaces of both the Faber-Krahn inequality and the Saint-Venant conjecture.

The importance of Cheeger's result (6.5) is that it allows him to derive a lower bound for λ_1 on an arbitrary Riemannian manifold in terms of purely geometric quantities associated with the manifold, such as curvature, volume and diameter. Better bounds of this nature were later obtained by Yau [1], also making use of isoperimetric ratios such as (6.6). (Yau's results, like Cheeger's, are for manifolds of arbitrary dimension, but he obtains much more detailed information in the case of surfaces.)

We conclude this section with three examples in which an isoperimetric inequality in combination with Cheeger's inequality (6.5) gives an explicit estimate for the lowest eigenvalue λ_1 . (For further illustrations, see Osserman [3], [4] and Hoffman [1]. Buser [1], [3], [5], [6] has recently contributed greatly to the understanding of the quantity h in (6.6) and has given further striking applications of Cheeger's inequality.)

THEOREM 6.1 (McKEAN [1]). *If D is a domain on a simply-connected surface with Gauss curvature $K \leq -\alpha^2 < 0$, then*

$$\lambda_1 \geq \alpha^2/4. \quad (6.8)$$

PROOF. Combining the isoperimetric inequality $L \geq \alpha A$ (see (4.33)) with (6.5) and (6.6) yields (6.8).

REMARK. In the n -dimensional case, if the sectional curvature is bounded above by $-\alpha^2$, $\alpha > 0$, then (5.26) implies that

$$\lambda_1 \geq [(n-1)\alpha/2]^2. \quad (6.8a)$$

This was also proved originally using other methods by McKean [1].

THEOREM 6.2 (CHENG [2, p. 187]). *Let M be a compact surface with Gauss curvature $K \geq 0$. Then*

$$\mu_1(M) \geq 1/4d^2 \quad (6.9)$$

where d is the diameter of M .

PROOF. By the Gauss-Bonnet theorem,

$$0 \leq \int_M K \, dA = 2\pi\chi, \quad (6.10)$$

so that the Euler characteristic χ of M is nonnegative. If $\chi = 0$, then again by Gauss-Bonnet, $K \equiv 0$, and M must be a flat torus or flat Klein bottle. For both of these the spectrum is completely known (see Berger, Gauduchon, and Mazet [1, p. 146]), and (6.9) may be verified directly. If $\chi = 1$, M is the projective plane, and by passing to the double covering, one can reduce the problem to the one remaining case when $\chi = 2$ and M is homeomorphic to the two-sphere. Then by a theorem of Cheng [3, Corollary 3.5] an eigenfunction corresponding to the first eigenvalue of μ_1 on M has a zero-set which is a smooth simple closed curve, dividing M into two simply-connected domains D_1, D_2 . Once more by Gauss-Bonnet, for at least one of these domains the total curvature is not more than 2π , and denoting that one by D , the given eigenfunction restricted to D is also the eigenfunction of the first eigenvalue of D , and in fact $\lambda_1(D) = \mu_1(M)$.

We now use the isoperimetric inequality (4.26) of Burago and Zalgaller, which says that for a simply-connected domain D'

$$rL \geq A + \left(\pi - \frac{1}{2} \iint_{D'} K^+ \, dA \right) r^2, \quad (6.11)$$

where r is the maximum distance from a point of D' to the boundary of D' . Since in our case, for every subdomain D' of D

$$\iint_{D'} K^+ \, dA \leq \iint_D K^+ \, dA = \iint_D K \, dA \leq 2\pi,$$

it follows that for every simply-connected subdomain D' ,

$$L/A \geq 1/r \geq 1/d. \quad (6.12)$$

The desired conclusion (6.9) would now follow from (6.5) and (6.6) *provided* that in the definition (6.6) of h one could restrict the competing domains D' to be simply-connected. That turns out to be true when the original domain D is simply-connected. In slightly more general form, one has the following lemma, which, when combined with the above reasoning, finishes the proof of Cheng's theorem.

LEMMA 6.3 (OSSERMAN [3]). *Let D be a domain homeomorphic to a plane domain of finite connectivity k . Define*

$$h_k = \inf_{D'} \frac{L}{A} \quad (6.13)$$

taken over all relatively compact subdomains D' of D whose connectivity is at most k . Then

$$\lambda_1(D) \geq \frac{1}{4} h_k^2. \quad (6.14)$$

The lemma follows from the observation that the subdomains used in

Cheeger's proof of (6.5) are the domains $D_t = \{p \in D: f(p) > t\}$, where f is the eigenfunction corresponding to the first eigenvalue λ_1 of D , and t may be chosen to be any regular value of f . It is known that f cannot change sign in D , and we may assume that $f \geq 0$. Then $\Delta f = -\lambda_1 f < 0$, so that f is superharmonic and takes its minimum on the boundary of any subdomain. Let D'_i be any connected component of D_t . If the connectivity of D'_i were greater than k , then the complement of D'_i would contain a component lying completely in D with points where $f < t$, whereas $f \equiv t$ on the boundary of this component. This contradiction establishes the lemma.

Combining Lemma 6.3 with the Burago-Zalgaller inequality (6.11), one has the following further result.

THEOREM 6.4 (OSSERMAN [3]). *Let D be a simply-connected domain with a Riemannian metric such that $\iint_D K^+ \leq 2\pi$. Let r be the supremum of the distance of points in D to the boundary. Then*

$$\lambda_1 \geq 1/4r^2. \quad (6.15)$$

The first result of this kind was due to Hayman [2] who showed that for simply-connected plane domains one has $\lambda_1 \geq 1/900r^2$. Hayman points out that an analogous result cannot hold in $\mathbf{R}^n$ for $n > 2$ without some further restriction, since narrow spikes pointed inward from the boundary of D would have a large effect on r , but little on λ_1 . However, he shows that his method does extend to domains in $\mathbf{R}^n$ provided a suitable restriction is placed on their boundaries. (For convex domains in $\mathbf{R}^n$, (6.15) holds: Osserman [4].)

Note that the value of $\lambda_1(D)$ is always bounded above by the value of λ_1 for any subdomain of D . Choosing as a subdomain an inscribed disk of radius r , and using the explicit value of λ_1 for such a disk in the plane, one has for arbitrary simply-connected plane domains that

$$1/2r < \Lambda \leq j/r \quad (6.16)$$

where $j \sim 2.4$ is the first zero of the Bessel function J_0 , and $\Lambda = \sqrt{\lambda_1}$ corresponds to the fundamental frequency of a drum in the shape of the domain D . As Hayman has pointed out, this leads to the surprising conclusion that one can tell the pitch of a (simply-connected) drum to within roughly an octave one way or the other simply from the knowledge of the largest circular disk it contains, regardless of any other considerations of its total size and shape.

B. Analysis and geometry. There would be no way to catalog all the applications that have been made of isoperimetric inequalities to problems ranging from the characterization of isometries (Firey [1], McMullen [1]) to the stability of minimal surfaces (Barbosa and doCarmo [2]). Going further afield, there would be interpretations of significant constants in terms of isoperimetric quantities (Finn [1], Huber [4]), and second-generation applications, such as that of Barbosa and doCarmo [1] using Peetre's [1] generalization of the Faber-Krahn inequality derived from the isoperimetric inequality on the sphere. See also Figiel, Lindenstrauss, and Milman [1] for an application of the isoperimetric inequality on S^n to the theory of Banach spaces.

We shall content ourselves here with just a few illustrations of ways that isoperimetric inequalities have been applied to some specific problems in geometry and analysis.

Our first example concerns the problem of type for simply-connected Riemann surfaces. By the Koebe uniformization theorem, every simply-connected Riemann surface is conformally equivalent to the sphere, the plane, or the unit disk. If the surface is not compact, then the first case cannot occur, and the "problem of type" is to give criteria for determining whether the surface is conformally the plane or the unit disk (referred to as *parabolic* and *hyperbolic* type respectively).

On the unit disk, $|z| < 1$, one has the *hyperbolic metric*

$$ds^2 = \frac{2}{1 - |z|^2} |dz|^2. \quad (6.17)$$

This metric has constant Gauss curvature $K \equiv -1$, and with it, the unit disk becomes a model for the hyperbolic plane. As we have seen in Part II of §4, the isoperimetric inequality then takes the form

$$L^2 \geq 4\pi A + A^2 \quad (6.18)$$

for simply-connected domains, and hence, for all such domains, one has

$$L > A. \quad (6.19)$$

It turns out that this property may be used to characterize hyperbolic Riemann surfaces. In fact, on an arbitrary Riemann surface one may consider *conformal metrics*, which are Riemannian metrics of the form

$$ds = \rho(z)|dz|$$

with respect to any local conformal parameter z .

THEOREM 6.5. *A simply-connected Riemann surface S is of hyperbolic type if and only if there exists a conformal metric on S such that (6.19) holds for every simply-connected domain on S .*

PROOF. If S is conformally the disk, then using the hyperbolic metric (6.17), (6.19) holds. Conversely, suppose there exists a conformal metric on S for which (6.19) holds. Then S is certainly not compact, since (6.19) would fail for the complement of a small disk. Hence S may be mapped conformally onto $|z| < R$, where $R < \infty$ or $R = \infty$ according as S is hyperbolic or parabolic. Any conformal metric on S is of the form $ds = \rho(z)|dz|$ in $|z| < R$. Let D_r be the domain $|z| < r$, and let $A(r)$ be the area of D_r , $L(r)$ the length of its boundary, both with respect to the given metric. Then

$$L(r) = \int_0^{2\pi} \rho(re^{i\theta})r \, d\theta,$$

$$A(r) = \int_0^r \int_0^{2\pi} \rho^2(te^{i\theta})t \, d\theta \, dt.$$

Applying (6.19) to the domain D_r , and using Schwarz' inequality gives

$$A(r)^2 < L(r)^2 \leq \int_0^{2\pi} \rho^2(re^{i\theta})r \, d\theta \int_0^{2\pi} r \, d\theta = 2\pi r \frac{dA}{dr}.$$

Thus, for $0 < r_0 < r_1 < R$, we have

$$\log \frac{r_1}{r_0} = \int_{r_0}^{r_1} \frac{1}{r} dr < 2\pi \int_{r_0}^{r_1} \left(\frac{dA}{dr} / A(r)^2 \right) dr = 2\pi \left[\frac{1}{A(r_0)} - \frac{1}{A(r_1)} \right],$$

whence

$$\log r_1 < \log r_0 + 2\pi / A(r_0)$$

for all $r_1 < R$. This leads to a contradiction in the case $R = \infty$, and shows that for a parabolic surface, there cannot exist *any* conformal metric such that (6.19) holds for every simply-connected domain.

Theorem 6.5 is a very special case of a theorem of Ahlfors [1, p. 188] describing relations between L and A that are compatible with the existence of a quasi-conformal map of a surface onto the entire plane.

Our next illustration is a theorem on conformal mapping of doubly-connected domains due to Carleman [1]. Carleman's original proof was by means of Laurent expansions. We give here a proof due to Szegő [1] based on the isoperimetric inequality.

THEOREM 6.6. *Consider the family of all doubly-connected plane domains bounded by an outer curve C_1 and an inner curve C_0 . For each domain D , let A_i be the area bounded by C_i , $i = 0, 1$. Then among all domains conformally equivalent to a given one, the minimum of A_1/A_0 is attained by a circular annulus.*

PROOF. Let $r_0 < |z| < r_1$, be a given annulus, and let D be its image under a conformal map $f(z)$. Let $L(r)$ be the length of the image of $|z| = r$, and $A(r)$ the area enclosed. Then

$$\begin{aligned} 4\pi A(r) &\leq L(r)^2 = \left[\int_0^{2\pi} |f'(re^{i\theta})| r d\theta \right]^2 \\ &\leq \int_0^{2\pi} |f'(re^{i\theta})|^2 r d\theta \int_0^{2\pi} r d\theta \\ &= 2\pi r A'(r), \end{aligned}$$

and

$$2/r \leq A'(r)/A(r), \quad r_0 < r < r_1.$$

Integrating from r_0 to r_1 , yields

$$\log \frac{r_1^2}{r_0^2} = 2 \log \frac{r_1}{r_0} \leq \log \frac{A(r_1)}{A(r_0)} = \log \frac{A_1}{A_0},$$

or

$$\pi r_1^2 / \pi r_0^2 \leq A_1 / A_0,$$

which proves the theorem.

Pólya and Szegő [1, p. 220] derived an analogous result for doubly-connected domains on a sphere, using the isoperimetric inequality (4.2) for domains on a sphere. Recently, Bandle [1, §2.1] used the general isoperimetric inequality (4.29) to obtain a simultaneous generalization of this Pólya-Szegő

result and the earlier one of Carleman to multiply-connected domains on general surfaces. Furthermore, she applies the result to obtain bounds on torsional rigidity [1, §2.5] and fundamental frequency [1, §3.3] of multiply-connected domains on surfaces. (See also in this connection, the papers of Pólya-Weinstein [1] and Gasser-Hersch [1].)

Aside from direct use of the isoperimetric inequality, there have been many applications of the method of symmetrization to conformal mapping, and to different areas of complex analysis. For older results, see the book of Hayman [1, Chapter IV]. Among more recent ones, we note in particular two beautiful and important areas of application. The first is to the theory of conformal capacity and conformal and quasiconformal mappings in n dimensions. The initial basic contributions were made by Gehring [1], [2] who showed that 3-dimensional conformal capacity decreases under circular symmetrization. He derives a number of consequences, including the fact that a quasi-conformal map of the open ball in $\mathbf{R}^3$ induces a quasi-conformal map on the boundary, and that the Liouville theorem, asserting that the only conformal maps in $\mathbf{R}^3$ are the Moebius transformations, holds with no differentiability assumptions. A fundamental paper of Mostow [1] extends Gehring's basic results to n dimensions, and goes on to show that compact Riemannian manifolds of dimension 3 or more with constant negative curvature must be conformally equivalent if they are diffeomorphic.

The other area of application was developed by Baernstein [1] who shows that a certain symmetrization process preserves subharmonic functions. He derives many consequences, including new extremal properties of the Koebe function. Some extensions of these results to higher dimensions are given by Baernstein and Taylor [1].

Two more very pretty applications of symmetrization to problems in analysis and geometry may be found in papers of Weinberger [1] and Moser [1]. For modern treatments of the symmetrization process itself, see Hilden [1], Sperner [1], [2], Spiegel [1].

We come next to the theory of minimal submanifolds and manifolds of prescribed mean curvature. In both the classical and more modern approaches to the subject, isoperimetric inequalities have played a significant role. A number of examples were cited in §§5B and 5C, and the references given there may be consulted concerning further applications.⁷ A quite different approach to surfaces of constant mean curvature, using the isoperimetric inequality for domains in $\mathbf{R}^3$, was introduced by Wente [1], [2]. Wente's method was extended to the study of surfaces of prescribed mean curvature by Steffen [1], [2] who shows that solutions of prescribed mean curvature Plateau problems exist whenever certain isoperimetric conditions are verified. (See also Radó [2, p. 534], for generalized isoperimetric inequalities in $\mathbf{R}^3$.) There are also related papers of Heinz [2], Hildebrandt and Wente [1], and Wente [3].

An important inequality in the theory of minimal surfaces is the following. Let M be an m -dimensional minimal submanifold in $\mathbf{R}^n$, and assume that the

⁷For a recent discussion of the important role of isoperimetric inequalities in the development of geometric measure theory see §8 of the 1977 Colloquium Lectures of Federer [2].

origin lies on M . Using the notation of §2, let

$$M_r = M \cap B_r^n, \quad (6.20)$$

and

$$V(r) = m\text{-dimensional measure of } M_r. \quad (6.21)$$

THEOREM 6.7 (FLEMING [1, p. 88]). *If M has no boundary points in B_r^n , then*

$$V(r) \geq \omega_n r^m. \quad (6.22)$$

We should note that this inequality was known earlier, and is much easier to prove, if M is not only minimal (in our sense of zero mean curvature), but actually area minimizing. In that connection, see Stolzenberg [1] for an excellent discussion of the relevance of (6.22) to the theory of complex analytic subvarieties. Allard [1] and Bombieri [1] derive an isoperimetric inequality on stationary varifolds and minimal graphs, respectively, using (6.22) and a covering theorem; (6.22) is in fact *equivalent* to the isoperimetric inequality (2.7) for domains on minimal submanifolds whose boundary lies on a sphere centered at some point of the submanifold. (See Alexander, Hoffman, and Osserman [1, Theorems 3.2 and 3.3].)

It is worth noting that a stronger form of (6.22) has recently been proved by H. Alexander (unpublished) and Yau [1, p. 506]. Namely, (6.22) holds also if by $V(r)$ we mean the m -dimensional measure of the geodesic ball of radius r centered at any point of M . It follows, for example, that if M is complete, it has infinite m -dimensional measure.

A combination of (6.22) with the isoperimetric-type inequality (5.18) has recently been used to show that even with real analytic boundary values on the unit ball in $\mathbf{R}^4$, there does not always exist a solution to the Dirichlet problem for the minimal surface equation in higher codimension. (See Lawson and Osserman [1, Theorem 6.1].)

We conclude our discussion of this circle of ideas with one more application.

THEOREM 6.8. *Let C be a rectifiable Jordan curve in $\mathbf{R}^n$, and let B be a set in $\mathbf{R}^n$ which links C . Let L be the length of C and let r be the distance between B and C . Then $L \geq 2\pi r$. Equality holds only when C is a euclidean circle of radius r .*

PROOF. Let S be a solution of Plateau's problem for C . That is, S is a simply-connected minimal surface spanning C . Since B and C are linked, it follows that $B \cap S \neq \emptyset$. Let p be a point of $B \cap S$. By a translation we may assume that p is the origin. By hypothesis, the boundary of S (which is the curve C) lies outside of B_r^n . We may therefore apply Theorem 6.7 and conclude that the area A of S satisfies

$$A \geq \pi r^2. \quad (6.23)$$

But then the isoperimetric inequality for minimal surfaces (Theorem 4.2(i) above) implies

$$L^2 \geq 4\pi A \geq 4\pi^2 r^2$$

which is the desired inequality. For equality to hold, it must also hold in the isoperimetric inequality, and that can only happen for a plane circle.

Theorem 6.8 was conjectured by Gehring [3] and first proved by M. Ortel (unpublished). The proof given above was suggested by Osserman [2]. Another proof has been given by Edelstein and Schwarz [1]. Gehring [4] has proved a higher-dimensional version of Theorem 6.8, but not with the best possible constant, which is presumably attained by linked spheres in orthogonal subspaces. A different proof which yields the best constant for a 2-sphere linked by a 1-sphere in $\mathbb{R}^4$ has been obtained by M. Gage [1].

As our last illustration of how the isoperimetric inequality may be applied to other areas of mathematics, we give an analytic inequality which can be used to prove the prime number theorem. The inequality in question and its application to the prime number theorem is contained in a paper of Bang [1].

THEOREM 6.9. *Let $f(t)$, $g(t)$ be real functions on $[0, \infty)$ satisfying*

$$|f(t)| < a, \quad |g(t)| < b, \quad (6.24)$$

and

$$|f'(t)| < 1, \quad |g'(t)| < 1. \quad (6.25)$$

Let

$$M = \limsup_{T \rightarrow \infty} \left| \frac{1}{T} \int_0^T f(t) g'(t) dt \right|.$$

Then

$$M \leq \sqrt{\frac{2ab}{\pi}}. \quad (6.26)$$

PROOF. For each $T > 0$, define a closed curve C as the image of the interval $[0, T + 1]$ as follows. On $[0, T]$ we set $x = f(t)$, $y = g(t)$. The image of $[T, T + 1]$ is the curve joining $(f(T), g(T))$ to $(f(0), g(0))$ by means of two horizontal segments, and one vertical segment along the y -axis. Then

$$\int_C x dy = \int_0^{T+1} f(t) g'(t) dt = \int_0^T f(t) g'(t) dt \quad (6.27)$$

since along the y -axis $f(t) \equiv 0$, and along the horizontal segments, $g'(t) \equiv 0$. By virtue of (6.24), the total length of the three line segments is at most $2a + 2b$. Hence by (6.25), the length L of C satisfies

$$\begin{aligned} L &< \int_0^T \sqrt{f'(t)^2 + g'(t)^2} dt + 2(a + b) \\ &< T\sqrt{2} + 2(a + b). \end{aligned} \quad (6.28)$$

As we have noted in §1, the complement of C consists of the union of domains D_k of area A_k , and if n_k denotes the winding number of C about a point of D_k , then

$$\begin{aligned}
\left| \int_0^T f(t) g'(t) dt \right| &= \left| \int_C x dy \right| = \left| \sum n_k A_k \right| \\
&\leq \sum |n_k| A_k \leq \left[\left(\sum n_k^2 A_k \right) \left(\sum A_k \right) \right]^{1/2} \\
&\leq \left[\frac{L^2}{4\pi} \cdot 4ab \right]^{1/2} = L \sqrt{\frac{ab}{\pi}}
\end{aligned} \tag{6.29}$$

where we have used (6.27), the Cauchy-Schwarz inequality, the Banchoff-Pohl inequality (1.10), and the fact that by (6.24), the curve C lies in a rectangle of sides $2a$ and $2b$, so that the sum of all the areas A_k for which $n_k \neq 0$ is at most $4ab$.

Finally, substituting (6.28) in (6.29) and using the definition of M , one obtains the inequality (6.26).

One comment concerning the above proof: (6.29) contains the isoperimetric-type inequality

$$A \leq L \sqrt{\frac{ab}{\pi}} \tag{6.30}$$

for a curve contained in a $2a \times 2b$ rectangle. One is led to pose the question: what is the best upper bound for A/L for a curve lying in a rectangle? (That question was raised in this context by Bombieri.) Several authors have considered versions of the same problem recently. See in particular Lin [1], Singmaster and Soupporis [1] and further references there; also an older related paper of Besicovitch [4]. Note that if $b > a$, then Bonnesen's inequality (4.5) gives

$$A/L < a \tag{6.31}$$

since in this case $\rho = a$. Thus one has a uniform upper bound independent of b , superceding (6.30) when $b > \pi a$. On the other hand, the application made by Bang uses only the case $b = a$, in which case (6.30) gives

$$\frac{A}{L} \leq \frac{a}{\sqrt{\pi}} \sim 0.564a. \tag{6.32}$$

But in that case (of a curve enclosed in a square) the optimal value is

$$\frac{A}{L} \leq \frac{4 - 2\sqrt{\pi}}{4 - \pi} a \sim 0.53a, \tag{6.33}$$

(see, for example, Lin [1, Theorem 2]), so that (6.30) is close to best possible.

BIBLIOGRAPHY

L. Ahlfors

1. *Zur Theorie der Überlagerungsflächen*, Acta Math. **65** (1935), 157–194.

A. D. Aleksandrov (A. D. Alexandrow)

1. *Isoperimetric inequalities for curved surfaces*, Dokl. Akad. Nauk USSR **47** (1945), 235–238.
2. *Die innere Geometrie der konvexen Flächen*, Akademie-Verlag, Berlin, 1955.
3. *Uniqueness theorems for surfaces in the large*. V, Vestnik Leningrad Univ. **13** (1958), 5–8; English transl., Amer. Math. Soc. Transl. (2) **21** (1962), pp. 412–416.

4. *A characteristic property of spheres*, Annali Mat. Pura Appl. **58** (1962), 303–315.
- A. D. Aleksandrov and V. V. Strel'tsov
1. *Estimates of the length of a curve on a surface*, Dokl. Akad. Nauk USSR **93** (1953), 221–224.
 2. *The isoperimetric problem and estimates of the length of a curve on a surface*, Proc. Steklov Inst. Math. **76** (1965), 67–80; English transl., Proc. Steklov Inst. Math. (1967), 81–99.
- A. D. Aleksandrov and V. A. Zalgaller
1. *Two-dimensional manifolds of bounded curvature*, Trudy Mat. Inst. Steklov **63** (1962); English transl., *Intrinsic geometry of surfaces*, Amer. Math. Soc., Providence, R. I., 1967.
 2. *Two-dimensional manifolds of bounded curvature*, Trudy Mat. Inst. Steklov **76** (1965); English transl., Proc. Steklov Inst. Math. (1967).
- H. Alexander, D. Hoffman and R. Osserman
1. *Area estimates for submanifolds of euclidean space*, INDAM Symposia Math. Vol. **14** (1974), 445–455.
- W. K. Allard
1. *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.
- F. J. Almgren, Jr.
1. *Three theorems on manifolds with bounded mean curvature*, Bull. Amer. Math. Soc. **71** (1965), 755–756.
 2. *The theory of varifolds*, (mimeographed notes), Princeton, N. J., 1965.
- T. Aubin
1. *Problèmes isopérimétriques et espaces de Sobolev*, C. R. Acad. Sci. Paris Sér. A **280** (1975), 279–281.
 2. *Le problème de Yamabe concernant la courbure scalaire*, C. R. Acad. Sci. Paris Sér. A **280** (1975), 721–724.
 3. *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry **11** (1976), 573–598.
 4. *Inégalités concernant la première valeur propre non nulle du laplacien pour certaines variétés riemanniennes*, C. R. Acad. Sci. Paris Sér. A **281** (1975), 979–982.
- A. Baernstein
1. *Integral means, univalent functions and circular symmetrization*, Acta Math. **133** (1974), 139–169.
- A. Baernstein and B. A. Taylor
1. *Spherical rearrangements, subharmonic functions, and $*$ -functions in n -space*, Duke Math. J. **43** (1976), 245–268.
- T. F. Banchoff and W. F. Pohl
1. *A generalization of the isoperimetric inequality*, J. Differential Geometry **6** (1971), 175–192.
- C. Bandle
1. *Konstruktion isoperimetrischer Ungleichungen der mathematischen Physik aus solchen der Geometrie*, Comment. Math. Helv. **46** (1971), 182–213.
 2. *Extremaleigenschaften von Kreissektoren und Halbkugeln*, Comment. Math. Helv. **46** (1971), 356–380.
 3. *A geometrical isoperimetric inequality and applications to problems of mathematical physics*, Comment. Math. Helv. **49** (1974), 496–511.
 4. *On a differential inequality and its applications to geometry*, Math. Z. **147** (1976), 253–261.
- Th. S. V. Bang
1. *An inequality for real functions of a real variable and its application to the prime number theorem*, On Approximation Theory (Proc. Conf., Oberwolfach, August 1963), P. L. Butzer and J. Korevaar, editors, Birkhäuser, Basel, 1964, pp. 155–160.
- J. L. Barbosa and M. P. do Carmo
1. *On the size of a stable minimal surface in $\mathbb{R}^3$* , Amer. J. Math. **98** (1976), 515–528.
 2. *Stable minimal surfaces in $\mathbb{R}^n$* (to appear).
 3. *A proof of a general isoperimetric inequality for surfaces* (to appear).
- E. F. Beckenbach and T. Radó
1. *Subharmonic functions and surfaces of negative curvature*, Trans. Amer. Math. Soc. **35** (1933), 662–674.

D. C. Benson

1. *Inequalities involving integrals of functions and their derivatives*, J. Math. Anal. Appl. **17** (1967), 292–308.
2. *Sharpened form of the isoperimetric inequality*, Amer. Math. Monthly **77** (1970), 29–34.

M. Berger

1. *Eigenvalues of the Laplacian*, pp. 121–125 of Global Analysis, Proc. Sympos. Pure Math. **16**, Amer. Math. Soc., Providence, R. I., 1970.
2. *Sur les premières valeurs propres des variétés Riemanniennes*, Compositio Math. **26** (1973), 129–149.
3. *Geometry of the spectrum*. I, pp. 129–152 of Differential Geometry, Proc. Sympos. Pure Math., vol. 27, part II, Amer. Math. Soc., Providence, R. I., 1975.

M. Berger, P. Gauduchon and E. Mazet

1. *Le spectre d'une variété Riemannienne*, Lecture Notes in Math., vol. 194, Springer-Verlag, Berlin, 1971.

F. Bernstein

1. *Über die isoperimetrische Eigenschaft des Kreises auf der Kugeloberfläche und in der Ebene*, Math. Ann. **60** (1905), 117–136.

A. S. Besicovitch

1. *On the definition and value of the area of a surface*, Quart. J. Math., Oxford Ser. **16** (1945), 86–102.
2. *A variant of a classical isoperimetric problem*, Quart. J. Math. Oxford Ser. **20** (1949), 84–94.
3. *Parametric surfaces*, Bull. Amer. Math. Soc. **56** (1950), 288–296.
4. *Variants of a classical isoperimetric problem*, Quart. J. Math. Oxford Ser. **3** (1952), 42–49.

W. Blaschke

1. *Kreis und Kugel*, 2nd ed., Gruyter, Berlin, 1956.
2. *Vorlesungen über Differentialgeometrie*. I, 3rd ed., Springer-Verlag, Berlin, 1930.
3. *Vorlesungen über Integralgeometrie*. I, Teubner, Leipzig, 1936; 3rd ed., VEB Deutscher Verlag, Berlin, 1955.
4. *Griechische und Anschauliche Geometrie*, Oldenbourg, Munich, 1953.

W. Blaschke and K. Leichtweiß

1. *Elementare Differentialgeometrie*, Springer-Verlag, Berlin, 1973.

W. Blaschke and H. Reichardt

1. *Einführung in die Differentialgeometrie*, 2nd ed., Springer, Berlin, 1960.

D. Bleecker and J. L. Weiner

1. *Extrinsic bounds on λ_1 of Δ on a compact manifold*, Comment. Math. Helv. **51** (1976), 601–609.

G. Bol

1. *Isoperimetrische Ungleichung für Bereiche auf Flächen*, Jber. Deutsch. Math.-Verein. **51** (1941), 219–257.

E. Bombieri

1. *Theory of minimal surfaces, and a counter-example to the Bernstein conjecture in higher dimensions*, Lecture notes, Courant Institute of Math., Spring, 1970.

E. Bombieri, E. deGiorgi, and M. Miranda

1. *Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche*, Arch. Rational Mech. Anal. **4** (1969), 255–267.

E. Bombieri and E. Giusti

1. *Local estimates for the gradient of non-parametric surfaces of prescribed mean curvature*, Comm. Pure Appl. Math. **26** (1973), 381–394.

T. Bonnesen

1. *Über eine Verschärfung der isoperimetrischen Ungleichheit des Kreises in der Ebene und auf der Kugeloberfläche nebst einer Anwendung auf eine Minkowskische Ungleichheit für konvexe Körper*, Math. Ann. **84** (1921), 216–227.
2. *Les problèmes des Isopérimètres et des Iséiphanes*, Gauthier-Villars, Paris, 1929.

T. Bonnesen and W. Fenchel

1. *Theorie der konvexen Körper*, Ergebnisse der Math. und ihrer Grenzgebiete, vol. 3, Springer-Verlag, Berlin, 1934; Berichtigter Reprint, 1974.

S. Breuer and J. J. Roseman

1. *A lower bound for the eigenvalues of the elliptic Dirichlet problem for a general domain in terms of its characteristic dimension*, SIAM J. Math. Anal. (to appear).

Yu. D. Burago

1. *Isoperimetric inequalities in the theory of surfaces of bounded external curvature*, Seminars in Math., V. A. Steklov Math. Inst., Leningrad, Vol. 10, 1968; English transl., Consultants Bureau, New York and London, 1970.
2. *Note on the isoperimetric inequality on two-dimensional surfaces*, Siberian Math. J. **14** (1973), 463–465.
3. *On the injectivity radius of surfaces with curvature bounded above*, Ukrain Geometr. Sb. **21** (1978), 10–14. (Russian)

Yu. D. Burago and V. A. Zalgaller

1. *Isoperimetric problems for regions on a surface having restricted width*, Two-Dimensional Manifolds of Bounded Curvature, Trudy Mat. Inst. Steklov **76** (1965); 81–87; English transl., Proc. Steklov Inst. **76** (1965), 109–141 (1967).

P. Buser

1. *Untersuchungen über den ersten Eigenwert des Laplace-Operators auf kompakten Flächen*, Ph. D. Thesis, Basel, 1976.
2. *Riemannsche Flächen mit Eigenwerten in $(0, \frac{1}{4})$* , Comment. Math. Helv. **52** (1977), 25–34.
3. *Ueber eine Ungleichung von Cheeger*, Math. Z. **158** (1978), 245–252.
4. *Riemannsche Flächen mit grosser Kragenweite*, preprint, Bonn, 1977.
5. *Ueber den ersten Eigenwert des Laplace-Operators auf kompakten Mannigfaltigkeiten nichtpositiver Krümmung*, preprint, Bonn, 1977.
6. *Beispiele für λ_1 auf kompakten Mannigfaltigkeiten*, preprint, Bonn, 1977.

T. Carleman

1. *Über ein Minimalproblem der mathematischen Physik*, Math. Z. **1** (1918), 208–212.
2. *Zur Theorie der Minimalflächen*, Math. Z. **9** (1921), 154–160.

G. D. Chakerian

1. *The isoperimetric theorem for curves on minimal surfaces*, Proc. Amer. Math. Soc. **69** (1978), 312–313.

I. Chavel

1. *On A. Hurwitz' method in isoperimetric inequalities* (to appear).

I. Chavel and E. A. Feldman

1. *An optimal Poincaré inequality for convex domains of non-negative curvature*, Arch. Rational Mech. Anal. **65** (1977), 263–273.
2. *Isoperimetric inequalities on curved surfaces* (to appear).

J. Cheeger

1. *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in Analysis, A Symposium in Honor of Salomon Bochner, Robert C. Gunning, editor, Princeton Univ. Press, Princeton, N. J., 1970, pp. 195–199.

S.-Y. Cheng

1. *Eigenvalue comparison theorems and its geometric applications*, Math. Z. **143** (1975), 289–297.
2. *Eigenfunctions and eigenvalues of Laplacian*, Differential Geometry, Proc. Sympos. Pure Math., vol. 27, part II, Amer. Math. Soc., Providence, R. I., 1975, pp. 185–193.
3. *Eigenfunctions and nodal sets*, Comment. Math. Helv. **51** (1976), 43–55.

O. Chisini

1. *Sulla teoria elementare degli isoperimetri*, Questioni riguardanti le matematiche elementari, vol. 3., Bologna, 1927, pp. 201–310.

R. Courant

1. *Dirichlet's principle, conformal mapping and minimal surfaces*, Interscience, New York, 1950. Springer-Verlag, New York, 1977 (reprint).

I. A. Danelich

1. *An estimate for the area of a surface of bounded absolute mean integral curvature in terms of its absolute mean integral curvature and the sum of the boundary curve lengths*, Siberian Math. J. **7** (1966), 951–955.

B. V. Dekster

1. *An inequality of the isoperimetric type for a domain in a Riemannian space*, Mat. Sb. **90** (1973); English transl., Math. USSR-Sb. **19** (1973), 257–274.

V. I. Diskant

1. *Bounds for the discrepancy between convex bodies in terms of the isoperimetric difference*, Siberian Math. J. **13** (1972), 529–532.
2. *Stability of the solution of the Minkowski equation*, Siberian Math. J. **14** (1973), 466–469.
3. *Strengthening of an isoperimetric inequality*, Siberian Math. J. **14** (1973), 608–611.
4. *A generalization of Bonnesen's inequalities*, Dokl. Akad. Nauk SSSR **213** (1973) = Soviet Math. Dokl. **14** (1973), 1728–1731.

M. P. doCarmo

1. *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, N. J., 1976.

M. Edelstein and B. Schwarz

1. *On the length of linked curves*, Israel J. Math. **23** (1976), 94–95.

C. Faber

1. *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitzungsber. Bayer. Akad. der Wiss. Math.-Phys., Munich, 1923, 169–172.

H. Federer

1. *Geometric measure theory*, Springer-Verlag, New York, 1969.
2. *Colloquium Lectures*, Amer. Math. Soc. Summer Meeting, Seattle, Wash., August 1977.

H. Federer and W. H. Fleming

1. *Normal and integral currents*, Ann. of Math. **72** (1960), 458–520.

J. Feinberg

1. *The isoperimetric inequality for doubly-connected minimal surfaces in $\mathbb{R}^n$* , Ph. D. Thesis, Stanford University, 1976.
2. *The isoperimetric inequality for doubly-connected minimal surfaces in $\mathbb{R}^n$* , J. d'Anal. Math. **32** (1977), 249–278.
3. *Some Wirtinger-like inequalities* (to appear).

L. Fejes Tóth

1. *Lagerungen in der Ebene, auf der Kugel, und in Raum*, Die Grundlehren der math. Wissenschaften, vol. 65, 2nd ed., Springer-Verlag, Berlin, 1972.

F. Fiala

1. *Le problème des isopérimètres sur les surfaces ouvertes à courbure positive*, Comment. Math. Helv. **13** (1940/41), 293–346.

R. Figiel, J. Lindenstrauss and V. D. Milman

1. *The dimension of almost spherical sections of convex bodies*, Acta Math. **139** (1977), 53–94.

R. Finn

1. *On a class of conformal metrics, with application to differential geometry in the large*, Comment. Math. Helv. **40** (1965), 1–30.

Wm. J. Firey

1. *Affinities which preserve lower dimensional volumes*, Amer. Math. Monthly **72** (1965), 645.

W. H. Fleming

1. *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo **11** (1962), 69–90.

L. E. Fraenkel and M. S. Berger

1. *A global theory of steady vortex rings in an ideal fluid*, Acta Math. **132** (1974), 13–51.

M. Gage

1. Ph. D. thesis, Stanford University, 1978.

G. Galilei

1. *Two new sciences*, translated, with Introduction and Notes, by Stillman Drake, Univ. of Wisconsin Press, Madison, Wisc., 1974.

P. R. Garabedian

1. *Partial differential equations*, Wiley, New York, 1964.

T. Gasser and J. Hersch

1. *Über Eigenfrequenzen einer mehrfach zusammenhängenden Membran: Erweiterung von isoperimetrischen Sätzen von Pólya und Szegő*, Z. Angew. Math. Phys. **19** (1968), 672–675.

F. W. Gehring

1. *Symmetrization of rings in space*, Trans. Amer. Math. Soc. **101** (1961), 499–519.
2. *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.
3. *Problem 7.22*, Symposium on Complex Analysis, Univ. of Kent (1973), Cambridge Univ. Press, Cambridge, 1974, p. 176.
4. *The Hausdorff measure of sets which link in euclidean space*, Contributions to Analysis, A collection of papers dedicated to Lipman Bers, Academic Press, New York, 1974, pp. 159–167.

H. Hadwiger

1. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag, Berlin, 1957.
2. *Gitterperiodische Punktmengen und Isoperimetrie*, Monatsch. Math. **76** (1972), 410–418.

K. Hanes

1. *Isoperimetric inequalities for manifolds with boundary*, J. Differential Geometry **7** (1972), 525–534.

G. H. Hardy, J. E. Littlewood, and G. Pólya

1. *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

P. Hartman

1. *Geodesic parallel coordinates in the large*, Amer. Math. **86** (1964), 705–727.
2. *Some characterizations of the euclidean sphere*, Nonlinear Analysis, Theory Methods and Applications **1** (1976), 37–48.

W. K. Hayman

1. *Multivalent functions*, Cambridge Univ. Press, Cambridge, 1958.
2. *Some bounds for principal frequency*, Applicable Analysis (to appear).

E. Heinz

1. *An inequality of isoperimetric type for surfaces of constant mean curvature*, Arch. Rational Mech. Anal. **33** (1969), 155–168.
2. *Elementare Bemerkung zur isoperimetrischen Ungleichung in $\mathbb{R}^3$* , Math. Z. **132** (1973), 319–322.

E. Heinz and S. Hildebrandt

1. *The number of branch points of surfaces of bounded mean curvature*, J. Differential Geometry **4** (1970), 227–235.

J. Hersch

1. *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C. R. Acad. Sci. Paris Sér. A **270** (1970), 1645–1648.

B. Herz and J. Kaapke

1. *Ein isoperimetrisches Problem mit Nebenbedingung*, Elem. Math. **28** (1973), 63–65.

S. Hildebrandt

1. *Über Flächen konstanter mittlerer Krümmung*, Math. Z. **112** (1969), 107–144.

S. Hildebrandt and H. Wente

1. *Variational problems with obstacles and a volume constraint*, Math. Z. **135** (1973/74), 55–68.

K. Hildén

1. *Symmetrization of functions in Sobolev spaces and the isoperimetric inequality*, Manuscripta Math. **18** (1976), 215–235.

D. Hoffman

1. *Lower bounds on the first eigenvalue of the Laplacian on Riemannian manifolds* (to appear).
2. *Remarks on a geometric constant of Yau* (to appear).

D. Hoffman and J. Spruck

1. *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Communications Pure Appl. Math. **27** (1974), 715–727; correction, *ibid.* **28** (1975), 765–766.

C. O. Horgan and L. T. Wheeler

1. *Isoperimetric inequalities for the Dirichlet Eigenvalue problem*, Quart. Appl. Math. **35** (1977), 406–409.

H. Hopf

1. *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Math. Nachr. **4** (1950–51), 232–249.

C. C. Hsiung

1. *Isoperimetric inequalities for two-dimensional Riemannian manifolds with boundary*, Ann. of Math. (2) **73** (1961), 213–220.

A. Huber

1. *On the isoperimetric inequality on surfaces of variable Gaussian curvature*, Ann. of Math. (2) **60** (1954), 237–247.
2. *Zur isoperimetrischen Ungleichung auf gekrümmten Flächen*, Acta. Math. **97** (1957), 95–101.
3. *Zum potentialtheoretischen Aspekt der Alexandrowschen Flächentheorie*, Comment. Math. Helv. **34** (1960), 99–126.
4. *Zur Isoperimeterproblem auf vollständigen Flächen mit summierbaren Gausscher Krümmung*, Arch. Rational Mech. Anal. **24** (1967), 173–192.

V. K. Ionin

1. *On isoperimetric and various other inequalities for a manifold of bounded curvature*, Siberian Math. J. **10** (1969), 233–243.
2. *Isoperimetric inequalities for surfaces of negative curvature*, Siberian Math. J. **13** (1972), 650–653.

H. H. Johnson and J. Osaka

1. *An isoperimetric inequality for polyhedra*, Amer. Math. Monthly **81** (1974), 58–61.

M. Kac

1. *Can one hear the shape of a drum?* Amer. Math. Monthly **73** (1966), 1–23.

H. Karcher

1. *Umkreise und Inkreise konvexer Kurven in der sphärischen und der hyperbolischen Geometrie*, Math. Ann. **177** (1968), 122–132.
2. *Anwendungen der Alexandrowschen Winkelvergleichssätze*, Manuscripta Math. **2** (1970), 77–102.

H. Kaul

1. *Isoperimetrische Ungleichung und Gauss-Bonnet-Formel für H-Flächen in Riemannschen Mannigfaltigkeiten*, Arch. Rational Mech. Anal. **45** (1972), 194–221.
2. *Ein Einschliessungssatz für H-Flächen in Riemannschen Mannigfaltigkeiten*, Manuscripta Math. **5** (1971), 103–112.
3. *Remarks on the isoperimetric inequality for multiply-connected H-surfaces*, Math. Z. **128** (1972), 271–276.

N. D. Kazarinoff

1. *Geometric inequalities*, New Mathematical Library, Random House, New York, and Yale University, 1961.

V. Klee

1. *Shapes of the future—some unsolved problems in geometry*. Part II: *Three dimensions* (Viewer's Manual to accompany film), 1972.

M.-Th. Kohler-Jobin

1. *Démonstration de l'inégalité isopérimétrique $P\lambda^2 > \pi j_0^4/2$, conjecturée par Pólya et Szegő*, C. R. Acad. Sci. Paris Sér. A **281** (1975), 119–121.
2. *Une inégalité isopérimétrique entre la fréquence fondamentale d'une membrane inhomogène et l'énergie d'équilibre du problème de Poisson correspondant*, C. R. Acad. Sci. Paris Sér. A **283** (1976), 65–68.
3. *Une propriété de monotonie isopérimétrique qui contient plusieurs théorèmes classiques*, C. R. Acad. Sci. Paris Sér. A **284** (1977), 917–920.

E. Krahn

1. *Über eine von Rayleigh formulierte Mininaleigenschaft des Kreises*, Math. Ann. **94** (1925), 97–100.
2. *Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen*, Acta Comm. Univ. Tartu (Dorpat) **A9** (1926), 1–44.

H. B. Lawson, Jr. and R. Osserman

1. *Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system*, Acta Math. **139** (1977), 1–17.

H. Lebesgue

1. *En marge du calcul des variations; Une introduction au calcul des variations et aux inégalités géométriques*, Institut de Mathématiques, Geneva, 1963.

H. Lewy

1. *Aspects of the Calculus of Variations* (notes by J. W. Green), Univ. of Calif. Press, Berkeley, Calif., 1939.

H. Liebmann

1. *Ueber die Verbiegung der geschlossenen Flächen positiver Krümmung*, Math. Ann. **53** (1900), 81–112.

T.-P. Lin

1. *Maximal area under constraint*, Math. Mag. **50** (1977), 32–34.

S. Lozinskii

1. *On subharmonic functions and their applications to the theory of surfaces*, Izv. Akad. Nauk USSR Ser. Mat. **8** (1944), 175–194.

J. Luttinger

1. *Generalized isoperimetric inequalities*. I, II, III, J. Mathematical Phys. **14** (1973), 586–593, 1444–1447, 1448–1450.

V. G. Maz'ya

1. *Classes of domains and imbedding theorems for function spaces*, Dokl. Akad. Nauk SSSR **133** (1960), 527–530; English transl., Soviet Math. Dokl. **1** (1960), 882–885.
2. *On certain integral inequalities for functions of many variables*, Problemy Matematicheskogo Analiza, No. 3: Integral'nyei Differentsial'nye Operatory. Differentsial'nye Uravneniya, Leningrad 1972, pp. 33–68; English transl., J. Soviet Math. **1** (1973), 205–234.

H. P. McKean

1. *An upper bound to the spectrum of Δ on a manifold of negative curvature*, J. Differential Geom. **4** (1970), 359–366.

P. McMullen

1. *Area preserving homeomorphisms*, Elem. Math. **30** (1975), 86–87.

J. H. Michael and L. M. Simon

1. *Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$* , Comm. Pure Appl. Math. **26** (1973), 361–379.

F. Minding

1. *Zur Theorie der Kurven kürzesten Umrings bei gegebenen Flächeninhalt auf krummen Flächen*, J. Reine Angew. Math. **86** (1879), 279–289.

M. Miranda

1. *Disuguaglianze di Sobolev sulle ipersuperfici minimali*, Rend. Sem. Mat. Univ. Padova **38** (1967), 69–79.

D. S. Mitrinović

1. *Analytic inequalities*, Springer, Berlin, 1970.

J. Moser

1. *A sharp form of an inequality by N. Trudinger*, Indiana Univ. J. Math. **20** (1970/71), 1077–1092.

G. D. Mostow

1. *Quasi-conformal mappings in n -space and the rigidity of hyperbolic space forms*, Inst. Hautes Etudes Sci. Publ. Math. **34** (1968), 53–104.

Z. Nehari

1. *On the principal frequencies of a membrane*, Pacific J. Math. **8** (1958), 285–293.

J. C. C. Nitsche

1. *The isoperimetric inequality for multiply-connected minimal surfaces*, Math. Ann. **160** (1965), 370–375.
2. *An isoperimetric property of surfaces with moveable boundaries*, Amer. Math. Monthly **77** (1970), 359–362.
3. *Vorlesungen über Minimalflächen*, Springer-Verlag, Berlin, 1975.

R. Osserman

1. *Isoperimetric and related inequalities*, Proc. Sympos. Pure Math. vol. 27, Amer. Math. Soc. Providence, R. I., 1975, pp. 207–215.
2. *Some remarks on the isoperimetric inequality and a problem of Gehring*, J. Analyse Math. **30** (1976), 404–410.
3. *A note on Hayman's Theorem on the bass note of a drum*, Comment. Math. Helv. **52** (1977), 545–555.

4. *Bonnesen-style isoperimetric inequalities*, Amer. Math. Monthly (to appear).
- R. Osserman and M. Schiffer
1. *Doubly-connected minimal surfaces*, Arch. Rational Mech. Anal. **58** (1975), 285–307.
- T. Otsuki
1. *A remark on the Sobolev inequality for Riemannian manifolds*, Proc. Japan Acad. **51** (1975), 785–789.
- L. E. Payne
1. *Isoperimetric inequalities and their applications*, SIAM Rev. **9** (1967), 453–488.
- L. E. Payne and H. F. Weinberger
1. *Some isoperimetric inequalities for membrane frequencies and torsional rigidity*, J. Math. Anal. Appl. **2** (1961), 210–216.
- J. Peetre
1. *A generalization of Courant's nodal line theorem*, Math. Scand. **5** (1957), 15–20.
- H. Poincaré
1. *Sur un théorème de M. Liapounoff relatif à l'équilibre d'une masse fluide*, C. R. Acad. Sci. Paris **104** (1887), 622–625.
 2. *Figures d'équilibre d'une masse fluide* (Leçons professées à la Sorbonne en 1900), edited by L. Dreyfus, Gauthier-Villars, Paris, 1902.
 3. *Oeuvres*, vol. 7.
- G. Pólya
1. *Über geometrische Wahrscheinlichkeiten*, Sitzungsber. Kaiserl. Akad. Wiss. Wien Math.-Natur. Kl., Abt. IIa **126** (1917), 1–10.
 2. *Induction and analogy in mathematics*, Mathematics and Plausible Reasoning, vol. I., Princeton Univ. Press, Princeton, N. J., 1954.
 3. *Two more inequalities between physical and geometrical quantities*, J. Indian Math. Soc. (N. S.) **24** (1960), 413–419.
- G. Pólya and G. Szegő
1. *Isoperimetric inequalities in mathematical physics*, Ann. of Math. Studies, no. 27, Princeton Univ. Press, Princeton, N. J., 1951.
- G. Pólya and A. Weinstein
1. *On the torsional rigidity of multiply connected domains*, Ann. of Math. (2) **52** (1950), 154–163.
- T. A. Porter
1. *A history of the classical isoperimetric problem*, Contributions to the Calculus of Variations 1931–1932, Univ. of Chicago Press, Chicago, Illinois 1933.
- Ptolemy
1. *The Almagest*
- T. Radó
1. *A lemma on the topological index*, Fund. Math. **27** (1936), 212–225.
 2. *The isoperimetric inequality and the Lebesgue definition of surface area*, Trans. Amer. Math. Soc. **41** (1947), 530–555.
 3. *Length and area*, Amer. Math. Soc. Colloq. Publ., vol. 30, Amer. Math. Soc., Providence, R. I., 1948.
- (Lord) Rayleigh
1. *The theory of sound*, MacMillan, New York, 1877, 1894; Dover, New York, 1945.
- W. T. Reid
1. *The isoperimetric inequality and associated boundary problems*, J. Math. Mech. **8** (1959), 897–906.
- R. C. Reilly
1. *Applications of the integral of an invariant of the Hessian*, Bull. Amer. Math. Soc. **82** (1976), 579–580.
 2. *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J. **26** (1977), 459–472.
 3. *On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space*, Comment. Math. Helv. **52** (1977), 525–533.
 4. *Geometric applications of the solvability of Neumann problems on a Riemannian manifold* (to appear).

H. Sachs

1. *Ungleichungen für Umfang, Flächeninhalt und Trägheitsmoment konvexer Kurven*, Acta. Math. Acad. Sci. Hungar. **11** (1960), 103–115.

L. A. Santaló

1. *On the circle of maximum radius contained in a domain*, Revista Un. Mat. Argentina **10** (1945), 155–162. (Spanish)
2. *Integral geometry on surfaces*, Duke Math. J. **16** (1949), 361–375.
3. *Introduction to integral geometry*, Actualités Sci. Indust. no. 1198, Hermann, Paris, 1953.
4. *Integral geometry and geometric probability*, Addison-Wesley, Reading, Mass., 1977.

E. Schmidt

1. *Über das isoperimetrische Problem im Raum von n Dimensionen*, Math. Z. **44** (1939), 689–788.
2. *Über die isoperimetrische Aufgabe im n -dimensionalen Raum konstanter negativer Krümmung. I. Die isoperimetrischen Ungleichungen in der hyperbolischen Ebene und für Rotationskörper im n -dimensionalen hyperbolischen Raum*, Math. Z. **46** (1940), 204–230.
3. *Die isoperimetrischen Ungleichungen auf der gewöhnlichen Kugel und für Rotationskörper im n -dimensionalen sphärischen Raum*, Math. Z. **46** (1940), 743–794.
4. *Über eine neue Methode zur Behandlung einer Klasse isoperimetrischer Aufgaben im Grossen*, Math. Z. **47** (1942), 489–642.
5. *Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und sphärischen Raum jeder Dimensionenzahl*, Math. Z. **49** (1943/44), 1–109.

R. Schoen

1. *Existence and regularity theorems for some geometrical variational problems*, Ph. D. Thesis, Stanford Univ., 1977.

R. Schoen, S. Wolpert and S.-T. Yau

1. *On the first eigenvalue of a compact Riemann surface* (to appear).

H. A. Schwarz

1. *Gesammelte Mathematische Abhandlungen*, Springer-Verlag, Berlin, 1890.

S. Z. Sefel

1. *On the intrinsic geometry of saddle surfaces*, Siberian Math. J. **5** (1964), 1382–1396.
2. *Saddle surfaces which are bounded by a rectifiable curve*, Soviet Math. Dokl. **6** (1965), 684–687.

J. Serrin

1. *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318.

M. Shiffman

1. *On the isoperimetric inequality for saddle surfaces with singularities*, Studies and Essays presented to R. Courant, Interscience, New York, 1948, pp. 383–394.

D. Singmaster and D. J. Soupporis

1. *A constrained isoperimetric problem*, Math. Proc. Cambridge Philos. Soc. **83** (1978), 73–82.

E. Sperner

1. *Zur Symmetrisierung von Funktionen auf Sphären*, Math. Z. **134** (1973), 317–327.
2. *Symmetrisierung für Funktionen mehrerer reeller Variablen*, Manuscripta Math. **11** (1974), 159–170.

W. Spiegel

1. *Über die Symmetrisierung stetiger Funktionen im euklidischen Raum*, Arch. Math. **24** (1973), 545–551.

K. Steffen

1. *Isoperimetric inequalities and the problem of Plateau*, Math. Ann. **222** (1976), 97–144.
2. *On the existence of surfaces with prescribed mean curvature and boundary*, Math. Z. **146** (1976), 113–135.

J. Steiner

1. *Sur le maximum et le minimum des figures dans le plan, sur la sphère et dans l'espace en général*, J. Reine Angew. Math. **24** (1842), 93–152.

G. Stolzenberg

1. *Volumes, limits, and extensions of analytic varieties*, Lecture notes in Math., vol. 19, Springer-Verlag, Berlin, 1966.

G. Szegő

1. *Über einige Extremalaufgaben der Potentialtheorie*, Math. Z. **31** (1930), 583–593.

G. Talenti

1. *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

V. A. Toponogov

1. *An isoperimetric inequality for surfaces whose Gaussian curvature is bounded above*, Siberian Math. J. **10** (1969), 104–113.

R. T. Waechter

1. *On hearing the shape of a drum; extensions to higher dimensions*, Proc. Cambridge Philos. Soc. **72** (1972), 439–447.

H. F. Weinberger

1. *Symmetrization in uniformly elliptic problems*, pp. 424–428 of Studies in Mathematical Analysis and Related Topics; Essays in Honor of G. Pólya, Stanford Univ. Press, Stanford, Calif., 1962.
2. *Remark on the preceding paper of Serrin*, Arch. Rational Mech. Anal. **43** (1971), 319–320.

J. L. Weiner

1. *A generalization of the isoperimetric inequality on the 2-sphere*, Indiana Univ. Math. J. **24** (1974), 243–248.

H. Wente

1. *An existence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. **26** (1969), 318–344.
2. *A general existence theorem for surfaces of constant mean curvature*, Math. Z. **120** (1971), 277–288.
3. *The differential equation $\Delta x = 2H(x_u \wedge x_v)$ with vanishing boundary values*, Proc. Amer. Math. Soc. **50** (1975), 131–137.

J. M. Wills

1. *Zum Verhältnis von Volumen zu Oberfläche bei konvexen Körpern*, Arch. Math. **21** (1970), 557–560.

S.-T. Yau

1. *Isoperimetric constants and the first eigenvalue of a compact manifold*, Ann. Sci. École Norm. Sup. **8** (1975), 487–507.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305