CHARACTERIZATION OF SUBSETS OF RECTIFIABLE CURVES IN $\mathbb{R}^n$

KATE OKIKIOLU

1. *Introduction*

By a cube Q in $\mathbb{R}^n$ we mean a closed cube with sides parallel to the axes. Let l_Q denote the sidelength of Q and for $\lambda > 0$ let λQ denote the cube concentric to Q with sidelength $l_{\lambda Q} = \lambda l_Q$. We say Q is a dyadic cube if

$$Q = \prod_{j=1}^{n} [m_j 2^{-k}, (m_j+1) 2^{-k}], \qquad k \in \mathbb{Z}, m_j \in \mathbb{Z}.$$

Let $\mathscr{D}$ denote the set of all dyadic cubes. Given a set $\Gamma \subseteq \mathbb{R}^n$ define the *cylinder radius* of Γ in Q, $r_Q = r_Q(\Gamma)$, to be the minimum radius of a cylinder containing $\Gamma \cap Q$, that is, the maximal distance from points of $\Gamma \cap Q$ to a best approximating line. Write $l(E)$ for the one-dimensional (outer) Hausdorff measure of a set E. In this paper we prove the following.

THEOREM. *If Γ is a connected set in $\mathbb{R}^n$ then*

$$\sum_{Q \in \mathscr{D}} \frac{r_{3Q}^2}{l_Q} \leqslant C l(\Gamma),$$

where $C = C(n)$.

The special case of $n = 2$ was proved by Peter Jones using complex analysis (see [3], for applications see [1, 2]) and the converse of the theorem is included in the following result (see [3]).

If $\Delta \subseteq \mathbb{R}^n$ then there exists a connected set Γ such that $\Delta \subseteq \Gamma$ and

$$l(\Gamma) \leqslant (1+\delta)\,\mathrm{diameter}(\Delta) + C \sum_{Q \in \mathscr{D}} \frac{r_{3Q}^2(\Delta)}{l_Q}$$

for $\delta > 0$, where $C = C(n, \delta)$.

It is well known that if Γ is a connected set in $\mathbb{R}^n$ then there is a tour of length $2l(\Gamma)$ that hits every point of Γ. So if Δ is any set in $\mathbb{R}^n$, the minimal length of a tour that hits every point of Δ is comparable to the minimal value of $l(\Gamma)$, where the minimum is taken over connected sets Γ containing Δ. By the results stated above, this is comparable to the quantity

$$\mathrm{diameter}(\Delta) + \sum_{Q \in \mathscr{D}} \frac{r_{3Q}^2(\Delta)}{l_Q}.$$

Received 14 September 1990; revised 15 April 1991.

1991 *Mathematics Subject Classification* 28A.

J. London Math. Soc. (2) 46 (1992) 336–348

I would like to thank John Garnett for a lot of very helpful advice.

Notation and outline of the proof of the theorem

We write

$$\begin{aligned}
&[x,y] = \{\lambda x+(1-\lambda)y : 0 \leqslant \lambda \leqslant 1\}, && x,y\in\mathbb{R}^n,\\
&B_\lambda(x) = \{y : |x-y| \leqslant \lambda\}, && x\in\mathbb{R}^n, \lambda\in\mathbb{R},\\
&E+x = \{e+x : e\in E\}, && x\in\mathbb{R}^n, E\subseteq\mathbb{R}^n,\\
&x\cdot E = \{x\cdot e : e\in E\}, && x\in\mathbb{R}^n, E\subseteq\mathbb{R}^n,\\
&x_j \text{ for the } j\text{th coordinate of } x, && x\in\mathbb{R}^n,\\
&E_j = \{e_j : e\in E\}, && E\subseteq\mathbb{R}^n,\\
&\partial E \text{ for the boundary of } E, && E\subseteq\mathbb{R}^n.
\end{aligned}$$

For $k\in\mathbb{Z}$, $\mathscr{D}_k$ is the set of dyadic cubes of sidelength 2^{-k}.

Let Q^0 be a cube in $\mathbb{R}^n$. Choose a new origin and coordinate axes in which $Q^0 = [0,1]^n$. We define the *dyadic decomposition of* Q^0 to be the set of dyadic cubes (with respect to the new coordinates) contained in Q^0 and we denote this set by $\langle Q^0\rangle$. We define the kth *generation of* Q^0 to be set of cubes in $\mathscr{D}_k$ contained in Q^0 and we denote this set by $\langle Q^0\rangle_k$.

Let $\lambda > 1$. In Lemma 1 part (b) we show how to associate to a cube Q^0 a finite number of larger cubes containing Q^0 such that if Q is a cube in the kth generation of Q^0, then λQ is contained in some cube Q^* in the kth generation of one of these larger cubes. Furthermore, the number of cubes Q in the dyadic decomposition of Q^0 giving rise to the same cube Q* under this association, is bounded. This association will be used several times during the proof. First we use it to reduce the theorem to proving the following:

If Γ *is a connected set in* $\mathbb{R}^n$ *and* Q^0 *is a cube in* $\mathbb{R}^n$ *then*

$$\sum_{Q\in\langle Q^0\rangle}\frac{r_Q^2}{l_Q} \leqslant Cl(\Gamma),$$

where $C = C(n)$.

To prove this result we write

$$\sum_{Q\in\langle Q^0\rangle}\frac{r_Q^2}{l_Q} = \sum_{Q\in\mathscr{A}}\frac{r_Q^2}{l_Q} + \sum_{Q\in\mathscr{B}}\frac{r_Q^2}{l_Q},$$

where $\mathscr{A}$ is the set of cubes $Q\in\langle Q^0\rangle$ such that $\Gamma\cap Q^*$ is 'almost' a union of two or more straight line segments with endpoints in ∂Q^* (where $Q^* = Q^*(Q, Q^0, \lambda)$, $\lambda = \lambda(n)$) and $\mathscr{B}$ is the set of cubes in $\langle Q^0\rangle$ which are not in $\mathscr{A}$. We now describe $\mathscr{A}$ precisely.

We may assume that Γ is closed. There exists an arclength preserving map $\gamma: T\to\Gamma$, where T is a circle with $l(T) = 2l(\Gamma)$, such that $\gamma(T) = \Gamma$ and γ hits almost every point of Γ twice. Let Q be a cube in $\mathbb{R}^n$. Let $\{T^\alpha : \alpha\in\Lambda_Q\}$ be the set of connected components of $\gamma^{-1}(Q)$, where Λ_Q is an indexing set.

Write

$$\Gamma^\alpha = \gamma(T^\alpha).$$

We have that $\Gamma \cap Q = \bigcup_{\alpha \in \Lambda_Q} \Gamma^\alpha$. Write

$$L^\alpha = \begin{cases} [\gamma(x), \gamma(y)] & \text{if } T^\alpha \neq T, \text{ where } x, y \text{ are the endpoints of the arc } T^\alpha, \\ \varnothing & \text{if } T^\alpha = T. \end{cases}$$

If $T^\alpha \neq T$ then Γ^α is (the image of) a curve with endpoints in ∂Q and L^α is the line segment joining these endpoints. Write

$$s_\alpha = \begin{cases} \sup_{z \in \Gamma^\alpha} \operatorname{dist}(z, L^\alpha) & \text{if } T^\alpha \neq T, \\ r_Q & \text{if } T^\alpha = T \end{cases}$$

and

$$s_Q = \sup_{\alpha \in \Lambda_Q} s_\alpha.$$

Notice that s_Q depends on our choice of γ. Also notice that if s_Q is very small then $\Gamma \cap Q$ is 'almost' a union of straight line segments with endpoints in ∂Q. We shall set

$$\mathscr{A} = \{Q \in \langle Q^0 \rangle : s_{Q^*} < \delta r_Q\},$$
$$\mathscr{B} = \{Q \in \langle Q^0 \rangle : s_{Q^*} \geqslant \delta r_Q\},$$

where $\delta = \delta(n) > 0$, $Q^* = Q^*(Q, Q^0, \lambda)$ and $\lambda = \lambda(n) > 0$. Lemmas 1 and 2 enable us to bound

$$\sum_{Q \in \mathscr{B}} \frac{r_Q^2}{l_Q}$$

while Lemmas 1 and 3 enable us to bound

$$\sum_{Q \in \mathscr{A}} \frac{r_Q^2}{l_Q}.$$

Lemma 2 states that if Γ is connected and $l(\Gamma) < \infty$ then

$$\sum_{Q \in \langle Q^0 \rangle} \frac{s_Q^2}{l_Q} \leqslant C l(\Gamma \cap Q^0),$$

where $C = C(n)$. The main ingredient in the proof (which follows Peter Jones [3]) is the Pythagorean theorem.

Lemma 3 states that

$$\sum_{Q \in \mathscr{A}} r_Q \leqslant C l(\Gamma \cap 2Q^0)$$

where $C = C(n)$.

In Section 2 we prove Lemmas 1 and 2 and put together Lemmas 1, 2 and 3 to prove the theorem. The proof of Lemma 3 is lengthy and is given in Section 3. To illustrate the proof consider the case where $\Gamma \cap \lambda Q^0$ is just a union of two straight segments with endpoints in $\partial \lambda Q^0$. Then for each cube $Q \in \langle Q^0 \rangle$ we can choose an interval $E_Q \subseteq \Gamma \cap 2Q$ such that

$$r_Q \leqslant c l(E_Q)$$

and such that any point of Γ is contained in E_Q for at most C cubes $Q \in \langle Q^0 \rangle$. Then

$$\sum_{Q \in \langle Q^0 \rangle} r_Q \leqslant c \sum_{Q \in \langle Q^0 \rangle} l(E_Q) \leqslant cCl(\Gamma \cap 2Q^0).$$

REMARK. The theorem is equivalent to the following result. If $\lambda' > \lambda > 0$, if $\Gamma \subseteq \mathbb{R}^n$ is a connected set and if $Q^0 \subseteq \mathbb{R}^n$ is a cube, then

$$\sum_{Q \in \langle Q^0 \rangle} \frac{r_{\lambda Q}^2}{l_Q} \leqslant Cl(\Gamma \cap \lambda' Q),$$

where $C = C(n, \lambda, \lambda')$. We do not prove this here.

2. *Proof of the theorem*

LEMMA 1. (a) *Let $\lambda > 0$. If $F \subseteq \mathbb{R}^n$ then for $k = 0, 1, 2, \ldots,$*

$$\#(\{Q \in \mathcal{D}_k : F \cap \lambda Q \neq \varnothing\}) \leqslant \left(\frac{\text{diameter}(F)}{2^{-k}} + \lambda + 1\right)^n.$$

(b) *Let $\lambda > 1$. If Q^0 is a cube in $\mathbb{R}^n$ then for $k = 0, 1, 2, \ldots$ and each cube $Q \in \langle Q^0 \rangle_k$ there exists a cube which we denote by $Q^* = Q^*(Q, Q^0, \lambda)$ such that*

$$\lambda Q \subseteq Q^* \in \bigcup_{e \in V} \langle Q^0(\lambda, e) \rangle_k,$$

where V is the set of the 2^n vertices of the cube $[0, 1]^n$ and

$$Q^0(\lambda, e) = 4\lambda Q^0 + \frac{4\lambda l_{Q^0}}{3} e.$$

If

$$\hat{Q} \in \bigcup_{e \in V} \langle Q^0(\lambda, e) \rangle_k$$

then

$$\#(\{Q \in \langle Q^0 \rangle : Q^* = \hat{Q}\}) \leqslant (4N)^n.$$

Proof. (a) Now F is contained in a cube, Q^F, of sidelength diameter (F). If $Q \in \mathcal{D}_k$ and $F \cap \lambda Q = \varnothing$ then Q is contained in the cube concentric to Q^F with sidelength equal to diameter$(F) + (\lambda + 1)\, 2^{-k}$.

(b) We prove (b) for

$$Q^0 = \left[\frac{1}{2} - \frac{1}{8N}, \frac{1}{2} + \frac{1}{8N}\right]^n$$

(so $4NQ^0 = [0, 1]^n$). The result follows for any other cube by dilation and translation.

Suppose that $k \in \{0, 1, 2, \ldots\}$ and $J \subseteq [\frac{1}{3}, 1]$ is an interval of length $l_J \leqslant 2^{-k}/3$. Then there exists an interval $I^* \in \langle [0, 1] \rangle_k$ such that

$$J \subseteq I^* + e/3 \tag{2.1}$$

where either $e = 0$ or $e = 1$. To see this, suppose that there is no interval $I^* \in \langle [0, 1] \rangle_k$ with $J \subseteq I^*$. Then there is an interval in $\langle [0, 1] \rangle_k$ with an endpoint $x \in J$. Since $J \subseteq [0, 1] + \frac{1}{3}$, there is an interval $I^* \in \langle [0, 1] \rangle_k$ with $x \in I^* + \frac{1}{3}$. Let y be an endpoint of $I^* + \frac{1}{3}$. Write $x = 2^{-k}p$, $y = 2^{-k}q + \frac{1}{3}$ with $p, q \in \mathbb{Z}$. Then

$$|y - x| = 2^{-k}\left|q - p + \frac{2^k}{3}\right| \geqslant \frac{2^{-k}}{3}.$$

Since $l_J \leqslant 2^{-k}/3$ we conclude that $J \subseteq I^* + \frac{1}{3}$. This proves (2.1).

Next, let $I^0 = [\frac{1}{2} - 1/8N, \frac{1}{2} + 1/8N]$ (so $4NI^0 = [0, 1]$). If $k \in \{0, 1, 2, \ldots\}$ and $I \in \langle I^0 \rangle_k$ then $NI \subseteq [\frac{1}{3}, 1]$ and $l_{NI} = 2^{-k}/4 < 2^{-k}/3$. By writing $J = NI$ in (2.1), we see that there exists $I^* \in \langle [0, 1] \rangle_k$ and $e = 0$ or 1 such that

$$NI \subseteq I^* + e/3. \tag{2.2}$$

Let $Q^0 = [\frac{1}{2} - 1/8N, \frac{1}{2} + 1/8N]^n$ and $Q \in \langle Q^0 \rangle_k$. By writing $I = Q_j$ in equation (2.2) for $1 \leqslant j \leqslant n$, we see that there exists $I_j^* \in \langle [0, 1] \rangle_k$ and $e_j = 0$ or 1 such that

$$(NQ)_j = NQ_j \subseteq I_j^* + e_j/3.$$

Thus

$$NQ \subseteq \prod_{j=1}^{n} \left(I_j^* + \frac{e_j}{3} \right) = \prod_{j=1}^{n} I_j^* + \frac{(e_1 \ldots e_n)}{3} = Q^* \in \left\langle [0, 1]^n + \frac{e}{3} \right\rangle_k,$$

where $e = (e_1, \ldots, e_n) \in V$.

Finally, if

$$\hat{Q} \in \bigcup_{e \in V} \left\langle [0, 1]^n + \frac{e}{3} \right\rangle$$

and $Q \in \langle Q^0 \rangle$ is such that $Q^* = \hat{Q}$ then $Q \subseteq \hat{Q}$ and $l_Q = l_{\hat{Q}}/4N$. There are at most $(4N)^n$ such cubes $Q \in \langle Q^0 \rangle$.

Proving the Theorem is now reduced to proving the following.

If Γ is a connected set in $\mathbb{R}^n$ and Q^0 is a cube in $\mathbb{R}^n$ then

$$\sum_{Q \in \langle Q^0 \rangle} \frac{r_Q^2}{l_Q} \leqslant Cl(\Gamma),$$

where $C = C(n)$.

To see this let Q^0 be a cube in $\mathbb{R}^n$. Then

$$\sum_{Q \in \langle Q^0 \rangle} \frac{r_{3Q}^2}{l_Q} \leqslant 12 \sum_{Q \in \langle Q^0 \rangle} \frac{r_{Q^*}^2}{l_{Q^*}} \leqslant (12)^{n+1} \sum_{e \in V} \sum_{Q \in \langle Q^0(3, e) \rangle} \frac{r_Q^2}{l_Q},$$

where $Q^* = Q^*(Q, Q^0, 3)$, by Lemma 1.

LEMMA 2. *If Γ is a connected set in $\mathbb{R}^n$ with $l(\Gamma) < \infty$ and Q^0 is a cube in $\mathbb{R}^n$ then*

$$\sum_{Q \in \langle Q^0 \rangle} \frac{s_Q^2}{l_Q} \leqslant l(\Gamma \cap Q^0),$$

where $C = C(n)$.

Proof. Let Q be a cube in $\mathbb{R}^n$ and let $\alpha \in \Lambda_Q$. Write

$$\Lambda_{\alpha, k} = \{\beta : T^\beta \subseteq T^\alpha, \beta \in \Lambda_{Q'} \text{ for some } Q' \in \langle Q \rangle_k\}, \qquad k = 0, 1, 2, \ldots,$$

$$t_\alpha = \begin{cases} \sup\limits_{\substack{x \in L^\beta \\ \beta \in \Lambda_{\alpha, 1}}} \operatorname{dist}(x, L^\alpha), & T^\alpha \neq T, \\ \sum\limits_{\beta \in \Lambda_{\alpha, 1}} l(L^\beta), & T^\alpha = T. \end{cases}$$

Let $\alpha(k)\in\Lambda_{\alpha,k}$ be an index such that

$$t_{\alpha(k)} = \sup_{\beta\in\Lambda_{\alpha,k}} t_\beta, \qquad k=0,1,2,\ldots.$$

Then

$$s_\alpha \leqslant \sum_{k=0}^{\infty} t_{\alpha(k)}. \tag{2.3}$$

To see this, suppose that α_k is a sequence such that $\alpha_0=\alpha, \alpha_{k+1}\in\Lambda_{\alpha_k,1}$. Then the sequence s_{α_k} is eventually non-increasing and $s_{\alpha_k}\to 0$, so

$$s_\alpha = \sum_{k=0}^{\infty}(s_{\alpha_k}-s_{\alpha_{k+1}}).$$

Let α_k be such that

$$s_{\alpha_{k+1}} = \sup_{\beta\in\Lambda_{\alpha_k,1}} s_\beta.$$

Then $s_{\alpha_k}-s_{\alpha_{k+1}}\leqslant t_{\alpha(k)}$, so we get equation (2.3).

We also have

$$\sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle}} \frac{t_\alpha^2}{l_Q} \leqslant Cl(\Gamma\cup Q^0),$$

where $C=C(n)$.

To see this, let $\alpha\in\Lambda_Q$. By the Pythagorean theorem,

$$\frac{t_\alpha^2}{l_Q}\leqslant 2\sqrt{n}\left(\sum_{\beta\in\Lambda_{\alpha,1}} l(L^\beta)-l(L^\alpha)\right).$$

Hence

$$\sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle_k}} \frac{t_\alpha^2}{l_Q}\leqslant C\left(\sum_{\substack{\beta\in\Lambda_Q\\ Q\in\langle Q^0\rangle_{k+1}}} l(L^\beta)-\sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle_k}} l(L^\alpha)\right)$$

for $k=0,1,2,\ldots$, where $C=C(n)$. Hence

$$\begin{aligned}\sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle}} \frac{t_\alpha^2}{l_Q} &\leqslant C\sup_k \sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle_k}} l(L^\alpha)\leqslant C\sup_k \sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle_k}} l(T^\alpha)\\ &= C\sup_k \int_{\gamma^{-1}(Q^0)} \sum_{Q\in\langle Q^0\rangle_k} \chi_Q(\gamma(x))\,dl(x)\\ &\leqslant 2^n Cl(\gamma^{-1}(Q^0)) = 2^{n+1}Cl(\Gamma\cap Q^0),\end{aligned}$$

where χ_Q denotes the characteristic function of the set Q and $C=C(n)$. Putting equations (2.3) and (2.4) together, we get

$$\begin{aligned}\left(\sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle}} \frac{s_\alpha^2}{l_Q}\right)^{\frac{1}{2}} &\leqslant \left(\sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle}} \frac{(\sum_{k=0}^{\infty} t_{\alpha(k)})^2}{l_Q}\right)^{\frac{1}{2}} \leqslant \sum_{k=0}^{\infty}\left(\sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle}} \frac{t_{\alpha(k)}^2}{l_Q}\right)^{\frac{1}{2}}\\ &= \sum_{k=0}^{\infty} 2^{-k/2}\left(\sum_{\substack{\alpha\in\Lambda_Q\\ Q\in\langle Q^0\rangle}} \frac{t_{\alpha(k)}^2}{2^{-k}l_Q}\right)^{\frac{1}{2}} \leqslant C(l(\Gamma\cap Q^0))^{\frac{1}{2}},\end{aligned}$$

where $C=C(n)$. From this, we get Lemma 2.

REMARK. Suppose that Γ is the image of a closed, rectifiable, chordarc Jordan curve with chordarc constant k. That is, suppose that there is a circle T and a length preserving bijection $\gamma: T \to \Gamma$ and if x and y are in Γ then the (shorter) arc between x and y has length bounded by $k|x-y|$. Let Q^0 be a cube in $\mathbb{R}^n$. If $Q \in \langle Q^0 \rangle$ and $Q^* = Q^*(Q, Q^0, \lambda)$, where $\lambda = k\sqrt{n}+1$, then at most one component of $\Gamma \cap Q^*$ meets Q and there is one arc in T mapped onto this component. Hence $r_Q \leqslant s_{Q^*} = s_{Q^*}(\gamma)$ and

$$\sum_{Q \in \langle Q^0 \rangle} \frac{r_Q^2}{l_Q} \leqslant \sum_{Q \in \langle Q^0 \rangle} \frac{s_{Q^*}^2}{l_Q} \leqslant Cl(\Gamma \cap Q^0),$$

where $C = C(n, K)$, by Lemmas 1 and 2. This proves the theorem for closed, chordarc, Jordan curves.

LEMMA 3. *If $\Gamma \subseteq \mathbb{R}^n$ is a connected set with $l(\Gamma) < \infty$ and $Q^0 \subset \mathbb{R}^n$ is a cube then*

$$\sum_{Q \in \mathscr{A}} r_Q \leqslant Cl(\Gamma \cap 2Q^0),$$

where $\mathscr{A} = \{Q \in \langle Q^0 \rangle : s_{Q^*} < \delta r_Q\}$, $Q^* = Q^*(Q, Q^0, \lambda)$, $\lambda = \lambda(n)$, $\delta = \delta(n)$ and $C = C(n)$.

Before proving Lemma 3 we shall now complete the proof of the theorem.

Proof. Let δ, λ be the constants in Lemma 3. Let

$$\mathscr{A} = \{Q \in \langle Q^0 \rangle : s_{Q^*} < \delta r_Q\},$$
$$\mathscr{B} = \{Q \in \langle Q^0 \rangle : s_{Q^*} \geqslant \delta r_Q\}.$$

Then

$$\sum_{Q \in \langle Q^0 \rangle} \frac{r_Q^2}{l_Q} \leqslant \sum_{Q \in \mathscr{A}} \frac{r_Q^2}{l_Q} + \sum_{Q \in \mathscr{B}} \frac{r_Q^2}{l_Q}.$$

Now

$$\sum_{Q \in \mathscr{A}} \frac{r_Q^2}{l_Q} \leqslant \frac{\sqrt{(n-1)}}{2} \sum_{Q \in \mathscr{A}} r_Q \leqslant Cl(\Gamma \cap 2Q^0),$$

where $C = C(n)$, by Lemma 3, and

$$\sum_{Q \in \mathscr{B}} \frac{r_Q^2}{l_Q} \leqslant \frac{1}{\delta^2} \sum_{Q \in \mathscr{B}} \frac{s_{Q^*}^2}{l_Q} \leqslant \frac{4\lambda}{\delta^2} \sum_{Q \in \langle Q^0 \rangle} \frac{s_{Q^*}^2}{l_{Q^*}}$$
$$\leqslant \frac{(4\lambda)^{n+1}}{\delta^2} \sum_{e \in V} \sum_{Q \in \langle Q^0(\lambda, e) \rangle} \frac{s_Q^2}{l_Q} \leqslant Cl(\Gamma \cap \lambda Q^0),$$

where $C = C(n)$, by Lemmas 1 and 2.

3. *Proof of Lemma* 3

First we give a brief outline of the proof beginning with some simple Euclidean geometry. Let $Q \subseteq \mathbb{R}^n$ be a cube and let $\mathscr{L}^\alpha$ and $\mathscr{L}^\beta$ be lines meeting $\frac{3}{2}Q$. We define two line segments

$$I^\alpha \subseteq \mathscr{L}^\alpha \cap \lambda Q \backslash \text{interior}(2Q)$$
$$I^\beta \subseteq \mathscr{L}^\beta \cap 2Q$$

having the same length, comparable to the maximum distance of points of $\mathscr{L}^\beta \cap 2Q$ to $\mathscr{L}^\alpha$.

In Lemma 4 part (a) we will show that there exists a unit vector w^0 such that the length of the interval $w \cdot I^\alpha \cap w \cdot I^\beta$ is comparable to this maximum distance for all unit vectors w close to w^0.

In Lemma 4 parts (b) and (c) we use this to show that if Γ is a connected set, Q^0 is a cube, Q is in the dyadic decomposition of Q^0 and s_{Q^*} is small (that is $\Gamma \cap Q^*$ is 'almost' a union of straight line segments), then we can find sets $\hat{I}^\alpha$ and $\hat{I}^\beta$ which are 'almost' straight line segments with

$$\hat{I}^\alpha \subseteq \Gamma \cap \lambda Q \backslash \text{interior}(2Q)$$
$$\hat{I}^\beta \subseteq \Gamma \cap 2Q$$

and a unit vector w^0 such that the length of the interval $w \cdot \hat{I}^\alpha \cap w \cdot \hat{I}^\beta$ bounds (up to a constant) r_Q for all unit vectors w close to w^0. Hence for such cubes Q, r_Q is bounded by the length of the interval

$$(w \cdot \Gamma \cap \lambda Q \backslash 2Q) \cap (w \cdot \Gamma \cap 2Q).$$

To complete the proof of Lemma 3, we show in Lemma 5 that we can sum the lengths of the above intervals over all cubes Q in the dyadic decomposition of Q^0, and the result will be bounded by a constant multiple of $l(\Gamma \cap 2Q^0)$. This is a slightly subtle fact which would be false if the number 2 were replaced by the number 3. The main ingredient in the proof (see in Lemma 5 part (a)) is the fact that if I is a dyadic interval (that is, a dyadic cube in $\mathbb{R}^1$) then the right endpoint of $2I$ is the *midpoint* of a dyadic interval of the same length as I. If however J is a dyadic interval longer than I, then the right endpoint of $2J$ is an *endpoint* of a dyadic interval of the same length as I. Hence these two right endpoints are separated by a distance at least half as long as I.

We now embark upon the full proof.

Let $Q \subseteq \mathbb{R}^n$ be a cube and let $\mathscr{L}^\alpha$ and $\mathscr{L}^\beta$ be lines meeting $\frac{3}{2}Q$. We define points $x^\alpha, y^\alpha \in \mathscr{L}^\alpha$ and $x^\beta, x^\beta \in \mathscr{L}^\beta$ as follows. Let u^α, v^α, u^β, $v^\beta \in \mathbb{R}^n$ be such that

$$\mathscr{L}^\alpha = \{u^\alpha + tv^\alpha : t \in \mathbb{R}\},$$
$$\mathscr{L}^\beta = \{u^\beta + tv^\beta : t \in \mathbb{R}\},$$
$$|v^\alpha| = |v^\beta| = 1, \quad v^\alpha \cdot v^\beta \geqslant 0,$$
$$(u^\alpha - u^\beta) \cdot v^\alpha = (u^\beta - u^\alpha) \cdot v^\beta.$$

By replacing (v^α, v^β) by $(-v^\alpha, -v^\beta)$ if necessary we can assume that

$$\left.\begin{aligned} \mathscr{L}^\alpha \cap 2Q &= [u^\alpha + s^\alpha v^\alpha, u^\alpha + t^\alpha v^\alpha], \\ \mathscr{L}^\beta \cap 2Q &= [u^\beta + s^\beta v^\beta, u^\beta + t^\beta v^\beta], \end{aligned}\right\} \tag{3.1}$$

where $s^\alpha \leqslant t^\alpha$, $s^\beta \leqslant t^\beta$ and $\max\{|s^\alpha|, |s^\beta|\} \leqslant \max\{t^\alpha, t^\beta\}$. Relabel $\mathscr{L}^\alpha$ and $\mathscr{L}^\beta$ if necessary so that $t^\alpha \leqslant t^\beta$. Write

$$\left.\begin{aligned} x^\alpha &= u^\alpha + t^\beta v^\alpha, & x^\beta &= u^\beta + t^\beta v^\beta, \\ y^\alpha &= u^\alpha + \left(t^\beta + \frac{l_Q}{2}\right) v^\alpha, & y^\beta &= u^\beta + \left(t^\beta - \frac{l_Q}{2}\right) v^\beta, \\ I^\alpha &= [x^\alpha, y^\alpha], & I^\beta &= [x^\beta, y^\beta]. \end{aligned}\right\} \tag{3.2}$$

Then

$$\operatorname{dist}(x^\beta, \mathscr{L}^\alpha) = \sup_{x \in \mathscr{L}^\beta \cap 2Q} \operatorname{dist}(x, \mathscr{L}^\alpha) \tag{3.3}$$

and

$$|x^\alpha - x^\beta| \leqslant \sqrt{2}\operatorname{dist}(x^\beta, \mathscr{L}^\alpha) \leqslant 2\sqrt{(2n)}\, l_Q.$$

So

$$I^\alpha \subseteq \lambda Q \backslash \operatorname{interior}(2Q), \tag{3.4}$$

where $\lambda = 4\sqrt{(2n)}+3$,

$$I^\beta \subseteq 2Q$$

and

$$l(I^\alpha) = l(I^\beta) \geqslant \frac{1}{4\sqrt{(2n)}}|x^\alpha - x^\beta|. \tag{3.5}$$

At this point we need two auxiliary lemmas.

LEMMA 4. (a) *Let $Q \subseteq \mathbb{R}^n$ be a cube and let $\mathscr{L}^\alpha$ and $\mathscr{L}^\beta$ be lines meeting $\frac{3}{2}Q$. Define x^α, x^β, I^α, I^β as in equations* (3.2). *Then there exists a unit vector $w^0 \in \mathbb{R}^n$ such that*

$$|x^\alpha - x^\beta| \leqslant Cl(w \cdot I^\alpha \cap w \cdot I^\beta)$$

for all $w \in B_c(w^0)$, where $C = C(n)$ and $c = c(n) > 0$.

(b) *Let $\lambda_0 > 1$. Let Q be a cube in $\mathbb{R}^n$, $\mathscr{L}$ a line meeting Q and z a point in $\mathscr{L}$. Given $\lambda \geqslant \lambda_0$, let $[x,y] = \mathscr{L} \cap \lambda Q$. We have that if $d = \min\{|x-z|, |y-z|\} > 0$, then*

$$B_{\varepsilon d}(z) \cap \partial(\lambda Q) = \varnothing,$$

where $\varepsilon = \varepsilon(n, \lambda_0) > 0$.

(c) *If Γ is a connected set in $\mathbb{R}^n$ with $l(\Gamma) < \infty$, if Q^0 is a cube in $\mathbb{R}^n$ and if $Q \in \langle Q^0 \rangle$ is such that $s_{Q^*} < \delta r_Q$ then there exists a unit vector $w^0 \in \mathbb{R}^n$ such that*

$$r_Q \leqslant Cl((w \cdot \Gamma \cap 2Q) \cap (w \cdot \Gamma \cap \lambda Q \backslash 2Q))$$

for all $w \in B_c(w^0)$, where $\delta = \delta(n) > 0$, $Q^* = Q^*(Q, Q^0, \lambda)$, $\lambda = \lambda(n)$, $C = C(n)$ and $c = c(n) > 0$.

Proof. (a) We can assume by translating that $x^\beta = 0$. We can assume that $x^\alpha \neq x^\beta$. Consider the orthogonal projection P onto a plane Π spanned by either

(i) v^α and v^β, or

(ii) $v^\alpha + v^\beta$ and $u^\alpha - u^\beta$.

For either of these choices of Π, P will satisfy

$$l(P(I^\alpha)) = l(P(I^\beta)) \geqslant \frac{l(I^\alpha)}{\sqrt{2}}$$

and for one of them, P will satisfy

$$\frac{|x^\alpha - x^\beta|}{\sqrt{2}} \leqslant |P(x^\alpha) - P(x^\beta)|.$$

Fix Π so that P satisfies this inequality.

Identify Π with $\mathbb{C}$ by choosing the orthonormal vectors $1, i \in \Pi$ such that

$$P(x^\alpha - x^\beta) = x > 0,$$
$$P(y^\beta) = Re^{-i\theta}, \quad R > 0, \tfrac{1}{4}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi.$$

Then $P(y^\alpha) = x + Re^{i\theta}$ and by equation (3.5), $0 < x \leqslant C_0 R$, where $C_0 = C_0(n)$. Write

$$r = \min(x, R), \quad I^0 = [0, re^{-i\theta}], \quad I^1 = [x, x + re^{i\theta}]. \tag{3.6}$$

Then $I^0 \subseteq P(I^\beta)$, $I^1 \subseteq P(I^\alpha)$ and

$$\tfrac{1}{4}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi, \quad 0 < r \leqslant x \leqslant \eta r, \tag{3.7}$$

where $\eta = \eta(n)$.

To prove (a) it suffices to show that if I^0 and I^1 are intervals in Π satisfying equations (3.6) and (3.7) then there exists a unit vector $w^0 \in \Pi$ such that

$$x \leqslant C_1(w \cdot I^0 \cap w \cdot I^1) \tag{3.8}$$

for all $w \in B_c(w^0)$, where $C_1 = C_1(\eta)$ and $c = c(\eta) > 0$.

Write $w^0 = e^{i\phi}$. Then

$$w^0 \cdot I^0 = [0, r\cos(\theta+\phi)],$$
$$w^0 \cdot I^1 = [x\cos\phi, x\cos\phi + r\cos(\theta-\phi)].$$

It is easy to check that if we choose ϕ to satisfy $x\cos\phi + r\cos(\theta-\phi) = 0$ then

$$x \leqslant C_2\, l(w^0 \cdot I^0 \cap w^0 \cdot I^1),$$

where $C_2 = C_2(\eta)$.

Let $w \in B_c(w^0)$. Let $j = 0$ or 1. If z is an endpoint of I^j then $|w \cdot z - w^0 \cdot z| \leqslant 2cx$. So $w \cdot I^j$ is an interval containing points within $2cx$ of the endpoints of $w^0 \cdot I^j$.

Hence

$$x \leqslant C_2(l(w \cdot I^0 \cap w \cdot I^1) + 4cx).$$

Choosing c sufficiently small, we get (3.8).

(b) The idea of this proof is that since $\mathscr{L}$ meets Q, $\mathscr{L}$ cannot meet any face of λQ (where $\lambda \geqslant \lambda_0 > 1$) at too small an angle.

Suppose that $z \in \lambda Q$ so $d \leqslant \tfrac{1}{2}\lambda\sqrt{n}\, l_Q$. Let F be a face of λQ. Choose new axes so that F lies in the hyper-plane $x_1 = 0$ and Q lies in the region $x_1 > 0$. Relabel x and y if necessary so that $x_1 \leqslant y_1$. If $x_1 \geqslant \tfrac{1}{4}(\lambda-1)\, l_Q$ then

$$z_1 \geqslant \frac{\lambda-1}{4} l_Q \geqslant \left(\frac{\lambda-1}{4}\right)\left(\frac{2}{\lambda\sqrt{n}}\right) d.$$

If $x_1 < \tfrac{1}{4}(\lambda-1)\, l_Q$ then let u be any point in $L \cap Q$ and write $v = (u-x)/|u-x|$, so v is parallel to $\mathscr{L}$. Then

$$v_1 \geqslant \frac{\tfrac{1}{2}(\lambda-1)\, l_Q - \tfrac{1}{4}(\lambda-1)\, l_Q}{\lambda\sqrt{n}\, l_Q} = \frac{\lambda-1}{4\lambda\sqrt{n}}$$

so

$$z_1 \geqslant x_1 + v_1 d > \frac{\lambda-1}{4\lambda\sqrt{n}} d.$$

If $z \notin \lambda Q$ then relabel x and y if necessary so that $y \notin [x, z]$ and let F be a face of λQ containing x. By choosing new axes and a point $u \in L \cap Q$ as we did above, we find that $z_1 \leqslant -d(\lambda-1)/2\lambda\sqrt{n}$.

(c) Let $0 < \delta < 1$ and $\lambda = 4\sqrt{(2n)} + 3$ (see (3.4)). Suppose that $s_{Q^*} < \delta r_Q$.

Let $\alpha \in \Lambda_{Q^*}$ be such that Γ^α meets Q. Then L^α meets

$$\left(1 + \frac{2s_{Q^*}}{l_Q}\right) Q$$

which is contained in $\tfrac{3}{2}Q$ if δ is sufficiently small. Let $\mathscr{L}^\alpha$ be the line containing L^α. Then

$$r_Q \leqslant \sup_{z \in \Gamma \cap Q} \operatorname{dist}(z, \mathscr{L}^\alpha). \tag{3.9}$$

Let $z \in \Gamma \cap Q$ attain this supremum and let $\beta \in \Lambda_{Q^*}$ be such that $z \in \Gamma^\beta$. Then

$$r_Q \leqslant \sup_{z \in L^\beta \cap \frac{3}{2}Q} \operatorname{dist}(z, \mathscr{L}^\alpha) + s_{Q^*}$$

so

$$r_Q \leqslant 2 \sup_{z \in L^\beta \cap \frac{3}{2}Q} \operatorname{dist}(z, \mathscr{L}^\alpha)$$

if δ is sufficiently small. By (3.3) and part (a), there exists a unit vector $w^0 \in \mathbb{R}^n$ such that

$$r_Q \leqslant Cl(w \cdot I^\alpha \cap w \cdot I^\beta)$$

for all $w \in B_c(w^0)$, where I^α and I^β are defined in (3.2) and α, β may have been interchanged. Now let $\varepsilon = \varepsilon(n, \frac{4}{3})$ be as in part (b). For $j = \alpha, \beta$ let $\hat{x}^j, \hat{y}^j \in \lambda Q$ be such that

$$|x^j - \hat{x}^j| = |y^j - \hat{y}^j| = s_{Q^*}/\varepsilon, \qquad [\hat{x}^j, \hat{y}^j] \subseteq [x^j, y^j],$$

where x^j, y^j are as in (3.2) and δ is small enough so that $2s_{Q^*}/\varepsilon < l(I^j)$.

Then

$$\begin{aligned} B_{s_{Q^*}}([\hat{x}^j, \hat{y}^j]) &= \{z : \operatorname{dist}(z, [\hat{x}^j, \hat{y}^j]) \leqslant s_{Q^*}\} \\ &\subseteq \begin{cases} \lambda Q \backslash 2Q & \text{if } j = \alpha, \\ 2Q & \text{if } j = \beta, \end{cases} \end{aligned}$$

by part (b). By a simple argument we can find a connected set $\hat{\Gamma}^j \subseteq \Gamma^j \cap B_{s_{Q^*}}([\hat{x}^j, \hat{y}^j])$ such that $\hat{\Gamma}^j$ meets $B_{s_{Q^*}}(\hat{x}^j)$ and $B_{s_{Q^*}}(\hat{y}^j)$. Since $\hat{\Gamma}^j$ contains points within $(1 + 1/\varepsilon) s_{Q^*}$ of the endpoints of I^j it follows that $w \cdot \hat{\Gamma}^j$ is an interval containing points within $|w|(1 + 1/\varepsilon) s_{Q^*}$ of the endpoints of $w \cdot I^j$. Hence by (3.9) and part (a)

$$r_Q \leqslant C(l(w \cdot \hat{\Gamma}^\alpha \cap w \cdot \hat{\Gamma}^\beta) + C_1 s_{Q^*})$$

for all $w \in B_c(w^0)$, where $C_1 = C_1(n)$. By choosing δ sufficiently small we get part (c).

Lemma 5. (a) *Let $\lambda > 2$, $x, y \in \mathbb{R}$. Write*

$$\begin{aligned} \mathscr{K} &= \mathscr{K}(x, y) \\ &= \{k \in \mathbb{Z} : \text{there exists an interval } I \in \mathscr{D}_k \text{ with } x \in 2I \text{ and } y \in \lambda I \backslash \text{interior } 2I)\}. \end{aligned}$$

Then $\#(\mathscr{K}) \leqslant 2 + \log_2(2\lambda)$.

(b) *Suppose that Q^0 is a cube in $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $F \subseteq \mathbb{R}^n$. Write*

$$\mathscr{E} = \mathscr{E}(x, F) = \{Q \in \langle Q^0 \rangle : x \in 2Q, F \cap 2Q = \varnothing, F \cap \lambda Q \neq \varnothing\}.$$

Then $\#(\mathscr{E}) \leqslant C$, where $C = C(n, \lambda)$.

(c) *Suppose that Γ is a closed subset of $\mathbb{R}^n$, Q^0 is a cube in $\mathbb{R}^n$ and w is a unit vector in $\mathbb{R}^n$. Then*

$$\sum_{Q \in \langle Q^0 \rangle} l((w \cdot \Gamma \cap 2Q) \cap (w \cdot \Gamma \cap \lambda Q \backslash 2Q)) \leqslant Cl(\Gamma \cap 2Q^0),$$

where $C = C(n)$.

Proof. (a) By replacing x, y by $-x, -y$ if necessary we can assume that $x \leqslant y$. If $I \in \mathscr{K}$ then the right endpoint of $2I$ lies in $[x, y]$. There exists $I' \in \mathscr{D}$ with $l_{I'} = l_I$ such that the right endpoint of $2I$ is the midpoint of I'. Now if $I, J \in \mathscr{D}$ and $I \neq J$ then

$$|\text{right endpoint of } I - \text{right endpoint of } J| \geqslant \tfrac{1}{2}\min\{l_I, l_J\}.$$

Hence there exists at most one interval $I \in \mathscr{K}$ with $2(y-x) < l_I$. If $I \in \mathscr{K}$ then $x, y \in \lambda I$ so $(y-x)/\lambda \leqslant l_I$.

(b) Let $y \in \text{closure}(F)$ be such that $|x-y| = \text{dist}(x, F)$. If $Q \in \mathscr{E}$ then

$$|\text{centre}(Q) - y| \leqslant |\text{centre}(Q) - x| + |x-y| < ((2+\lambda)\sqrt{n})\, l_Q.$$

So $y \in 2((2+\lambda)\sqrt{n})\, l_Q = \lambda_1 Q$. Thus

$$\mathscr{E} \subseteq \{Q \in \langle Q^0 \rangle : x \in 2Q, y \in \lambda_1 Q \backslash \text{interior}(2Q)\}. \tag{3.10}$$

Recall that $Q_j = \{q_j : q \in Q\}$. If Q is in the right-hand side of equation (3.10) then there exists j, $1 \leqslant j \leqslant n$, such that $x_j \in 2Q_j$, $y_j \in \lambda_1 Q_j \backslash \text{interior}(2Q_j)$. Hence by part (a) we have

$$\#\{k : \mathscr{E} \cap \langle Q^0 \rangle_k \neq \varnothing\} \leqslant n(2 + \log_2(2\lambda_1)).$$

For each k,

$$\mathscr{E} \cap \langle Q^0 \rangle_k \subseteq \{Q \in \langle Q^0 \rangle_k : x \in 2Q\}$$

so by Lemma 1 part (a),

$$\#(\mathscr{E} \cap \langle Q^0 \rangle_k) \leqslant 3^n.$$

(c) Let $x \in \mathbb{R}^n$. For $1 \leqslant j \leqslant n$ write

$$F^j(x) = \{y \in \Gamma : w \cdot y = w \cdot x, y_j > x_j\},$$
$$F^{-j}(x) = \{y \in \Gamma : w \cdot y = w \cdot x, y_j < x_j\}.$$

Let Q be a cube in $\mathbb{R}^n$. For $1 \leqslant |j| \leqslant n$ write

$$\Gamma(j, Q) = \{x \in \Gamma \cap 2Q : F^j(x) \cap 2Q = \varnothing, F^j(x) \cap \lambda Q \neq \varnothing\}.$$

Then

$$(w \cdot \Gamma \cap 2Q) \cap (w \cdot \Gamma \cap \lambda Q \backslash 2Q) \subseteq \bigcup_{1 \leqslant |j| \leqslant n} w \cdot \Gamma(j, Q). \tag{3.11}$$

To see (3.11) write

$$Q = \Pi_{j=1}^n [m_{-j}, m_j].$$

Suppose that $x \in \Gamma \cap 2Q$, $y \in \Gamma \cap \lambda Q \backslash 2Q$ and $w \cdot x = w \cdot y$ is a point in the left-hand side of (3.11). There exists j, $1 \leqslant j \leqslant n$, with $y_j > m_j$ or $y_j < m_{-j}$. Suppose that $y_j > m_j$. Let $z \in \{z' \in \Gamma \cap 2Q : w \cdot z' = w \cdot x\} = E$ be such that $z_j = \sup E_j$. Then

$$F^j(z) \cap 2Q = \varnothing, \qquad F^j(z) \cap \lambda Q \backslash 2Q \neq \varnothing.$$

So $z \in \Gamma(j, Q)$ and $w \cdot x = w \cdot z$ belongs to the right-hand side of (3.11). If $y_j < m-j$ the argument is similar. Now

$$\begin{aligned} \sum_{Q \in \langle Q^0 \rangle} l((w \cdot \Gamma \cap 2Q) \cap (w \cdot \Gamma \cap \lambda Q \backslash 2Q)) &\leqslant \sum_{1 \leqslant |j| \leqslant n} \sum_{Q \in \langle Q^0 \rangle} l(\Gamma(j, Q)) \\ &\leqslant \sum_{1 \leqslant |j| \leqslant n} \int_{\Gamma \cap 2Q^0} \sum_{Q \in \langle Q^0 \rangle} \chi_{\Gamma(j, Q)}(x)\, dl(x) \\ &\leqslant Cl(\Gamma \cap 2Q^0), \end{aligned}$$

where $\chi_{\Gamma(j, Q)}$ denotes the characteristic function of the set $\Gamma(j, Q)$ and $C = C(n)$. The last inequality holds because the function in the integrand is uniformly bounded by part (b).

We can now complete the proof of Lemma 3.

Proof. Let $c = c(n)$ be as in Lemma 4 part (c). Since the unit sphere S^{n-1} is compact there exists a finite set $W \subseteq S^{n-1}$ such that

$$S^{n-1} \subseteq \bigcup_{w \in W} B_c(w).$$

Let $\delta = \delta(n)$ and $\lambda = \lambda(n)$ be as in Lemma 4 part (c). Then if $Q \in \mathscr{A}$,

$$r_Q \leqslant C \sum_{w \in W} l((w \cdot \Gamma \cap 2Q) \cap (w \cdot \Gamma \cap \lambda Q \backslash 2Q)),$$

where $C = C(n)$. Hence

$$\sum_{Q \in \mathscr{A}} r_Q \leqslant C \sum_{w \in W} \sum_{Q \in \langle Q^0 \rangle} l((w \cdot \Gamma \cap 2Q) \cap (w \cdot \Gamma \cap \lambda Q \backslash 2Q)) \leqslant C_1 l(\Gamma \cap 2Q^0),$$

where $C_1 = C_1(n)$, by Lemma 5 part (c).

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Department of Mathematics
UCLA
Los Angeles CA 90024
USA

Present address:
Department of Mathematics
Princeton University
Princeton NJ 08544–1000
USA