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ENERGY OF A KNOT

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§0. INTRODUCTION

IN THIS paper, we define a real valued functional, the *energy* E, which is well-behaved for embedded circles in the 3 dimensional Euclidean space, and which blows up for curves with self-intersections.

Let $E_{\lambda B}$ be the sum of the energy E and the total squared curvature functional. We show that for any real number α, there are only finitely many ambient isotopy classes of embeddings (i.e. *knot types*) with the value of $E_{\lambda B}$ not greater than α.

There have been studied the total curvature (Fary [1], Fenchel [2], Milnor [5]), the total squared curvature (Langer and Singer [4]), and the Gauss integral of the linking number for a single curve, which, with the total torsion, leads to the notion of the self linking number (Pohl [7]) as functionals on the space of closed curves in $\mathbb{R}^3$ with suitable conditions. But these functionals do not have the above properties. They do not blow up for curves with self-intersections, and we can not in general show the finiteness of knot types by them, though we can distinguish the trivial knot from non-trivial knots by the total curvature, and hence, by the total squared curvature ([1], [5]).

"Energy" of polygonal knots which is something like electrostatic energy was studied by Fukuhara [3], and "energy" of geodesic links in S^3 which is defined by the principal angles was studied by Sakuma [9]. This work was motivated by [8].

In §1, we define the energy E, show the continuity of E, give a lower bound of E and the formulation for E by the double integral, and state a fundamental property of E. In §2, we show the finiteness of the knot types under the bounded value of $E_{\lambda B}$.

Throughout this paper, we always consider the embeddings from S^1 into $\mathbb{R}^3$ of class C^2 such that the norm of the derivative is always one. We use the notation $|\cdot|$ for the standard norm of $\mathbb{R}^3$.

§1. DEFINITIONS AND PROPERTIES OF ENERGY E

(1) Definitions and well-definedness

Definition 1.1. Let $f: S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be an embedding of class C^2 such that $|f'(t)| = 1$ for all $t \in S^1$, where $|\cdot|$ denotes the standard norm of $\mathbb{R}^3$. Let ε be a real number with $0 < \varepsilon < 1/2$. We define $V_\varepsilon(f, x)$ for $x \in S^1$ and $E_\varepsilon(f)$ by the following formulas.

$$(1)\quad V_\varepsilon(f, x) = \int_{x+\varepsilon}^{1+x-\varepsilon} |f(y) - f(x)|^{-2}\, dy,$$

$$(2)\quad E_\varepsilon(f) = \frac{1}{2}\int_0^1 V_\varepsilon(f, x)\, dx.$$

THEOREM 1.2. *Let $f: S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be an embedding of class C^2 such that $|f'(t)| = 1$ for all $t \in S^1$. Then the following holds.*

(1) *There exists $\lim_{\varepsilon \to 0} (V_\varepsilon(f, x) - 2/\varepsilon)$ for any $x \in S^1$. Let it be denoted by $V(f, x)$.*

(2) *There exists $\lim_{\varepsilon \to 0} (E_\varepsilon(f) - 1/\varepsilon)$. Let it be denoted by $E(f)$. Then,*

$$E(f) = \frac{1}{2} \int_{S^1} V(f, x)\,dx.$$

Definition 1.3. We call $E(f)$ the *energy* of f.

Proof. (1) For the given f, let a positive number K be the maximum of the curvature $|f''(t)|$. To show (1), it is enough to show the existence of $V(f, 0)$. For $0 \leqq y \leqq \pi/K$, define $\theta(y) \geqq 0$ by $\cos\theta(y) = (f'(y), f'(0))$, where $(\,,\,)$ denotes the standard inner product of $\mathbb{R}^3$. Then, for $0 \leqq y \leqq \pi/K$, we have $\theta(y) \leqq Ky$, and hence we have the following estimate of $|f(y) - f(0)|$;

$$y \geqq |f(y) - f(0)| \geqq \int_0^y \cos\theta(t)\,dt \geqq \int_0^y \cos Kt\,dt = K^{-1}\sin Ky \cdots \tag{1.2.1}$$

Therefore for any $\varepsilon_1, \varepsilon_2$ with $0 < \varepsilon_1 < \varepsilon_2 < \pi/K$, we have:

$$2\int_{\varepsilon_1}^{\varepsilon_2} y^{-2}\,dy \leqq V_{\varepsilon_1}(f, 0) - V_{\varepsilon_2}(f, 0) \leqq 2\int_{\varepsilon_1}^{\varepsilon_2} (K^{-1}\sin Ky)^{-2}\,dy,$$

and hence,

$$\begin{aligned} 0 &\leqq (V_{\varepsilon_1}(f, 0) - 2/\varepsilon_1) - (V_{\varepsilon_2}(f, 0) - 2/\varepsilon_2) \\ &\leqq 2\{(K\cot K\varepsilon_1 - 1/\varepsilon_1) - (K\cot K\varepsilon_2 - 1/\varepsilon_2)\}. \end{aligned}$$

Since there exists $\lim_{t \to 0} (K\cot Kt - 1/t)$, there exists the limit $\lim_{\varepsilon \to 0} (V_\varepsilon(f, 0) - 2/\varepsilon)$.

(2) Note that in the above proof, $V_\varepsilon(f, x) - 2/\varepsilon$ converges to $V(f, x)$ uniformly on x. As $V_\varepsilon(f, x)$ is a continuous function of x for all ε, $V(f, x)$ is a continuous function of x, and hence is integrable. The assertion follows directly from the uniform convergence of $V(f, x)$. (q.e.d.)

(2) Continuity

Put $\mathscr{L} = \{f: S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$; embedding of class C^2 such that $|f'(t)| = 1$ for all $t \in S^1\}$.

PROPOSITION 1.4. *The energy functional $E: \mathscr{L} \to \mathbb{R}$ is continuous with respect to the C^2-topology.*

Proof. Suppose $f \in \mathscr{L}$. We have to show that for a given positive number ε there exist positive numbers d_0, d_1, d_2 such that if $g \in \mathscr{L}$ satisfies $|f(x) - g(x)| < d_0$, $|f'(x) - g'(x)| < d_1$, and $|f''(x) - g''(x)| < d_2$ for all $x \in S^1$, then $|E(f) - E(g)| < \varepsilon$. Put $K = \max_{s \in S^1} |f''(s)|$. Put $d_2 = K$, then $|g''(x)| \leqq 2K$ for all x. There exists a positive number δ_1 such that if $0 < t \leqq \delta_1$ then

$$|2K\cot 2Kt - 1/t| < \varepsilon/4 \quad \text{and} \quad |K\cot Kt - 1/t| < \varepsilon/4.$$

Therefore for any $x \in S^1$ we have

$$|V(f, x) - (V_{\delta_1}(f, x) - 2/\delta_1)| \leqq \varepsilon,$$

and

$$|V(g, x) - (V_{\delta_1}(g, x) - 2/\delta_1)| \leqq \varepsilon.$$

Put

$$\delta_2 = \inf_{|x-y| \geqq \delta_1} \{|f(x) - f(y)|\} = \min_{|x-y| \geqq \delta_1} \{|f(x) - f(y)|\} > 0,$$

and d_0 such that $(\delta_2 - 2d_0)^{-2} - \delta_2^{-2} < \varepsilon$. Then $|V_{\delta_1}(f, x) - V_{\delta_1}(g, x)| < \varepsilon$ for all x. Therefore $|V(f, x) - V(g, x)| < 3\varepsilon$ for all x, and hence $|E(f) - E(g)| < 3\varepsilon/2$. (q.e.d.)

(3) Lower bound of V and E

PROPOSITION 1.5. *Let $f: S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be an embedding of class C^2 such that $|f'(t)| = 1$ for all $t \in S^1$. Then we have $V(f, x) \geqq -4$ for all $x \in S^1$ and $E(f) \geqq -2$.*

Proof. $|f(x) - f(x \pm s)| \leqq s$ for all x and for all s satisfying $0 \leqq s \leqq 1/2$. The assertion follows immediately. (q.e.d.)

(4) Example, the standard S^1

Define an embedding $i: S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ by

$$i(t) = (1/2\pi \cos 2\pi t,\ 1/2\pi \sin 2\pi t,\ 0).$$

Then,

$$V_\varepsilon(i, x) = \int_{x+\varepsilon}^{1+x-\varepsilon} \pi^2 \{\sin \pi(y - x)\}^{-2} \mathrm{d}y = 2\pi \cot \pi\varepsilon,$$

therefore, $V(i, x) = \lim_{\varepsilon \to 0} (V_\varepsilon(i, x) - 2/\varepsilon) = 0$. Hence $E(i) = 0$.

(5) Formulation for E by double integral

Let $f: S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be an embedding of class C^2 such that $|f'(t)| = 1$ for all $t \in S^1$. Using the above example we get;

$$V(f, x) = \lim_{\varepsilon \to 0} (V_\varepsilon(f, x) - 2/\varepsilon) = \lim_{\varepsilon \to 0} (V(f, x) - V(i, x))$$

$$= \lim_{\varepsilon \to 0} \int_{x+\varepsilon}^{1+x-\varepsilon} |f(x) - f(y)|^{-2} - \pi^2 \{\sin \pi(y - x)\}^{-2} \mathrm{d}y.$$

CLAIM 1.6. *Let f be as before. If we define $F(f; x, y)$ by*

$$F(f; x, y) = |f(x) - f(y)|^{-2} - \pi^2 \{\sin \pi(y - x)\}^{-2}$$

for $x \neq y$, then F is continuous and bounded on $S^1 \times S^1$-diagonal.

Proof. Put $K = \max_{s \in S^1} |f''(s)|$ and $t = \min\{|x - y|, 1 - |x - y|\}$. If $0 < t < \pi/K$ we have

$$t^{-2} \leqq |f(x) - f(y)|^{-2} \leqq K^2 (\sin Kt)^{-2},$$

and hence,

$$t^{-2} - \pi^2 (\sin \pi t)^{-2} \leqq F(f; x, y) \leqq K^2 (\sin Kt)^{-2} - \pi^2 (\sin \pi t)^{-2}.$$

The left term goes to $-\pi^2/3$ and the right term to $K^2/3 - \pi^2/3$ as t goes to $+0$. Hence F is bounded on $S^1 \times S^1$-diagonal. (q.e.d.)

Let us put $F(f;x,x)=\lim_{y\to x}F(f;x,y)$ if exists, or else $F(f;x,x)=0$. Then,

$$V(f,x)=\lim_{\varepsilon\to 0}\int_{x+\varepsilon}^{1+x-\varepsilon}F(f;x,y)\,dy=\int_0^1 F(f;x,y)\,dy.$$

Since $E(f)=\frac{1}{2}\int_0^1 V(f,x)\,dx$, we get;

THEOREM 1.7. *Let $f\colon S^1=\mathbb{R}/\mathbb{Z}\to\mathbb{R}^3$ be an embedding of class C^2 such that $|f'(t)|=1$ for all $t\in S^1$. Then we have;*

$$E(f)=\frac{1}{2}\iint_{S^1\times S^1}|f(x)-f(y)|^{-2}-\pi^2\{\sin\pi(y-x)\}^{-2}\,dx\,dy.$$

(6) The fundamental property of E

We want to show the following statement; "Two distinct points in S^1 can not approach each other so close if the energy E is bounded above."

Let us consider the following condition.

(#) For some $x,y\in S^1$ with $\min\{|x-y|,1-|x-y|\}=\delta$, $|f(x)-f(y)|=\sigma$.

We can calculate a lower bound of the energy E under the above condition (#) by integrating over S^1 the minimum of V at each point of S^1 under the condition (#). In this section, we will omit the letter "f" in V. We may assume $x=0$ and $y=\delta$. Put $\gamma=1-\delta$. For $0\leqq t\leqq(\delta-\sigma)/2$, we have

$$V_\varepsilon(\delta-t),\ V_\varepsilon(t)\geqq\left(\int_\varepsilon^{(\delta+\sigma)/2}+\int_\varepsilon^{(\gamma+\sigma)/2+t}+\int_{t+\sigma}^{(\delta+\sigma)/2}+\int_{t+\sigma}^{(\gamma+\sigma)/2+t}\right)s^{-2}\,ds.$$

(See Fig. 1.9.) Hence, we have

$$V(\delta-t),\ V(t)\geqq 2((t+\sigma)^{-1}-2(\delta+\sigma)^{-1}-2(\gamma+\sigma+2t)^{-1}).$$

For $0\leqq t\leqq(\gamma-\sigma)/2$ we have similarly

$$V(\delta+t),\ V(-t)\geqq 2((t+\sigma)^{-1}-2(\gamma+\sigma)^{-1}-2(\delta+\sigma+2t)^{-1}).$$

For $(\delta-\sigma)/2\leqq t\leqq(\delta+\sigma)/2$ and $\delta+(\gamma-\sigma)/2\leqq t\leqq\delta+(\gamma+\sigma)/2$ we have $V(t)\geqq-4$ by Proposition 1.5. Hence under the condition (#), we have

$$\begin{aligned}E(f)=\frac{1}{2}\int_{S^1}V&\geqq\frac{1}{2}(2\sigma)(-4)+2\int_0^{(\delta-\sigma)/2}\{(t+\sigma)^{-1}-2(\delta+\sigma)^{-1}-2(\gamma+\sigma+2t)^{-1}\}\,dt\\&\quad+2\int_0^{(\gamma-\sigma)/2}\{(t+\sigma)^{-1}-2(\gamma+\sigma)^{-1}-2(\delta+\sigma+2t)^{-1}\}\,dt\\&=-4\sigma-2(\delta-\sigma)/(\delta+\sigma)-2(\gamma-\sigma)/(\gamma+\sigma)-4\log 2+4\log(\delta+\sigma)\\&\quad+4\log(\gamma+\sigma)-4\log\sigma\\&\geqq-4-8\log 2+4\log\delta-4\delta-4\log\sigma.\end{aligned}$$

Therefore,

$$\log\sigma\geqq-E/4-1-\log 4-\delta+\log\delta.$$

Summarizing these up, we have

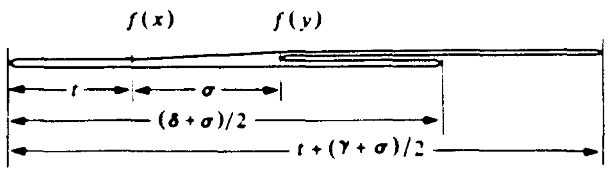

Fig. 1.9.

THEOREM 1.8. *Given real numbers* $\alpha \geqq -2$ *and* $\delta > 0$. *Let* $f\colon S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ *be an embedding of class* C^2 *such that* $|f'(t)| = 1$ *for all* $t \in S^1$. *Suppose* $E(f) \leqq \alpha$ *and* $min\{|x-y|, 1-|x-y|\} = \delta$ *for* $x, y \in S^1$, *then we have*:

$$|f(x) - f(y)| \geqq \frac{\delta}{4e\cdot\exp(\alpha/4)\cdot\exp\delta} > \frac{\delta}{18\exp(\alpha/4)}.$$

§2. FINITENESS OF KNOT TYPES

(1) Bending energy

To make things easier, we consider the sum of the energy E and the total squared curvature. The latter physically corresponds to the bending energy of the elastic rod. Let $f\colon S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be an embedding of class C^2 such that $|f'(t)| = 1$ for all $t \in S^1$. We define the bending energy of f by

$$B(f) = \frac{1}{4\pi^2}\int_0^1 |f''(t)|^2\,dt.$$

We give the lower bound of the bending energy B, which we use later in Theorem 2.4.

THEOREM 2.2. (Fenchel [2]) *Let* f *be as above. Then* $B(f) \geqq 1$.

(2) $E_{\lambda B}$ *and the finiteness of knot types*

Definition 2.3. Let $f\colon S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be an embedding of class C^2 such that $|f'(t)| = 1$ for all $t \in S^1$. For a positive number λ, define $E_{\lambda B}(f)$ by

$$\begin{aligned} E_{\lambda B}(f) &= E(f) + \lambda B(f) \\ &= \frac{1}{2}\iint_{S^1\times S^1} \{|f(x)-f(y)|^{-2} - \pi^2\{\sin\pi(y-x)\}^{-2} \\ &\quad + \lambda/4\pi^2(|f''(x)|^2 + |f''(y)|^2)\}\,dx\,dy. \end{aligned}$$

Let f be as before. We have $E(f) \geqq -2$ by Proposition 1.5 and $B(f) \geqq 1$ by Theorem 2.2. Given a real number $\alpha \geqq \lambda - 2$, then $E_{\lambda B}(f) \leqq \alpha$ means $E(f) \leqq \alpha - \lambda$ and $B(f) \leqq \lambda^{-1}(\alpha + 2)$. (Remark that $E_{\lambda B}(f) \leqq \alpha$ for some f means necessarily $\alpha \geqq \lambda - 2$.)

Now we can have a main theorem.

THEOREM 2.4. *Given real numbers* $\lambda > 0$ *and* $\alpha \geqq \lambda - 2$. *Then there exist only finitely many ambient isotopy classes of embeddings (knot types) which can be represented by an embedding* f *of class* C^2 *from* S^1 *into* $\mathbb{R}^3$ *such that* $|f'(t)| = 1$ *for all* $t \in S^1$ *and that* $E_{\lambda B}(f) \leqq \alpha$.

LEMMA 2.5. *For given $\lambda > 0$ and $\alpha \geqq \lambda - 2$, there exists a positive number $r = r(\lambda, \alpha)$ with a following property.*

Let $f: S^1 \to \mathbb{R}^3$ be an embedding of class C^2 such that $|f'(t)| = 1$ for all $t \in S^1$ and that $E_{\lambda B}(f) \leqq \alpha$, X be a point in $f(S^1)$, and $B_r(X)$ be a 3-ball with center X and radius r. Then $B_r(X) \cap f(S^1)$ is an arcwise connected unknotted curve segment, that is, it can pass into a straight line segment by an isotopy of $B_r(X)$ which keeps the boundary sphere fixed.

We show that Lemma 2.5 implies Theorem 2.4.

Proof of Theorem 2.4. For given $\lambda > 0$ and $\alpha \geqq \lambda - 2$, take a positive number $r = r(\lambda, \alpha)$ of the above Lemma, and put the natural number $N = N(\lambda, \alpha) = [1/2r] + 1$, where [ ] is Gauss's symbol. Suppose $E_{\lambda B}(f) \leqq \alpha$. Then $f(S^1)$ can be covered by N balls with radius r in each of which $f(S^1)$ is an unknotted curve segment. Hence $f(S^1)$ is ambient isotopic to a PL knot with N vertices. Now the proof follows directly from the fact that there exist only finitely many knot types of PL knots with N vertices. (q.e.d.)

Proof of Lemma 2.5. We show that

$$r = 1/18 \exp((\alpha - \lambda)/4) \cdot (2 - \sqrt{2})\lambda/4\pi^2(\alpha + 2).$$

has the property of Lemma 2.5.

Suppose $E_{\lambda B}(f) \leqq \alpha$. Then we have $E(f) \leqq \alpha - \lambda$ and $B(f) \leqq (\alpha + 2)/\lambda$, may assume $X = f(0) = 0$ and $f'(0) = (1, 0, 0)$. Put $f = (f_1, f_2, f_3)$ and $t_0 = (2 - \sqrt{2})\lambda/4\pi^2(\alpha + 2)$.

For $0 \leqq t \leqq t_0$ we have the following estimates:

$$\begin{aligned} |f'(t) - f'(0)|^2 &= \left| \int_0^t f''(s)\,\mathrm{d}s \right|^2 \\ &\leqq \left(\int_0^t |f''(s)|\,\mathrm{d}s \right)^2 \\ &\leqq t \int_0^t |f''(s)|^2\,\mathrm{d}s \\ &\leqq t4\pi^2(\alpha + 2)/\lambda \leqq 2 - \sqrt{2}, \end{aligned}$$

and therefore $f_1'(t) \geqq 1/\sqrt{2}$ and $f_1(t)/|f(t)| > 1/\sqrt{2}$ since $f_1(t) > t/\sqrt{2}$.

Hence for $0 \leqq t \leqq t_0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(|f(t)|^2) = 2(f(t), f'(t)) > 0.$$

Besides, for $t_0 \leqq t \leqq 1/2$ we have $|f(t)| > r$ by Theorem 1.9. Summarizing these up, we have the proof. (q.e.d.)

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