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Riemannian geometries on spaces of plane curves

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Abstract. We study some Riemannian metrics on the space of smooth regular curves in the plane, viewed as the orbit space of maps from S^1 to the plane modulo the group of diffeomorphisms of S^1, acting as reparametrizations. In particular we investigate the metric, for a constant $A > 0$,

$$G_c^A(h,k) := \int_{S^1} (1 + A\kappa_c(\theta)^2)\langle h(\theta), k(\theta)\rangle |c'(\theta)|\, d\theta$$

where κ_c is the curvature of the curve c and h, k are normal vector fields to c. The term $A\kappa^2$ is a sort of geometric Tikhonov regularization because, for $A = 0$, the geodesic distance between any two distinct curves is 0, while for $A > 0$ the distance is always positive. We give some lower bounds for the distance function, derive the geodesic equation and the sectional curvature, solve the geodesic equation with simple endpoints numerically, and pose some open questions. The space has an interesting split personality: among large smooth curves, all its sectional curvatures are ≥ 0, while for curves with high curvature or perturbations of high frequency, the curvatures are ≤ 0.

1. Introduction

This paper arose from the attempt to find the simplest Riemannian metric on the space of 2-dimensional 'shapes'. By a shape we mean a compact simply connected region in the plane whose boundary is a simple closed curve. By requiring that the boundary curve has various degrees of smoothness, we get not just one space but a whole hierarchy of spaces. All these spaces will include, however, a core, namely the space of all shapes with C^∞ boundary curves. We expect that the most natural shape spaces will arise as the completions of this core space in some metric, hence we take this core as our basic space. Note that it is the orbit space

$$B_e(S^1, \mathbb{R}^2) = \mathrm{Emb}(S^1, \mathbb{R}^2)/\,\mathrm{Diff}(S^1)$$

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of the space of all C^∞ embeddings of S^1 in the plane, under the action by composition from the right by diffeomorphisms of the circle. The space $\mathrm{Emb}(S^1, \mathbb{R}^2)$ is a smooth manifold, in fact an open subset of the Fréchet space $C^\infty(S^1, \mathbb{R}^2)$, and it is the total space of a smooth principal bundle with base $B_e(S^1, \mathbb{R}^2)$.

In fact, most of our results carry over to the bigger orbit space of immersions mod diffeomorphisms:

$$B_i(S^1, \mathbb{R}^2) = \mathrm{Imm}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1).$$

This action is not quite free (see 2.4 and 2.5), hence this orbit space is an orbifold (see 2.5) and not quite a manifold. There is the slightly smaller space Imm_f (see 2.1) of immersed curves where diffeomorphisms act freely, the total space of a principal fiber bundle with a natural connection admitting parallel transport. Existence of horizontal curves, however, holds also in the big space Imm (see 2.5) which will be one of the weapons in our hunt for geodesics on B_i.

The second author was led to study the space B_e from its relevance to computer vision. To understand an image of the world, one needs to identify the most salient objects present in this image. In addition to readily quantifiable properties like color and area, objects in the world and their projections depicted by 2D images possess a 'shape' which is readily used by human observers to distinguish, for example, cats from dogs, BMW's from Hondas, etc. In fact people are not puzzled by what it means to say two shapes are *similar* but rather find this a natural question. This suggests that we construct, on some crude level, a mental metric which can be used to recognize familiar objects by the similarity of their shapes and to cluster categories of related objects like cats. Incidentally, immersions also arise in vision when a 3D object partially occludes itself from some viewpoint, hence its full 2D contour has visible and invisible parts which, together, form an immersed curve in the image plane.

It is a central problem in computer vision to devise algorithms by which computers can similarly recognize and cluster shapes. Many types of metrics have been proposed for this purpose [7]. For example, there are L^1-type metrics such as the area of the symmetric difference of the interiors of two shapes. And there are L^∞-type metrics such as the Hausdorff metric: the maximum distance of points on either shape from the points on the other or of points outside one shape from points outside the other. These metrics will come up below, but the starting point of this investigation was whether one could use the manifold structure on the space of shapes and define an L^2-type metric by introducing a Riemannian structure on the space.

Such questions have also arisen in Teichmüller theory and string theory, where the so-called Weil–Peterssen metric on the space of shapes (also called the 'universal Teichmüller space') has been much studied. In a second part of this paper, we will compare our metric to this remarkable (homogeneous!) metric.

In this paper, we sought the absolutely simplest Riemannian metric that the space B_i supports. The most obvious $\mathrm{Diff}(S^1)$-invariant weak Riemannian metric on the space of immersions is the H^0-metric:

$$G^0_c(h, k) = \int_{S^1} \langle h(\theta), k(\theta)\rangle |c'(\theta)|\, d\theta$$

where $c : S^1 \to \mathbb{R}^2$ is an embedding defining a point in B_e and h, k are vector fields along the image curve, defining two tangent vectors to $\mathrm{Imm}(S^1, \mathbb{R}^2)$ at c. This induces a $\mathrm{Diff}(S^1)$-invariant weak Riemannian metric on the space of all immersions and on $\mathrm{Emb}(S^1, \mathbb{R}^2)$, and for the latter space it induces a weak Riemannian metric on the base manifold B_e.

Surprisingly, the Riemannian distance defined as the infimum of the arc length of paths connecting two points in $B_e(S^1, \mathbb{R}^2)$ turns out to be 0 (see 3.10)! This seems to be one of the first examples where this purely infinite-dimensional phenomenon actually appears.

Motivated by the proof of this result 3.10 we are led to consider the invariant Riemannian metric 3.2.6, for a constant $A > 0$,

$$G_c^A(h, k) := \int_{S^1} (1 + A\kappa_c(\theta)^2)\langle h(\theta), k(\theta)\rangle |c'(\theta)|\, d\theta$$

where $\kappa_c(\theta)$ is the curvature of c at $c(\theta)$. We will argue that this induces a reasonable metric on $B_e(S^1, \mathbb{R}^2)$, as the infimum of the arc lengths of paths connecting distinct points is always positive. Another reason is that the length function $\ell : B_e(S^1, \mathbb{R}^2) \to \mathbb{R}_{\geq 0}$ has the following Lipschitz estimate 3.3.2 with respect to this Riemannian distance:

$$\sqrt{\ell(C_1)} - \sqrt{\ell(C_0)} \leq \frac{1}{2\sqrt{A}}\, \mathrm{dist}_{G^A}^{B_e}(C_1, C_2).$$

In fact, one can bound the Fréchet distance between two curves in terms of this metric (see 3.5). The completion of the space of smooth curves in this metric contains all curves whose curvature exists weakly as a finite signed measure (e.g. piecewise C^2 curves) and is contained in the space of Lipschitz maps from S^1 to $\mathbb{R}^2$ modulo a suitable equivalence relation (see 2.11).

The geodesic equation for the metric G^A on $\mathrm{Emb}(S^1, \mathbb{R}^2)$ and on $B_e(S^1, \mathbb{R}^2)$ can be found in 4.1.1: It is a highly non-linear partial differential equation of order 4 with degenerate symbol, but which nonetheless seems to have a hypoelliptic linearization. If $A = 0$, the equation reduces to a non-linear second order hyperbolic PDE, which gives a well defined local geodesic spray. For any A, the sectional curvature on $B_e(S^1, \mathbb{R}^2)$ has an elegant expression which can be found in 4.6.2 and 4.6.4. It is non-negative if $A = 0$ and, for general A, becomes strictly negative only if the curve has large curvature or the plane section has high frequency. Of course we would have liked to solve the problem of existence and uniqueness of geodesics for $A > 0$. We can, however, translate the minimization of path length in our metric into an anisotropic Plateau-like problem: In 3.12 we show that a curve projects onto a geodesic in $B_e(S^1, \mathbb{R}^2)$ if and only if its graph in $[0, 1] \times \mathbb{R}^2$ is a surface with given boundary at $\{0\} \times \mathbb{R}^2$ and $\{1\} \times \mathbb{R}^2$ which is critical for the anisotropic area functional 3.12.3.

In 5.1 we determine the geodesic running through concentric circles and the equation for Jacobi vector fields along this geodesic. The solution of the ordinary differential equation 5.1.1 describing this geodesic can be written in terms of elliptic functions. This geodesic is no longer globally minimizing when the radius of the circles is large compared to $\sqrt{A}$ and has conjugate points when it hits this positive curvature zone. In 5.2

we study geodesics connecting arbitrary distant curves, hence requiring long translations. The middle parts of such geodesics appear to be approximated by a uniformly translating 'cigar'-like curve with semi-circular ends of radius $\sqrt{A}$ connected by straight line segments parallel to the direction of translation. These figures were found by numerically minimizing a discrete form of the energy functional 3.12.1.

Finally, in 5.3 and 5.4, we have some further pictures of geodesics. First we examine the formation of singularities when a small perturbation is propagated forward and $A = 0$. Then we look at some geodesic triangles in B_e whose vertices are ellipses with the same eccentricity and center but different orientations. For various values of A, we find that these triangles have angle sums greater and less than π.

2. The manifold of immersed closed curves

2.1. Conventions. It is often convenient to use the identification $\mathbb{R}^2 \cong \mathbb{C}$, giving us:

$$\bar{x}y = \langle x, y\rangle + i\,\det(x, y), \quad \det(x, y) = \langle ix, y\rangle.$$

We shall use the following spaces of C^∞ (smooth) diffeomorphisms and curves, and we give the shorthand and the full name:

$\mathrm{Diff}(S^1)$, the regular Lie group ([6], 38.4) of all diffeomorphisms $S^1 \to S^1$ with its connected components $\mathrm{Diff}^+(S^1)$ of orientation preserving diffeomorphisms and $\mathrm{Diff}^-(S^1)$ of orientation reversing diffeomorphisms.

$\mathrm{Diff}_1(S^1)$, the subgroup of diffeomorphisms fixing $1 \in S^1$. We have diffeomorphically $\mathrm{Diff}(S^1) = \mathrm{Diff}_1(S^1) \times S^1 = \mathrm{Diff}_1^+(S^1) \times (S^1 \rtimes \mathbb{Z}_2)$.

$\mathrm{Emb} = \mathrm{Emb}(S^1, \mathbb{R}^2)$, the manifold of all smooth embeddings $S^1 \to \mathbb{R}^2$. Its tangent bundle is given by $T\,\mathrm{Emb}(S^1, \mathbb{R}^2) = \mathrm{Emb}(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2)$.

$\mathrm{Imm} = \mathrm{Imm}(S^1, \mathbb{R}^2)$, the manifold of all smooth immersions $S^1 \to \mathbb{R}^2$. Its tangent bundle is given by $T\,\mathrm{Imm}(S^1, \mathbb{R}^2) = \mathrm{Imm}(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2)$.

$\mathrm{Imm}_f = \mathrm{Imm}_f(S^1, \mathbb{R}^2)$, the manifold of all smooth free immersions $S^1 \to \mathbb{R}^2$, i.e., those with trivial isotropy group for the right action of $\mathrm{Diff}(S^1)$ on $\mathrm{Imm}(S^1, \mathbb{R}^2)$.

$B_e = B_e(S^1, \mathbb{R}^2) = \mathrm{Emb}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1)$, the manifold of 1-dimensional connected submanifolds of $\mathbb{R}^2$ (see 2.3).

$B_i = B_i(S^1, \mathbb{R}^2) = \mathrm{Imm}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1)$, an infinite-dimensional 'orbifold'; its points are, roughly speaking, smooth curves with crossings and multiplicities (see 2.5).

$B_{i,f} = B_{i,f}(S^1, \mathbb{R}^2) = \mathrm{Imm}_f(S^1, \mathbb{R}^2)/\,\mathrm{Diff}(S^1)$, a manifold, the base of a principal fiber bundle (see 2.4.3).

We want to avoid referring to a path in our infinite-dimensional spaces like Imm or B_e as a curve, because it is then a 'curve of curves' and confusion arises when you refer to a curve. So we will always talk of *paths* in the infinite-dimensional spaces, not curves. Curves will be in $\mathbb{R}^2$. Moreover, if $t \mapsto (\theta \mapsto c(t, \theta))$ is a path, its t^{th} curve will be denoted by $c(t) = c(t, \ )$. By c_t we shall denote the derivative $\partial_t c$, and $c_\theta = \partial_\theta c$.

2.2. Length and curvature on $\operatorname{Imm}(S^1, \mathbb{R}^2)$**.** The volume form on S^1 induced by c is given by

$$\operatorname{vol} : \operatorname{Emb}(S^1, \mathbb{R}^2) \to \Omega^1(S^1), \qquad \operatorname{vol}(c) = |c_\theta|\, d\theta, \tag{1}$$

and its derivative is

$$d \operatorname{vol}(c)(h) = \frac{\langle h_\theta, c_\theta \rangle}{|c_\theta|}\, d\theta. \tag{2}$$

We shall also use the *normal unit field*

$$n_c = i\, \frac{c_\theta}{|c_\theta|}.$$

The length function is given by

$$\ell : \operatorname{Imm}(S^1, \mathbb{R}^2) \to \mathbb{R}, \qquad \ell(c) = \int_{S^1} |c_\theta|\, d\theta, \tag{3}$$

and its differential is

$$\begin{aligned} d\ell(c)(h) &= \int_{S^1} \frac{\langle h_\theta, c_\theta \rangle}{|c_\theta|}\, d\theta = -\int_{S^1} \Big\langle h, \frac{c_{\theta\theta}}{|c_\theta|} - \frac{\langle c_{\theta\theta}, c_\theta \rangle}{|c_\theta|^3} c_\theta \Big\rangle\, d\theta \\ &= -\int_{S^1} \langle h, \kappa(c) \cdot i c_\theta \rangle\, d\theta = -\int_{S^1} \langle h, n_c \rangle \kappa(c) \operatorname{vol}(c). \end{aligned} \tag{4}$$

The curvature mapping is given by

$$\kappa : \operatorname{Imm}(S^1, \mathbb{R}^2) \to C^\infty(S^1, \mathbb{R}), \qquad \kappa(c) = \frac{\det(c_\theta, c_{\theta\theta})}{|c_\theta|^3} = \frac{\langle i c_\theta, c_{\theta\theta} \rangle}{|c_\theta|^3}, \tag{5}$$

and is equivariant so that $\kappa(c \circ f) = \pm \kappa(c) \circ f$ for $f \in \operatorname{Diff}^{\pm}(S^1)$. Its derivative is given by

$$d\kappa(c)(h) = \frac{\langle i h_\theta, c_{\theta\theta} \rangle}{|c_\theta|^3} + \frac{\langle i c_\theta, h_{\theta\theta} \rangle}{|c_\theta|^3} - 3\kappa(c) \frac{\langle h_\theta, c_\theta \rangle}{|c_\theta|^2}. \tag{6}$$

With some work, this can be shown to equal

$$d\kappa(c)(h) = \frac{\langle h, c_\theta \rangle}{|c_\theta|^2} \kappa_\theta + \frac{\langle h, i c_\theta \rangle}{|c_\theta|} \kappa^2 + \frac{1}{|c_\theta|} \left(\frac{1}{|c_\theta|} \left(\frac{\langle h, i c_\theta \rangle}{|c_\theta|} \right)_\theta \right)_\theta. \tag{7}$$

To verify this, note that both the left and right hand side are equivariant with respect to $\operatorname{Diff}(S^1)$, hence it suffices to check it for constant speed parametrizations, i.e. $|c_\theta|$ is constant and $c_{\theta\theta} = \kappa |c_\theta| i c_\theta$. By linearity, it is enough to take the two cases $h = a i c_\theta$ and $h = b c_\theta$. If we substitute these into formulas (6) and (7), the result is straightforward.

2.3. The principal bundle of embeddings $\mathrm{Emb}(S^1, \mathbb{R}^2)$**.** We recall some basic results whose proof can be found in [6]:

(A) *The set* $\mathrm{Emb}(S^1, \mathbb{R}^2)$ *of all smooth embeddings* $S^1 \to \mathbb{R}^2$ *is an open subset of the Fréchet space* $C^\infty(S^1, \mathbb{R}^2)$ *of all smooth mappings* $S^1 \to \mathbb{R}^2$ *with the* C^∞*-topology. It is the total space of a smooth principal bundle* $\pi : \mathrm{Emb}(S^1, \mathbb{R}^2) \to B_e(S^1, \mathbb{R}^2)$ *with structure group* $\mathrm{Diff}(S^1)$*, the smooth regular Lie group of all diffeomorphisms of* S^1*, whose base* $B_e(S^1, \mathbb{R}^2)$ *is the smooth Fréchet manifold of all submanifolds of* $\mathbb{R}^2$ *of type* S^1*, i.e., the smooth manifold of all simple closed curves in* $\mathbb{R}^2$ ([6, 44.1]).

(B) *This principal bundle admits a smooth principal connection described by the horizontal bundle whose fiber* $\mathcal{N}_c$ *over* c *consists of all vector fields* h *along* c *such that* $\langle h, c_\theta \rangle = 0$*. The parallel transport for this connection exists and is smooth* ([6, 39.1 and 43.1]).

See 2.4.3 for a sketch of proof of the first part in a slightly more general situation. See also 3.2.2 and 3.2.3 for the horizontal bundle $\mathcal{N}_c$. Here we want to sketch the use of the second part. Suppose that $t \mapsto (\theta \mapsto c(t, \theta))$ is a path in $\mathrm{Emb}(S^1, \mathbb{R}^2)$. Then $\pi \circ c$ is a smooth path in $B_e(S^1, \mathbb{R}^2)$. Parallel transport over it with initial value $c(0, \cdot)$ is now a path f in $\mathrm{Emb}(S^1, \mathbb{R}^2)$ which is horizontal, i.e., we have $\langle f_t, f_\theta \rangle = 0$. This argument will play an important role below. In 2.5 below we will prove this property for general immersions.

2.4. Free immersions. The manifold $\mathrm{Imm}(S^1, \mathbb{R}^2)$ of all immersions $S^1 \to \mathbb{R}^2$ is an open set in the manifold $C^\infty(S^1, \mathbb{R}^2)$ and thus itself a smooth manifold. An immersion $c : S^1 \to \mathbb{R}^2$ is called *free* if $\mathrm{Diff}(S^1)$ acts freely on it, i.e., $c \circ \varphi = c$ for $\varphi \in \mathrm{Diff}(S^1)$ implies $\varphi = \mathrm{Id}$. We have the following results:

(1) *If* $\varphi \in \mathrm{Diff}(S^1)$ *has a fixed point and if* $c \circ \varphi = c$ *for some immersion* c *then* $\varphi = \mathrm{Id}$. This is [2, 1.3].

(2) *If for* $c \in \mathrm{Imm}(S^1, \mathbb{R}^2)$ *there is a point* $x \in c(S^1)$ *with only one preimage then* c *is a free immersion.* This is [2, 1.4]. There exist free immersions without such points: Consider a figure eight consisting of two touching ovals, and map S^1 to this by first transversing the upper oval three times and then the lower oval two times. This is a free immersion.

(3) **The manifold** $B_{i,f}(S^1, \mathbb{R}^2)$ ([2, 1.5])**.** *The set* $\mathrm{Imm}_f(S^1, \mathbb{R}^2)$ *of all free immersions is open in* $C^\infty(S^1, \mathbb{R}^2)$ *and thus a smooth submanifold. The projection*

$$\pi : \mathrm{Imm}_f(S^1, \mathbb{R}^2) \to \frac{\mathrm{Imm}_f(S^1, \mathbb{R}^2)}{\mathrm{Diff}(S^1)} =: B_{i,f}(S^1, \mathbb{R}^2)$$

onto a Hausdorff smooth manifold is a smooth principal fibration with structure group $\mathrm{Diff}(S^1)$*. By* [6, 39.1 and 43.1] *this fibration admits a smooth principal connection described by the horizontal bundle with fiber* $\mathcal{N}_c$ *consisting of all vector fields* h *along* c *such that* $\langle h, c_\theta \rangle = 0$*. This connection admits a smooth parallel transport over each smooth curve in the base manifold.*

We might view $\operatorname{Imm}_f(S^1,\mathbb{R}^2)$ as the nonlinear Stiefel manifold of parametrized curves in $\mathbb{R}^2$ and consequently $B_{i,f}(S^1,\mathbb{R}^2)$ as the nonlinear Grassmannian of unparametrized simple closed curves.

Sketch of proof. See also [2] for a slightly different proof with more details. For $c \in \operatorname{Imm}_f(S^1,\mathbb{R}^2)$ and $s=(s_1,s_2) \in \mathcal{V}(c) \subset C^\infty(S^1,\mathbb{R}\times S^1)$ consider

$$\varphi_c(s) : S^1 \to \mathbb{R}^2, \qquad \varphi_c(s)(\theta) = c(s_2(\theta)) + s_1(s_2(\theta)) \cdot n_c(s_2(\theta)),$$

where $\mathcal{V}(c)$ is a C^∞-open neighborhood of $(0, \operatorname{Id}_{S^1})$ in $C^\infty(S^1,\mathbb{R}\times S^1)$ chosen in such a way that:

- $s_2 \in \operatorname{Diff}(S^1)$ for each $s \in \mathcal{V}(c)$.
- $\varphi_c(s)$ is a free immersion for each $s \in \mathcal{V}(c)$.
- For $(s_1,s_2) \in \mathcal{V}(c)$ and $\alpha \in \operatorname{Diff}(S^1)$ we have $(s_1, s_2\circ\alpha) \in \mathcal{V}(c)$.

Obviously $\varphi_c(s_1,s_2)\circ\alpha = \varphi_c(s_1,s_2\circ\alpha)$ and s_2 is uniquely determined by $\varphi_c(s_1,s_2)$ since this is a free immersion. Thus the inverse of φ_c is a smooth chart for the manifold $\operatorname{Imm}_f(S^1,\mathbb{R}^2)$. Moreover, we consider the mapping (which will be important in Section 4 below)

$$\begin{aligned}
&\psi_c : C^\infty(S^1,(-\varepsilon,\varepsilon)) \to \operatorname{Imm}_f(S^1,\mathbb{R}^2), \qquad \mathcal{Q}(c) := \psi_c(C^\infty(S^1,(-\varepsilon,\varepsilon))),\\
&\psi_c(f)(\theta) = c(\theta) + f(\theta) n_c(\theta) = \varphi_c(f, \operatorname{Id}_{S^1})(\theta),\\
&\pi\circ\psi : C^\infty(S^1,(-\varepsilon,\varepsilon)) \to B_{i,f}(S^1,\mathbb{R}^2),
\end{aligned}$$

where ε is small. Then (an open subset of) $\mathcal{V}(c)$ splits diffeomorphically into

$$C^\infty(S^1,(-\varepsilon,\varepsilon)) \times \operatorname{Diff} S^1$$

and thus its image under φ_c splits into $\mathcal{Q}(c) \times \operatorname{Diff}(S^1)$. So the inverse of $\pi\circ\psi_c$ is a smooth chart for $B_{i,f}(S^1,\mathbb{R}^2)$. That the chart changes induced by the mappings φ_c and ψ_c contructed here are smooth is shown by writing them in terms of compositions and projections only and applying the setting of [6]. ◻

2.5. Non-free immersions. Any immersion is proper since S^1 is compact and thus by [2, 2.1] the orbit space $B_i(S^1,\mathbb{R}^2) = \operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$ is Hausdorff. Moreover, by [2, 3.1 and 3.2] for any immersion c the isotropy group $\operatorname{Diff}(S^1)_c$ is a finite cyclic group which acts as group of covering transformations for a finite covering $q_c : S^1 \to S^1$ such that c factors over q_c to a free immersion $\bar c : S^1 \to \mathbb{R}^2$ with $\bar c \circ q_c = c$. Thus the subgroup $\operatorname{Diff}_1(S^1)$ of all diffeomorphisms φ fixing $1 \in S^1$ acts freely on $\operatorname{Imm}(S^1,\mathbb{R}^2)$. Moreover, for each $c \in \operatorname{Imm}$ the submanifold $\mathcal{Q}(c)$ from the proof of 2.4.3 (dropping the freeness assumption) is a slice in a strong sense:

- $\mathcal{Q}(c)$ is invariant under the isotropy group $\operatorname{Diff}(S^1)_c$.
- If $\mathcal{Q}(c)\circ\varphi \cap \mathcal{Q}(c) \neq \emptyset$ for $\varphi \in \operatorname{Diff}(S^1)$ then φ is already in the isotropy group $\operatorname{Diff}(S^1)_c$.

- $\mathcal{Q}(c) \circ \operatorname{Diff}(S^1)$ is an invariant open neighborhood of the orbit $c \circ \operatorname{Diff}(S^1)$ in $\operatorname{Imm}(S^1, \mathbb{R}^2)$ which admits a smooth retraction r onto the orbit. The fiber $r^{-1}(c \circ \varphi)$ equals $\mathcal{Q}(c \circ \varphi)$.

Note that also the action

$$\operatorname{Imm}(S^1, \mathbb{R}^2) \times \operatorname{Diff}(S^1) \to \operatorname{Imm}(S^1, \mathbb{R}^2) \times \operatorname{Imm}(S^1, \mathbb{R}^2), \qquad (c, \varphi) \mapsto (c, c \circ \varphi),$$

is proper so that all assumptions and conclusions of Palais' slice theorem [8] hold. These results show that the orbit space $B_i(S^1, \mathbb{R}^2)$ has only very simple singularities of the type of a cone $\mathbb{C}/\{e^{2\pi k/n} : 0 \le k < n\}$ times a Fréchet space. We may call the space $B_i(S^1, \mathbb{R}^2)$ an infinite-dimensional *orbifold*. The projection $\pi : \operatorname{Imm}(S^1, \mathbb{R}^2) \to B_i(S^1, \mathbb{R}^2) = \operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$ is a submersion off the singular points and has only mild singularities at the singular strata. The normal bundle $\mathcal{N}_c$ mentioned in 2.3 is well defined and is a smooth vector subbundle of the tangent bundle. We do not have a principal bundle and thus no principal connections, but we can prove the main consequence, the existence of horizontal paths, directly:

Proposition. *For any smooth path c in $\operatorname{Imm}(S^1, \mathbb{R}^2)$ there exists a smooth path φ in $\operatorname{Diff}(S^1)$ with $\varphi(0, \ ) = \operatorname{Id}_{S^1}$ depending smoothly on c such that the path e given by $e(t, \theta) = c(t, \varphi(t, \theta))$ is horizontal: $e_t \perp e_\theta$.*

Proof. Let us write $e = c \circ \varphi$ for $e(t, \theta) = c(t, \varphi(t, \theta))$, etc. We look for φ as the integral curve of a time dependent vector field $\xi(t, \theta)$ on S^1, given by $\varphi_t = \xi \circ \varphi$. We want the following expression to vanish:

$$\begin{aligned}
\langle \partial_t(c \circ \varphi), \partial_\theta(c \circ \varphi) \rangle &= \langle c_t \circ \varphi + (c_\theta \circ \varphi)\, \varphi_t, (c_\theta \circ \varphi)\, \varphi_\theta \rangle \\
&= (\langle c_t, c_\theta \rangle \circ \varphi)\, \varphi_\theta + (\langle c_\theta, c_\theta \rangle \circ \varphi)\, \varphi_\theta\, \varphi_t \\
&= ((\langle c_t, c_\theta \rangle + \langle c_\theta, c_\theta \rangle\, \xi) \circ \varphi)\, \varphi_\theta.
\end{aligned}$$

Using the time dependent vector field $\xi = -\langle c_t, c_\theta \rangle / |c_\theta|^2$ and its flow φ achieves this. □

2.6. The manifold of immersions with constant speed. Let $\operatorname{Imm}_a(S^1, \mathbb{R}^2)$ be the space of all immersions $c : S^1 \to \mathbb{R}^2$ which are parametrized by scaled arc length, so that $|c_\theta|$ is constant.

Proposition. *The space $\operatorname{Imm}_a(S^1, \mathbb{R}^2)$ is a smooth manifold. There is a diffeomorphism $\operatorname{Imm}(S^1, \mathbb{R}^2) = \operatorname{Imm}_a(S^1, \mathbb{R}^2) \times \operatorname{Diff}_1^+(S^1)$ which respects the splitting $\operatorname{Diff}(S^1) = \operatorname{Diff}_1^+(S^1) \ltimes (S^1 \ltimes \mathbb{Z}_2)$. There is a smooth action of the rotation and reflection group $S^1 \ltimes \mathbb{Z}_2$ on $\operatorname{Imm}_a(S^1, \mathbb{R}^2)$ with orbit space $\operatorname{Imm}_a(S^1, \mathbb{R}^2)/(S^1 \ltimes \mathbb{Z}_2) = B_i(S^1, \mathbb{R}^2)$.*

Proof. For $c \in \operatorname{Imm}(S^1, \mathbb{R}^2)$ we put

$$\begin{aligned}
&\sigma_c \in \operatorname{Diff}_1(S^1), \qquad \sigma_c(\theta) = \exp\left(\frac{2\pi i \int_1^\theta |c'(u)|\, du}{\int_{S^1} |c'(u)|\, du} \right), \\
&\alpha : \operatorname{Imm}(S^1, \mathbb{R}^2) \to \operatorname{Imm}_a(S^1, \mathbb{R}^2), \qquad \alpha(c)(\theta) := c(\sigma_c^{-1}(\theta)).
\end{aligned}$$

By the fundamentals of manifolds of mappings [6] the mapping α is smooth from $\mathrm{Imm}(S^1,\mathbb{R}^2)$ into itself and we have $\alpha\circ\alpha=\alpha$.

Now we show that $\mathrm{Imm}_a(S^1,\mathbb{R}^2)$ is a manifold. We use the notation from the proof of 2.4.3 with the freeness assumption dropped. For $c\in\mathrm{Imm}_a(S^1,\mathbb{R}^2)$ we use the following mapping as the inverse of a chart:

$$\begin{aligned}C^\infty(S^1,(-\varepsilon,\varepsilon))\times S^1\to\bigcup_{\theta\in S^1}\mathcal{Q}(c(\ \ +\theta))\xrightarrow{\alpha}\mathrm{Imm}_a(S^1,\mathbb{R}^2),\\ (f,\theta)\mapsto\psi_{c(\ \ +\theta)}(f(\ \ +\theta))\mapsto\alpha(\psi_{c(\ \ +\theta)}(f(\ \ +\theta))).\end{aligned}$$

The chart changes are smooth: If for $(f_i,\theta_i)\in C^\infty(S^1,(-\varepsilon,\varepsilon))\times S^1$ we have $\alpha(\psi_{c_1(\ \ +\theta_1)}(f_1(\ \ +\theta_1)))=\alpha(\psi_{c_2(\ \ +\theta_2)}(f_2(\ \ +\theta_2)))$ then the initial points agree and both curves are equally oriented so $c_1(\theta+\theta_1)+f_1(\theta+\theta_1)n_{c_1}(\theta+\theta_1)=c_2(\varphi(\theta)+\theta_2)+f_2(\varphi(\theta)+\theta_2)n_{c_2}(\varphi(\theta)+\theta_2)$ for all θ. From this one can express (f_2,θ_2) smoothly in terms of (f_1,θ_1).

For the latter assertion one has to show that a smooth path through e_1 in $\mathcal{Q}(c_1)$ is mapped to a smooth path in $\mathrm{Diff}_1(S^1)$. This follows from the finite-dimensional implicit function theorem. The mapping α is now smooth into $\mathrm{Imm}_a(S^1,\mathbb{R}^2)$ and the diffeomorphism $\mathrm{Imm}(S^1,\mathbb{R}^2)\to\mathrm{Imm}_a(S^1,\mathbb{R}^2)\times\mathrm{Diff}_1(S^1)$ is given by $c\mapsto(\alpha(c),\sigma_c)$ with inverse $(e,\varphi)\mapsto e\circ\varphi^{-1}$. Only the group $S^1\ltimes\mathbb{Z}_2$ of rotations and reflections of S^1 then still acts on $\mathrm{Imm}_a(S^1,\mathbb{R}^2)$ with orbit space $B_i(S^1,\mathbb{R}^2)$. The rest is clear. □

2.7. Tangent space, length, curvature, and Frenet–Serret formulas on $\mathrm{Imm}_a(S^1,\mathbb{R}^2)$**.** A smooth curve $t\mapsto c(\ \ ,t)\in\mathrm{Imm}(S^1,\mathbb{R}^2)$ lies in $\mathrm{Imm}_a(S^1,\mathbb{R}^2)$ if and only if $|\partial_\theta c|^2=|c_\theta|^2$ is constant in θ, i.e., $\partial_\theta|c_\theta|^2=2\langle c_\theta,c_{\theta\theta}\rangle=0$. Thus $h=\partial_t|_0c\in T_c\mathrm{Imm}(S^1,\mathbb{R}^2)=C^\infty(S^1,\mathbb{R}^2)$ is tangent to $\mathrm{Imm}_a(S^1,\mathbb{R}^2)$ at the foot point c if and only if $\langle h_\theta,c_{\theta\theta}\rangle+\langle h_{\theta\theta},c_\theta\rangle=\langle h_\theta,c_\theta\rangle_\theta=0$, i.e., $\langle h_\theta,c_\theta\rangle$ is constant in θ. For $c\in\mathrm{Imm}_a(S^1,\mathbb{R}^2)$ the volume form is constant in θ since $|c_\theta|=\ell(c)/2\pi$. Thus for the curvature we have

$$\begin{aligned}\kappa:\mathrm{Imm}_a(S^1,\mathbb{R}^2)\to C^\infty(S^1,\mathbb{R}),\\ \kappa(c)=\left(\frac{2\pi}{\ell(c)}\right)^3\det(c_\theta,c_{\theta\theta})=\left(\frac{2\pi}{\ell(c)}\right)^3\langle ic_\theta,c_{\theta\theta}\rangle,\end{aligned}$$

and for the derivative of the length function we get

$$d\ell(c)(h)=\int_{S^1}\frac{\langle h_\theta,c_\theta\rangle}{|c_\theta|}\,d\theta=\frac{(2\pi)^2}{\ell(c)}\langle h_\theta(1),c_\theta(1)\rangle.$$

Since $c_{\theta\theta}$ is orthogonal to c_θ we have (Frenet formulas)

$$\begin{aligned}c_{\theta\theta}&=\left(\frac{2\pi}{\ell(c)}\right)^2\langle ic_\theta,c_{\theta\theta}\rangle ic_\theta=\frac{\ell(c)}{2\pi}\kappa(c)ic_\theta,\\ c_{\theta\theta\theta}&=\frac{\ell(c)}{2\pi}\kappa(c)_\theta\,ic_\theta+\frac{\ell(c)}{2\pi}\kappa(c)ic_{\theta\theta}=\frac{\ell(c)}{2\pi}\kappa(c)_\theta\,ic_\theta-\left(\frac{\ell(c)}{2\pi}\right)^2\kappa(c)^2c_\theta.\end{aligned}$$

The derivative of the curvature thus becomes

$$d\kappa(c)(h) = -2\Big(\frac{2\pi}{\ell(c)}\Big)^2 \langle h_\theta, c_\theta\rangle \kappa(c) + \Big(\frac{2\pi}{\ell(c)}\Big)^3 \langle ic_\theta, h_{\theta\theta}\rangle.$$

2.8. Horizontality on $\operatorname{Imm}_a(S^1,\mathbb{R}^2)$. Let us denote by $\operatorname{Imm}_{a,f}(S^1,\mathbb{R}^2)$ the splitting submanifold of Imm consisting of all constant speed free immersions. From 2.6 and 2.4.3 we conclude that the projection $\operatorname{Imm}_{a,f}(S^1,\mathbb{R}^2) \to B_f(S^1,\mathbb{R}^2)$ is a principal fiber bundle with structure group $S^1 \ltimes \mathbb{Z}_2$, and it is a reduction of the principal fibration $\operatorname{Imm}_f \to B_f$. The principal connection described in 2.4.3 is not compatible with this reduction. But we can easily find some principal connections. The one we will use is described by the horizontal bundle with fiber $\mathcal{N}_{a,c}$ consisting of all vector fields h along c such that $\langle h_\theta, c_\theta\rangle_\theta = 0$ (tangent to Imm_a) and $\langle h(1), c_\theta(1)\rangle = 0$ for $1 \in S^1$ (horizontality). This connection admits a smooth parallel transport; but we can even do better, beyond the principal bundle, in the following proposition whose proof is similar and simpler than that of Proposition 2.5.

Proposition. *For any smooth path c in $\operatorname{Imm}_a(S^1,\mathbb{R}^2)$ there exists a smooth curve φ_c in S^1 with $\varphi_c(0) = 1$ depending smoothly on c such that the path e given by $e(t,\theta) = c(t, \varphi_c(t)\theta)$ is horizontal: $e_t(1) \perp e_\theta(1)$.* □

2.9. The degree of immersions. Recall that the degree of an immersion $c : S^1 \to \mathbb{R}^2$ is the winding number with respect to 0 of the tangent $c' : S^1 \to \mathbb{R}^2$. Since this is invariant under isotopies of immersions, the manifold $\operatorname{Imm}(S^1,\mathbb{R}^2)$ decomposes into the disjoint union of the open submanifolds $\operatorname{Imm}^k(S^1,\mathbb{R}^2)$ for $k \in \mathbb{Z}$ according to the degree k. We shall also need the space $\operatorname{Imm}^k_a(S^1,\mathbb{R}^2)$ of all immersions of degree k with constant speed.

2.10. Theorem.

(1) *The manifold $\operatorname{Imm}^k(S^1,\mathbb{R}^2)$ of immersed curves of degree k contains the subspace $\operatorname{Imm}^k_a(S^1,\mathbb{R}^2)$ as smooth strong deformation retract.*
(2) *For $k \neq 0$ the manifold $\operatorname{Imm}^k_a(S^1,\mathbb{R}^2)$ of immersed constant speed curves of degree k contains S^1 as a strong smooth deformation retract.*
(3) *For $k \neq 0$ the manifold $B^k_i(S^1,\mathbb{R}^2) := \operatorname{Imm}^k(S^1,\mathbb{R}^2)/\operatorname{Diff}^+(S^1)$ is contractible.*

Note that for $k \neq 0$, Imm^k is invariant under the action of the group $\operatorname{Diff}^+(S^1)$ of orientation preserving diffeomorphism only, and that any orientation reversing diffeomorphism maps Imm^k to Imm^{-k}.

The non-trivial S^1 in Imm^k appears in two ways: (a) by rotating each curve around $c(0)$ so that $c'(0)$ rotates, and (b) also by acting $S^1 \ni \beta \mapsto (c(\theta) \mapsto c(\beta\theta))$. The two corresponding elements a and b in the fundamental group are then related by $a^k = b$, which explains our failure to describe the topological type of B^0_i.

Proof of Theorem 2.10. (1) is a consequence of 2.6 since $\operatorname{Diff}^+_1(S^1)$ is contractible.

The general proof is inspired by the proof of the Whitney–Graustein theorem, [9], [4], [3]. We shall view curves here as 2π-periodic plane-valued functions. For any curve c we consider its *center of mass*

$$C(c) = \mathrm{Center}(c) := \frac{1}{\ell(c)} \int_0^{2\pi} c(u)|c'(u)|\,du \in \mathbb{R}^2,$$

which is invariant under $\mathrm{Diff}(S^1)$. We shall also use $\alpha(c) = c'(0)/|c'(0)|$.

The case $k \neq 0$**.** We first embed S^1 into $\mathrm{Imm}(S^1, \mathbb{R}^2)$ in the following way. For $\alpha \in S^1 \subset \mathbb{C} = \mathbb{R}^2$ and $k \neq 0$ we set $e_\alpha(\theta) = \alpha e^{ik\theta}/ik$, a circle of radius $1/|k|$ traversed k times in the direction indicated by the sign of k. Note that we have $\mathrm{Center}(e_\alpha) = 0$ and $e'_\alpha(0) = \alpha$.

Since the isotopies to be constructed later will destroy the property of having constant speed, we shall first construct a smooth deformation retraction $A : [0,1] \times \mathrm{Imm}^k \to \mathrm{Imm}^k_{1,0}$ onto the subspace $\mathrm{Imm}^k_{1,0}$ of unit speed degree $k \neq 0$ curves with center 0.

Let $c : \mathbb{R} \to \mathbb{R}^2$ be an arbitrary constant speed immersion of degree k, period 2π, and length $\ell(c)$. Let $s_c(v) = \int_0^v |c'(u)|\,du$ be the arc length function of c and put

$$A(c,t,u) = \left(1 - t + t\frac{2\pi}{\ell(c)}\right) \cdot \left(c\left((1-t)u + t \cdot s_c^{-1}\left(\frac{\ell(c)}{2\pi}u\right)\right) - t \cdot C(c)\right).$$

Then A_c is an isotopy between c and $c_1 := A(c, 1, \ )$ depending smoothly on c. The immersion c_1 has unit speed, length 2π, and $\mathrm{Center}(c_1) = 0$. Moreover, for the winding number w_0 around 0 we have

$$w_0(c_1'|_{[0,2\pi]}) = \deg(c_1) = \deg(c) = k = \deg(e_{\alpha(c)}) = w_0(e'_{\alpha(c)}|_{[0,2\pi]}).$$

Thus Imm^k contains the space $\mathrm{Imm}^k_{1,0}$ of unit speed immersions with center of mass 0 and degree k as smooth strong deformation retract.

For $c \in \mathrm{Imm}^k_{1,0}$ a unit speed immersion with center 0 we now construct an isotopy $t \mapsto H^1(c,t,\ )$ between c and a suitable curve e_α. It will destroy the unit speed property, however. For $d\arg = \frac{-x\,dy + y\,dx}{\sqrt{x^2+y^2}}$ we put

$$\begin{aligned}
\varphi_c(u) &:= \int_{c'|_{[0,u]}} d\arg, \quad \text{so that} \quad c'(u) = c'(0)\, e^{i\varphi_c(u)},\\
\alpha(c) &:= \frac{1}{2\pi}\int_0^{2\pi} (\varphi_c(v) - kv)\,dv,\\
\psi_c(t,u) &:= (1-t)\varphi_c(u) + t(ku + \alpha(c)),\\
h(c,t,u) &:= \int_0^u e^{i\psi_c(t,v)}\,dv - \frac{u}{2\pi}\int_0^{2\pi} e^{i\psi_c(t,v)}\,dv,\\
H^1(c,t,u) &:= c'(0)(h(c,t,u) - \mathrm{Center}(h(c,t,\ ))).
\end{aligned}$$

Then $H^1(c,t,u)$ is smooth in all variables, 2π-periodic in u, with center of mass at 0, $H^1(1,c,u)$ equals one the e_α's, and $H^1(0,c,u) = c(u)$. But $H^1(c,t,\ )$ is no longer of unit speed in general. And we still have to show that $t \mapsto h(c, t, \quad)$ (and hence H^1) is an isotopy. Now

$$\partial_u h(c, t, u) = e^{i\psi_c(t,u)} - \frac{1}{2\pi} \int_0^{2\pi} e^{i\psi_c(t,v)}\, dv,$$

and

$$\left| \frac{1}{2\pi} \int_0^{2\pi} e^{i\psi_c(t,v)}\, dv \right| \le 1. \tag{4}$$

If the last inequality is strict we have $\partial_u h(t, u) \neq 0$ so that h is an isotopy. If we have equality then $\psi_c(t, v)$ is constant in v, which leads to a contradiction as follows: If $k \neq 0$ then $\psi_c(t, 2\pi) - \psi_c(t, 0) = 2\pi k$ so it cannot be constant for any t.

Let us finally check how this construction depends on the choice of the base point $c(0)$. We have

$$\begin{aligned}
\varphi_{c(\beta+\quad)}(u) &= \varphi_c(\beta + u) - \varphi_c(\beta),\\
\alpha(c(\beta + \quad)) &= \alpha(c) + k\beta - \varphi_c(\beta),\\
\psi_{c(\beta+\quad)}(t, u) &= \psi_c(t, u + \beta) - \varphi_c(\beta),\\
h(c(\beta + \quad), t, u) &= e^{-i\varphi_c(\beta)}(h(c, t, \beta + u) - h(c, t, \beta)),\\
H^1(c(\beta + \quad), t, u) &= H^1(c, t, \beta + u).
\end{aligned}$$

Let us now deform H^1 back into $\mathrm{Imm}^k_{1,0}$. For $c \in \mathrm{Imm}^k_{1,0}$ we consider

$$\begin{aligned}
H^2(c, t, u) &:= A(1, H^1(c, t, \quad), u),\\
H^3(c, t, u) &:= H^2(c, t, u + \varphi_{H^2(c)}(t)),
\end{aligned}$$

where the φ_f for a unit speed path f is from Proposition 2.8, so that $H^3(c)$ is a horizontal path of unit speed curves of length 2π (i.e., $\partial_t H^3(c, t, 0) \perp \partial_u|_0 H^3(c, t, u)$).

The isotopy A reacts in a complicated way to rotations of the parameter, but we have $A(c(\beta + \quad), 1, u) = A(c, 1, \frac{2\pi}{\ell(c)} s_c(\beta) + u)$. Thus $H^3(c(\quad + \beta), t, u) = H^3(c, t, u + \beta)$, so H^3 is equivariant under the rotation group $S^1 \subset \mathrm{Diff}(S^1)$. For $k \neq 0$ we get an equivariant smooth strong deformation retract within $\mathrm{Imm}^k_{1,0}$ onto the subset $\{e_\alpha : \alpha \in S^1\} \subset \mathrm{Imm}^k_{1,0}$ which is invariant under the rotation group $S^1 \subset \mathrm{Diff}(S^1)$. It factors to a smooth contraction on B^k_i. This proves assertions (2) and (3) for $k \neq 0$. □

2.11. Bigger spaces of 'immersed' curves. We want to introduce a larger space containing $B_i(S^1, \mathbb{R}^2)$, which is complete in a suitable metric. This will serve as an ambient space which will contain the completion of $B_i(S^1, \mathbb{R}^2)$. Let $\mathrm{Cont}(S^1, \mathbb{R}^2)$ be the space of all *continuous* functions $c : S^1 \to \mathbb{R}^2$. Instead of a group operation and its associated orbit space, we introduce an equivalence relation on $\mathrm{Cont}(S^1, \mathbb{R}^2)$. Define a subset $R \subset S^1 \times S^1$ to be a *monotone correspondence* if it is the image of a map

$$x \mapsto (h(x) \bmod 2\pi, k(x) \bmod 2\pi),$$

where $h, k : \mathbb{R} \to \mathbb{R}$ are non-decreasing continuous functions such that

$$h(x + 2\pi) \equiv h(x) + 2\pi, \quad k(x + 2\pi) \equiv k(x) + 2\pi.$$

In words, this is an orientation preserving homeomorphism from S^1 to S^1 which is allowed to have intervals where one or the other variable remains constant while the other continues to increase. (These correspondences arise naturally in computer vision in comparing the images seen by the right and left eyes, see [1].) Then we define the equivalence relation on $\mathrm{Cont}(S^1, \mathbb{R}^2)$ by $c \sim d$ if and only if there is a monotone correspondence R such that for all $\theta, \varphi \in R$, $c(\theta) = d(\varphi)$. It is easily seen that any non-constant $c \in \mathrm{Cont}(S^1, \mathbb{R}^2)$ is equivalent to an c_1 which is not constant on any intervals in S^1 and that for such c_1's and d_1's, the equivalence relation amounts to $c_1 \circ h \equiv d_1$ for some homeomorphism h of S^1. Let $B_i^{\mathrm{cont}}(S^1, \mathbb{R}^2)$ be the quotient space by this equivalence relation. We call these *Fréchet curves*.

The quotient metric on $B_i^{\mathrm{cont}}(S^1, \mathbb{R}^2)$ is called the *Fréchet* metric, a variant of the *Hausdorff* metric mentioned in the Introduction, both being L^∞ type metrics. Namely, define

$$\begin{aligned} d_\infty(c, d) &= \inf_{\text{monotone corresp. } R} \Big(\sup_{(\theta,\varphi)\in R} |c(\theta) - d(\varphi)| \Big) \\ &= \inf_{\text{homeomorph. } h: S^1 \to S^1} \|c \circ h - d\|_\infty . \end{aligned}$$

It is straightforward to check that this makes $B_i^{\mathrm{cont}}(S^1, \mathbb{R}^2)$ into a complete metric space.

Another very natural space is the subset $B_i^{\mathrm{lip}}(S^1, \mathbb{R}^2) \subset B_i^{\mathrm{cont}}(S^1, \mathbb{R}^2)$ given by the non-constant *Lipschitz* maps $c : S^1 \to \mathbb{R}^2$. The great virtue of Lipschitz maps is that their images are rectifiable curves and thus each of them is equivalent to a map d in which θ is proportional to arc length, as in the previous section. More precisely, if c is Lipschitz, then c_θ exists almost everywhere and is bounded and we can reparametrize by

$$h(\theta) = \int_0^\theta |c_\theta| \, d\theta \Big/ \int_0^{2\pi} |c_\theta| \, d\theta,$$

obtaining an equivalent d for which $|d_\theta| \equiv L/2\pi$. This d will be unique up to rotations, i.e. the action of S^1 in the previous section.

This subspace of rectifiable Fréchet curves is the subject of a nice compactness theorem due to Hilbert, namely that the set of all such curves in a closed bounded subset of $\mathbb{R}^2$ and whose length is bounded is compact in the Fréchet metric. This can be seen as follows: we can lift all such curves to specific Lipschitz maps c whose Lipschitz constants are bounded. This set is an equicontinuous set of functions by the bound on the Lipschitz constant. By the Ascoli–Arzelà theorem the topology of pointwise convergence equals then the topology of uniform convergence on S^1. So this set is a closed subset in a product of S^1 copies of a large ball in $\mathbb{R}^2$; this product is compact. The Fréchet metric is coarser than the uniform metric, so our set is also compact.

3. Metrics on spaces of curves

3.1. Need for invariance under reparametrization. The pointwise metric on the space of immersions $\mathrm{Imm}(S^1, \mathbb{R}^2)$ is given by

$$G_c(h, k) := \int_{S^1} \langle h(\theta), k(\theta)\rangle \, d\theta.$$

This Riemannian metric is not invariant under reparametrizations of the variable θ and thus does not induce a sensible metric on the quotient space $B_i(S^1, \mathbb{R}^2)$. Indeed, it induces the zero metric since *for any two curves $C_0, C_1 \in B_i(S^1, \mathbb{R}^2)$ the infimum of the arc lengths of curves in $\mathrm{Imm}(S^1, \mathbb{R}^2)$ which connect embeddings $c_0, c_1 \in \mathrm{Imm}(S^1, \mathbb{R}^2)$ with $\pi(c_i) = C_i$ turns out to be zero.* To see this, take any c_0 in the $\mathrm{Diff}(S^1)$-orbit over C_0. Take the following variation $c(\theta, t)$ of c_0: for θ outside a small neighborhood U of length ε of 1 in S^1, $c(\theta, t) = c_0(\theta)$. If $\theta \in U$, then the variation for $t \in [0, 1/2]$ moves the small part of c_0 so that $c(\theta, 1/2)$ for θ in U takes off C_0, goes to C_1, traverses nearly all of C_1, and returns to C_0. Now in the orbit through $c(\cdot, 1/2)$, reparametrize in such a way that the new curve is diligently traversing C_1 for $\theta \notin U$, and for $\theta \in U$ it travels back to C_0, runs along C_0, and comes back to C_1. This reparametrized curve is then varied for $t \in [1/2, 1]$ in such a way that the part for $\theta \in U$ is moved towards C_2. It is clear that the length of both variations is bounded by a constant (depending on the distance between C_0 and C_1 and the lengths of both C_0 and C_1) times ε.

3.2. The simplest Riemannian metric on B_i. Let $h, k \in C^\infty(S^1, \mathbb{R}^2)$ be two tangent vectors with foot point $c \in \mathrm{Imm}(S^1, \mathbb{R}^2)$. The induced volume form is $\mathrm{vol}(c) = \langle \partial_\theta c, \partial_\theta c\rangle^{1/2} \, d\theta = |c_\theta| \, d\theta$. We consider first the simple H^0 weak Riemannian metric on $\mathrm{Imm}(S^1, \mathbb{R}^2)$:

$$G_c(h, k) := \int_{S^1} \langle h(\theta), k(\theta)\rangle |c'(\theta)| \, d\theta, \tag{1}$$

which is invariant under $\mathrm{Diff}(S^1)$. This makes the map $\pi : \mathrm{Imm}(S^1, \mathbb{R}^2) \to B_i(S^1, \mathbb{R}^2)$ into a *Riemannian submersion* (off the singularities of $B_i(S^1, \mathbb{R}^2)$), which is very convenient. We call this the *H^0-metric*.

Now we can determine the bundle $\mathcal{N} \to \mathrm{Imm}(S^1, \mathbb{R}^2)$ of tangent vectors which are normal to the $\mathrm{Diff}(S^1)$-orbits. The tangent vectors to the orbits are $T_c(c \circ \mathrm{Diff}(S^1)) = \{g \cdot c_\theta : g \in C^\infty(S^1, \mathbb{R})\}$. Inserting this for k into the expression (1) of the metric we see that

$$\begin{aligned}\mathcal{N}_c &= \{h \in C^\infty(S^1, \mathbb{R}^2) : \langle h, c_\theta\rangle = 0\} \\ &= \{a i c_\theta \in C^\infty(S^1, \mathbb{R}^2) : a \in C^\infty(S^1, \mathbb{R})\} \\ &= \{b n_c \in C^\infty(S^1, \mathbb{R}^2) : b \in C^\infty(S^1, \mathbb{R})\},\end{aligned} \tag{2}$$

where n_c is the normal unit field along c.

A tangent vector $h \in T_c \operatorname{Imm}(S^1, \mathbb{R}^2) = C^\infty(S^1, \mathbb{R}^2)$ has an orthonormal decomposition

$$\begin{aligned}
h &= h^\top + h^\perp \in T_c(c \circ \operatorname{Diff}^+(S^1)) \oplus \mathcal{N}_c \quad \text{where}\\
h^\top &= \frac{\langle h, c_\theta\rangle}{|c_\theta|^2} c_\theta \in T_c(c \circ \operatorname{Diff}^+(S^1)), \qquad (3)\\
h^\perp &= \frac{\langle h, ic_\theta\rangle}{|c_\theta|^2} ic_\theta \in \mathcal{N}_c,
\end{aligned}$$

into smooth tangential and normal components.

Since the Riemannian metric G on $\operatorname{Imm}(S^1, \mathbb{R}^2)$ is invariant under the action of $\operatorname{Diff}(S^1)$ it induces a metric on the quotient $B_i(S^1, \mathbb{R}^2)$ as follows. For any $C_0, C_1 \in B_i$, consider all liftings $c_0, c_1 \in \operatorname{Imm}$ such that $\pi(c_0) = C_0$, $\pi(c_1) = C_1$ and all smooth curves $t \mapsto (\theta \mapsto c(t, \theta))$ in $\operatorname{Imm}(S^1, \mathbb{R}^2)$ with $c(0, \cdot) = c_0$ and $c(1, \cdot) = c_1$. Since the metric G is invariant under the action of $\operatorname{Diff}(S^1)$ the arc length of the curve $t \mapsto \pi(c(t, \cdot))$ in $B_i(S^1, \mathbb{R}^2)$ is given by

$$\begin{aligned}
L_G^{\mathrm{hor}}(c) &:= L_G(\pi(c(t,\cdot))) = \int_0^1 \sqrt{G_{\pi(c)}(T_c\pi \cdot c_t, T_c\pi \cdot c_t)}\, dt = \int_0^1 \sqrt{G_c(c_t^\perp, c_t^\perp)}\, dt\\
&= \int_0^1 \left(\int_{S^1} \left\langle \frac{\langle c_t, ic_\theta\rangle}{|c_\theta|^2} ic_\theta, \frac{\langle c_t, ic_\theta\rangle}{|c_\theta|^2} ic_\theta \right\rangle |c_\theta|\, d\theta \right)^{1/2} dt\\
&= \int_0^1 \left(\int_{S^1} \langle c_t, n_c\rangle^2 |c_\theta|\, d\theta \right)^{1/2} dt = \int_0^1 \left(\int_{S^1} \langle c_t, ic_\theta\rangle^2 \frac{d\theta}{|c_\theta|} \right)^{1/2} dt. \qquad (4)
\end{aligned}$$

The metric on $B_i(S^1, \mathbb{R}^2)$ is defined by taking the infimum of this over all paths c (and all lifts c_0, c_1):

$$\operatorname{dist}_G^{B_i}(C_1, C_2) = \inf_c L_G^{\mathrm{hor}}(c).$$

Unfortunately, we will see below that this metric is too weak: the distance that it defines turns out to be identically zero! For this reason, we will mostly study in this paper a family of stronger metrics. These are obtained by the most minimal change in G. We want to preserve two simple properties of the metric: that it is local and that it has no derivatives in it. The standard way to strengthen the metric is go from an H^0-metric to an H^1-metric. But when we work out the natural H^1-metric, picking out those terms which are local and do not involve derivatives leads us to our chosen metric.

We consider next the H^1 weak Riemannian metric on $\operatorname{Imm}(S^1, \mathbb{R}^2)$:

$$G_c^1(h,k) := \int_{S^1} \left(\langle h(\theta), k(\theta)\rangle + A \frac{\langle h_\theta, k_\theta\rangle}{|c_\theta|^2} \right) |c_\theta|\, d\theta, \qquad (5)$$

which is invariant under $\operatorname{Diff}(S^1)$. Thus $\pi : \operatorname{Imm}(S^1, \mathbb{R}^2) \to B_i(S^1, \mathbb{R}^2)$ is again a *Riemannian submersion* off the singularities of $B_i(S^1, \mathbb{R}^2)$. We call this the H^1*-metric* on B_i.

To understand this metric better, we assume $h = k = aic_\theta/|c_\theta| + bc_\theta/|c_\theta|$. Moreover, for any function $f(\theta)$, we write $f_s = f_\theta/|c_\theta|$ for the derivative with respect to arc length. Then

$$h_s = \frac{h_\theta}{|c_\theta|} = (aic_s + bc_s)_s = (a_s + \kappa b)ic_s + (b_s - \kappa a)c_s.$$

Therefore

$$\begin{aligned}G^1_c(h,h) &= \int_{S^1}(a^2 + b^2 + A(a_s+\kappa b)^2 + A(b_s - \kappa a)^2)\,ds\\ &= \int_{S^1}(a^2(1+A\kappa^2) + Aa_s^2) + 2A\kappa(a_s b - b_s a) + (b^2(1+A\kappa^2) + Ab_s^2)\,ds.\end{aligned}$$

Letting T_1 and T_2 be the differential operators $T_1 = I + A\kappa^2 - A(\frac{d}{ds})^2$, $T_2 = A(\kappa_s + 2\kappa\frac{d}{ds})$, then integrating by parts on S^1, we get

$$G^1_c(h,h) = \int_{S^1}(T_1(a)\cdot a + 2T_2(a)\cdot b + T_1(b)\cdot b)\,ds.$$

Note that T_1 is a positive definite self-adjoint operator on functions on c, hence it has an inverse given by a Green's function which we write T_1^{-1}. Completing the square and using the fact that T_1 is self-adjoint, we simplify the metric to

$$G^1_c(h,h) = \int_c (T_1(a)\cdot a - T_1^{-1}(T_2(a))\cdot T_2(a) + T_1(b + T_1^{-1}(T_2(a)))\cdot(b + T_1^{-1}(T_2(a))))\,ds.$$

If we fix a and minimize this in b, we get the bundle $\mathcal{N}^1 \to \mathrm{Imm}(S^1,\mathbb{R}^2)$ of tangent vectors which are G^1-normal to the $\mathrm{Diff}(S^1)$-orbits. In other words

$$\mathcal{N}^1_c = \{h \in C^\infty(S^1,\mathbb{R}^2) : h = aic_s + bc_s,\ b = -T_1^{-1}(T_2(a))\}$$

and on horizontal vectors of this type

$$G^1_c(h,h) = \int_c((1+A\kappa^2)a^2 + Aa_s^2)\,ds - \int_c T_1^{-1}(T_2(a))\cdot T_2(a)\,ds.$$

If we drop terms involving a_s, say because we assume $|a_s|$ is small, then what remains is just the integral of $(1+A\kappa^2)a^2$ plus the integral of $T_1^{-1}(\kappa_s a)\kappa_s a$. The second is a non-local regular integral operator, so dropping this we are left with the main metric of this paper:

$$G^A_c(h,h) = \int_c (1+A\kappa^2)a^2\,ds, \qquad h = aic_s,$$

which we call the *H^0_κ-metric with curvature weight A*. For further reference, on $\mathrm{Imm}(S^1,\mathbb{R}^2)$, for a constant $A \ge 0$, it is given by

$$G^A_c(h,k) := \int_{S^1}(1 + A\kappa_c(\theta)^2)\langle h(\theta), k(\theta)\rangle |c'(\theta)|\,d\theta, \tag{6}$$

which is again invariant under $\mathrm{Diff}(S^1)$. Thus $\pi : \mathrm{Imm}(S^1, \mathbb{R}^2) \to B_i(S^1, \mathbb{R}^2)$ is again a *Riemannian submersion* off the singularities. Note that for the metric (6), the bundle $\mathcal{N} \subset T\,\mathrm{Imm}(S^1, \mathbb{R}^2)$ is the same as for $A = 0$, as described in (2). The arc length of a curve $t \mapsto \pi(c(t, \cdot))$ in $B_i(S^1, \mathbb{R}^2)$ is given by the analog of (4),

$$\begin{aligned} L^{\mathrm{hor}}_{G^A}(c) &:= L_{G^A}(\pi(c(t,\cdot))) = \int_0^1 \sqrt{G^A_{\pi(c)}(T_c\pi \cdot c_t, T_c\pi \cdot c_t)}\,dt = \int_0^1 \sqrt{G^A_c(c_t^\perp, c_t^\perp)}\,dt \\ &= \int_0^1 \left(\int_{S^1} (1 + A\kappa_c^2)\langle c_t, n_c\rangle^2 |c_\theta|\,d\theta\right)^{1/2} dt \\ &= \int_0^1 \left(\int_{S^1} (1 + A\kappa_c^2)\langle c_t, ic_\theta\rangle^2 \,\frac{d\theta}{|c_\theta|}\right)^{1/2} dt. \end{aligned} \tag{7}$$

The metric on $B_i(S^1, \mathbb{R}^2)$ is defined by taking the infimum of this over all paths c (and all lifts c_0, c_1):

$$\mathrm{dist}^{B_i}_{G^A}(C_1, C_2) = \inf_c L^{\mathrm{hor}}_{G^A}(c).$$

Note that if a path $\pi(c)$ in $B_i(S^1, \mathbb{R}^2)$ is given, then one can choose its lift to a path c in $\mathrm{Imm}(S^1, \mathbb{R}^2)$ to have various good properties. Firstly, we can choose the lift $c(0,\ )$ of the inital curve to have a parametrization of constant speed, i.e. if its length is ℓ, then $|c_\theta|(\theta, 0) = \ell/2\pi$ for all $\theta \in S^1$. Secondly, we can make the tangent vector to c everywhere horizontal, i.e. $\langle c_t, c_\theta\rangle \equiv 0$, by 2.5. Thirdly, we can reparametrize the coordinate t on the path of length L so that the path is traversed at constant speed, i.e.

$$\int_{S^1} (1 + A\kappa_c^2)\langle c_t, ic_\theta\rangle^2\, d\theta/|c_\theta| \equiv L^2 \qquad \text{for all } 0 \le t \le 1.$$

3.3. A Lipschitz bound for arc length in G^A. We apply the Cauchy–Schwarz inequality to the derivative 2.2.4 of the length function along a path $t \mapsto c(t,\ )$:

$$\begin{aligned} \partial_t \ell(c) = d\ell(c)(c_t) &= -\int_{S^1} \kappa(c)\langle c_t, n_c\rangle |c_\theta|\,d\theta \le \left|\int_{S^1} \kappa(c)\langle c_t, n_c\rangle |c_\theta|\,d\theta\right| \\ &\le \left(\int_{S^1} 1^2 |c_\theta|\,d\theta\right)^{1/2} \left(\int_{S^1} \kappa(c)^2\langle c_t, n_c\rangle^2 |c_\theta|\,d\theta\right)^{1/2} \\ &\le \ell(c)^{1/2} \frac{1}{\sqrt{A}} \left(\int_{S^1} (1 + A\kappa(c)^2)\langle c_t, n_c\rangle^2 |c_\theta|\,d\theta\right)^{1/2}. \end{aligned}$$

Thus

$$\partial_t(\sqrt{\ell(c)}) = \frac{\partial_t \ell(c)}{2\sqrt{\ell(c)}} \le \frac{1}{2\sqrt{A}} \left(\int_{S^1} (1 + A\kappa(c)^2)\langle c_t, n_c\rangle^2 |c_\theta|\,d\theta\right)^{1/2}$$

and by using 3.2.7 we get

$$\begin{aligned}\sqrt{\ell(c_1)}-\sqrt{\ell(c_0)}&=\int_0^1 \partial_t(\sqrt{\ell(c)})\,dt\\ &\le \frac{1}{2\sqrt{A}}\int_0^1\Big(\int_{S^1}(1+A\kappa(c)^2)\langle c_t,n_c\rangle^2|c_\theta|\,d\theta\Big)^{1/2}dt\\ &=\frac{1}{2\sqrt{A}}L^{\text{hor}}_{G^A}(c). \qquad (1)\end{aligned}$$

If we take the infimum over all paths connecting c_0 with the $\operatorname{Diff}(S^1)$-orbit through c_1 we get:

Lipschitz continuity of $\sqrt{\ell}\,:\,B_i(S^1,\mathbb{R}^2)\to\mathbb{R}_{\ge 0}$**.** *For C_0 and C_1 in $B_i(S^1,\mathbb{R}^2)=\operatorname{Imm}(S^1,\mathbb{R}^2)/\operatorname{Diff}(S^1)$ we have, for $A>0$,*

$$\sqrt{\ell(C_1)}-\sqrt{\ell(C_0)}\le\frac{1}{2\sqrt{A}}\operatorname{dist}^{B_i}_{G^A}(C_1,C_2). \qquad (2)$$

3.4. Bounding the area swept by a path in B_i. Secondly, we want to bound the area swept out by a path starting from C_0 to reach any curve C_1 nearby in our metric. First we use the Cauchy–Schwarz inequality in the Hilbert space $L^2(S^1,|c_\theta(t,\theta)|\,d\theta)$ to get

$$\begin{aligned}&\int_{S^1}1\cdot|c_t(t,\theta)|\,|c_\theta(t,\theta)|\,d\theta=\langle 1,|c_t|\rangle_{L^2}\\ &\qquad\le\|1\|_{L^2}\|c_t\|_{L^2}=\Big(\int_{S^1}|c_\theta(t,\theta)|\,d\theta\Big)^{1/2}\Big(\int_{S^1}|c_t(t,\theta)|^2|c_\theta(t,\theta)|\,d\theta\Big)^{1/2}.\end{aligned}$$

Now we assume that the variation $c(t,\theta)$ is horizontal, so that $\langle c_t,c_\theta\rangle=0$. Then $L_{G^A}(c)=L^{\text{hor}}_{G^A}(c)$. We use this inequality and then the intermediate value theorem of integral calculus to obtain

$$\begin{aligned}L^{\text{hor}}_{G^A}(c)=L_{G^A}(c)&=\int_0^1\sqrt{G^A_c(c_t,c_t)}\,dt\\ &=\int_0^1\Big(\int_{S^1}(1+A\kappa(c)^2)|c_t(t,\theta)|^2|c_\theta(t,\theta)|\,d\theta\Big)^{1/2}dt\\ &\ge\int_0^1\Big(\int_{S^1}|c_t(t,\theta)|^2|c_\theta(t,\theta)|\,d\theta\Big)^{1/2}dt\\ &\ge\int_0^1\Big(\int_{S^1}|c_\theta(t,\theta)|\,d\theta\Big)^{-1/2}\int_{S^1}|c_t(t,\theta)|\,|c_\theta(t,\theta)|\,d\theta\,dt\\ &=\Big(\int_{S^1}|c_\theta(t_0,\theta)|\,d\theta\Big)^{-1/2}\int_0^1\int_{S^1}|c_t(t,\theta)|\,|c_\theta(t,\theta)|\,d\theta\,dt\\ &\qquad\qquad\text{for some intermediate value } 0\le t_0\le 1,\\ &=\frac{1}{\sqrt{\ell(c(t_0,\cdot))}}\int_{[0,1]\times S^1}|\det dc(t,\theta)|\,d\theta\,dt.\end{aligned}$$

Area swept out bound. *If c is any path from C_0 to C_1, then*

$$\begin{pmatrix}\textit{area of the region swept}\\ \textit{out by the variation } c\end{pmatrix} \leq \max_t \sqrt{\ell(c(t,\cdot))}\cdot L^{\text{hor}}_{G^A}(c). \tag{1}$$

This result enables us to compare the double cover $B_i^{\text{or}}(S^1,\mathbb{R}^2)$ of our metric space $B_i(S,\mathbb{R}^2)$ consisting of oriented unparametrized curves to the fundamental space of geometric measure theory. Note that there is a map h_1 from B_i^{or} to the space of 1-currents $\mathcal{D}_1'$ given by

$$\langle h_1(c \bmod \operatorname{Diff}^+(S^1)),\omega\rangle = \int_{S^1} c^*\omega, \qquad c\in \operatorname{Imm}(S^1,\mathbb{R}^2).$$

The image $h_1(C)$ is, in fact, closed. For any C, define the integer-valued measurable function w_C on $\mathbb{R}^2$ by

$$w_C((x,y)) = \text{winding number of } C \text{ around } (x,y).$$

Then it is easy to see that, as currents, $h_1(C)=\partial(w_C\,dxdy)$, hence $\partial h_1(C)=0$.

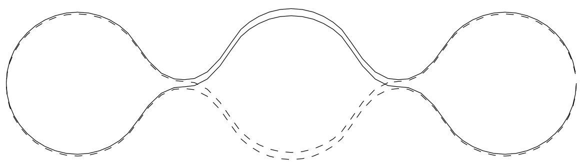

Fig. 1. Two distinct immersions of S^1 in the plane whose underlying currents are equal. One curve is solid, the other dashed.

Although h_1 is obviously injective on the space B_e, it is not injective on B_i as illustrated in Figure 1. The image of this mapping lies in the basic subset $\mathcal{I}_{1,c}\subset\mathcal{D}_1'$ of closed *integral* currents, namely those which are both closed and countable sums of currents defined by Lipschitz mappings $c_i:[0,1]\to\mathbb{R}^2$ of finite total length. Integral currents carry what is called the *flat* metric, which, for closed 1-currents, reduces (by the isoperimetric inequality) to the area distance

$$d^\flat(C_1,C_2) = \iint_{\mathbb{R}^2} |w_{C_1}-w_{C_2}|\,dx\,dy. \tag{2}$$

To connect this with our 'area swept out bound', note that if we have any path c in $\operatorname{Imm}(S^1,\mathbb{R}^2)$ joining C_1 and C_2, this path defines a 2-current $w(c)$ such that $\partial w(c) = h_1(C_1)-h_1(C_2)$ and

$$\int_{\mathbb{R}^2}|w(c)|\,dx\,dy \leq \int_0^1\int_{S^1}|\det c|\,d\theta\,dt,$$

which is what we are calling the area swept out. But $\partial(w_{C_1} - w_{C_2}) = h_1(C_1) - h_1(C_2)$ too, so $w(c) = w_{C_1} - w_{C_2}$. Thus

$$d^\flat(C_1, C_2) \le \min_{\text{all paths } c \text{ joining } C_1, C_2} [\text{area swept out by } c]. \tag{3}$$

Finally, we recall the fundamental compactness result of geometric measure theory in this simple case: the space of integral 1-currents of bounded length is compact in the flat metric. This implies that our 'area swept out bound' above has the property:

Corollary.

- *If $\{C_n\}$ is any Cauchy sequence in B_i for the metric dist_{G^A}, then $\{h_1(C_n)\}$ is a Cauchy sequence in $\mathcal{I}_{1,c}$ on which length is bounded.*
- *Hence h_1 extends to a continuous map from the completion $\overline{B}_i$ of B_i in the metric G^A to $\mathcal{I}_{1,c}$.*

3.5. Bounding how far curves move in small paths in B_i. We want to bound the maximum distance a curve C_0 can move on any path whose length is small in G^A metric. Fix the initial curve C_0 and let ℓ be its length. The result is:

Maximum distance bound. *Let $\epsilon < \min\{2\sqrt{A\ell}, \ell^{3/2}\}/8$ and let $\eta = 4(\ell^{3/4}A^{-1/4} + \ell^{1/4})\sqrt{\epsilon}$. Then for any path c starting at C_0 whose length is ϵ, the final curve lies in the tubular neighborhood of C_0 of width η. More precisely, if we choose the path $c(t,\theta)$ to be horizontal, then $\max_\theta |c(0,\theta) - c(1,\theta)| < \eta$.*

Proof. For all of this proof, we assume the path in B_i has been lifted to a horizontal path $c \in \mathrm{Imm}(S^1, \mathbb{R}^2)$ with $|c_\theta|(0,\theta) \equiv \ell/2\pi$, so that $\langle c_t, c_\theta\rangle \equiv 0$, and also that $\int_{S^1}(1 + A\kappa_c^2)|c_t|^2|c_\theta|\, d\theta \equiv \epsilon^2$. The first step in the proof is to refine the Lipschitz bound on the length of a curve to a local estimate. Note that by horizontality

$$\frac{\partial}{\partial t}\sqrt{|c_\theta|} = \frac{\langle c_{\theta t}, c_\theta\rangle}{2|c_\theta|^{3/2}} = -\frac{\langle c_t, c_{\theta\theta}\rangle}{2|c_\theta|^{3/2}} = -\frac{\langle c_t, ic_\theta\rangle}{2|c_\theta|}\kappa_c\,|c_\theta|^{1/2} = \mp\tfrac{1}{2}\kappa_c\,|c_t|\,|c_\theta|^{1/2},$$

hence

$$\int_{S^1}\left(\frac{\partial}{\partial t}\sqrt{|c_\theta|}\right)^2 ds \le \frac{\epsilon^2}{4A}.$$

Now we make the key definition:

$$\widetilde{|c_\theta|}(t,\theta) = \min_{0\le t_1\le t} |c_\theta|(t_1,\theta).$$

Note that the t-derivative of $\widetilde{|c_\theta|}$ is either 0 or equal to that of $|c_\theta|$ and is ≤ 0. Thus

$$\begin{aligned}\int_{S^1}\left(\sqrt{\frac{\ell}{2\pi}} - \sqrt{\widetilde{|c_\theta|}(1,\theta)}\right) d\theta &\le \int_0^1\int_{S^1} -\frac{\partial}{\partial t}\sqrt{\widetilde{|c_\theta|}}\, d\theta\, dt \le \int_0^1\int_{S^1}\left|\frac{\partial}{\partial t}\sqrt{|c_\theta|}\right| d\theta\, dt\\ &\le \int_0^1\left(\int_{S^1} d\theta\right)^{1/2}\left(\int_{S^1}\left|\frac{\partial}{\partial t}\sqrt{|c_\theta|}\right|^2 d\theta\right)^{1/2} dt\\ &\le \sqrt{2\pi}\,\frac{\epsilon}{2\sqrt{A}}.\end{aligned}$$

To make use of this inequality, let $E = \{\theta : |\widetilde{c_\theta}|(1,\theta) \le (1-(A\ell)^{-1/4}\sqrt{\varepsilon})\ell/2\pi\}$. Our assumption on ε gives $(A\ell)^{-1/4}\sqrt{\varepsilon} < 1/2$, hence on $S^1 \setminus E$ we have $|\widetilde{c_\theta}| > \ell/4\pi$. On E we have also $(|\widetilde{c_\theta}|)^{1/2} \le (1-(A\ell)^{-1/4}\sqrt{\varepsilon}/2)\sqrt{\ell/2\pi}$. Combining this with the previous inequality, we get (where $\mu(E)$ is the measure of E)

$$\mu(E)\frac{1}{2\sqrt{2\pi}}\left(\frac{\ell}{A}\right)^{1/4}\sqrt{\varepsilon} \le \sqrt{2\pi}\,\frac{\varepsilon}{2\sqrt{A}}, \quad \text{hence} \quad \mu(E) \le 2\pi\,\frac{\sqrt{\varepsilon}}{(A\ell)^{1/4}} < \pi.$$

We now use the lower bound on $|c_\theta|$ on $S^1 - E$ to control $c(1,\theta) - c(0,\theta)$:

$$\begin{aligned}
\int_{S^1-E} |c(1,\theta)-c(0,\theta)|\, d\theta &\le \int_0^1 \int_{S^1-E} |c_t|\, d\theta\, dt \\
&\le \sqrt{2\pi}\int_0^1 \left(\int_{S^1-E} |c_t|^2\, d\theta\right)^{1/2} dt \\
&\le \frac{\sqrt{2\pi}}{\sqrt{\ell/4\pi}}\int_0^1 \left(\int_{S^1-E} |c_t|^2|c_\theta|\, d\theta\right)^{1/2} dt \le \frac{2\sqrt{2\pi}}{\sqrt{\ell}}\varepsilon.
\end{aligned}$$

Again, introduce a small exceptional set $F = \{\theta : \theta \notin E \text{ and } |c(1,\theta)-c(0,\theta)| \ge \ell^{1/4}\sqrt{\varepsilon}\}$. By the inequality above, we get

$$\mu(F)\cdot \ell^{1/4}\sqrt{\varepsilon} \le \frac{2\sqrt{2\pi}\,\varepsilon}{\sqrt{\ell}}, \quad \text{hence} \quad \mu(F) \le \frac{2\sqrt{2\pi}\,\sqrt{\varepsilon}}{\ell^{3/4}} < \pi.$$

The last inequality follows from the second assumption on ε. Knowing $\mu(E)$ and $\mu(F)$ gives us the lengths $|c(0,E)|$ and $|c(0,F)|$ in $\mathbb{R}^2$. But we need the lengths $|c(1,E)|$ and $|c(1,F)|$ too. We get these using the fact that the whole length of C_1 cannot be too large, by 3.3:

$$\sqrt{|C_1|} \le \sqrt{\ell} + \frac{\varepsilon}{2\sqrt{A}}, \quad \text{hence} \quad |C_1| \le \ell + 2\varepsilon\sqrt{\frac{\ell}{A}} \le \ell + \sqrt{\varepsilon}\cdot\frac{\ell^{3/4}}{A^{1/4}}.$$

On $S^1 \setminus E$ we have $|\widetilde{c_\theta}| > (1-(A\ell)^{-1/4}\sqrt{\varepsilon})\ell/2\pi$, thus we get

$$\begin{aligned}
|c(1, E\cup F)| &= |C_1| - |c(1, S^1 \setminus (E\cup F))| \\
&\le \ell + \sqrt{\varepsilon}\frac{\ell^{3/4}}{A^{1/4}} - \left(1 - \frac{\sqrt{\varepsilon}}{(A\ell)^{1/4}}\right)\frac{\ell}{2\pi}(2\pi - \mu(E\cup F)) \\
&\le \sqrt{\varepsilon}\left(3\frac{\ell^{3/4}}{A^{1/4}} + \sqrt{2}\ell^{1/4}\right).
\end{aligned}$$

Finally, we can get from $c(0,\theta)$ to $c(1,\theta)$ by going via $c(0,\theta')$ and $c(1,\theta')$ where $\theta' \in S^1 \setminus (E\cup F) \neq \emptyset$. Thus

$$\begin{aligned}
\max_\theta |c(0,\theta) - c(1,\theta)| &\le |c(0, E\cup F)| + \ell^{1/4}\sqrt{\varepsilon} + |c(1, E\cup F)| \\
&\le 4(\ell^{3/4}A^{-1/4} + \ell^{1/4})\sqrt{\varepsilon}. \qquad \square
\end{aligned}$$

Combining this with the Lipschitz continuity of the square root of arc length, we get:

3.6. Corollary. *For any $A > 0$, the map from $B_i(S^1, \mathbb{R}^2)$ the dist_{G^A} metric to the space $B_i^{\mathrm{cont}}(S^1, \mathbb{R}^2)$ in the Fréchet metric is continuous, and, in fact, uniformly continuous on every subset where the length ℓ is bounded. In particular, dist_{G^A} is a separating metric on $B_i(S^1, \mathbb{R}^2)$. Moreover, the completion $\overline{B}_i(S^1, \mathbb{R}^2)$ of $B_i(S^1, \mathbb{R}^2)$ in this metric can be identified with a subset of $B_i^{\mathrm{lip}}(S^1, \mathbb{R}^2)$.*

If we iterate this bound, then we get the following:

3.7. Corollary. *Consider all paths in B_i joining curves C_0 and C_1. Let L be the length of such a path in the dist_{G^A} metric and let $\ell_{\min}, \ell_{\max}$ be the minimum and maximum of the arc lengths of the curves in this path. Then there are parametrizations c_0, c_1 of C_0 and C_1 such that*

$$\max_\theta |c_0(\theta) - c_1(\theta)| \le 50 \max(LF^*, \sqrt{\ell_{\max} LF^*}), \quad \textit{where}$$
$$F^* = \max\left(\frac{1}{\sqrt{\ell_{\min}}}, \sqrt{\frac{\ell_{\max}}{A}}\right).$$

To prove this, you need only break up the path into a minimum number of pieces for which the maximum distance bound 3.5 holds and add together the estimates for each piece. We will only sketch this proof which is straightforward. The constant 50 is just what comes out without attempting to optimize the bound. The second option for bound, $50\sqrt{\ell_{\max} LF^*}$ is just a rephrasing of the bound already in the theorem for short paths. If the path is too long to satisfy the condition of the theorem, we break the path at intermediate curves C_i of length ℓ_i such that each begins a subpath with length $\varepsilon_i = \min(\sqrt{A\ell_i}, \ell_i^{3/2})/8$ and which do not overlap for more than 2:1. Thus $\sum_i \varepsilon_i \le 2L$. Then apply the maximum distance bound 3.5 to each piece, letting η_i be the bound on how far points move in this subpath *or any parts thereof* and satisfy

$$\eta_i \le 2\sqrt{2}\ell_i \le 16\sqrt{2}\varepsilon_i F^*,$$

from which we get what we need by summing over i.

3.8. A final corollary shows that if we parametrize any path appropriately, we get explicit equicontinuous continuity bounds on the parametrization depending only on L, $\ell_{\max}$ and $\ell_{\min}$. This is a step towards establishing the existence of weak geodesics. The idea is this: instead of the horizontal parametrization $\langle c_t, c_\theta\rangle \equiv 0$, we parametrize each curve at constant speed $|c_\theta| \equiv \ell(t)/2\pi$ where $\ell(t)$ is the length of the t^{th} curve and ask only that $\langle c_t, c_\theta\rangle(0, t) \equiv 0$ for some base point $0 \in [0, 2\pi]$ (see 2.8). Then we get:

Corollary. *If a path $c(t, \theta)$, $0 \le t \le 1$ satisfies*

$$|c_\theta(t,\theta)| \equiv \ell(t)/2\pi \quad \textit{and} \quad \langle c_t, c_\theta\rangle(t, 0) \equiv 0 \quad \textit{for all } \theta, t,$$
$$\int_{C_t} (1 + A\kappa_{C_t}^2)|\langle c_t, ic_\theta\rangle|^2\, d\theta/|c_\theta| \equiv L^2 \quad \textit{for all } t,$$

then

$$|c(t_1,\theta_1)-c(t_2,\theta_2)| \le \frac{\ell_{\max}}{2\pi}|\theta_1-\theta_2| + 7(\ell_{\max}^{3/4}/A^{1/4}+\ell_{\max}^{1/4})\sqrt{L(t_1-t_2)}$$

whenever $|t_1-t_2| \le \min(2\sqrt{A\ell_{\min}}, \ell_{\min}^{3/2})/(8L)$.

Proof. We need to compare the constant speed parametrization here with the horizontal parametrization—call it c^*—used in the maximum distance bound 3.5. Under the horizontal parametrization, let the point (t_1,θ_1) on C_{t_1} correspond to (t_2,θ_1^*) on C_{t_2}, i.e. $c(t_2,\theta_1^*) = c^*(t_2,\theta_1)$. Let $C = (\ell_{\max}^{3/4}/A^{1/4}+\ell_{\max}^{1/4})$. Then we know from 3.5 that

$$|c(t_1,\theta_1)-c(t_2,\theta_1^*)| \le 4C\sqrt{L(t_1-t_2)}.$$

To compare θ_1 and θ_1^*, we use the properties of the set E in the proof of 3.5 to estimate

$$\begin{aligned}
\frac{(\theta_1^*-\theta_1)\ell_2}{2\pi} &= \int_0^{\theta_1} |c_\theta^*(t_2,\varphi)|\,d\varphi - \frac{\theta_1\ell_2}{2\pi} \\
&\ge \left(1-\frac{\sqrt{L(t_1-t_2)}}{(A\ell_1)^{1/4}}\right)(\theta_1-\mu(E))\frac{\ell_1}{2\pi} - \frac{\theta_1\ell_2}{2\pi} \\
&\ge -2\ell_1\frac{\sqrt{L(t_1-t_2)}}{(A\ell_1)^{1/4}} - |\ell_1-\ell_2|
\end{aligned}$$

and similarly

$$\begin{aligned}
\frac{((2\pi-\theta_1^*)-(2\pi-\theta_1))\ell_2}{2\pi} &= \int_{\theta_1}^{2\pi} |c_\theta^*(t_2,\varphi)|\,d\varphi - \frac{(2\pi-\theta_1)\ell_2}{2\pi} \\
&\ge -2\ell_1\frac{\sqrt{L(t_1-t_2)}}{(A\ell_1)^{1/4}} - |\ell_1-\ell_2|.
\end{aligned}$$

Combining these and using the Lipschitz property of length, we get

$$\begin{aligned}
\frac{|\theta_1^*-\theta_1|\ell_2}{2\pi} &\le 2C\sqrt{L(t_1-t_2)} + 2|\sqrt{\ell_1}-\sqrt{\ell_2}|\sqrt{\ell_{\max}} \\
&\le 2C\sqrt{L(t_1-t_2)} + \sqrt{\ell_{\max}}\frac{L(t_1-t_2)}{\sqrt{A}} \le \frac{5}{2}C\sqrt{L(t_1-t_2)}.
\end{aligned}$$

Thus, finally

$$\begin{aligned}
|c(t_1,\theta_1)-c(t_2,\theta_2)| &\le |c(t_1,\theta_1)-c(t_2,\theta_1^*)| \\
&\qquad + |c(t_2,\theta_1^*)-c(t_2,\theta_1)| + |c(t_2,\theta_1)-c(t_2,\theta_2)| \\
&\le 4C\sqrt{L(t_1-t_2)} + \frac{5}{2}C\sqrt{L(t_1-t_2)} + \frac{\ell_{\max}}{2\pi}|\theta_1-\theta_2|. \qquad \square
\end{aligned}$$

3.9. One might also ask whether the maximum distance bound 3.5 can be strengthened to assert that the 1-jets of such curves C must be close to the 1-jets of C_0. The answer is NO, as is easily seen from looking at small wavelet-type perturbations of C_0. Specifically, calculate the length of the path $c(t,\theta) = c_0(\theta) + t \cdot af(\theta/a) \cdot i(c_0)_\theta(\theta)$, $0 \le t \le 1$, where f is an *arbitrary* C^2 function with compact support and a is very small. We claim the length of this path is $O(\sqrt{a})$, while the 1-jet at the point $\theta = 0$ of the final curve of the path approaches $(1 + if'0)(c_0)_\theta(0)$.

We sketch the proof, which is straightforward. Let $C_{a,t}$ be the curves on this path. Then $\sup|c_t| = O(a)$, $\sup|\kappa_{C_{a,t}}| = O(1/a)$, $A \le |c_\theta| \le B$ for suitable $A, B > 0$ and $\ell(\operatorname{supp}(c_t)) = O(a)$. Then the integral $\int_{S^1}(1 + A\kappa_c^2)(c_t, ic_\theta)^2 \frac{d\theta}{|c_\theta|}$ breaks up into two pieces, the first being $O(a^2)$, the second being $O(1)$ and the integral vanishing outside an interval of length $O(a)$. Thus the total distance is $O(\sqrt{a})$.

3.10. The H^0-distance on $B_i(S^1, \mathbb{R}^2)$ vanishes. Let $c_0, c_1 \in \operatorname{Imm}(S^1, \mathbb{R}^2)$ be two immersions, and suppose that $t \mapsto (\theta \mapsto c(t,\theta))$ is a smooth curve in $\operatorname{Imm}(S^1, \mathbb{R}^2)$ with $c(0,\cdot) = c_0$ and $c(1,\cdot) = c_1$.

The arc length for the H^0-metric of the curve $t \mapsto \pi(c(t,\cdot))$ in $B_i(S^1, \mathbb{R}^2)$ is given by 3.2.7 as

$$L^{\text{hor}}_{G^0}(c) = \int_0^1 \left(\int_{S^1} \langle c_t, ic_\theta \rangle^2 \frac{d\theta}{|c_\theta|} \right)^{1/2} dt. \tag{1}$$

Theorem. *For $c_0, c_1 \in \operatorname{Imm}(S^1, \mathbb{R}^2)$ there always exists a path $t \mapsto c(t,\cdot)$ with $c(0,\cdot) = c_0$ and $\pi(c(1,\cdot)) = \pi(c_1)$ such that $L^{\text{hor}}_{G^0}(c)$ is arbitrarily small.*

Heuristically, the reason for this is that if the curve is made to zig-zag wildly, say with teeth at an angle α, then the length of the curve goes up by a factor $1/\cos(\alpha)$ but the *normal* component of the motion of the curve goes down by the factor $\cos(\alpha)$—and this normal component is squared, hence it dominates.

Proof. Take a path $c(t,\theta)$ in $\operatorname{Imm}(S^1, \mathbb{R}^2)$ from c_0 to c_1 and make it horizontal using 2.5 so that $\langle c_t, c_\theta \rangle = 0$; this forces a reparametrization on c_1.

Now let us view c as a smooth mapping $c : [0,1] \times [0,1] \to \mathbb{R}^2$. We shall use the piecewise linear reparametrization $(\varphi(t,\theta), \theta)$ of the square shown above, which for $0 \le t \le 1/2$ deforms the straight line into a zig-zag of height 1 and period $n/2$ connecting the two end-curves, and then removes the teeth for $1/2 \le t \le 1$. In detail: Let $\tilde{c}(t,\theta) = c(\varphi(t,\theta), \theta)$ where

$$\varphi(t,\theta) = \begin{cases} 2t(2n\theta - 2k) & \text{for } 0 \le t \le 1/2,\ \frac{2k}{2n} \le \theta \le \frac{2k+1}{2n}, \\ 2t(2k + 2 - 2n\theta) & \text{for } 0 \le t \le 1/2,\ \frac{2k+1}{2n} \le \theta \le \frac{2k+2}{2n}, \\ 2t - 1 + 2(1-t)(2n\theta - 2k) & \text{for } 1/2 \le t \le 1,\ \frac{2k}{2n} \le \theta \le \frac{2k+1}{2n}, \\ 2t - 1 + 2(1-t)(2k + 2 - 2n\theta) & \text{for } 1/2 \le t \le 1,\ \frac{2k+1}{2n} \le \theta \le \frac{2k+2}{2n}. \end{cases}$$

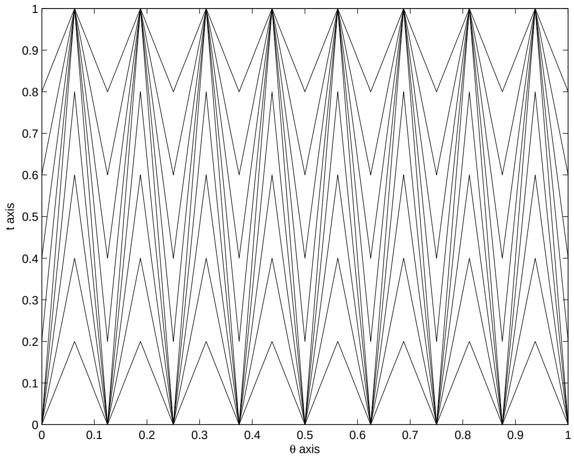

Fig. 2. The reparametrization of a path of curves used to make its length arbitrarily small.

Then we get $\tilde{c}_\theta = \varphi_\theta \cdot c_t + c_\theta$ and $\tilde{c}_t = \varphi_t \cdot c_t$ where

$$\varphi_\theta = \begin{cases} +4nt, \\ -4nt, \\ +4n(1-t), \\ -4n(1-t), \end{cases} \qquad \varphi_t = \begin{cases} 4n\theta - 4k, \\ 4k+4-4n\theta, \\ 2-4n\theta+4k, \\ -(2-4n\theta+4k). \end{cases}$$

Also, $\langle c_t, c_\theta\rangle = 0$ implies $\langle \tilde{c}_t, i\tilde{c}_\theta\rangle = \varphi_t \cdot |c_t| \cdot |c_\theta|$ and $|\tilde{c}_\theta| = |c_\theta|\sqrt{1+\varphi_\theta^2(|c_t|/|c_\theta|)^2}$. Thus

$$\begin{aligned} L^{\text{hor}}(\tilde{c}) &= \int_0^1 \left(\int_0^1 \langle \tilde{c}_t, i\tilde{c}_\theta\rangle^2 \frac{d\theta}{|\tilde{c}_\theta|}\right)^{1/2} dt = \int_0^1 \left(\int_0^1 \frac{\varphi_t^2 |c_t|^2 |c_\theta|}{\sqrt{1+\varphi_\theta^2 (\frac{|c_t|}{|c_\theta|})^2}}\, d\theta\right)^{1/2} dt \\ &= \int_0^{1/2} \Bigg(\sum_{k=0}^{n-1} \Bigg(\int_{\frac{2k}{2n}}^{\frac{2k+1}{2n}} \frac{(4n\theta-4k)^2 |c_t(\varphi,\theta)|^2 |c_\theta(\varphi,\theta)|}{\sqrt{1+(4nt)^2(\frac{|c_t(\varphi,\theta)|}{|c_\theta(\varphi,\theta)|})^2}}\, d\theta \\ &\quad + \int_{\frac{2k+1}{2n}}^{\frac{2k+2}{2n}} \frac{(4k+4-4n\theta)^2 |c_t(\varphi,\theta)|^2 |c_\theta(\varphi,\theta)|}{\sqrt{1+(4nt)^2(\frac{|c_t(\varphi,\theta)|}{|c_\theta(\varphi,\theta)|})^2}}\, d\theta\Bigg)\Bigg)^{1/2} dt \\ &\quad + \int_{1/2}^1 \Bigg(\sum_{k=0}^{n-1} \Bigg(\int_{\frac{2k}{2n}}^{\frac{2k+1}{2n}} \frac{(2-4n\theta+4k)^2 |c_t(\varphi,\theta)|^2 |c_\theta(\varphi,\theta)|}{\sqrt{1+(4n)^2(1-t)^2(\frac{|c_t(\varphi,\theta)|}{|c_\theta(\varphi,\theta)|})^2}}\, d\theta \\ &\quad + \int_{\frac{2k+1}{2n}}^{\frac{2k+2}{2n}} \frac{(2-4n\theta+4k)^2 |c_t(\varphi,\theta)|^2 |c_\theta(\varphi,\theta)|}{\sqrt{1+(4n)^2(1-t)^2(\frac{|c_t(\varphi,\theta)|}{|c_\theta(\varphi,\theta)|})^2}}\, d\theta\Bigg)\Bigg)^{1/2} dt. \end{aligned}$$

The function $|c_\theta(\varphi,\theta)|$ is uniformly bounded above and away from 0, and $|c_t(\varphi,\theta)|$ is uniformly bounded. Thus we may estimate

$$\sum_{k=0}^{n-1}\int_{\frac{2k}{2n}}^{\frac{2k+1}{2n}} \frac{(4n\theta-4k)^2|c_t(\varphi,\theta)|^2|c_\theta(\varphi,\theta)|}{\sqrt{1+(4nt)^2(\frac{|c_t(\varphi,\theta)|}{|c_\theta(\varphi,\theta)|})^2}}\,d\theta$$
$$\le O(1)\sum_{k=0}^{n-1}\int_0^{\frac{1}{2n}} \frac{4n^2\theta^2|c_t(\varphi(t,\frac{2k}{2n}+\theta),\frac{2k}{2n}+\theta)|^2}{\sqrt{1+(4nt)^2|c_t(\varphi(t,\frac{2k}{2n}+\theta),\frac{2k}{2n}+\theta)|^2}}\,d\theta.$$

We estimate as follows. Fix $\varepsilon>0$. First we split off the integral $\int_0^\varepsilon$ which is $O(\varepsilon)$ uniformly in n; so for the rest we have $t\ge\varepsilon$. The last sum of integrals is now estimated as follows: Consider first the set of all θ such that $|c_t(\varphi(t,\frac{2k}{2n}+\theta),\frac{2k}{2n}+\theta)|<\varepsilon$ which is a countable disjoint union of open intervals. There we get the estimate

$$O(1)\cdot n\cdot 4n^2\cdot\varepsilon^2(\theta^3/3)|_{\theta=0}^{\theta=1/2n}=O(\varepsilon),$$

uniformly in n. On the complementary set of all θ where $|c_t(\varphi(t,\frac{2k}{2n}+\theta),\frac{2k}{2n}+\theta)|\ge\varepsilon$ we use also $t\ge\varepsilon$ and estimate by

$$O(1)\cdot n\cdot 4n^2\cdot\tfrac{1}{4n\varepsilon^2}\cdot(\theta^3/3)|_{\theta=0}^{\theta=1/2n}=O\Big(\frac{1}{\varepsilon^2 n}\Big).$$

The other sums of integrals can be estimated similarly, thus $L^{\text{hor}}(\tilde c)$ goes to 0 as $n\to\infty$. It is clear that one can approximate φ by a smooth function without changing the estimates essentially. □

3.11. Non-smooth curves in the completion of B_i. We have seen in 3.6 that the completion of B_i in the metric G^A lies in the space of Lipschitz maps $c:S^1\to\mathbb{R}^2$ mod monotone correspondences, that is, rectifiable Fréchet immersed curves. But how big is it really? We cannot answer this, but we show, in this section, that certain non-smooth curves are in the completion. To be precise, if c is rectifiable, then we can assume c is parametrized at constant speed $|c_\theta|\equiv L/2\pi$ where L is the length of the curve. Therefore $c_\theta=(L/2\pi)e^{i\alpha(\theta)}$ for some measurable function $\alpha(\theta)$ giving the orientation of the tangent line at almost every point. We will say that a rectifiable curve c is *1-BV* if the function α is of bounded variation. Note that this means that the derivative of α exists as a finite signed measure, hence the curvature of c—which is $(2\pi/L)\alpha'$—is also a finite signed measure. In particular, there are a countable set of 'vertices' on such a curve, points where α has a discontinuity and the measure giving its curvature has an atomic component. Note that α has left and right limits everywhere and vertices can be assigned angles, namely $\alpha_+(\theta)-\alpha_-(\theta)$.

Theorem. *All 1-BV rectifiable curves are in the completion of B_i with respect to the metric G^A.*

Proof. This is proven using the following lemma:

Lemma. *Let $c(t,\theta)$, $0<t\le 1$, be an open path of smooth curves $c(t)$ and let $\alpha(t,\theta)=\arg(c_\theta(t,\theta))$. Assume that*

(1) *the length of all curves $c(t)$ is bounded by C_1,*
(2) *$|c_t|\le C_2$, for all (t,θ),*
(3) *for all t, the total variation in θ of $\alpha(t,\theta)$ is bounded by C_3,*
(4) *the curvature of $c(t)$ satisfies $|\kappa_{c(t)}(t,\theta)|\le C_4/t$ for all θ.*

Then the length of this path is bounded by $C_2(\sqrt{C_1}+2\sqrt{AC_3C_4})$.

To prove the lemma, let s_t be arc length on $c(t)$ and estimate the integral

$$\begin{aligned}\int_{c(t)}(1+A\kappa(c(t))(t,\theta)^2)\langle c_t, ic_\theta/|c_\theta|\rangle^2|c_\theta|\,d\theta &\le C_2^2\Big(C_1+A\int_{c(t)}\kappa_{c(t)}^2\,ds_t\Big)\\ &= C_2^2\Big(C_1+A\int_{c(t)}\kappa_{c(t)}\,\frac{d\alpha}{ds_t}\,ds_t\Big)\\ &\le C_2^2\Big(C_1+A\frac{C_4}{t}C_3\Big).\end{aligned}$$

Taking the square root of both sides and integrating from 0 to 1, we get the result.

We apply this lemma to the simplest possible smoothing of a 1-BV rectifiable curve c_0:

$$c(t,\theta)=\frac{1}{\sqrt{2\pi}\,t}\int_{\mathbb{R}}c_0(\theta-\varphi)e^{-\varphi^2/2t^2}\,d\varphi=\frac{1}{\sqrt{2\pi}\,t}\int_{\mathbb{R}}c_0(\varphi)e^{-(\theta-\varphi)^2/2t^2}\,d\varphi$$

for $0<t\le 1$. Note that t is the standard deviation of the Gaussian, *not* the variance. We assume c_0 has a constant speed parametrization and $c_0'=(L/2\pi)e^{i\alpha}$ as above, where α' is a finite signed measure. Thus,

$$\begin{aligned}c_\theta &= \frac{L}{(2\pi)^{3/2}t}\int_{\mathbb{R}}e^{i\alpha(\theta-\varphi)-\varphi^2/2t^2}\,d\varphi,\\ c_{\theta\theta} &= \frac{iL}{(2\pi)^{3/2}t}\int_{\mathbb{R}}e^{i\alpha(\varphi)-(\theta-\varphi)^2/2t^2}\,\alpha'(d\varphi).\end{aligned}$$

Moreover, using the second expression for the convolution and the heat equation for the Gaussian, we see that $c_t=tc_{\theta\theta}$. We now estimate:

$$\begin{aligned}|c_\theta| &\le L/2\pi, \quad \text{hence} \quad \text{length}(C_t)\le L\\ |c_{\theta\theta}| &\le \frac{L}{(2\pi)^{3/2}t}\int_{S^1}\sum_n e^{-(\theta-\varphi-nL)^2/2t^2}|\alpha'|(d\varphi)\\ &\le \sup_x\Big(\sum_n e^{-(x-nL)^2/2t^2}\Big)\frac{L\cdot\text{Var}(\arg(c_0'))}{(2\pi)^{3/2}t}=O(1/t),\\ \int_{S^1}|c_{\theta\theta}|\,d\theta &\le \frac{L}{2\pi}\Big(\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}\,t}e^{-\theta^2/2t^2}\,d\theta\Big)\Big(\int_{S^1}|\alpha'(d\varphi)|\Big)=\frac{L}{2\pi}\text{Var}(\arg(c_0')),\\ |c_t| &= t|c_{\theta\theta}|=O(1).\end{aligned}$$

To finish the proof, all we need to do is get a lower bound on $|c_\theta|$. However, $|c_\theta|$ can be very small if the curve c_0 has corners with small angles. In fact, c_0 can even double back on itself, giving a ‘corner’ with angle π. We need to treat this as a special case. When all the vertex angles of c_0 are less than π, we can get a lower bound for $|c_\theta|$ as follows. We start with the estimate

$$\begin{aligned}|c_\theta(\theta)| &= \left|\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{i\alpha(\theta-\varphi)-\varphi^2/2t^2}\,d\varphi/e^{i\alpha(\theta)}\right|\\ &\geq \left|\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}\cos(\alpha(\theta-\varphi)-\alpha(\theta))e^{-\varphi^2/2t^2}\,d\varphi\right|.\end{aligned}$$

We break up the integral over $\mathbb{R}$ into 3 intervals $(-\infty,\theta-\delta/2]$, $[\theta-\delta/2,\theta+\delta/2]$, $[\theta+\delta/2,+\infty)$ for a suitable δ. If t is sufficiently small, the integral of the Gaussian over the first and third intervals goes uniformly to 0 and, on the middle interval, goes to 1. Thus it suffices to estimate the cos in the middle interval. We use a remark on BV functions:

Lemma. *For any BV function f and any $C>0$, there is a $\delta>0$ such that on every interval I of length less than δ, either $f|_I$ has a single jump of size $\geq C$, or we have* $\max(f|_I)-\min(f|_I)\leq C$.

In fact, let $C-\varepsilon$ be the size of the largest jump in f less than C and break up the domain of f into intervals J_i on each of which the variation of f is less than $\epsilon/2$, big jumps being on their boundaries. If δ is less than the minimum of the lengths of the J_i, we get what we want.

Now let $\pi-\beta$ be the largest vertex angle of the curve c_0. Using the lemma, choose a δ so that for every interval I in the θ-line of length less than δ, either I contains a single vertex with exterior angle $\geq\beta/3$ or $\max\alpha|_I-\min\alpha|_I\leq\beta/3$. Now if there is no vertex in $[\theta-\delta/2,\theta+\delta/2]$, then $|\alpha(\theta-\varphi)-\alpha(\theta)|\leq\beta/3$ on this interval and our lower bound is

$$|c_\theta(\theta)|\geq\cos(\beta/3)-o(t).$$

On the other hand, if there is such a vertex, say at $\bar\theta$, then α varies by at most $\beta/3$ in $[\theta-\delta/2,\bar\theta)$, jumps by at most $\pi-\beta$ at $\bar\theta$ and then varies by at most $\beta/3$ on $(\bar\theta,\theta+\delta/2]$. Assume $\theta<\bar\theta$ (the case $\theta>\bar\theta$ is similar). Then

$$\cos(\alpha(\theta-\varphi)-\alpha(\theta))\geq\begin{cases}\cos(\beta/3) & \text{if } \varphi\in(\theta-\bar\theta,\theta+\delta/2],\\ \cos(\pi-\beta+\beta/3)=-\cos(2\beta/3) & \text{if } \varphi\in[\theta-\delta/2,\theta-\bar\theta).\end{cases}$$

Thus

$$|c_\theta(\theta)|\geq\tfrac{1}{2}(\cos(\beta/3)-\cos(2\beta/3))-o(t),$$

hence, if t is sufficiently small, we get a uniform lower bound on $|c_\theta|$. Since $|\kappa_{C_t}|\leq|c_{\theta\theta}|/|c_\theta|^2$, we get the required upper bound both on $|\kappa_{C_t}|$ and on the variation of α_{C_t}, i.e. $\int_{S^1}|\kappa_{C_t}|$ and all the requirements of the lemma are satisfied.

If c_0 has a vertex with angle π, we need to add an extra argument. c_0 certainly has at most a finite number of such vertices and we can construct a new curve by drawing a circle of radius t around each of these vertices and letting $c_0^{(t)}$ be the curve which follows c_0 until

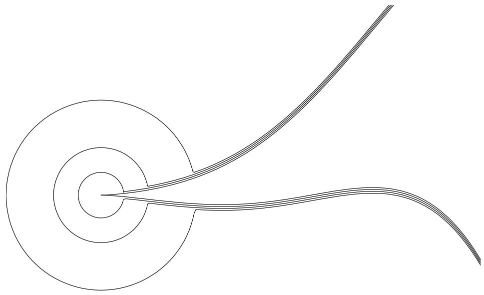

Fig. 3. Approximating 1-BV curves with zero angle vertices by curves with positive angle vertices.

it hits one of these circles and then replaces the vertex with a circuit around the circle: see Figure 3. Each of the curves $c_0^{(t)}$ is in the completion of B_i by the previous argument and the path formed by the $c_0^{(t)}$'s also has finite length, hence c_0 is in the completion. We omit the details which are straightforward.

3.12. The energy of a path as 'anisotropic area' of its graph in $\mathbb{R}^3$. Consider a path $t \mapsto c(t, \cdot)$ in the manifold $\mathrm{Imm}(S^1, \mathbb{R}^2)$. It projects to a path $\pi \circ c$ in $B_i(S^1, \mathbb{R}^2)$ whose energy is

$$\begin{aligned} E_{G^A}(\pi \circ c) &= \frac{1}{2} \int_a^b G^A_{\pi(c)}(T_c\pi \cdot c_t, T_c\pi \cdot c_t)\, dt \\ &= \frac{1}{2} \int_a^b G^A_c(c_t^\perp, c_t^\perp)\, dt = \frac{1}{2} \int_a^b \int_{S^1} (1 + A\kappa(c)^2) \langle c_t^\perp, c_t^\perp \rangle |c_\theta|\, d\theta\, dt \\ &= \frac{1}{2} \int_a^b \int_{S^1} (1 + A\kappa(c)^2) \left\langle \frac{\langle c_t, ic_\theta\rangle}{|c_\theta|^2} ic_\theta, \frac{\langle c_t, ic_\theta\rangle}{|c_\theta|^2} ic_\theta \right\rangle |c_\theta|\, d\theta\, dt \\ &= \frac{1}{2} \int_a^b \int_{S^1} (1 + A\kappa(c)^2) \langle c_t, ic_\theta\rangle^2 \frac{d\theta}{|c_\theta|}\, d\theta\, dt. \end{aligned} \tag{1}$$

If the path c is horizontal, i.e., $\langle c_t, c_\theta\rangle = 0$, then $\langle c_t, ic_\theta\rangle = |c_t| \cdot |c_\theta|$ and we have

$$E^{\mathrm{hor}}_{G^A}(c) = \frac{1}{2} \int_a^b \int_{S^1} (1 + A\kappa(c)^2) |c_t|^2 |c_\theta|\, d\theta\, dt, \qquad \langle c_t, c_\theta\rangle = 0. \tag{2}$$

which is just the usual energy of c.

Let $c(t, \theta) = (x(t, \theta), y(t, \theta))$ be still horizontal and consider the graph

$$\Phi(t, \theta) = (t, x(t, \theta), y(t, \theta)) \in \mathbb{R}^3.$$

We also have $|x_t y_\theta - x_\theta y_t| = |\det(c_t, c_\theta)| = |c_t| \cdot |c_\theta|$ and for the vector product we get $\Phi_t \times \Phi_\theta = (x_t y_\theta - x_\theta y_t, -y_\theta, x_\theta)$, so

$$|\Phi_t \times \Phi_\theta|^2 = (x_t y_\theta - x_\theta y_t)^2 + y_\theta^2 + x_\theta^2 = (x_\theta^2 + y_\theta^2)(x_t^2 + y_t^2 + 1) = |c_\theta|^2 (|c_t|^2 + 1).$$

We express now $E^{\text{hor}}(c)$ as an integral over the immersed surface $S \subset \mathbb{R}^3$ parametrized by Φ in terms of the surface area $d\mu_S = |\Phi_t \times \Phi_\theta|\, d\theta\, dt$ as follows:

$$\begin{aligned} E^{\text{hor}}_{G^A}(c) &= \frac{1}{2}\int_a^b \int_{S^1} (1 + A\kappa(c)^2) \frac{|c_t|^2 |c_\theta|}{|\Phi_t \times \Phi_\theta|} |\Phi_t \times \Phi_\theta|\, d\theta\, dt \\ &= \frac{1}{2}\int_{[a,b]\times S^1} (1 + A\kappa(c)^2) \frac{|c_t|^2}{\sqrt{|c_t|^2 + 1}}\, d\mu_S. \end{aligned}$$

Next we want to express the integrand as a function γ of the unit normal $n_S = (\Phi_t \times \Phi_\theta)/|\Phi_t \times \Phi_\theta|$. Let $e_0 = (1, 0, 0)$. Then the absolute value of the t-component n^0_S of the unit normal n_S is

$$|n^0_S| := |\langle e_0, n_S\rangle| = \frac{|c_t|}{\sqrt{|c_t|^2 + 1}}, \quad \text{and} \quad \frac{|c_t|^2}{\sqrt{|c_t|^2 + 1}} = \frac{|n^0_S|^2}{\sqrt{1 - |n^0_S|^2}}.$$

Thus for horizontal c (i.e., with $c_t \perp c_\theta$) we have

Horizontal energy as anisotropic area.

$$\boxed{E^{\text{hor}}_{G^A}(c) = \frac{1}{2}\int_{[a,b]\times S^1} (1 + A\kappa(c)^2) \frac{|n^0_S|^2}{\sqrt{1 - |n^0_S|^2}}\, d\mu_S.} \tag{3}$$

Here the final expression is only in terms of the surface S and does not depend on the curve c being horizontal. This anisotropic area functional has to be minimized in order to prove that geodesics exists between arbitrary curves (of the same degree) in $B_i(S^1, \mathbb{R}^2)$. Thus we are led to

Question. *For immersions $c_0, c_1 : S^1 \to \mathbb{R}^2$ does there exist an immersed surface $S = (\text{ins}_{[0,1]}, c) : [0, 1] \times S^1 \to \mathbb{R} \times \mathbb{R}^2$ such that the functional* (3) *is critical at S?*

A first step is:

Bounding the area. *For any path $[a, b] \ni t \mapsto c(t, \ \ )$ the area of the graph surface $S = S(c)$ is bounded as follows:*

$$\text{Area}(S) = \int_{[a,b]\times S^1} d\mu_S \le 2E^{\text{hor}}_{G^A}(c) + \max_t \ell(c(t, \ \ ))(b - a). \tag{4}$$

Proof. Writing the unit normal $n_S = (n^0_S, n^1_S, n^2_S) \in S^2$ according to the coordinates (t, x, y) we have

$$|n^1_S| + |n^2_S| + \frac{|n^0_S|^2}{\sqrt{1 - |n^0_S|^2}} \ge |n^1_S|^2 + |n^2_S|^2 + |n^0_S|^2 = 1.$$

Since $|n^1_S| d\mu_S$ is the area element of the projection of S onto the (t, y)-plane we have

$$\begin{aligned}\operatorname{Area}(S) &= \int_{[a,b]\times S^1} d\mu_S \le \int_{[a,b]\times S^1}(1+A\kappa(c)^2)\left(|n_S^1|+|n_S^2|+\frac{|n_S^0|^2}{\sqrt{1-|n_S^0|^2}}\right)d\mu_S\\ &\le 2E^{\text{hor}}_{G^A}(c)+\max_t \ell(c(t,\ \ ))(b-a).\end{aligned}$$

□

4. Geodesic equations and sectional curvatures

4.1. Geodesics on $\operatorname{Imm}(S^1,\mathbb{R}^2)$**.** The energy of a curve $t \mapsto c(t,\cdot)$ in the space $\operatorname{Imm}(S^1,\mathbb{R}^2)$ is

$$E_{G^A}(c)=\frac{1}{2}\int_a^b\int_{S^1}(1+A\kappa_c^2)\langle c_t,c_t\rangle|c_\theta|\,d\theta\,dt.$$

By calculating its first variation, we get the equation for a geodesic:

Geodesic equation.

$$\boxed{((1+A\kappa^2)|c_\theta|\cdot c_t)_t=\left(\frac{-1+A\kappa^2}{2}\cdot\frac{|c_t|^2}{|c_\theta|}\cdot c_\theta+A\frac{(\kappa|c_t|^2)_\theta}{|c_\theta|^2}\cdot ic_\theta\right)_\theta.}\tag{1}$$

Proof. From 2.2 we have

$$\kappa(c)_s=\frac{\langle ic_{s\theta},c_{\theta\theta}\rangle}{|c_\theta|^3}+\frac{\langle ic_\theta,c_{s\theta\theta}\rangle}{|c_\theta|^3}-3\kappa\frac{\langle c_{s\theta},c_\theta\rangle}{|c_\theta|^2}$$

and

$$c_{\theta\theta}=\frac{\langle c_{\theta\theta},c_\theta\rangle}{|c_\theta|^2}c_\theta+\frac{\langle c_{\theta\theta},ic_\theta\rangle}{|c_\theta|^2}ic_\theta=\frac{|c_\theta|_\theta}{|c_\theta|}c_\theta+\kappa(c)|c_\theta|ic_\theta.$$

Now we compute

$$\begin{aligned}
\partial_s|_0E(c)&=\frac{1}{2}\partial_s|_0\int_a^b\int_{S^1}(1+A\kappa^2)\langle c_t,c_t\rangle|c_\theta|\,d\theta\,dt\\
&=\int_a^b\int_{S^1}\left(A\kappa\kappa_s|c_\theta|\,|c_t|^2+(1+A\kappa^2)\langle c_{st},c_t\rangle|c_\theta|+\frac{1+A\kappa^2}{2}|c_t|^2\frac{\langle c_{s\theta},c_\theta\rangle}{|c_\theta|}\right)d\theta\,dt\\
&=\int_a^b\int_{S^1}\Bigg(A\kappa\langle ic_{s\theta},c_{\theta\theta}\rangle\frac{|c_t|^2}{|c_\theta|^2}+A\kappa\langle ic_\theta,c_{s\theta\theta}\rangle\frac{|c_t|^2}{|c_\theta|^2}-3A\kappa^2\langle c_{s\theta},c_\theta\rangle\frac{|c_t|^2}{|c_\theta|}\\
&\qquad-\left\langle c_s,((1+A\kappa^2)|c_\theta|c_t)_t+\left(\frac{1+A\kappa^2}{2}\frac{|c_t|^2}{|c_\theta|}c_\theta\right)_\theta\right\rangle\Bigg)d\theta\,dt\\
&=\int_a^b\int_{S^1}\Bigg(\left\langle c_s,A\left(\kappa\frac{|c_t|^2}{|c_\theta|^2}ic_{\theta\theta}\right)_\theta\right\rangle+\left\langle c_s,A\left(\kappa\frac{|c_t|^2}{|c_\theta|^2}ic_\theta\right)_{\theta\theta}\right\rangle+\left\langle c_s,3A\left(\kappa^2\frac{|c_t|^2}{|c_\theta|}c_\theta\right)_\theta\right\rangle\\
&\qquad-\left\langle c_s,((1+A\kappa^2)|c_\theta|c_t)_t+\left(\frac{1+A\kappa^2}{2}\frac{|c_t|^2}{|c_\theta|}c_\theta\right)_\theta\right\rangle\Bigg)d\theta\,dt\\
&=\int_a^b\int_{S^1}\langle c_s,-((1+A\kappa^2)|c_\theta|c_t)_t+F_\theta\rangle\,d\theta\,dt
\end{aligned}$$

where

$$F = A\kappa\frac{|c_t|^2}{|c_\theta|^2}ic_{\theta\theta} + A(\kappa|c_t|^2)_\theta\frac{ic_\theta}{|c_\theta|^2} - 2A\kappa|c_t|^2\frac{|c_\theta|_\theta ic_\theta}{|c_\theta|^3} + A\kappa\frac{|c_t|^2}{|c_\theta|^2}ic_{\theta\theta}$$
$$+ 3A\kappa^2\frac{|c_t|^2}{|c_\theta|}c_\theta - \frac{1+A\kappa^2}{2}\frac{|c_t|^2}{|c_\theta|}c_\theta.$$

If we substitute the expression for $c_{\theta\theta}$ and simplify, this reduces to

$$F = \frac{-1+A\kappa^2}{2}\frac{|c_t|^2}{|c_\theta|}c_\theta + A(\kappa|c_t|^2)_\theta\frac{ic_\theta}{|c_\theta|^2},$$

which gives the required formula for geodesics.

Putting $A=0$ in 4.1.1 we get the geodesic equation for the H^0-metric on $\operatorname{Imm}(S^1,\mathbb{R}^2)$,

$$(|c_\theta|c_t)_t = -\frac{1}{2}\left(\frac{|c_t|^2c_\theta}{|c_\theta|}\right)_\theta. \tag{2}$$

4.2. Geodesics on $B_i(S^1,\mathbb{R}^2)$. We may also restrict to geodesics which are perpendicular to the orbits of $\operatorname{Diff}(S^1)$, i.e. $\langle c_t, c_\theta\rangle \equiv 0$, obtaining the geodesics in the quotient space $B_i(S^1,\mathbb{R}^2)$. To write this in the simplest way, we introduce the 'velocity' a by setting $c_t = iac_\theta/|c_\theta|$ (so that $|c_t|^2 = a^2$). When we substitute this into the above geodesic equation, the equation splits into a multiple of c_θ and a multiple of ic_θ. The former vanishes identically and the latter gives

$$((1+A\kappa^2)|c_\theta|a)_t\frac{ic_\theta}{|c_\theta|} = \frac{-1+A\kappa^2}{2}a^2\left(\frac{c_\theta}{|c_\theta|}\right)_\theta + A\left(\frac{(\kappa a^2)_\theta}{|c_\theta|}\right)_\theta\frac{ic_\theta}{|c_\theta|}, \quad \text{or}$$
$$((1+A\kappa^2)|c_\theta|a)_t = \frac{-1+A\kappa^2}{2}\kappa|c_\theta|a^2 + A\left(\frac{(\kappa a^2)_\theta}{|c_\theta|}\right)_\theta.$$

If we use derivatives with respect to arc length instead of θ and write these with the subscript s, so that $f_s = f_\theta/|c_\theta|$, this simplifies. We need

$$|c_\theta|_t = \frac{\langle c_\theta, c_{t\theta}\rangle}{|c_\theta|} = -\frac{\langle c_{\theta\theta}, c_t\rangle}{|c_\theta|} = -a\frac{\langle c_{\theta\theta}, ic_\theta\rangle}{|c_\theta|^2} = -a\kappa|c_\theta|.$$

which gives us a simple form for the equation for geodesics on $B_i(S^1,\mathbb{R}^2)$:

$$((1+A\kappa^2)a)_t = \frac{1+3A\kappa^2}{2}\kappa a^2 + A(\kappa a^2)_{ss}. \tag{1}$$

Finally, we may expand the t-derivatives on the left hand side, using the formula $\kappa_t = a\kappa^2 + a_{ss}$ noted in 2.2.7; we also collect all constraint equations that we chose along the way:

$$\boxed{\begin{aligned} 0 &= \langle c_t, c_s\rangle, \quad c_t = aic_s, \quad \kappa = \langle c_{ss}, ic_s\rangle, \\ a_t &= \frac{\frac{1}{2}\kappa a^2 + A(a^2(\kappa_{ss} - \frac{1}{2}\kappa^3) + 4\kappa_s a a_s + 2\kappa a_s^2)}{1+A\kappa^2}. \end{aligned}} \tag{2}$$

Handle this with care: Going to unit speed parametrization (so that f_s is really a holonomic partial derivative) destroys the first constraint 'horizontality'. This should be seen as a gauge fixing.

4.3. Geodesics on $B_i(S^1,\mathbb{R}^2)$ for $A = 0$. Let us now set $A = 0$. We keep looking at horizontal geodesics, so that $\langle c_t, c_\theta\rangle = 0$ and $c_t = iac_\theta/|c_\theta|$ for $a \in C^\infty(S^1)$. We use the functions a, $s = |c_\theta|$, and κ. We use equations from 4.2 but we do not use the anholonomic derivative:

$$s_t = -a\kappa s, \quad a_t = \tfrac{1}{2}\kappa a^2, \quad \kappa_t = a\kappa^2 + \frac{1}{s}\left(\frac{a_\theta}{s}\right)_\theta = a\kappa^2 + \frac{a_{\theta\theta}}{s^2} - \frac{a_\theta s_\theta}{s^3}. \tag{1}$$

We may assume that $s|_{t=0}$ is constant. Let $v(\theta) = a(0,\theta)$ be the initial value for a. Then from equations (1) we get

$$\frac{s_t}{s} = -a\kappa = -2\frac{a_t}{a}, \quad \text{hence} \quad \log(sa^2)_t = 0,$$

so that sa^2 is constant in t,

$$s(t,\theta)a(t,\theta)^2 = s(0,\theta)a(0,\theta)^2 = v(\theta)^2, \tag{2}$$

a smooth family of conserved quantities along the geodesic. This leads to the substitutions

$$s = \frac{v^2}{a^2}, \quad \kappa = 2\frac{a_t}{a^2},$$

which transform the last equation (1) to

$$a_{tt} - 4\frac{a_t^2}{a} - \frac{a^6 a_{\theta\theta}}{2v^4} + \frac{a^6 a_\theta v_\theta}{v^5} - \frac{a^5 a_\theta^2}{v^4} = 0, \quad a(0,\theta) = v(\theta), \tag{3}$$

a non-linear hyperbolic second order equation. Note that (2) implies that wherever $v = 0$ then also $a = 0$ for all t. For that reason, let us transform equation (3) into a less singular form by substituting $a = vb$. Note that $b = 1/\sqrt{s}$. The outcome is

$$(b^{-3})_{tt} = -\frac{v^2}{2}(b^3)_{\theta\theta} - 2vv_\theta(b^3)_\theta - \frac{3vv_{\theta\theta}}{2}b^3, \quad b(0,\theta) = 1. \tag{4}$$

4.4. The induced metric on $B_{i,f}(S^1,\mathbb{R}^2)$ in a chart. We also want to compute the curvature of $B_i(S^1,\mathbb{R}^2)$ in this metric. For this, we need second derivatives and the most convenient way to calculate these seems to be to use a local chart. Consider the smooth principal bundle $\pi : \mathrm{Imm}_f(S^1,\mathbb{R}^2) \to B_{i,f}(S^1,\mathbb{R}^2)$ with structure group $\mathrm{Diff}(S^1)$ described in 2.4.3. We shall describe the metric in the following chart near $C \in B_{i,f}(S^1,\mathbb{R}^2)$: Let $c \in \mathrm{Imm}_f(S^1,\mathbb{R}^2)$ be parametrized by arc length with $\pi(c) = C$ of length L, with unit normal n_c. We assume that the parameter θ runs in the scaled circle S^1_L below. As in the proof of 2.4.3 we consider the mapping

$$\begin{aligned}
&\psi : C^\infty(S^1_L, (-\varepsilon, \varepsilon)) \to \mathrm{Imm}_f(S^1_L, \mathbb{R}^2), \qquad \mathcal{Q}(c) := \psi(C^\infty(S^1_L, (-\varepsilon, \varepsilon))),\\
&\psi(f)(\theta) = c(\theta) + f(\theta)n_c(\theta) = c(\theta) + f(\theta)ic'(\theta),\\
&\pi \circ \psi : C^\infty(S^1_L, (-\varepsilon, \varepsilon)) \to B_{i,f}(S^1, \mathbb{R}^2),
\end{aligned}$$

where ε is so small that $\psi(f)$ is an embedding for each f. By 2.4.3 the mapping $(\pi \circ \psi)^{-1}$ is a smooth chart on $B_{i,f}(S^1_L, \mathbb{R}^2)$. Note that:

$$\begin{aligned}
\psi(f)' &= c' + f'ic' + fic'' = (1 - f\kappa_c)c' + f'ic',\\
\psi(f)'' &= c'' + f''ic' + 2f'ic'' + fic''' = -(2f'\kappa_c + f\kappa_c')c' + (\kappa_c + f'' - f\kappa_c^2)ic',\\
n_{\psi(f)} &= \frac{1}{\sqrt{(1 - f\kappa_c)^2 + f'^2}}((1 - f\kappa_c)ic' - f'c'),\\
T_f\psi \cdot h &= h \cdot ic' \in C^\infty(S^1, \mathbb{R}^2) = T_{\psi(f)} \mathrm{Imm}_f(S^1_L, \mathbb{R}^2)\\
&= \frac{h(1 - f\kappa_c)}{\sqrt{(1 - f\kappa_c)^2 + f'^2}} n_{\psi(f)} + \frac{hf'}{(1 - f\kappa_c)^2 + f'^2}\psi(f)',\\
(T_f\psi \cdot h)^\perp &= \frac{h(1 - f\kappa_c)}{\sqrt{(1 - f\kappa_c)^2 + f'^2}} n_{\psi(f)} \in \mathcal{N}_{\psi(f)},\\
\kappa_{\psi(f)} &= \frac{1}{((1 - f\kappa_c)^2 + f'^2)^{3/2}} \langle i\psi(f)', \psi(f)''\rangle\\
&= \frac{\kappa_c + f'' - 2f\kappa_c^2 - ff''\kappa_c + f^2\kappa_c^3 + 2f'^2\kappa_c + ff'\kappa_c'}{((1 - f\kappa_c)^2 + f'^2)^{3/2}}.
\end{aligned}$$

Let G^A denote also the induced metric on $B_{i,f}(S^1_L, \mathbb{R}^2)$. Since π is a Riemannian submersion, $T_{\psi(f)}\pi : (\mathcal{N}_{\psi(f)}, G^A_{\psi(f)}) \to (B_{i,f}(S^1_L, \mathbb{R}^2), G^A_{\pi(\psi(f))})$ is an isometry. Then we compute, for $f \in C^\infty(S^1_L, (-\varepsilon, \varepsilon))$ and $h, k \in C^\infty(S^1_L, \mathbb{R})$,

$$\begin{aligned}
((\pi \circ \psi)^* G^A)_f(h, k) &= G^A_{\pi(\psi(f))}(T_f(\pi \circ \psi)h, T_f(\pi \circ \psi)k)\\
&= G^A_{\psi(f)}((T_f\psi \cdot h)^\perp, (T_f\psi \cdot k)^\perp)\\
&= \int_{S^1_L} (1 + A\kappa^2_{\psi(f)})\langle (T_f\psi \cdot h)^\perp, (T_f\psi \cdot k)^\perp\rangle |\psi(f)'| \; d\theta\\
&= \int_{S^1_L} (1 + A\kappa^2_{\psi(f)}) \frac{hk(1 - f\kappa_c)^2}{\sqrt{(1 - f\kappa_c)^2 + f'^2}} \; d\theta.
\end{aligned}$$

This is the expression from which we have to compute the geodesic equation in the chart on $B_{i,f}(S^1_L, \mathbb{R}^2)$.

4.5. Computing the Christoffel symbols in $B_{i,f}(S^1_L, \mathbb{R}^2)$ at $C = \pi(c)$. We have to compute second derivatives in f of the expression of the metric in 4.2. For that we expand the two main contributing expressions in f to order 2, where we put $\kappa = \kappa_c$:

$$\begin{aligned}
&\kappa_{\psi(f)}\\
&= (1 - 2f\kappa + f^2\kappa^2 + f'^2)^{-3/2}(\kappa + f'' - 2f\kappa^2 - ff''\kappa + f^2\kappa^3 + 2f'^2\kappa + ff'\kappa')\\
&= \kappa + (f'' + f\kappa^2) + (f^2\kappa^3 + \tfrac{1}{2}f'^2\kappa + ff'\kappa' + 2ff''\kappa) + O(f^3),
\end{aligned}$$

$$(1 - f\kappa)^2(1 - 2f\kappa + f^2\kappa^2 + f'^2)^{-1/2} = 1 - f\kappa - \tfrac{1}{2}f'^2 + O(f^3).$$

Thus

$$\begin{aligned}
(1 + A\kappa^2_{\psi(f)})\frac{(1 - f\kappa_c)^2}{\sqrt{(1 - f\kappa_c)^2 + f'^2}} &= 1 + A\kappa^2 + 2Af''\kappa + Af\kappa^3 - f\kappa\\
&\quad - \tfrac{1}{2}f'^2 + Af^2\kappa^4 + A\tfrac{1}{2}f'^2\kappa^2 + 2Aff'\kappa\kappa' + Af''^2 + 4Aff''\kappa^2
\end{aligned}$$

and finally

$$\begin{aligned}
G^A_f(h,k) &= ((\pi\circ\psi)^*G^A)_f(h,k)\\
&= \int_{S^1_L} hk((1 + A\kappa^2) + (2Af''\kappa + Af\kappa^3 - f\kappa) - \tfrac{1}{2}f'^2\\
&\qquad + A(4ff''\kappa^2 + f^2\kappa^4 + \tfrac{1}{2}f'^2\kappa^2 + 2ff'\kappa\kappa' + f''^2) + O(f^3))\,d\theta. \qquad (1)
\end{aligned}$$

We differentiate the metric:

$$\begin{aligned}
dG^A(f)(l)(h,k) = \int_{S^1_L} &hk(2Al''\kappa + (A\kappa^3 - \kappa)l + 4Alf''\kappa^2 + 4Afl''\kappa^2\\
&+ 2Afl\kappa^4 + (A\kappa^2 - 1)f'l' + 2Alf'\kappa\kappa' + 2Afl'\kappa\kappa' + 2Af''l'' + O(f^2))\,d\theta
\end{aligned}$$

and compute the Christoffel symbols:

$$\begin{aligned}
-2G^A_f(\Gamma_f(h,k),l) = &-dG^A(f)(l)(h,k) + dG^A(f)(h)(k,l) + dG^A(f)(k)(l,h)\\
= &\int_{S^1_L} l((A\kappa^3 - \kappa + 2A\kappa\kappa' f' + 4A\kappa^2 f'' + 2A\kappa^4 f)kh\\
&+ (2A\kappa + 4A\kappa^2 f + 2Af'')(h''k + hk'')\\
&+ (A\kappa^2 f' - f' + 2A\kappa\kappa' f)(h'k + hk') + O(f^2))\,d\theta\\
&- \int_{S^1_L} (l'(A\kappa^2 f'hk - f'hk + 2A\kappa\kappa' fhk)\\
&+ l''(2A\kappa hk + 4A\kappa^2 fhk + 2Af''hk) + O(f^2))\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_{S^1_L} l((A\kappa^3 - \kappa - 2A\kappa'')hk - 4A\kappa'(h'k + hk') - 4A\kappa h'k' \\
&\quad + (-2Af^{(4)} - f'' + 2A\kappa^4 f - 6A\kappa'^2 f - 6A\kappa\kappa'' f - 10A\kappa\kappa' f' + A\kappa^2 f'')hk \\
&\quad - (2f' + 4Af''' + 12A\kappa\kappa' f + 6A\kappa^2 f')(h'k + hk') \\
&\quad - 2(4A\kappa^2 f + 2Af'')h'k' + O(f^2))\, d\theta.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&G^A_f(\Gamma_f(h,k), l) \\
&\qquad = \int_{S^1_L} l((\tfrac{1}{2}\kappa - \tfrac{1}{2}A\kappa^3 + A\kappa'')hk + 2A\kappa'(h'k + hk') + 2A\kappa h'k' \\
&\qquad\quad - (-Af^{(4)} - \tfrac{1}{2}f'' + A\kappa^4 f - 3A\kappa'^2 f - 3A\kappa\kappa'' f - 5A\kappa\kappa' f' + \tfrac{1}{2}A\kappa^2 f'')hk \\
&\qquad\quad + (f' + 2Af''' + 6A\kappa\kappa' f + 3A\kappa^2 f')(h'k + hk') \\
&\qquad\quad + (4A\kappa^2 f + 2Af'')h'k' + O(f^2))\, d\theta.
\end{aligned}
$$

At the center of the chart, for $f = 0$, we get

$$
\begin{aligned}
&G^A_0(\Gamma_0(h,k), l) \\
&\qquad = \int_{S^1_L} l((\tfrac{1}{2}\kappa - \tfrac{1}{2}A\kappa^3 + A\kappa'')hk + 2A\kappa'(h'k + hk') + 2A\kappa h'k')\, d\theta \\
&\qquad = \int_{S^1_L} l\left(\frac{(\tfrac{1}{2}\kappa - \tfrac{1}{2}A\kappa^3 + A\kappa'')hk + 2A\kappa'(h'k + hk') + 2A\kappa h'k'}{1 + A\kappa^2}\right)(1 + A\kappa^2)\ \ d\theta \\
&\qquad = G^A_0\left(\frac{(\tfrac{1}{2}\kappa - \tfrac{1}{2}A\kappa^3 + A\kappa'')hk + 2A\kappa'(h'k + hk') + 2A\kappa h'k'}{1 + A\kappa^2}, l\right)
\end{aligned}
$$

so that

$$
\Gamma_0(h,k) = \frac{(\tfrac{1}{2}\kappa - \tfrac{1}{2}A\kappa^3 + A\kappa'')hk + 2A\kappa'(h'k + hk') + 2A\kappa h'k'}{1 + A\kappa^2}. \tag{2}
$$

If we let $h = k = f_t$, this leads to the geodesic equation, valid at $f = 0$:

$$
f_{tt} = \frac{(\tfrac{1}{2}\kappa - \tfrac{1}{2}A\kappa^3 + A\kappa'')f_t^2 + 4A\kappa' f_t f_t' + 2A\kappa(f_t')^2}{1 + A\kappa^2}.
$$

If we substitute a for f_t and a_t for f_{tt}, this is the same as the previous geodesic equation derived in 4.2 by variational methods. There is a subtle point here, however: why is it ok to identify the second derivatives a_t and f_{tt} with each other? To check this let $c(\theta) + (ta_1(\theta) + \frac{t^2}{2}a_2(\theta))ic'(\theta)$ be a 2-jet in our chart. Then if we reparametrize the nearby curves by substituting $\theta - \frac{t^2}{2}a_1 a_1'$ for θ, letting

$$\begin{aligned}c(t,\theta) &= c\Big(\theta - \frac{t^2}{2}a_1a_1'\Big) + \Big(ta_1\Big(\theta - \frac{t^2}{2}a_1a_1'\Big) \\ &\quad + \frac{t^2}{2}a_2\Big(\theta - \frac{t^2}{2}a_1a_1'\Big)\Big)ic\Big(\theta - \frac{t^2}{2}a_1a_1'\Big)' \\ &\equiv c(\theta) - \Big(\frac{t^2}{2}a_1a_1'\Big)c'(\theta) + \Big(ta_1(\theta) + \frac{t^2}{2}a_2(\theta)\Big)ic'(\theta) \bmod t^3,\end{aligned}$$

then $\langle c', c_t\rangle \equiv 0 \bmod t^2$, hence this 2-jet is horizontal and $\langle c_{tt}, ic'\rangle \equiv a_2 \bmod t$ as required.

4.6. Computation of the sectional curvature in $B_{i,f}(S^1_L, \mathbb{R}^2)$ at C. We now go further. We use the following formula which is valid in a chart:

$$\begin{aligned}2R_f(m,h,m,h) &= 2G^A_f(R_f(m,h)m,h) \\ &= -2d^2G^A(f)(m,h)(h,m) + d^2G^A(f)(m,m)(h,h) + d^2G^A(f)(h,h)(m,m) \\ &\quad -2G^A(\Gamma(h,m),\Gamma(m,h)) + 2G^A(\Gamma(m,m),\Gamma(h,h)). \qquad (1)\end{aligned}$$

The sectional curvature at the two-dimensional subspace $P_f(m,h)$ of the tangent space which is spanned by m and h is then given by

$$k_f(P(m,h)) = -\frac{G^A_f(R(m,h)m,h)}{\|m\|^2\|h\|^2 - G^A_f(m,h)^2}. \qquad (2)$$

We compute this directly for $f = 0$. From the expansion up to order 2 of $G^A_f(h,k)$ in 4.5.1 we get

$$\begin{aligned}\frac{1}{2!}d^2G^A(0)(m,l)(h,k) &= \int_{S^1_L} hk(-\tfrac12 m'l' \\ &\quad + A(2(ml'' + m''l)\kappa^2 + ml\kappa^4 + \tfrac12 m'l'\kappa^2 + (ml' + m'l)\kappa\kappa' + m''l''))\,d\theta. \qquad (3)\end{aligned}$$

Thus we have

$$\begin{aligned}&-d^2G^A(0)(m,h)(h,m) + \tfrac12 d^2G^A(0)(m,m)(h,h) + \tfrac12 d^2G^A(0)(h,h)(m,m) \\ &\quad = -2\int_{S^1_L} hm(-\tfrac12 m'h' \\ &\qquad +A(2(mh'' + m''h)\kappa^2 + mh\kappa^4 + \tfrac12 m'h'\kappa^2 + (mh' + m'h)\kappa\kappa' + m''h''))\,d\theta \\ &\qquad + \int_{S^1_L} hh(-\tfrac12 m'^2 + A(4mm''\kappa^2 + m^2\kappa^4 + \tfrac12 m'^2\kappa^2 + 2mm'\kappa\kappa' + m''^2))\,d\theta \\ &\qquad + \int_{S^1_L} mm(-\tfrac12 h'h' + A(4hh''\kappa^2 + hh\kappa^4 + \tfrac12 h'h'\kappa^2 + 2hh'\kappa\kappa' + h''h''))\,d\theta \\ &\quad = \int_{S^1_L} (\tfrac12(A\kappa^2 - 1)(mh' - m'h)^2 + A(mh'' - m''h)^2)\,d\theta.\end{aligned}$$

For the second part of the curvature we have

$$
\begin{aligned}
&-G_0(\Gamma_0(h,m),\Gamma_0(m,h)) + G_0(\Gamma_0(m,m),\Gamma_0(h,h))\\
&\qquad = \int_{S^1_L} -((\tfrac12\kappa - \tfrac12 A\kappa^3 + A\kappa'')hm + 2A\kappa'(h'm+m'h) + 2A\kappa h'm')^2 \,\frac{d\theta}{1+A\kappa^2}\\
&\qquad\quad + \int_{S^1_L} ((\tfrac12\kappa - \tfrac12 A\kappa^3 + A\kappa'')m^2 + 4A\kappa' mm' + 2A\kappa m'^2)\\
&\qquad\quad \cdot ((\tfrac12\kappa - \tfrac12 A\kappa^3 + A\kappa'')h^2 + 4A\kappa' hh' + 2A\kappa h'^2)\,\frac{d\theta}{1+A\kappa^2}\\
&\qquad = \int_{S^1_L} ((A\kappa^2 - A^2\kappa^4 + 2A^2\kappa\kappa'' - 4A^2\kappa'^2)(mh'-m'h)^2)\,\frac{d\theta}{1+A\kappa^2}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
R_0(m,h,m,h) &= G_0^A(R_0(m,h)m,h)\\
&= \int_{S^1_L} (\tfrac12(A\kappa^2-1)(mh'-m'h)^2 + A(mh''-m''h)^2)\,d\theta\\
&\quad + \int_{S^1_L} ((A\kappa^2 - A^2\kappa^4 + 2A^2\kappa\kappa'' - 4A^2\kappa'^2)(mh'-m'h)^2)\,\frac{d\theta}{1+A\kappa^2}.
\end{aligned}
$$

Letting $W = mh' - hm'$ be the Wronskian of m and h and simplifying, we have

$$
\boxed{
\begin{aligned}
&R_0(m,h,m,h)\\
&\quad = \int_{S^1_L}\left(\frac{-(A\kappa^2-1)^2 + 4A^2\kappa\kappa'' - 8A^2\kappa'^2}{2(1+A\kappa^2)}\right) W^2\,d\theta + \int_{S^1_L} AW'^2\,d\theta.
\end{aligned}
} \tag{4}
$$

What does this formula say? First of all, if $\operatorname{supp}(m)\cap\operatorname{supp}(h)=\emptyset$, the sectional curvature in the plane spanned by m and h is 0. Secondly, we can divide the curve c into two parts:

$$
\begin{aligned}
c_A^+ &= \text{ set of points where } \kappa\kappa'' < 2(\kappa')^2 + \left(\frac{A^{-1}-\kappa^2}{2}\right)^2,\\
c_A^- &= \text{ set of points where } \kappa\kappa'' > 2(\kappa')^2 + \left(\frac{A^{-1}-\kappa^2}{2}\right)^2.
\end{aligned}
$$

Note that if A is sufficiently small, $c_A^- = \emptyset$ and even if A is large, c_A^- need not be non-empty. But if $\operatorname{supp}(m), \operatorname{supp}(h) \subset c_A^-$, the sectional curvature is always negative. The interesting case is when $\operatorname{supp}(m), \operatorname{supp}(h) \subset c_A^+$. We may introduce the self-adjoint differential operator on $L^2(S^1)$:

$$
Sf = f'' + \frac{(A\kappa^2-1)^2 - 4A^2\kappa\kappa'' + 8A^2\kappa'^2}{2A(1+A\kappa^2)} f
$$

so that $R = -A\langle SW, W\rangle$. The eigenvalues of S tend to $-\infty$, hence S has a finite number of positive eigenvalues. If we take, for example, $m = 1$ and h such that h' is in the span of the positive eigenvalues, the corresponding sectional curvature will be positive. In general, the condition that the sectional curvature be positive is that the Wronskian W have a sufficiently large component in the positive eigenspace of S. The special case where c is the unit circle may clarify the picture: then

$$Sf = f'' + \frac{(A-1)^2}{2A(1+A)} f$$

and the eigenfunctions are linear combinations of sines and cosines. It is easy to see that for any A, a plane spanned by m and h of pure frequencies k and l will have positive curvature if and only if k and l are sufficiently near each other (asymptotically $|k-l| < |A-1|/\sqrt{A+a^2}$), hence 'beat' at a low frequency.

4.7. The sectional curvature for the induced H^0-metric on $B_{i,f}(S^1_L, \mathbb{R}^2)$ in a chart. In the setting of 4.2 we have, for $f \in C^\infty(S^1_L, (-\varepsilon, \varepsilon))$ and $h, k \in C^\infty(S^1_L, \mathbb{R})$,

$$\begin{aligned} G^0_f(h,k) &= ((\pi \circ \psi)^* G^0)_f(h,k) = G^0_{\pi(\psi(f))}(T_f(\pi\circ\psi)h, T_f(\pi\circ\psi)k) \\ &= G^0_{\psi(f)}((T_f\psi \cdot h)^\perp, (T_f\psi\cdot k)^\perp) = \int_{S^1_L} \frac{hk(1-f\kappa_c)^2}{\sqrt{(1-f\kappa_c)^2 + f'^2}}\, d\theta. \end{aligned} \tag{1}$$

At the center of the chart described in 4.4, i.e., for $f = 0$, the Christoffel symbol 4.5.2 for $A = 0$ becomes

$$\Gamma_0(h,k) = \tfrac{1}{2}\kappa_c hk. \tag{2}$$

The curvature 4.6.4 at $f = 0$ for $A = 0$ becomes

$$\begin{aligned} R_0(m,h,m,h) &= G_0(R_0(m,h)m,h) \\ &= -\frac{1}{2}\int_{S^1_L} (h'm - hm')^2\, d\theta = -\frac{1}{2}\int_{S^1_L} W(m,h)^2\, d\theta \end{aligned} \tag{3}$$

and the sectional curvature $k_0(P(m,h))$ from 4.6.2 for $A = 0$ and $f = 0$ is non-negative.

In the full chart 4.2, starting from the metric 4.6.1, we managed to compute the full geodesic equation not just for $f = 0$ but for general f, so long as $A = 0$. The outcome is

$$\begin{aligned} \Gamma_f(h,h) = {} & \frac{\kappa_c h^2}{1 - f\kappa_c} + \frac{-\frac{1}{2}\kappa_c(1-f\kappa_c)h^2 + (\frac{1}{2}h^2 f'' + 2hh'f')}{((1-f\kappa_c)^2 + f'^2)} \\ & - \frac{\kappa_c h^2 f'^2}{(1-f\kappa_c)((1-f\kappa_c)^2 + f'^2)} + \frac{\frac{3}{2}\kappa_c(1-f\kappa_c)h^2 f'^2 - \frac{3}{2}h^2 f'^2 f''}{((1-f\kappa_c)^2 + f'^2)^2}. \end{aligned} \tag{4}$$

The geodesic equation is thus

$$\begin{aligned} f_{tt} = {} & -\frac{\kappa_c f_t^2}{1 - f\kappa_c} - \frac{-\frac{1}{2}\kappa_c(1-f\kappa_c)f_t^2 + (\frac{1}{2}f_t^2 f_{\theta\theta} + 2f_t f_{t\theta} f_\theta)}{((1-f\kappa_c)^2 + f_\theta^2)} \\ & + \frac{\kappa_c f_t^2 f_t h^2}{(1-f\kappa_c)((1-f\kappa_c)^2 + f_\theta^2)} - \frac{\frac{3}{2}\kappa_c(1-f\kappa_c)f_t^2 f_\theta^2 - \frac{3}{2}f_t^2 f_\theta^2 f_{\theta\theta}}{((1-f\kappa_c)^2 + f_\theta^2)^2}. \end{aligned} \tag{5}$$

For $A > 0$ we were unable to get the analogous result.

5. Examples and numerical results

5.1. The geodesics running through concentric circles. The simplest possible geodesic in B_i is given by the set of all circles with common center. Let C_r be the circle of radius r with center the origin. Consider the path of such circles $C_{r(t)}$ given by the parametrization $c(t,\theta) = r(t)e^{i\theta}$, where $r(t)$ is a smooth increasing function $r : [0,1] \to \mathbb{R}_{>0}$. Then $\kappa_c(t,\theta) = 1/r(t)$. If we vary r then the horizontal energy and the variation of this curve are

$$E^{\text{hor}}_{G^A}(c) = \frac{1}{2}\int_0^1 \int_{S^1} (1 + A/r^2) r_t^2 r \, d\theta \, dt,$$

$$\partial_s|_{s=0} E^{\text{hor}}_{G^A}(c) = \int_0^1 \int_{S^1} \left(1 + \frac{A}{r^2}\right) r_s \left(-r_{tt} - \frac{(1 - A/r^2)}{2(r + A/r)} r_t^2\right) r \, d\theta \, dt,$$

so that c is a geodesic if and only if

$$r_{tt} + \frac{(1 - A/r^2)}{2(r + A/r)} r_t^2 = 0. \tag{1}$$

Also the geodesic equation 4.1.1 reduces to (1) for c of this form.

The solution of (1) can be written in terms of the inverse of a complete elliptic integral of the second kind. More important is to look at what happens for small and large r. As $r \to 0$, the ODE reduces to

$$r_{tt} - \frac{r_t^2}{2r} = 0,$$

whose general solution is $r(t) = C(t - t_0)^2$ for some contants C, t_0. In other words, at one end, the path ends in finite time with the circles imploding at their common center. Note that $r' \to 0$ as $r \to 0$ but not fast enough to prevent the collapse. On the other hand, as $r \to \infty$, the ODE becomes

$$r_{tt} + \frac{r_t^2}{2r} = 0,$$

whose general solution is $r(t) = C(t - t_0)^{2/3}$ for some constants C, t_0. Thus at the other end of the geodesic, the circles expand forever but with decreasing speed.

An interesting point is that this geodesic has conjugate points on it, so that it is an extremal path but not a local minimum for length over all intervals. This is a concrete reflection of the collapse of the metric when $A = 0$. To work this out, take any $f(\theta)$ such that $\int_0^{2\pi} f \, d\theta = 0$ and any function $a(t)$. Then $X = f(\theta)a(t)\partial/\partial r$ is a vector field along the geodesic, i.e. a family of tangent vectors to B_e at each circle $C_{r(t)}$ normal to the tangent to the geodesic. Its length is easily seen to be

$$\|X\|^2_{C_{r(t)}} = \left(r(t) + \frac{A}{r(t)}\right) a(t)^2 \int_0^{2\pi} f(\theta)^2 \, d\theta.$$

We need to work out its covariant derivative:

$$\nabla_{d/dt}(X) = f(\theta) a_t \frac{\partial}{\partial r} + \Gamma_{C_r}\left(r_t \frac{\partial}{\partial r}, f(\theta) a \frac{\partial}{\partial r}\right).$$

Using a formula for the Christoffel symbols which we get from 4.2.2 by polarizing, and noting that $\kappa \equiv 1/r$, $\kappa_s \equiv 0$, we get

$$\begin{aligned}\nabla_{d/dt}(X) &= f(\theta)a_t\frac{\partial}{\partial r} + f(\theta)ar_t\left(\frac{1-A/r^2}{2(r+A/r)}\right)\frac{\partial}{\partial r}\\ &= f(\theta)(r+A/r)^{-1/2}((r+A/r)^{1/2}a)_t\frac{\partial}{\partial r}.\end{aligned}$$

(This formula also follows from the fact that the vectors $(r+A/r)^{-1/2}\partial/\partial r$ have length independent of t, hence covariant derivative zero.) Jacobi's equation is therefore

$$f(\theta)(r+A/r)^{-1/2}((r+A/r)^{1/2}a)_{tt}\frac{\partial}{\partial r} + R\left(X, r_t\frac{\partial}{\partial r}\right)\left(r_t\frac{\partial}{\partial r}\right) = 0, \tag{2}$$

where R is the curvature tensor. For later purposes, it is convenient to write this equation using r as the independent variable along the geodesic rather than t and think of a as a function of r. Note that for any function b along the geodesic, $b_t = b_r r_t$ and

$$b_{tt} = b_{rr}r_t^2 + b_r r_{tt} = \left(b_{rr} - \frac{(1-A/r^2)}{2(r+A/r)}b_r\right)r_t^2.$$

Then a somewhat lengthy bit of algebra shows that:

$$(r+A/r)^{-1/2}((r+A/r)^{1/2}a)_{tt} = (r+A/r)^{-1/4}((r+A/r)^{1/4}a)_{rr}r_t^2 + F(r)ar_t^2,$$

where

$$F(r) = -\frac{5}{16}\left(\frac{1-A/r^2}{r+A/r}\right)^2 + \frac{A}{2r^3(r+A/r)}.$$

To work out the structure of R in this case, use the fact that the circles C_r and the vector field $\partial/\partial r$ are invariant under rotations. This means that the map $f \mapsto R(\partial/\partial r, f\partial/\partial r)(\partial/\partial r)$ has the two properties: it commutes with rotations and it is symmetric. The only such maps are diagonal in the Fourier basis, i.e. there are real constants λ_n such that

$$R\left(\partial/\partial r, \begin{Bmatrix}\cos(n\theta)\partial/\partial r\\ \sin(n\theta)\partial/\partial r\end{Bmatrix}\right)(\partial/\partial r) = \lambda_n\begin{Bmatrix}\cos(n\theta)\partial/\partial r\\ \sin(n\theta)\partial/\partial r\end{Bmatrix}.$$

To evaluate λ_n, we take the inner product with $\cos(n\theta)$ (or $\sin(n\theta)$) and use our calculation of $R_0(m,h,m,h)$ in Section 4.6 to show

$$\begin{aligned}\left\langle R\left(\frac{\partial}{\partial r}, \cos(n\theta)\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial r}\right), \cos(n\theta)\frac{\partial}{\partial r}\right\rangle &= R_0\left(\frac{\partial}{\partial r}, \cos(n\theta)\frac{\partial}{\partial r}, \frac{\partial}{\partial r}, \cos(n\theta)\frac{\partial}{\partial r}\right)\\ &= \int_0^{2\pi}\left(-\frac{(1-A/r^2)^2}{2(1+A/r^2)}W^2 + AW'^2\right)r\,d\theta\end{aligned}$$

where

$$W = 1\cdot\frac{d}{ds}\cos(n\theta) = -n\frac{\sin(n\theta)}{r} \quad \text{and} \quad W' = \frac{d}{ds}W = -n^2\frac{\cos(n\theta)}{r^2}.$$

Simplifying gives

$$\lambda_n \left\| \cos(n\theta)\frac{\partial}{\partial r}\right\|^2 = \int_0^{2\pi}\left(-\frac{(1-A/r^2)^2}{2(r+A/r)}n^2\sin^2(n\theta)+\frac{A}{r^3}n^4\cos^2(n\theta)\right)d\theta$$
$$= -\frac{(1-A/r^2)^2}{2(r+A/r)}n^2\pi+\frac{A}{r^3}n^4\pi,$$

hence

$$\lambda_n = -\frac{(1-A/r^2)^2}{2(r+A/r)^2}n^2+\frac{A}{r^3(r+A/r)}n^4.$$

Thus for $X=\cos(n\theta)a_n(t)\partial/\partial r$, if we combine everything, Jacobi's equation reads

$$(r+A/r)^{-1/4}((r+A/r)^{1/4}a_n)_{rr}$$
$$= \left(-\frac{(1-A/r^2)^2}{2(r+A/r)^2}(n^2-\tfrac{5}{8})+\frac{A}{r^3(r+A/r)}(n^4-\tfrac{1}{2})\right)a_n. \qquad (3)$$

Calling the right hand side the *potential* of Jacobi's equation, we can check that for each n, the potential is positive for small r, negative for large r and it has one zero, approximately at $\sqrt{2An}$ for large n. Thus, for small r, these perturbations diverge from

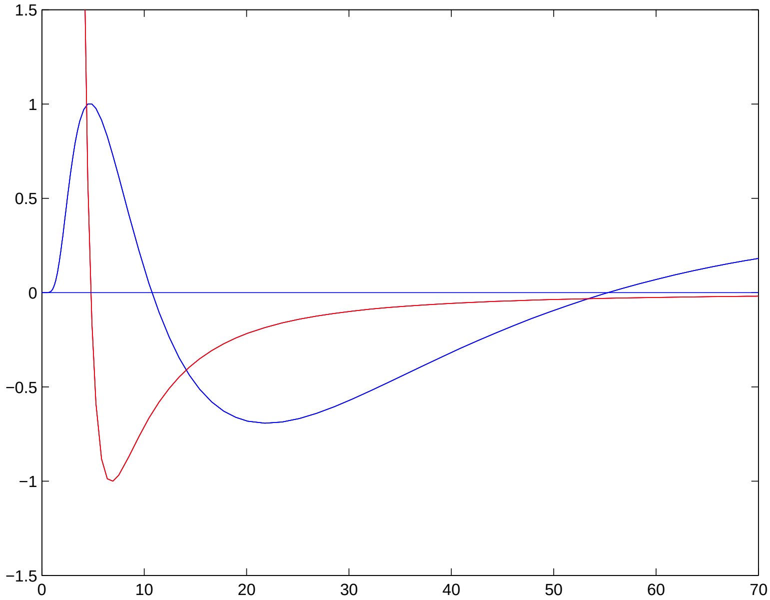

Fig. 4. The potential in the Jacobi ODE and its solution for an infinitesimal triangular perturbation of the circles in the geodesic of concentric circles. Note the first conjugate point at $10.77\sqrt{A}$.

the geodesic of circles. For large r, if we write $b_n = (r + A/r)^{1/4} a_n$, then Jacobi's equation approaches

$$(b_n)_{rr} \approx -\frac{n^2 - 0.625}{2r^2} b_n.$$

This is solved by $b_n = cx^\lambda + c' x^{\lambda'}$ where λ, λ' are solutions of $\lambda^2 - \lambda = -(n^2 - 0.625)/2$. For $n = 1$, λ, λ' are real and b_n has no zeros; but for $n > 1$, λ, λ' have an imaginary part, say $i\gamma_n$, and

$$b_n \approx \sqrt{r}(c \cos(\gamma_n \log(r)) + c' \sin(\gamma \log(r)))$$

with infinitely many zeros.

Figure 4 shows the solution for $n = 3$ which approaches 0 as $r \to 0$. The first zero of this solution is about $10.77\sqrt{A}$, making it a conjugate point of $r = 0$. For other n, the first such conjugate point appears to be bigger, so we conclude: on any segment $0 < r_1 < r_2 < 10.77\sqrt{A}$, the geodesic of circles is locally (and presumably globally) minimizing.

5.2. The geodesic connecting two distant curves. For any two distant curves C_1, C_2, one can construct paths from one to the other by (a) changing C_1 to some auxiliary curve D near C_1, (b) translating D without modifying it to a point near C_2 and (c) changing the translated curve D to C_2. If C_1 and C_2 are very far from each other, the energy of the translation will dominate the energy required to modify them both to D. Thus we expect that a geodesic between distant curves will asymptotically utilize a curve D which is optimized for least energy translation. To find such curves D, heuristically we may argue that it should be a curve such that the path given by all its translates in a fixed direction is a geodesic.

Such geodesics can be found as special cases of the general geodesic. We fix $e = (1, 0)$ as the direction of translation and assume that the path $\{D + te\}$ is a geodesic. We need to express this geodesic up to order $O(t^2)$ in the chart used in Section 4.4. Let $c(s)$ be the arc length parametrization of D and $\theta(s)$ be the orientation of D at point $c(s)$, i.e. $c_s = \cos(\theta) + i \sin(\theta)$. Then a little calculation shows that if we reparametrize nearby curves via $\tilde{s} = s - \langle e, c_s \rangle t$, then the path of translates in direction e is just

$$\begin{aligned} c(\tilde{s}) + te &= c(s) + \left(t\langle e, ic_s\rangle + \frac{t^2}{2}\langle e, c_s\rangle^2 \kappa + O(t^3) \right) ic_s \\ &= c(s) + \left(-\sin(\theta(s))t + \frac{t^2}{2}\cos^2(\theta(s)\kappa) + O(t^3) \right) ic_s. \end{aligned}$$

Thus, in the notation of 4.2, $a = -\sin(\theta)$, hence $a_s = -\cos(\theta)\kappa$ and, moreover, $a_t = \cos^2(\theta)\kappa$. Substituting this in the geodesic formula 4.2.1, we get

$$\begin{aligned} &(1 + A\kappa^2)\cos^2(\theta)\kappa \\ &\quad = \frac{\kappa \sin^2(\theta)}{2} + A((\kappa_{ss} - \kappa^3/2)\sin^2(\theta) + 4\cos(\theta)\sin(\theta)\kappa\kappa_s + 2\kappa^3\cos^2(\theta)). \end{aligned}$$

Since $\kappa = \theta_s$, this becomes, after some manipulation, a singular third order equation for $\theta(s)$:

$$\theta_{sss} = 4\cot(\theta)\theta_s\theta_{ss} + (\tfrac{1}{2} - \cot^2(\theta))\theta_s(\theta_s^2 - 1/A).$$

One solution of this equation is $\theta(s) \equiv 1/\sqrt{A}$, i.e. a circle of radius $\sqrt{A}$. In fact, this seems to be the only simple closed curve which solves this equation. However, if we drop smoothness, a weak solution of this equation is given by the C^1, piecewise C^2 curve made up of two semi-circles of radius $\sqrt{A}$ joined by two straight line segments parallel to the vector e and separated by the distance $2\sqrt{A}$ (as in Figure 5). Note that such 'cigar'-shaped curves can be made with line segments of any length.

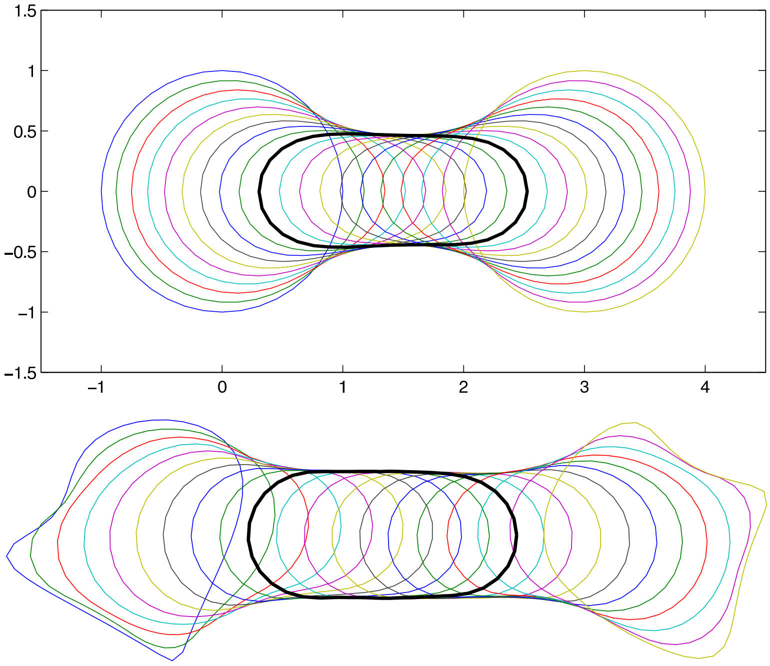

Fig. 5. Top: the geodesic joining circles of radius 1 at distance 3 apart with $A = .1$ (using 20 time samples and a 40-gon for the circle). Bottom: the geodesic joining 2 'random' shapes of size about 1 at distance 5 apart with $A = .25$ (using 20 time samples and a 48-gon approximation for all curves). In both cases the middle curve which is highlighted.

A numerical approach to minimize $E^{\text{hor}}_{G^1}(c)$ for variations c with initial and end curves circles at a certain distance produced the two such geodesics shown in Figure 5. Note that the middle curve is indeed close to such a 'cigar'-shape. However, the width of this shape is somewhat greater than $2\sqrt{A}$: this is presumably because the end-curves of this path are not sufficiently far apart. Thus experiments as well as the theory suggest strongly that geodesics joining any two curves sufficiently far apart compared to their size asymptotically approach a constant ‘cigar’-shaped C^1 intermediate curve made up of 2 semi-circles of radius $\sqrt{A}$ and 2 parallel line segments. We conjecture that this is true.

5.3. The growth of a ‘bump’ on a straight line when $A = 0$. We have seen above that the geodesic spray is locally well defined when $A = 0$. To understand this spray and see whether it appears to have global solutions, we take that the initial curve contains a segment with curvature identically zero, i.e. contains a line segment, and that the initial velocity a is set to a smooth function with compact support contained in this segment. For simplicity, we take the velocity a to be a cubic B-spline, i.e. a piecewise cubic which is C^2 with five non-C^3 knots approximating a Gaussian blip. The result of integrating is shown in Figure 6. Note several things: first, where the curvature is zero, the curve moves with constant velocity if we follow the orthogonal trajectories. Secondly, where the curve is moving opposite to its curvature (like an expanding circle, the part in the middle), it is decelerating; but where it is moving with its curvature (like a contracting circle, the parts at the two ends), it is accelerating. This acceleration at the two ends creates higher and higher curvature until a corner forms. In the figure, the simulation is stopped just before the curvature explodes. In the middle, the curve appears to be getting more and more circular. As the corners form, the curve is approaching the boundary of our space. Perhaps, with the right entropy condition, one can prolong the solution past the corners with a suitable piecewise C^1 curve.

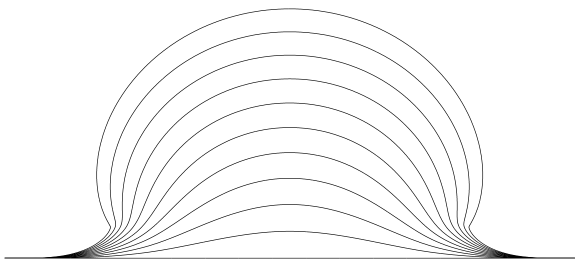

Fig. 6. The forward integration of the geodesic equation when $A = 0$, starting from a straight line in the direction given by a smooth bump-like vector field. Note that two corner-like singularities with curvature going to ∞ are about to form.

Although this calculation assumes $A = 0$, one will find very similar geodesics when A is much smaller than $1/\kappa^2$, $1/(\kappa_s \log(a)_s)$ and κ/κ_{ss}, so that the dominant terms in the geodesic equation are those without an A. In other words, geodesics between large smooth curves are basically the same as those with $A = 0$.

5.4. Several geodesic triangles in B_e. We have examined dilations, translations and the evolution of blips. We look next at rotations. To get a pure rotational situation, we consider ellipses centered at $(0, 0)$ with the same eccentricity 3 and maximum radius 1, but differently oriented. We take three such ellipses, with orientations differing by 60° and 120° degrees. Joining each pair by a geodesic, we get a triangle in B_e.

We wanted to examine whether along the geodesic joining two such ellipses (a) one ellipse rotates into the other or (b) the initial ellipse shrinks towards a circle, while the final ellipse grows, independently of one another. It turns out that, depending on the value of A, both can happen. Note that we get similar geodesics by either changing A or making the ellipses smaller or larger with A held fixed. For each A, we get an absolute distance scale with unit $1/\sqrt{A}$ and, if the ellipses are bigger than this, (b) dominates, while, if smaller, (a) dominates.

The results are shown in Figure 7. We have taken the three values $A = 1, 0.1$ and 0.01. For each value, on the top, we show the geodesic joining two of the ellipses as a sequence of curves in their common ambient $\mathbb{R}^2$. Below this, we show the triple of geodesics as a triangle, by displaying the intermediate curves as small shapes along lines joining the ellipses. This Euclidean triangle is being used purely for display, to indicate that the computed structure is a triangle in B_e. Note that for $A = 1$, the intermediate shapes are very close to ellipses, whose axes are rotating; while for $A = 0.01$, the bulges in one ellipse shrink while those of the other grow.

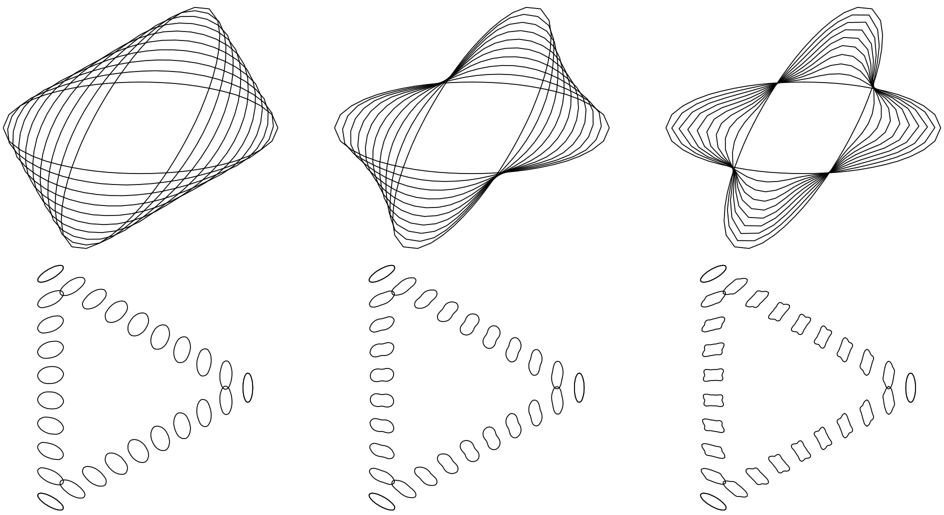

Fig. 7. Top row: Geodesics in three metrics joining the same two ellipses. The ellipses have eccentricity 3, the same center and are at 60° degree angles to each other. At left, $A = 1$; in middle $A = 0.1$; on right $A = 0.01$. Bottom row: Geodesic triangles in B_e formed by joining three ellipses at angles of 0, 60 and 120 degrees, for the same three values of A. Here the intermediate shapes are just rotated versions of the geodesic in the top row but are laid out on a plane triangle for visualization purposes.

We can also compute the angles in B_e between the sides of this triangle. They turn out to be 34° when $A = 1$, i.e. the angle sum for the triangle is 102°, much less than π radians, showing strong negative sectional curvature in the plane containing this triangle. But if $A = 0.1$ or 0.01, the angle is 77° and 69° respectively, giving more than π radians in the triangle. Thus the sectional curvature is positive for such small values of A.

5.5. Notes on the numerical simulations. All simulations in this paper were carried out in MatLab. The forward integration for the geodesic equation for $A = 0$ was carried out by the simplest possible finite difference scheme. This seems very stable and reliable. Solving for the geodesics was done using the MatLab minimization routine `fminunc` using both its medium and large scale modes. This, however, was quite unstable due to discretization artifacts. A general path between two curves was represented by a matrix of points in $\mathbb{R}^2$, approximating each curve by a polygon and sampling the path discretely. The difficulty is that when the polygons have very acute angles, the discretization tends to be highly inaccurate because of the high curvature localized at one vertex. Initially, in order to minimize the number of variables in the problem, we tried to use small numbers of samples and higher order accurate discrete approximations to the derivatives. In all these attempts, the discrete approximation 'cheated' by finding minima to the energy of the path with polygons with very small angles. The only way we got around this was to use first order accurate expressions for the derivatives and relatively large numbers of samples (e.g. 48 points on each curve, 20 samples along the geodesic, hence $2 \times 20 \times 48 = 1920$ variables in the expression for the energy).

Another problem is that the energy only depends on the path of unparametrized curves and is independent of the parametrization. To solve this, we added a term to the energy which is minimized by constant speed parametrizations. This still leaves a possibly wandering basepoint, and we added ϵ times another term which asked that all points on each curve should move as normally as possible. In practice, if the initialization was reasonable, this term was not needed. The final discrete energy that was minimized was this. Let $x_{i,j}$ be the i^{th} sample point on the j^{th} curve C_j. For each (i, j), estimate the sum of the squared curvature of C_j plus the squared acceleration of the parametrization by

$$k(i,j) = \frac{1}{2}\left(\frac{1}{\|x_{i-1,j} - x_{i,j}\|^4} + \frac{1}{\|x_{i,j} - x_{i+1,j}\|^4}\right) \cdot \|x_{i-1,j} - 2x_{i,j} + x_{i+1,j}\|^2.$$

(The harmonic mean of the segment lengths is used here to further force the parametrization to be uniform.) Then, for each (i, j), the *four* triangles $t = \{a = (i, j),\ b = (i \pm 1, j),\ c = (i, j \pm 1)\}$ around (i, j) are considered and the energy is taken to be

$$\sum_{i,j,t}\left(\frac{\langle (x_a - x_b), (x_a - x_c)^{\perp}\rangle^2 + \epsilon\langle (x_a - x_b), (x_a - x_c)\rangle^2}{\|x_a - x_b\|}\right)(1 + Ak(a)).$$

We make no guarantees about the accuracy of this simulation! The results, however, seem to be stable and reasonable.

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