Asymptotic Behavior of Area-Minimizing Currents in Hyperbolic Space

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Introduction

Our study of the asymptotic behavior of area-minimizing locally rectifiable currents in hyperbolic space consists of two parts. In this first part, we shall concern ourselves with the aspect of geometric measure theory of the problem. The p.d.e. aspect of the problem will be treated in the second part.

In the upper half-space model, $(n + k)$-dimensional hyperbolic space is given as the set

$$\mathbb{H}^{n+k} = \{(x, y) \in \mathbb{R}^{n+k-1} \times \mathbb{R}_+\}, \qquad k \geqq 1, \quad n \geqq 2,$$

equipped with the hyperbolic metric $y^{-2}(dx^2 + dy^2)$. A standard compactification of $\mathbb{H}^{n+k}$ involves adding the boundary $(\mathbb{R}^{n+k-1} \times \{0\}) \cup (*)$ so that $\overline{\mathbb{H}}^{n+k}$ is simply the one-point compactification of the Euclidian closed half-space $\mathbb{R}^{n+k-1} \times \{0, \infty)$. Suppose $0 \leqq \alpha \leqq 1$, and Γ is a compact $(n - 1)$-dimensional $C^{1,\alpha}$ smooth submanifold of $\mathbb{R}^{n+k-1} \times \{0\}$. In [4], M. Anderson proved that there exists an area-minimizing locally rectifiable n-dimensional current T, complete without boundary, in $\mathbb{H}^{n+k}$ asymptotic to Γ at infinity (see also [5]). By the interior regularity theory due to F. Almgren [2], the support M of any such hyperbolic-area-minimizing T is a relatively closed subset of $\mathbb{H}^{n+k}$ which is a real analytic submanifold away from a relatively closed singular set of Hausdorff dimension at most $n - 2$. Anderson's construction gives M for which, in the ordinary Euclidian metric, $\overline{M} \sim M = \Gamma$. The question of the behavior of M near Γ was raised in [4] and, for the hypersurface case (i.e., $k = 1$), was first studied in [11]. In that paper one proved the "boundary regularity at infinity" result that, for any such hyperbolic-area-minimizing T, the union $M \cup \Gamma$, in the ordinary Euclidian metric, is, near Γ, a finite union of $C^{1,\alpha}$ hyper-surfaces with boundary Γ; these have disjoint analytic interiors and meet $\mathbb{R}^n \times \{0\}$ orthogonally at Γ. (For the particular T constructed by Anderson only one hypersurface will occur.) It follows that, for $n \leqq 6$, $M \cup \Gamma$ has finite topological type, and for $n \geqq 7$, any interior singularities of M must remain in a bounded region of hyperbolic space $\mathbb{H}^{n+1}$. Near points of Γ (in the Euclidian topology), $M \cup \Gamma$ may thus be locally described as the graph of a function. This function is the solution of an elliptic partial differential equation that becomes degenerate along the part of the boundary corresponding to Γ. The author had recently established in [13] a

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boundary higher-regularity result of this equation, which implies, in particular, that $M \cup \Gamma$ is, near Γ, a $C^{l,\alpha}$ smooth hypersurface with boundary Γ provided that Γ is a $C^{l,\alpha}$ submanifold for $0 < \alpha < 1$ and for $l = 2, 3, \cdots, \infty$.

The asymptotic behavior of an n-dimensional area minimizing locally rectifiable current M in $\mathbb{H}^{n+k}$ at its asymptotic boundary Γ has been unknown since the arguments in [11] cannot be generalized to the higher codimensions. The main difficulties involved are the following:

(i) the interior mass bound for certain indecomposable, n-dimensional area-minimizing integral currents in $\mathbb{R}^{n+k}$, $k \geqq 2$ (see the open problem 1 in Section 3 below);

(ii) the local regularity theory for area-minimizing submanifolds of $\mathbb{R}^{n+k}$ (see the open problem 2 in Section 3 for details).

These are crucial in [11].

The purpose of this paper is to show the following

THEOREM. *Let Γ be a compact $(n-1)$-dimensional $C^{1,\alpha}$. $0 \leqq \alpha \leqq 1$, smooth submanifold of $\mathbb{R}^{n+k-1} \times \{0\}$. Then there exists a complete, area-minimizing locally rectifiable n-dimensional current T in $\mathbb{H}^{n+k}$ asymptotic to Γ at infinity. Moreover, the set* $\operatorname{spt}(T) \cup \Gamma$, *in the ordinary Euclidian metric, is, near Γ, a $C^{1,\alpha}$ submanifold with boundary Γ which meets $\mathbb{R}^{n+k-1} \times \{0\}$ orthogonally at Γ.*

In the second part [14], we shall show, in particular, that $\operatorname{spt}(T) \cup \Gamma$ is smooth near Γ if Γ is smooth.

Unlike the case in [11], where one has the "boundary regularity at infinity" for such hyperbolic-area-minimizing hypersurfaces which have smooth asymptotic boundary (which may even have higher multiplicity), we are in fact only able to show here that the solutions we constructed are smooth near the asymptotic boundary. Γ may in general bound more than one area-minimizing locally rectifiable current in $\mathbb{H}^{n+k}$. In fact, even when Γ is a smooth Jordan curve in $\mathbb{R}^2 \times \{0\}$, Γ may still bound at least two distinct area-minimizing surfaces in $\mathbb{H}^3$ (see e.g. [5]). The study of stationary currents (even in the Euclidian space) is much more difficult than that of minimal hypersurfaces. The nonexistence, nonuniqueness and irregularity of solutions to the Dirichlet problem for the minimal surface system were shown by B. Lawson and R. Osserman [15]. On the other hand, W. Allard [1] proved a general boundary regularity theorem, which implies in particular that stationary varifolds near smooth, extreme boundaries are smooth submanifolds with boundary. Due to the convexity of asymptotic boundaries in hyperbolic space $\mathbb{H}^{n+k}$, our regularity theorem was in some sense expected.

The paper is organized in the following way: In Section 1, after some preliminary discussions, we show that the proof of the $C^{1,\alpha}$ regularity theorem of [11] can be modified to obtain the same $C^{1,\alpha}$ estimate for any area-minimizing flat chain modulo 2 in $\mathbb{H}^{n+k}$ with $C^{1,\alpha}$-asymptotic boundary. This uses, however, a universal interior mass bound for area minimizing flat chains modulo 2 obtained recently by F. Morgan [16]. In Section 2, we use area-minimizing flat chains modulo 2 as comparison surfaces to show that the constructed solutions satisfy a uniform *mass ratio bound* up to the asymptotic boundary. In proving this *mass ratio bound*, we also need to solve a free boundary problem. Finally, in Section 3, we list a few open problems.

For convenience, we adopt the following convention: All unspecified statements about metric and topology will refer to the usual Euclidian metric on $\mathbb{R}^{n+k-1} \times \mathbb{R}$, and the word "hyperbolic" will be stated explicitly when appropriate.

1. Boundary Estimate for Area-Minimizing Modulo 2

Let p be an integer, $p \geqq 2$. The definition of various quantities associated with a *flat chain modulo p* can be found in [6], 4.2.26. By the compactness theorem of Federer-Fleming (see [6], 4.2.26) and by the monotonicity formula for the volume growth of stationary varifolds (see [1], [4]), the proof of the existence theorem in [4] can be easily generalized to obtain

THEOREM 1.1 (Existence). *Let Γ be an $(n-1)$-dimensional $C^{1,\alpha}$ submanifold of $\mathbb{R}^{n+k-1} \times \{0\}$, $0 \leqq \alpha \leqq 1$, $k \geqq 1$. Then there exists a complete hyperbolic area-minimizing flat chain modulo p, $p \geqq 2$, T in $\mathbb{H}^{n+k}$ without interior boundary and asymptotic to Γ at infinity.*

In this paper only flat chains modulo 2 will be used. For $p = 2$, by the interior regularity theory of geometric measure theory, the $\operatorname{spt}^2(T)$ of any such hyperbolic-area-minimizing flat chain modulo 2, T, is a relatively closed subset of $\mathbb{H}^{n+k}$ which is a real analytic submanifold away from a relatively closed singular set of Hausdorff dimension at most $n-2$ (see [7]). Here we shall show the "boundary regularity at infinity".

THEOREM 1.2 (Regularity). *For any such hyperbolic area-minimizing flat chain modulo 2, T, the set $\operatorname{spt}^2(T) \cup \Gamma$, in the ordinary Euclidian metric, is, near Γ, a $C^{1,\alpha}$ submanifold with boundary Γ, and meets $\mathbb{R}^{n+k-1} \times \{0\}$ orthogonally at Γ.*

The proof of this theorem is based on the following lemmas.

LEMMA 1.3 (Mass bound). *For $1 \leqq n, k < \infty$, there is a positive constant $C_1(n,k)$ such that if T is an n-dimensional hyperbolic area-minimizing flat chain modulo p in the open ball $B_1^{n+k}(0,2)$ of $\mathbb{H}^{n+k}$ with $\partial T \llcorner B_1^{n+k}(0,2) \equiv 0 \pmod p$, then*

$$M^p\left(T \llcorner B_{1/2}^{n+k}(0,2)\right) \leqq p C_1(n,k).$$

Proof: A hyperbolic area may be described by the parametric integrand (see [6], 5.1.1) $\Phi((x, y), \xi) = |y|^{-n}|\xi|$. On the ball $B_1^{n+k}(0,2) = \{(x, y) \in \mathbb{R}^{n+k-1} \times R: |x|^2 + |y-2|^2 < 1\}$, this integrand is (with respect to the Euclidian metric) elliptic (see [6], 5.1.2) with ellipticity bound 3^{-n}. Since T is a hyperbolic area-minimizing flat chain modulo p, it has locally finite mass, and, for all $0 < r < 1$, we may define

$$T_r = T \llcorner B_r^{n+k}(0,2), \qquad f(r) = M^p(T_r). \tag{1.1}$$

Since $f(r)$ is increasing, it is differentiable for almost all r. Slicing theory (see [6], 4.2.26) with $u(x, y) = (|x|^2 + |y-2|^2)^{1/2}$ implies that, for almost all $0 < r < 1$,

$$\begin{aligned} f'(r) &\geqq M^p\langle T_r, u, r^+\rangle \\ &= M^p\big(\partial T_r - (\partial T) \llcorner B_r^{n+k}(0,2)\big) \\ &= M^p(\partial T_r); \end{aligned} \tag{1.2}$$

the last equality holds since $\partial T \llcorner B_1^{n+k}(0,2) \equiv 0 \pmod p$. Moreover,

$$f(r)^{k+1} = M^p(T_r)^{k+1} \leqq M^p(S_r)^{k+1} 3^{(k+1)n}; \tag{1.3}$$

here S_r is an area-minimizing flat chain modulo p with $\partial S_r \equiv \partial T_r \pmod p$. Thus

$$\begin{aligned} f(r)^{k+1} &\leqq 3^{(k+1)n} M^p(S_r)^{k+1} \\ &\leqq c_0 3^{(k+1)n} p M^p(\partial T_r)^k \quad \text{for some} \quad c_0 = c_0(n, k), \end{aligned} \tag{1.4}$$

by the isoperimetric inequality of [16], 2.5.

Combining (1.2) and (1.4) we see that

$$f'(r) f(r)^{-(k+1)/k} \geqq p^{-1/k}\big(c_0 3^{(k+1)n}\big)^{-1/k} \quad \text{for a.e.} \quad 0 < r < 1, \tag{1.5}$$

and hence, integrating from $\frac{1}{2}$ to 1 we obtain

$$f(\tfrac{1}{2})^{-1/k} - f(1)^{-1/k} \geqq \tfrac{1}{2} k p^{-1/k}\big(c_0 3^{(k+1)n}\big)^{-1/k}. \tag{1.6}$$

In particular,

$$M^p\big(T \llcorner B_{1/2}^{n+k}(0,2)\big) = f(\tfrac{1}{2}) \leqq p c_1(n, k).$$

We shall sometimes identify $\mathbb{R}^{n+k-1}$ with $\mathbb{R}^{n-1} \times \mathbb{R}^k$ and use the projection $P\colon \mathbb{R}^{n-1} \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^{n-1} \times \mathbb{R}$,

$$P((w,z),y) = (w,y). \tag{1.7}$$

LEMMA 1.4 (Interior regularity). *For any positive number ε, there exists a positive δ such that if S is an n-dimensional hyperbolic area-minimizing flat chain modulo 2 in $\mathbb{H}^{n+k}$ with*

$$\operatorname{spt}^2(S) \subset B_2^{n-1}(0) \times B_\delta^k(0) \times \left[\tfrac{1}{2},4\right],$$

$$\operatorname{spt}^2(\partial S) \subset P^{-1}\partial\left(B_2^{n-1}(0) \times \left[\tfrac{1}{2},4\right]\right),$$

and

$$P_\# S = \left(E^n \llcorner B_2^{n-1}(0) \times \left[\tfrac{1}{2},4\right]\right)^{(2)},$$

then $S \llcorner P^{-1}(B_1^{n-1}(0) \times [1,2])$ is a graph of a real analytic function $z = u(w,y)$ with $\|u\|_{C^{1,1}} \leqq \varepsilon$.

Proof: The parametric integrand $\Phi((x,y),\xi) = |y|^{-n}|\xi|$ on the region $\mathbb{R}^{n+k-1} \times [\frac{1}{2},4]$ is elliptic with ellipticity bound 4^{-n}. Since S is Φ-minimizing modulo 2, one obtains, from Lemma 1.3, that

$$M^2\left(S \llcorner P^{-1}\left(B_{3/2}^{n-1}(0) \times \left[\tfrac{2}{3},3\right]\right)\right) \leqq c_2(n,k). \tag{1.8}$$

Hence $S \llcorner P^{-1}(B_{3/2}^{n-1}(0) \times [\frac{2}{3},3])$ is also a representative modulo 2 (see [6], pp. 430–431), that is, S has density at most 1 almost everywhere, so that

$$\begin{gathered} M\left(S \llcorner P^{-1}\left(B_{3/2}^{n-1}(0) \times \left[\tfrac{2}{3},3\right]\right)\right) = M^2\left(S \llcorner P^{-1}\left(B_{3/2}^{n-1}(0) \times \left[\tfrac{2}{3},3\right]\right)\right), \\ \operatorname{Spt}(S) = \operatorname{Spt}^2(S). \end{gathered} \tag{1.9}$$

By the upper semi-continuity of the density of a varifold with bounded first variation, we also conclude that S has density greater than or equal to 1 everywhere.

Now we can apply the standard squashing deformation (see [12], 3.2)

$$F((w,z),y) = ((w,\mu(w,y)z),y), \tag{1.10}$$

here $\mu\colon \mathbb{R}^n \to [0,1]$ is a C^1-function, and such that

$$\mu(w,y) = \begin{cases} 0 & \text{for} \quad (w,y) \in B_{5/4}^{n-1}(0) \times \left[\frac{5}{4},\frac{9}{4}\right], \\ 1 & \text{for} \quad (w,y) \notin B_{3/2}^{n-1}(0) \times \left[\frac{2}{3},3\right]; \end{cases} \tag{1.11}$$

and that

$$|D\mu| \leqq 9,$$

to obtain

$$\begin{aligned}
&M\left(S \llcorner P^{-1}\left(B_{5/4}^{n-1}(0) \times [\tfrac{5}{4}, \tfrac{9}{4}]\right)\right) - M\left(B_{5/4}^{n-1}(0) \times [\tfrac{5}{4}, \tfrac{9}{4}]\right) \\
&\quad = M\left(S \llcorner P^{-1}\left(B_{3/2}^{n-1}(0) \times [\tfrac{3}{2}, 3]\right)\right) - M\left(S \llcorner P^{-1}(A)\right) \\
&\qquad - M\left(F_{\#}\left(S \llcorner P^{-1}\left(B_{5/4}^{n-1}(0) \times [\tfrac{5}{4}, \tfrac{9}{4}]\right)\right)\right) \\
(1.12) \qquad &\quad \leqq M\left(F_{\#}\left(S \llcorner P^{-1}\left(B_{3/2}^{n-1}(0) \times [\tfrac{3}{2}, 3]\right)\right)\right) - M\left(S \llcorner P^{-1}(A)\right) \\
&\qquad - M\left(F_{\#}\left(S \llcorner P^{-1}\left(B_{5/4}^{n-1}(0) \times [\tfrac{5}{4}, \tfrac{9}{4}]\right)\right)\right) \\
&\quad = M\left(F_{\#}\left(S \llcorner P^{-1}(A)\right)\right) - M\left(S \llcorner P^{-1}(A)\right),
\end{aligned}$$

where $A = B_{3/2}^{n-1}(0) \times [\frac{3}{2}, 3] - B_{5/4}^{n-1}(0) \times [\frac{5}{4}, \frac{9}{4}]$. In view of the computations in [12], 3.2, and (1.8), we have

$$(1.13) \qquad M\left(F_{\#}\left(S \llcorner P^{-1}(A)\right)\right) - M\left(S \llcorner P^{-1}(A)\right) \leqq c_3(n, k)\delta.$$

Combining (1.12) and (1.13) with [18], we conclude that there is a positive number $\delta = \delta(\varepsilon, n, k)$ which satisfies the assertion of Lemma 1.4.

Remark 1.5. Let $M = \operatorname{spt}^2 T$. As in [11], Section 1, one has the following facts about the location of M near Γ:

(i) if $x \in \mathbb{R}^{n+k-1}$ and $0 < r < d(x) = \operatorname{dist}(x, \Gamma)$, then $M \cap B_r(x, 0) = \varnothing$, where $B_r(x, 0) = \{(w, y) \in \mathbb{R}^{n+k-1} \times \mathbb{R}, |w - x|^2 + y^2 < 1\}$;

(ii) there is a positive number ρ_Γ (depending only on the C^1-character of Γ) such that $M \cap \{y < \rho_\Gamma\}$ is contained in the set

$$W = \left[\mathbb{R}^{n+k-1} \times (0, \rho_\Gamma)\right] \sim \bigcup_{d(x) > 2\rho_\Gamma} B_{2\rho_\Gamma}(x, 0) \sim \bigcup_{0 < d(x) \leqq 2\rho_\Gamma} B_{d(x)}(x, 0);$$

(iii) at each edge point $(a, 0) \in \Gamma$, the tangent cone of the containing set W equals the vertical half n-plane $\tan(\Gamma, a) \times (0, \infty)$.

Let N_k be the normal bundle of Γ in $\mathbb{R}^{n+k-1}$ and, for $(a, 0) \in \Gamma$ and $r > 0$, let

$$\delta(a, r) = \min_{v \in N_k(a)} \{d(a + rv), |v| = 1\}.$$

Then one has that

(i) $r^{-1}\delta(a, r) \to 1$ as $r \to 0$ if Γ is differentiable at $(a, 0)$;
(ii) $\sup_{a\in\Gamma}\{1 - r\delta(a, r)\} \to 0$ as $r \to 0$ if Γ is C^1;
(iii) $\sup_{a\in\Gamma}[1 - r^{-1}\delta(a, r)] \leqq c_\Gamma r^\alpha$ if Γ is $C^{1,\alpha}$ for $0 < \alpha < 1$;
(iv) $r^{-1}\delta(a, r) \equiv 1$ for $0 < r < 1/\|\text{max. principle curve}\|_{L^\infty(\Gamma)}$ if Γ is $C^{1,1}$.

We also note that if $(x, y) \in W$, and $(a, 0) \in \Gamma$ with $|x - a| = d(x)$, then $x = a + d(x)v(a)$ for some $v(a) \in N_k(a)$, $|v(a)| = 1$. Let $r = d(x) + y^2/d(x)$; we see that the three points $(a, 0)$, (x, y) and $(a + rv(a), 0)$ are vertices of a right triangle. Let δ be the distance between the latter two vertices; then

$$\frac{d(x)}{y} = \frac{[r^2 - \delta^2]^{1/2}}{\delta} = \left[\left(\frac{r}{\delta}\right)^2 - 1\right]^{1/2},$$

since $r \leqq 2d(x)$, $\delta(a, r) \leqq d(a + rv(a)) \leqq \delta \leqq r$.
We finally conclude that $d(x)/y \to 0$ as $(x, y) \to \Gamma \times \{0\}$ if Γ is C^1.

1.6. Proof of Theorem 1.2: We first want to show:

(∗) If Γ is C^1, then there exists a positive $\rho < \rho_\Gamma$ so that $(M \cup \Gamma) \cap \{y < \rho\}$ is a C^1-submanifold, and that $M \cup \Gamma$ meets $\mathbb{R}^{n+k-1} \times \{0\}$ orthogonally at Γ.

To see this, we choose an arbitrary point $(a, 0) \in \Gamma$. For convenience, we may assume that $a = 0$ and that $\tan(\Gamma, 0) = \mathbb{R}^{n-1}$. For each sufficiently small positive $\rho < \rho_\Gamma$, one may choose a map Q that projects $\Gamma \cap [B^{n-1}_{2\rho}(0) \times B^k_{\delta\rho}(0) \times \{0\}]$ C^1-diffeomorphically onto $B^{n-1}_{2\rho}(0) \times \{0\}$. It then follows from Remark 1.5 that $M \cap [B^{n-1}_{2\rho}(0) \times \partial B^k_{\delta\rho}(0) \times (0, 4\rho)] = \varnothing$ for some sufficiently small positive δ. Also one knows that

$$Q_\#\Big(T \llcorner \big(B^{n-1}_{2\rho}(0) \times B^k_{\delta\rho}(0) \times (0, 4\rho)\big)\Big) = \big(E^n \llcorner B^{n-1}_{2\rho}(0) \times (0, 4\rho)\big)^{(2)}.$$

Next we scale T by a factor $1/\rho$ about $(0, 0)$. Since the homothety map μ_ρ that sends (x, y) to $(1/\rho)(x, y)$ induces a hyperbolic isometry of $\mathbb{H}^{n+k}$, the flat chain

$$S = \mu_{\rho\#}\big[T \llcorner B^{n-1}_{2\rho}(0) \times B^k_{\delta\rho}(0) \times (0, 4\rho)\big] \llcorner P^{-1}\big[B^{n-1}_{2\rho}(0) \times \left(\tfrac{1}{2}\rho, 4\rho\right)\big]$$

is a hyperbolic area-minimizing flat chain modulo 2. Moreover, one verifies that S satisfies the hypothesis of Lemma 1.4, from which we may conclude that $(M \cup \Gamma) \cap C_\rho$ is a C^1-graph over a subregion of the vertical cylinder $\Gamma \times \mathbb{R}$, where $C_\rho = \{(x, y) \in \mathbb{R}^{n+k-1} \times \mathbb{R}_+,\ |x| \leqq 2y \leqq \rho\}$. Since the point $(0, 0) \in \Gamma$ is chosen arbitrarily, by the compactness of Γ we obtain the statement (∗).

Let $0 < \alpha \leqq 1$; we wish to show now that $(M \cup \Gamma) \cap \{y < \rho\}$ is, in fact, a $C^{1,\alpha}$ submanifold.

If Γ is C^1, then as in the first part of our proof we may view $(M \cup \Gamma) \cap \{y < \rho\}$ locally, near $(0,0) \in \Gamma$, as lying in the graph of a vector-valued function $z = u(w, y)$ that is C^1 on a region $B_\rho^{n-1}(0) \times (0, \rho)$.

For any compact $K \subset B_\rho^{n-1}(0) \times (0, \rho)$, the hyperbolic area of the graph $U|_K$ is

$$\int_K (\det(g_{ij}))^{1/2} y^{-n}\, dw\, dy,$$

where $(t_1, \cdots, t_{n-1}, t_n) = (w_1, \cdots, w_{n-1}, y)$, $U_i = \partial U/\partial t_i$, and

$$g_{ij} = \delta_{ij} + U_i \cdot U_j \quad \text{for} \quad i \leqq j,\ j \leqq n.$$

The minimality of T leads to the Euler-Lagrange equation

$$(**) \qquad g^{-1/2} \frac{\partial}{\partial t_i}\left[g^{ij} g^{1/2} U_j\right] - \frac{n}{y} g^{in} U_i = 0.$$

Here $g = \det(g_{ij})$, (g^{ij}) is the inverse matrix of (g_{ij}), and we have used the summation convention for repeated indices. Thus $U = u(w, y)$ is a C^1-solution of $(**)$ on the region $B_\rho^{n-1}(0) \times (0, \rho)$. By Remark 1.5, one obtains, as in [11], Section 3, that $|U(w, y)| \leqq c_\Gamma(|w| + y)^{\alpha+1}$ for $(w, y) \in B_\rho^{n-1}(0) \times [0, \rho)$, $0 < \alpha \leqq 1$. The $C^{1,\alpha}$ estimate follows from arguments in [11], Section 3 and the interior estimates of [8], Chapter VI.

2. Boundary Estimate for Hyperbolic Area-Minimizing Currents

Let Γ be a compact $(n-1)$-dimensional $C^{1,\alpha}$, $0 \leqq \alpha \leqq 1$, submanifold of $\mathbb{R}^{n+k-1} \times \{0\}$, and let S be hyperbolic area-minimizing modulo 2 with asymptotic boundary Γ. By Theorem 1.2, $\operatorname{spt}^2(S) \cup \Gamma$ is, near Γ, a $C^{1,\alpha}$ submanifold with boundary Γ, and meets $\mathbb{R}^{n+k-1} \times \{0\}$ orthogonally at Γ. From the orientation of Γ one thus obtains an orientation for $\operatorname{spt}^2(S \llcorner \{y < 2\rho_\Gamma\})$, for some positive number ρ_Γ (depending only on the C^1-character of Γ). Using this orientation we obtain a multiplicity 1 rectifiable current $\tilde{S}$ with $\operatorname{spt}(\tilde{S}) = \operatorname{spt}^2(S \llcorner \{y < 2\rho_\Gamma\})$.

Now let $\{\varepsilon_i\}$ be a sequence of positive numbers which tend to zero, and let $\Gamma_i = \langle \tilde{S}, y, \varepsilon_i \rangle$, for $\varepsilon_i < 2\rho_\Gamma$. We consider a sequence of solutions T_i of the oriented Plateau problem in $\mathbb{H}^{n+k}$ with given boundaries Γ_i. As in [4] one can verify easily that $T_i \to T$ in $F^{\mathrm{loc}}(\mathbb{H}^{n+k})$ (by taking a subsequence if necessary). Moreover, T is a local hyperbolic area-minimizing rectifiable current in

$\mathbb{H}^{n+k}$ with asymptotic boundary Γ. We wish to show that $\operatorname{spt}(T)$ is, near Γ, a $C^{1,\alpha}$ submanifold with boundary Γ. For this purpose, we consider a point $(a,0) \in \Gamma$. For simplicity we may assume that $a = 0$, $\tan(\Gamma, 0) = \mathbb{R}^{n-1}$, and let P: $(\mathbb{R}^{n-1} \times \mathbb{R}^k) \times \mathbb{R} \to \mathbb{R}^{n-1} \times \mathbb{R}$, $P((w,z),y) = (w,y)$; then, by the above construction, by the convex hull property of stationary varifolds in $\mathbb{H}^{n+k}$ (see [2], [11]), and by the constancy theorem (see [6], 4.1.7), we have

$$(2.1)\qquad P_{\#}\big(T \llcorner B_\rho^{n-1}(0) \times (\rho,\rho] \times B_{\rho_\Gamma}^k(0)\big) = E^n \llcorner B_\rho^{n-1}(0) \times (0,\rho] \quad \text{for} \quad 0 < \rho < \rho_\Gamma.$$

Note also that $\operatorname{spt}(T) \cap \{y < \rho_\Gamma\}$ is contained in the set W defined in Remark 1.5. In particular, one sees that

$$(2.2)\qquad \operatorname{spt}(T) \cap B_\rho^{n-1}(0) \times (0,\rho) \times \partial B_\rho^k(0) = \varnothing,$$

and that

$$(2.3)\quad T \llcorner \big(B_\rho^{n-1}(0) \times (0,\rho) \times B_\rho^k(0)\big) = T \llcorner \big(B_\rho^{n-1}(0) \times (0,\rho) \times B_{\rho_\Gamma}^k(0)\big)$$

for $\rho < \rho_\Gamma$.

The following lemma is the key point in proving the theorem stated in the introduction.

LEMMA 2.1. *There is a positive number ρ_0 (depending only on the C^1-character of Γ) such that, if*

$$A_\delta = B_\delta^{n-1}(0) \times (\delta, 3\delta) \times B_\delta^k(0),$$

then

$$(2.4)\qquad M(T \llcorner A_\delta) \leq C(n,k,\Gamma)\delta^n \quad \textit{for} \quad 0 < \delta < \rho_0,$$

and for some constant C(n, k, Γ).

Proof: The proof is divided into three steps.

Step 1. To show (2.4) it suffices to show, for all sufficiently small ε_i, that $M(T_i \llcorner A_\delta) \leq c(n,k,\Gamma)\delta^n$ (see [6], 5.4.2). For this purpose, we let

$$(2.5)\qquad R_i = \tilde{S} \llcorner \{\varepsilon_i < y < \rho_\Gamma\} + \tilde{T}_i,$$

where $\tilde{T}_i$ is a solution of the oriented Plateau problem in $\mathbb{H}^{n+k}$ with

$$\partial \tilde{T}_i = -\Gamma_i + \partial\left(\tilde{S} \llcorner \{\varepsilon_i < y < \rho_\Gamma\}\right)$$

$$= \langle \tilde{S}, y, \rho_\Gamma \rangle .$$

Since $M_H(\partial \tilde{T}_i) \leqq C_1(n, k, \Gamma)$, one can use cone comparison to obtain $M_H(\tilde{T}_i) \leqq c_2(n, k, \Gamma)$. Then

$$\begin{aligned} M_H(S \llcorner \{y > \varepsilon_i\}) &\leqq M_H(T_i) \leqq M_H(R_i) \\ &\leqq M_H\left(\tilde{S} \llcorner (\varepsilon_i < y < \rho_\Gamma\right) + c_2(n, k, \Gamma) \\ &\leqq M_H\left(S \llcorner (y > \varepsilon_i)\right) + c_2(n, k, \Gamma) \end{aligned} \tag{2.6}$$

(note that R_i is an integral current), and

$$0 \leqq M_H(T_i) - M_H(S \llcorner \{y > \varepsilon_i\}) \leqq c_2(n, k, \Gamma), \tag{2.7}$$

for all sufficiently small ε_i.

Step 2. Since the homothety map μ_δ which sends $(x, y) \in \mathbb{R}^{n+k-1} \times \mathbb{R}_+$ to $\delta^{-1}(x, y)$ is an isometry in $\mathbb{H}^{n+k}$, one obtains, from (2.7),

$$0 \leqq M_H(S_i) - M_H(S_i^*) \leqq c_2(n, k, \Gamma), \tag{2.8}$$

where $S_i = \mu_{\delta\#} T_i$, $S_i^* = \mu_{\delta\#}(S \llcorner \{y > \varepsilon_i\})$.

Next note that

$$M(S_i^* \llcorner A_1) \leqq c_3(n, k, \Gamma). \tag{2.9}$$

In fact (for $\varepsilon_i < \delta$), $S_i^* \llcorner A_1$ can be represented by a graph of a smooth function $z = u_i(w, y)$ on the region $B_1^{n-1}(0) \times (1, 3)$, with $\|u_i\|_{C^{1,1}} \leqq c_4(\alpha, n, k, \Gamma)\delta^\alpha$ provided that $\delta < \frac{1}{3}\rho_\Gamma$ and that Γ is $C^{1,\alpha}$ (here $0 \leqq \alpha \leqq 1$, $\alpha = 0$ means that $c_4(\alpha, n, k, \Gamma)$ can be arbitrarily small for sufficiently small δ).

Note also that inequality (2.4) will follow, by scaling, from $M_H(S_i \llcorner A_1) \leqq c_4(n, k, \Gamma)$ because

$$M(S_i \llcorner A_1) \leqq 3^n M_H(S_i \llcorner A_1). \tag{2.10}$$

Combining (2.8), (2.9) and (2.10) we see that it now suffices to show that

$$M_H(S_i^* \llcorner A_1^c) - M_H(S_i \llcorner A_1^c) \leqq c_5(n, k, \Gamma). \tag{2.11}$$

To do so, we consider a solution Q_i to the following partial free-boundary hyperbolic area-minimizing modulo 2 problem:

$$\text{(2.12)}\quad \begin{aligned} &\text{Minimize } M_H(Q), \text{ for all } Q \text{ satisfying}\\ &\text{(i) } Q \text{ is a flat chain modulo 2 in } \mathbb{H}^{n+k},\\ &\text{(ii) } \partial Q \equiv \mu_{\delta\#}\Gamma_i + \tilde{\Gamma}_i \pmod 2,\\ &\quad\ \text{where}\\ &\text{(iii) } \operatorname{spt}^2(\tilde{\Gamma}_i) \subset (\mu_{\delta\#}W) \cap \left[\partial\left(B_1^{n-1}(0) \times (1,3)\right) \times B_1^k(0)\right]\\ &\quad\ \text{and } P_\#(\tilde{\Gamma}_i) = \left(E^{n-1} \llcorner \partial\left(B_1^{n-1}(0) \times (1,3)\right)\right)^{(2)}. \end{aligned}$$

(W is the set in Remark 1.5.) The existence of a solution Q_i to (2.12) follows again from the standard direct method of the calculus of variations and the compactness theorem (see [6], 4.2.26).

Since $S_i \llcorner A_1^c$ and $S_i^* \llcorner A_1^c$ satisfy (i), (ii) and (iii), it is clear that

$$M_H(Q_i) \leqq \min\{M_H(S_i \llcorner A_1^c), M_H(S_i^* \llcorner A_1^c)\}.$$

Therefore, (2.11) will follow from

$$\text{(2.13)}\qquad M_H(S_i^* \llcorner A_1^c) - M_H(Q_i) \leqq c_6(n,k,\Gamma).$$

We *claim* that $Q_i \llcorner (B_2^{n-1}(0) \times [\frac{1}{4}, 4] \times B_2^k(0) - B_{3/2}^{n-1}(0) \times [\frac{1}{2}, 2] \times B_2^k(0))$ is a graph of a smooth function $z = \tilde{u}_i(w,y)$ on a subregion of $B_2^{n-1}(0) \times [\frac{1}{4}, 4]$ with $\|\tilde{u}_i\|_{C^{1,1}} \leqq c_7(\alpha, n, k, \Gamma)\delta^\alpha$. By Theorem 1.2, we may also assume that $S_i^* \llcorner B_2^{n-1}(0) \times [\frac{1}{4}, 4] \times B_2^k(0)$ is a graph of a smooth function $z = u_i(w,y)$ on $B_2^{n-1}(0) \times [\frac{1}{4}, 4]$ with $\|u_i\|_{C^{1,1}} \leqq c_4(\alpha, n, k, \Gamma)\delta^\alpha$. This implies, in particular, that both $Q_i \llcorner B^c$ and $S_i^* \llcorner B^c$ are hyperbolic area-minimizing flat chains modulo 2 with boundaries $\mu_{\delta\#}\Gamma_i + \tilde{\Gamma}_i$ and $\mu_{\delta\#}\Gamma_i + \Gamma_i^*$, respectively. Here $B = B_{7/4}^{n-1}(0) \times [\frac{3}{8}, \frac{15}{4}] \times B_2^k(0)$ and $\tilde{\Gamma}_i, \Gamma_i^*$ are graphs of functions $\tilde{u}_i$, and u_i over the set $\partial(B_{7/4}^{n-1}(0) \times [\frac{3}{8}, \frac{15}{4}])$, respectively. Hence

$$\text{(2.14)}\qquad \begin{aligned} &M_H(S_i^* \llcorner A_1^c) - M_H(Q_i)\\ &\quad \leqq M_H(S_i^* \llcorner B^c) - M_H(Q_i \llcorner B^c) + c_7(n,k,\Gamma). \end{aligned}$$

Finally, we also have

$$\text{(2.15)}\qquad M_H(S_i^* \llcorner B^c) - M_H(Q_i \llcorner B^c) \leqq c_8(n,k,\Gamma)$$

which follows from a simple comparison (by adding a vertical piece resulting from the homotopy formula to the 2-graphs $\{(1-t)\tilde{u}_i(w,y) + tu_i(w,y),\ 0 \leqq t \leqq 1,\ (w,y) \in \partial(B_{7/4}^{n-1}(0) \times [\frac{3}{8}, \frac{15}{4}])\}$).

Step 3. We wish to show the *claim* in Step 2. Consider $Q_i \llcorner (B_3^{n-1}(0) \times [0,5] \times B_2^k(0))$. It is a hyperbolic area-minimizing flat chain modulo 2 such that

$$\operatorname{spt}^2\big(Q_i \llcorner \big(B_3^{n-1}(0) \times [0,5] \times B_2^k(0)\big)\big) \subset B_3^{n-1}(0) \times (0,5) \times B_\eta^k(0).$$

Here $\eta \leqq c_9(\alpha, n, k, \Gamma)\delta^\alpha$ if $0 < \alpha \leqq 1$, and η may be chosen arbitrarily small if $\alpha = 0$ and if δ is sufficiently small (which depends only on Γ). This, in fact, follows from Remark 1.5 about the containing set W and a rescaling.

One may apply Lemmas 1.3 and 1.4 to see that, for all sufficiently small ε_i, the *claim* is true.

Remark 2.2. The proof of the regularity theorem which is stated in the introduction is similar to that of Section 1 and [11]. One uses (2.1) and Lemma 2.1 instead of Lemmas 1.3 and 1.4.

3. Open Problems

We would like to conclude the paper by discussing a few open problems which arise very naturally along the lines of our study.

PROBLEM 1 (Interior mass bound). Let T be an indecomposable, n-dimensional area-minimizing integral current in $\mathbb{R}^{n+k}$, $k \geqq 1$, and let

$$D_r = B_r^n(0) \times \{0\} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k, |x| < r, y = 0\},$$

$\mathbb{C}_r = B_r^n(0) \times \mathbb{R}^k$, and $P = \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ which is defined by $P(x, y) = x$, for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k$. Suppose that

(i) $P_\# T = mE^n \llcorner D_1$ (m is the multiplicity of the projection, $m \geqq 1$),

(ii) $\partial T \llcorner C_1 = \varnothing$,

(iii) the Hausdorff distance $\rho(D_1, \operatorname{spt} T) = \delta \ll 1$.

Is there a positive number $\delta_0 = \delta_0(m, n, k)$ such that, for any such indecomposable area-minimizing current T which satisfies (i), (ii) and (iii) with $\delta \leqq \delta_0$, $M(T \llcorner C_{1/2}) \leqq C(m, n, k)$?

Even if one assumes that $m = 1$, and that T is a smooth graph over D_1, the answer to the above question is unknown. We should point out, indecomposable currents in higher codimension may be rather complicated. For example, one can easily construct an integral 2-dimensional indecomposable current T in $\mathbb{R}^4$ which satisfies (i), (ii) and (iii) (for any given positive integer m, and any $\delta > 0$), and such that $T \llcorner C_{1/2} = m\mathbb{E}^n \llcorner D_{1/2}$, and that $T \llcorner (\mathbb{C}_1 \sim \mathbb{C}_r)$ is a smooth embedded surface (which is also a graph over $D_1 \sim D_r$), for any $r > \frac{1}{2}$.

PROBLEM 2 (Local regularity). Let T be as in Problem 1. Suppose

$$M(T \llcorner \mathbb{C}_1) \leqq (m+1)M(D_1);$$

then, is it true that $T \llcorner \mathbf{C}_{1/2}$ can be represented as a graph of a $C^{1,\alpha}$ m-valued function, for sufficiently small δ? For $m = 1$ the answer is yes, and it is well known (see for example [6], 5.3) when $m \geqq 2$, Almgren gave several counterexamples in [2].

PROBLEM 3 (Boundary regularity). Let T be an n-dimensional, complete, area-minimizing locally rectifiable current in $\mathbb{H}^{n+k}$, $k \geq 1$, with smooth, embedded submanifold Γ as its asymptotic boundary. Suppose, in addition, that T is a normal current in the ordinary Euclidian topology with $\partial T = \Gamma$. Is $\operatorname{spt}(T) \cup \Gamma$ smooth near Γ in the Euclidean metric?

It seems most likely that the answer will be yes. In general, the additional hypothesis in the above question may be replaced by the following *multiplicity* 1 condition.

Let $C(\Gamma)$ be the convex hull of Γ in $\mathbb{H}^{n+k}$, and, near Γ, $P = C(\Gamma) \to \Gamma \times \mathbb{R}$, the horizontal nearest point projection. One defines $\partial T = m\Gamma$ if $P_{\#}T = m\Gamma \times \mathbb{R}$ (in a neighborhood of Γ). The *multiplicity* 1 condition means simply that $m = 1$.

PROBLEM 4 (Higher multiplicity boundary). Let T be an n-dimensional, complete, area-minimizing locally rectifiable current in $\mathbb{H}^{n+k}$, $k \geqq 2$, with $\partial T = m\Gamma$. Here Γ is a smooth submanifold of $\mathbb{R}^{n+k-1} \times \{0\}$. Is $\operatorname{spt} T \cup \Gamma$, near Γ, a union of at most m distinct smooth submanifolds with boundary Γ in the Euclidian metric?

If $n = 2$, and if the total Euclidian mass of T is bounded, then one can easily verify that the interior *mass ratio* of T is uniformly bounded up to Γ provided that Γ is $C^{1,1}$. In general, one can construct an area-minimizing n-dimensional integral current T in $\mathbb{H}^{n+k}$ whose asymptotically smooth, embedded boundary is Γ with multiplicity m (i.e., $\partial T = m\Gamma$) via Anderson's construction. Moreover, such a construction will imply automatically that the Euclidian mass of T is bounded.

A related question for the hypersurface has been studied by B. White (see [19]). The following is, however, unknown:

PROBLEM 5. Let Γ be an $(n-1)$-dimensional smooth, embedded (extreme) submanifold of $\mathbb{R}^{n+k}$, and let T be an area-minimizing integral current modulo $2p$. Suppose that $\partial T = p\Gamma \pmod{2p}$. Is $\operatorname{spt}(T)$, near Γ, a union of at most p distinct smooth submanifolds with boundary?

If the answer to the above is yes, then how about the solutions of the oriented Plateau problem with higher multiplicity boundary?

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