Efficient Shape Modeling: ⋮ -Entropy, Adaptive Coding, and Boundary Curves -vs- Blum's Medial Axis

Article *in* International Journal of Computer Vision · August 2007

DOI: 10.1007/s11263-006-0010-3 · Source: dx.doi.org

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Abstract

We propose efficiency of representation as a criterion for evaluating shape models, then apply this criterion to compare the boundary curve representation with the medial axis. We estimate the ε-entropy of two compact classes of curves. We then construct two adaptive encodings for non-compact classes of shapes, one using the boundary curve and the other using the medial axis, and determine precise conditions for when the medial axis is more efficient. Along the way we construct explicit near-optimal boundary-based approximations for compact classes of shapes, construct an explicit compression scheme for non-compact classes of shapes based on the medial axis, and derive some new results about the medial axis.

1 Introduction

One of the great lessons of computer vision research is that, while humans are quite expert at isolating and recognizing shapes, the notion of shape does not lend itself to precise quantification. As a result, there now exist many competing shape models supported by factions of researchers whose reasons for support range from the practical: compatibility with another phase of image analysis (e.g., [19, 28, 5, 27]), to near-religious fervor: "shape simply *is* [insert model of your choice]". In this work, we attempt to sidestep that minefield by evaluating shape models on purely intrinsic and quantitative data. The idea behind the work is that some shapes are naturally suited to being modeled as boundary curves while others are suited for regions, etc., in the sense that modeling a particular shape in one way will be simpler than any other. Borrowing from information theory, we take efficiency of representation as a quantitative measure of that simplicity.

1.1 Goals

The goals of this work are threefold. First, we wish to introduce the approach of efficiency of representation as a way to evaluate deterministic shape models. This is not to suggest that the efficiency criterion should take precedence over all other criteria, but rather that in the absence of other practical considerations, a quantitative basis for model selection is preferable to a religious one. Even more, we argue that understanding which shape model is more efficient for a particular shape or class of shapes gives important insights into the geometry and structure of that shape or class even if, for a particular task, another model is more convenient.

Second, we wish to address one of the current shape model debates: are boundary curves or skeletons better as shape models? Current views toward skeletal models such as the medial axis are fairly dismissive, but this paper intends to demonstrate that the skeleton has a defensible position as a shape model.

Finally, we are interested in theoretical properties of spaces of shapes. In the course of evaluating the efficiency of boundary and skeletal models, we construct efficient coverings of shape space. The nature and the size of these coverings contributes to the currently growing body of mathematical understanding of these highly non-linear, infinite-dimensional spaces.

The paper as a whole realizes the first goal, and three theorems realize the second and third. Theorem 33 gives a criterion in terms of medial data for when the medial axis is a more efficient shape model than the boundary curve, and we provide examples of shapes better modeled by each. Theorem 19 gives a tight estimate for the ε-entropy of certain compact classes of curves, and a near-tight estimate for boundary curves, while Theorem 20 gives an indication of what an optimal covering should look like in the non-compact case. Along the way, we derive some interesting properties of the medial axis and explicitly construct what are, to our knowledge, the first provably optimal coverings of a non-linear space.

1.2 Motivation

Shape seems to be garnering serious attention of late. One mathematically rigorous approach to shape theory, taken by Mumford & Michor [17], Miller & Younes [1], Srivastava & Mio [10], and others, is to place shape-related tasks on shape manifolds. A primary thrust of that work is the search for Riemannian metrics on the shape manifold that result in geodesics between shapes that are computationally tractable. Even better would be to define a probability measure on the manifold in order to do statistical inference there. These problems are very hard, and have therefore generated a general interest in the structure of shape spaces. This gives a theoretical motivation for our work.

A more practical motivation is the desire to construct a method for conclusively determining the intrinsic suitability of shape models for particular shapes. Most researchers would agree that shapes can be extracted from certain images only by region-growing techniques (e.g., the spotty dalmation image), and from others only by edge-based techniques (e.g., a sketchy line drawing). The distinction depends on properties of the image, such as relative grey levels or textures—not on properties of the shape. We wish to determine appropriate properties of shape that will allow us to make similar judgements. Are certain shapes suited to certain models, and can we take measurements to aid in model distinction? Here we analyze two models, the boundary curve and the medial axis, and we hope our work will inspire similar analysis of other models.

The boundary curve is an obvious selection for one of the models, but we justify our choice of the medial axis. Blum first introduced the medial axis in the context of mathematical morphology in the early Seventies [2]. Since then, the medial axis has been applied to many shape-related problems, such as recognition [30], animation [23] and medical imaging [29]. Giblin and Kimia have explored the medial axis extensively as a tool for shape reconstruction [8] and shape matching [20], where its ability to decompose an object into parts is quite useful. Kimia has also used it to impose a discrete structure on shape space [24], decomposing the space into cells based on medial topology and defining a shape similarity based on the cost of moves between cells. One of the most powerful aspects of the medial axis representation is the relationship that exists between the geometry of the boundary curve and the geometry of the medial pair. In Section 6, we introduce these and other known properties of the medial axis pair, then derive some new results. It is well-known that two arbitrarily close shapes can have tremendously different medial structures; this contributes to the divisive nature of the medial axis as a shape model. Researchers such as Zhu [30] and Pizer & Damon [4, 9] have modified the definition of the medial axis in their work to make the axis more robust, but as a consequence have sacrificed some of these wonderful geometric properties. We choose to work with the original medial axis construction, avoiding the instability issue by focusing on the reverse relationship: when two medial curves are arbitrarily close, their corresponding shapes must be, too. Such bounds are among the new results we present.

While this work emerges from the shape community, it owes a tremendous debt to information theory. Our entire philosophy, inspired by work of Rissanen [12, 18] and Kolmogorov [11], depends on the notion that, given an encoding of a shape, the encoding requiring the fewest bits to describe is the preferable one. When an object to be encoded belongs to a compact class, Kolmogorov's ε-entropy gives the optimal bit rate for a fixed-length encoding, where each element of the class requires the same number of bits. In a non-compact class, Rissanen's minimum description length (MDL) principle says that when trying to choose between two probability models for a class of objects, look at the codelength in a variable-length encoding, where the number of bits required depend on the particular element being encoded. The model resulting in a shorter expected value for the codelength is the better model. We transfer this principle to a deterministic setting by choosing the model with the shorter codelength on a shape-by-shape basis.

2 Metric Spaces of Curves

We will identify a shape with the boundary curve of its silhouette. In order to define the information theoretic quantities of interest for shape model comparison, we must work in a metric space of curves. It is desirable to have translation and rotation invariance in shape descriptors, and our spaces of curves will therefore assume a canonical location and orientation for each curve.

Notation.

(a) A plane curve will be denoted by γ, and its tangent angle function by θ_γ.

(b) $\mathcal{C}$ denotes the collection of curves $\gamma(s_\gamma) : [0, L_\gamma] \to \mathbb{R}^2$, for $L_\gamma \geq 0$, satisfying:

(i) $\gamma \in C^1$.

(ii) $\left|\frac{d\gamma}{ds_\gamma}\right| \equiv 1$.

(iii) $\gamma(0) = (0, 0)$, $\theta_\gamma(0) = 0$.

(c) $\mathcal{I} \subset \mathcal{C}$ denotes the collection of immersions of S^1 into $\mathbb{R}^2$ passing through the origin with horizontal tangent direction, i.e., the collection of curves $\gamma \in \mathcal{C}$ so that $\gamma(0) = \gamma(L_\gamma)$.

Abusing notation, we will drop the subscript γ from s_γ and will instead assume that the curve $\gamma(s)$ is arclength parameterized with the domain of s varying from curve to curve unless explicitly stated otherwise. Within these larger classes of curves lie relevant compact classes:

(a) $\mathcal{C}_K^L \subset \mathcal{C}$ is the collection of curves of length at most L, so that for $\gamma \in \mathcal{C}_K^L$, $|\theta_\gamma(s_1) - \theta_\gamma(s_2)| \leq K|s_1 - s_2|$ where s is arclength on γ.

(b) $\mathcal{I}_K^L \subset \mathcal{I}$ is the collection of curves of length at most L, so that for $\gamma \in \mathcal{I}_K^L$, $|\theta_\gamma(s_1) - \theta_\gamma(s_2)| \leq K|s_1 - s_2|$ where s is arclength on γ.

We define a C^1-type metric on $\mathcal{C}$. We include a term for orientation because there is some indication that it plays an important role in human shape perception [15]. The L^∞ framework is desirable because we will be constructing minimal ε-covers, and the box-like L^∞ balls stack efficiently. For $\gamma_1, \gamma_2 \in \mathcal{C}$ and $\lambda > 0$ a dimension-normalizing constant, define:

$$\rho(\gamma_1, \gamma_2) = \sup_{i=1,2} \left\{ \sup_{\substack{s_j \\ j \neq i}} \inf_{s_i} \frac{1}{\lambda} |\gamma_i(s_i) - \gamma_j(s_j)| + |\theta_i(s_i) - \theta_j(s_j)| \right\}.$$

Note that as $\lambda \to \infty$, the distance between any two closed curves goes to zero. A suitable choice for λ for $\mathcal{C}_K^L$, for example, would be $\lambda < \frac{1}{K}$.

Given a sequence of curves $\{\gamma_i\}$, a straightforward argument on the sequence of pairs $\{(\theta_i, L_i)\}$ where L_i is the length of γ_i and θ_i is the tangent angle function gives the following result:

Proposition 1. *$\mathcal{C}_K^L$ and $\mathcal{I}_K^L$ are compact in the metric ρ.*

3 Fixed Codelengths: Compact Classes and ε-entropy

The ε-entropy of a totally bounded metric space, defined below, essentially counts the minimum number of ε-balls required to cover the space. There is a direct connection between ε-entropy and fixed length encoding. Suppose one wishes to ε-represent each element in a compact metric space with a fixed number of bits in the most efficient way possible. Given an ε-cover $\{U_i\}_1^n$ for the space, each element can be represented by the ε-ball to which it belongs. Enumerating these balls gives a binary representation for each element of the space, requiring $\lceil \log n \rceil$ bits per ball. If the covering is minimal, the number of bits will be given by the ε-entropy of the space.

In this section, we construct a minimal covering for $\mathcal{C}_K^L$ and a near-minimal covering for $\mathcal{I}_K^L$. In other words, the boundary curve attains optimal efficiency for fixed-length encodings of shapes. While we do not present the overly technical details here (see [14]), we have also constructed a medial-axis-based covering for a class of curves closely related to $\mathcal{I}_K^L$. The end result is that while the medial covering is of the correct order, it is less efficient than the boundary curve for fixed-length encodings. We present the argument for the optimality of the boundary curve below, saving discussion of the medial axis for when it becomes interesting in Section 5.

Before tackling the nonlinear spaces of curves, we introduce the concept of ε-entropy and give an example of an ε-entropy calculation for a linear function class. The example will play a key role in the estimation of ε-entropy for curves given in the next section, as we will derive entropy estimates for classes of curves by applying results for functions to classes of tangent angle functions to those curves. In this way, we exploit the structure of linear classes of functions to obtain the desired information about nonlinear classes of curves.

3.1 ε-entropy

Kolmogorov invented the notion of ε-entropy as a way of quantifying the massiveness of infinite-dimensional metric spaces by capturing the exponent of the number of balls in a minimal ε-covering of compact subsets of the space [11]. In the first section, we introduce ε-entropy and discuss some relevant properties. In the second section, we provide an important example of an ε-entropy estimate.

3.1.1 Definitions and preliminaries

Consider a subset $X \subset (M, \rho)$, where M is a metric space with metric ρ. A system of sets U_α, $\alpha \in A$, such that $X \subseteq \bigcup_{\alpha \in A} U_\alpha$ and the diameter d of U_α satisfies $d \leq 2\varepsilon$, is called an **ε-cover** for X. A set of points $\{x_\alpha\} \subset M$ is an **ε-net** for X when for any $x \in X$, there exists an x_α so that $\rho(x, x_\alpha) \leq \varepsilon$. Note that any ε-net gives rise to an ε-cover, but not necessarily conversely (when X is a *centered* space, the two are equivalent).

From a compression standpoint, the most desirable ε-cover is one with the fewest balls. This leads to the definition of ε-entropy. Let $\mathcal{N}_\varepsilon$ be the cardinality of a minimal ε-cover for a totally bounded set $X \subset (M, \rho)$. Then the **ε-entropy** of X is $\mathcal{H}_\varepsilon(X, \rho) = \log_2 \mathcal{N}_\varepsilon$.

A set $U \subset X$ is **ε-separated** when $\rho(x_1, x_2) \geq \varepsilon$ for any $x_1 \neq x_2 \in U$. For $\mathcal{M}_\varepsilon$ equal to the maximal number of elements in an ε-separated set $U \subset X$, the **ε-capacity** of X is $\mathcal{C}_\varepsilon(X, \rho) = \log_2 \mathcal{M}_\varepsilon$. The following theorem gives the relationship between these quantities and provides the foundation for the ε-entropy estimates.

Theorem 2. *[11] For every totally bounded set X contained in a metric space M, the following inequalities hold:*

1. $\mathcal{M}_{2\varepsilon}(X) \leq \mathcal{N}_\varepsilon(X) \leq \mathcal{M}_\varepsilon(X)$
2. $\mathcal{C}_{2\varepsilon}(X) \leq \mathcal{H}_\varepsilon(X) \leq \mathcal{C}_\varepsilon(X)$.

3.1.2 Example: Kolmogorov, L^∞, and Lipschitz Functions

The classical method of estimating ε-entropy is to construct an ε-cover with K_ε elements and a 2ε-separated set with $L_{2\varepsilon}$ elements so that $\lim_{\varepsilon\to 0} \frac{K_\varepsilon}{L_{2\varepsilon}} = 1$, or at worst is some non-zero constant. The following theorem of Kolmogorov and Tikhomirov applies this technique. Estimates for ε-entropy are not usually as clean as this example might indicate; the use of the L^∞ metric allows the ε-balls to stack very neatly. In fact, the result in Theorem 3 produces a rare example where $\mathcal{C}_{2\varepsilon}(X) = \mathcal{H}_\varepsilon(X)$ (up to ± 1). We provide an explanation of Kolmogorov's construction in the one-dimensional case where this miraculous equality holds, but consult [11] for rigorous proofs and full generality.

Theorem 3. *For $I = [a, b]$, define:*

$$\mathcal{F}_1^I(C) = \{f : I \to \mathbb{R} \mid f(a) = 0,\ |f(x) - f(x')| \leq C|x - x'|,\ \forall x, x' \in I\},$$

and $\rho_\infty(f, g) = \sup_{x\in I} |f(x) - g(x)|$. Then:

$$\mathcal{H}_\varepsilon(\mathcal{F}_1^I(C), \rho_\infty) = \begin{cases} \frac{|b-a|C}{\varepsilon} - 1 & \frac{|b-a|C}{\varepsilon} \in \mathbb{Z}^+ \\ \left[\frac{|b-a|C}{\varepsilon}\right] & else. \end{cases}$$

Proof.

(i) First, construct an efficient ε-covering for $\mathcal{F}_1^I(C)$ and count the number of balls. Divide the interval $[a, b]$ into $(n + 1) = (\left[\frac{|b-a|C}{\varepsilon}\right] + 1)$ subintervals $I_k = [a + (k - 1)\varepsilon, k\varepsilon]$, $k = 1, .., n$, and $I_{n+1} = [n\varepsilon, b]$.

Let $\phi_i(t)$ be a function so that $\phi_i(a) = 0$ and ϕ_i is linear with slope $\pm C$ on each subinterval I_k. See Figure 1. To each ϕ_i, associate an 2ε-corridor:

$$K(\phi_i) = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} -x \leq y \leq x & x \in I_1 \\ \phi_i(x) - 2\varepsilon \leq y \leq \phi_i(x) & \text{otherwise} \end{matrix} \right\}.$$

The diameter of each $K(\phi_i)$ is at most 2ε, and each distinct ϕ_i gives rise to a distinct corridor. Furthermore, one may see that every $f \in (F)_I^1$ belongs to such a corridor.

Figure 1: Epsilon corridors, I. [11]

A short calculation shows that the total number of $\{\phi_i\}$ is at most $2^n + 1$, giving an upper bound for $\mathcal{H}_\varepsilon(\mathcal{F}_1^I, \rho_\infty)$ of $\left(\left[\frac{|b-a|C}{\varepsilon}\right] + 1\right)$.

(ii) To find the lower bound, construct a 2ε-separated set similar to the $\{\phi_i\}$. Again taking $n = \left[\frac{|b-a|C}{\varepsilon}\right]$, divide I into n equal subintervals, J_k, where $|J_k| = \frac{|b-a|C}{n} \geq \varepsilon$. Now consider the set Ψ_n of functions ψ_i which are linear with slope $\pm C$ on each J_k and vanish at a. These functions are 2ε-separated, and there are 2^n of them, which gives the appropriate lower bound for $\mathcal{H}_\varepsilon(\mathcal{F}_1^I, \rho_\infty)$.

□

Corollary 4. *Define*

$$\mathcal{F}_1^I(B,C) = \{f : I = [a,b] \to \mathbb{R} \mid |f(x)| \leq B, \text{ and } |f(x) - f(x')| \leq C|x - x'|, \forall x, x' \in I\}.$$

Then:

$$H_\varepsilon(\mathcal{F}_1^I(B,C), \rho_\infty) = \left[\frac{|b-a|C}{\varepsilon}\right] + \left[\log\frac{B}{\varepsilon}\right] + O(1).$$

The corollary is proved by applying the constructions of functions ϕ_i and ψ_i as defined in the proof of Theorem 3 to each of the starting points $(-\varepsilon, 2k\varepsilon)$, for $k = -[\frac{B}{2\varepsilon}], \ldots, [\frac{B}{2\varepsilon}]$ to obtain functions $\phi_{i,k}$ and $\psi_{i,k}$. See Figure 2.

Figure 2: Epsilon corridors, II [11].

3.2 Adaptations of Kolmogorov's Theorem

In the next two theorems, we explore variations of Kolmogorov's Theorem 3. The first adaptation is a simple one where we consider functions whose starting point and ending point are the same, foreshadowing applications to closed curves. The second is more fundamental, modifying both the space and the metric. It provides the functional analogue for the estimates we obtain for curves.

Theorem 5. *For $I = [a,b]$, define:*

$$\tilde{\mathcal{F}}_1^I(C) = \{f : I \to \mathbb{R} \mid f(a) = f(b) = B \text{ and } |f(x) - f(x')| \leq C|x - x'|, \forall x, x' \in I\}.$$

Then:

$$\mathcal{H}_\varepsilon(\tilde{\mathcal{F}}_1^I(C), \rho_\infty) \sim \frac{|b-a|C}{\varepsilon}.$$

The proof of Theorem 3 requires only a slight modification: we require that the $\{\phi_i(t)\}$ satisfy $\phi(a) = \phi_i(b) = B$. There will be, in the above notation, $\binom{n}{n/2} \leq 2^n$ such functions when n is even; take $n+1$ if n is odd. On the other hand, to find a 2ε-separated set, we will find $\binom{m}{m/2}$ elements for $m = n-2$ or $m = n-3$, whichever is even, and $\frac{1}{m+1}2^m \leq \binom{m}{m/2}$ [3]. The corollary immediately follows:

Corollary 6. *Define*

$$\tilde{\mathcal{F}}_1^I(B,C) = \{f : I \to \mathbb{R} \mid |f(x)| \leq B, f(a) = f(b), |f(x) - f(x')| \leq C|x - x'|, \forall x, x' \in I\}.$$

Then:

$$H_\varepsilon(\tilde{\mathcal{F}}_1^I(B,C), \rho_\infty) \sim \frac{|b-a|C}{\varepsilon}.$$

Next, we adapt Theorem 3 by modifying the class of functions and the metric to include derivative information. This new function class mirrors the classes of curves $\mathcal{C}_K^L$ and $\mathcal{I}_K^L$. Define a class of functions whose first derivatives are Lipschitz:

$$\mathcal{G}^I = \{f : I \to \mathbb{R} \mid f(0) = f'(0) = 0, |f'(x) - f'(x')| \leq C|x - x'|, \forall x, x' \in I\}.$$

We introduce a C^1 metric on $\mathcal{G}^I$:

$$\rho_{C^1}(f,g) = \sup_{x \in I}\left(\frac{1}{\lambda}|f(x) - g(x)| + |f'(x) - g'(x)|\right),$$

for $\lambda > 0$ a dimension-normalizing constant. We will construct a covering in this metric space by covering the space of derivatives, then lifting that cover to the space of primitives, $\mathcal{G}^I$. The lifted cover is then refined to give an ε-cover of $\mathcal{G}^I$. As it turns out, this process produces the critical insight for our results on curves: the high order term of the ε-cover comes from the covering of the derivative space; refining the lifted cover requires only lower order terms. Hence the leading term in covering spaces of curves will come from covering the *linear* space of tangent angle functions.

Theorem 7. $\mathcal{H}_\varepsilon(\mathcal{G}^I, \rho_{C^1}) \sim \frac{|b-a|C}{\varepsilon}$.

Proof. Take $\mathcal{F}^I$ to be the collection of first derivatives for functions in $\mathcal{G}^I$, equipped with the L^∞ metric.

(i) For $\delta_1 > 0$, we may apply Theorem 3 to obtain an L^∞ δ_1-cover $\{U_i'\}$ for $\mathcal{F}$, with centers $\{f_i'\}$, $i = 1, \ldots, 2^{\left[\frac{|b-a|C}{\delta_1}\right]}$. Define $U_i \subset \mathcal{G}^I$ to be the collection of primitives of all the elements in U_i', so $U_i = \{f(x) = \int_0^x f'(t)dt \mid f' \in U_i'\}$. The ρ_{C^1}-diameter of U_i is $\frac{|b-a|}{\lambda}\delta_1 + \delta_1$. Fix a ξ with $1 < \xi < 2$. We will construct a $(\frac{\delta_1^\xi}{\lambda} + \delta_1)$-cover for each U_i in the metric ρ_{C^1}.

Divide the interval $[a, b]$ into subintervals $I_k = [x_{k-1}, x_k]$ of width $\Delta = \frac{\delta_1^{\xi-1}}{2}$ except possibly the last one which might be shorter, so $k = 1, \ldots, \left(\left[\frac{2|b-a|}{\delta_1^{\xi-1}}\right] + 1\right)$. For each U_i, construct a collection of piecewise Lipschitz functions $g_{i,j}$, where $g_{i,j}' = f_i'$ on the interior of each I_k, but $g_{i,j}$ jumps by $\pm\frac{\delta_1^\xi}{2}$ at each interval endpoint x_k for all but the last k. See Figure 3. We will associate a $g_{i,j}$ to each $f \in U_i$ by choosing the function $g_{i,j}$ whose values at the x_k minimize $|g_{i,j}(x_k) - f(x_k)|$. Denote the collection of $f \in U_i$ associated to $g_{i,j}$ by $V_{i,j}$.

Figure 3: Jumps to correct the location of approximating functions $g_{i,j}$.

We claim the ρ_{C^1}-diameter of $V_{i,j}$ is at most $2(\frac{\delta_1^\xi}{\lambda} + \delta_1)$ in the metric ρ_{C^1}. By construction, $\sup_x |f'(x) - g_{i,j}'(x)| \le \delta_1$ and $|f(0) - g_{i,j}(0)| = 0 \le \frac{\delta_1^\xi}{2}$ for $f \in V_{i,j}$. In fact, $\sup_x |f(x) - g_{i,j}(x)| \le \delta_1^\xi$. Assume $|f(x_{k-1}) - g_{i,j}(x_{k-1})| \le \frac{\delta_1^\xi}{2}$ for some k. Then, for $x \in I_k$:

$$\begin{aligned} |f(x) - g_{i,j}(x)| &\le \int_{I_k} \left|f'(t) - g_{i,j}'(t)\right| \, dt + \frac{\delta_1^\xi}{2} \\ &\le \delta_1\Delta + \frac{\delta_1^\xi}{2} \\ &= \delta_1^\xi. \end{aligned}$$

At the point $x = x_k$, $g_{i,j}$ jumps by $\frac{\delta_1^\xi}{2}$, giving $|f(x_k) - g_{i,j}(x_k)| \le \frac{\delta_1^\xi}{2}$. Therefore, given $f, \bar{f} \in V_i, j$, $\sup_x |f(x) - \bar{f}(x)| \le 2\delta_1^\xi$, and the diameter of $V_{i,j}$ is as claimed.

At each x_k but the last, the functions $g_{i,j}$ may jump positively or negatively, giving a total of $2^{\left[\frac{2|b-a|}{\delta_1^{\xi-1}}\right]}$ functions, and therefore the same number of balls $V_{i,j}$ for each i. This gives the total number of balls in the $\left(\frac{\delta_1^\xi}{\lambda} + \delta_1\right)$-cover as $K_{\delta_1} = 2^{\left[\frac{|b-a|C}{\delta_1}\right] + \left[\frac{2|b-a|}{\delta_1^{\xi-1}}\right]}$.

(ii) For $\delta_2 > 0$, apply Theorem 3 to obtain a $2\delta_2$-separated set in L^∞ for $\mathcal{F}$ with at least $L_{2\delta_2} = 2^{\left[\frac{|b-a|C}{\delta_2}\right]}$ elements h_i'. Taking the collection of primitives $h_i(x) = \int_0^x h_i'(t)\ dt$, we obtain a $2\left(\frac{\delta_2^2}{2C\lambda}+\delta_2\right)$-separated set in the metric ρ_{C^1}. Certainly, for each $i \neq j$, there exists some x_{ij} so that $|h_i'(x_{ij}) - h_j'(x_{ij})| \geq 2\delta_2$. Then, by definition of h_i and h_j, we have that $|h_i(x_{ij}) - h_j(x_{ij})| \geq \int_0^{\frac{\delta_2}{C}} 2Ct\, dt = \frac{\delta_2^2}{C}$, as desired.

(iii) Fix an $\varepsilon > 0$. To estimate $\mathcal{H}_\varepsilon(\mathcal{G}^I, \rho_{C^1})$, we select $\xi = \frac{3}{2}$, δ_1 so that $\varepsilon = \delta_1 + \frac{\delta_1^{\frac{3}{2}}}{\lambda}$, and δ_2 so that $\varepsilon = \delta_2 + \frac{\delta_2^2}{2C\lambda}$. Then from (i), we have an ε-cover with at most K_{δ_1} elements, and from (ii), we have a 2ε-separated set with at least $L_{2\delta_2}$ elements, giving:

$$\log L_{2\delta_2} \leq \mathcal{C}_{2\varepsilon} \leq \mathcal{H}_\varepsilon \leq \log K_{\delta_1}.$$

But then for $0 < \delta_i < \varepsilon$:

$$\lim_{\varepsilon\to 0} \frac{\log L_{2\delta_2}}{\frac{|b-a|C}{\varepsilon}} = \lim_{\varepsilon\to 0} \frac{\frac{|b-a|C}{\delta_2}}{\frac{|b-a|C}{\varepsilon}} = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta_2} = \lim_{\delta_2\to 0} \frac{\delta_2 + \frac{\delta_2^2}{2C\lambda}}{\delta_2} = 1$$

and similarly

$$\lim_{\varepsilon\to 0} \frac{\log K_{\delta_1}}{\frac{|b-a|C}{\varepsilon}} = \lim_{\delta_1\to 0} \frac{\delta_1 + \frac{\delta_1^{\frac{3}{2}}}{\lambda}}{\delta_1} = 1,$$

which gives

$$\lim_{\varepsilon\to 0} \frac{\mathcal{H}_\varepsilon(\mathcal{G}^I, \rho_{C^1})}{\frac{|b-a|C}{\varepsilon}} = 1.$$

□

Note that because the centers for these balls have jump discontinuities, they are no longer in a space of functions with Lipschitz derivatives. What has happened is that whereas a ball of derivatives and its balls of primitives have centers in the appropriate space, the derivative of a primitive center is not the center of the derivative ball, and so we fail to obtain a center in the appropriate space for the C^1 metric.

4 Estimation of ε-entropy for spaces of curves

We next apply techniques developed in the linear setting of classes of functions to the nonlinear setting of classes of plane curves, specifically curves with Lipschitz tangent angle as a function of arclength. We obtain tight bounds on ε-entropy for classes of curves containing both open and closed curves in Section 4.1, and slightly weaker bounds for closed curves in Section 4.2. Together, these results comprise Theorem 19.

4.1 Estimation of $H_\varepsilon(\mathcal{C}_K^L, \rho)$

4.1.1 Upper Bound

To construct an ε-cover for curves, we will mimic the techniques of Theorem 7. As before, the ε-cover for curves does not correspond to an ε-net for the space: we can find centers in the product space $\mathcal{C}_K^L \times S^1$, but for each center, the tangent angle function paired with the curve is not the tangent angle for that curve. Instead, centers for $\mathcal{C}_K^L$ will have jump discontinuities.

To refine the lifted cover for derivatives in the case of functions, the direction of adjustment necessary for the refinement was clear; it was parallel to the vertical axis (c.f. Figure 3. In the curve case, however, the direction of correction must be specified. The next two lemmas address that issue; proofs are in the Appendix.

Lemma 8. *Given $\tilde{\gamma}(s) : [0, L] \to \mathbb{R}^2$, $\tilde{\gamma} \in C^1$ with tangent angle function $\tilde{\theta}(s)$, define:*

$$\Gamma_\delta(\tilde{\gamma}) = \left\{ \gamma : [0, L] \to \mathbb{R}^2 \ \mid \ \gamma \in C^1, \ |\gamma(0) - \tilde{\gamma}(0)| \le \tfrac{\delta^\xi}{2}, \ |\theta(s) - \tilde{\theta}(s)| \le \delta \right\},$$

where s is the arclength parameter for all curves. There exists a $(\delta + \frac{\delta^\xi}{\lambda})$-cover for $(\Gamma_\delta(\tilde{\gamma}), \rho)$ with at most $2^{\log 3 \left\lceil \frac{2L}{\delta^{\xi-1}} \right\rceil}$ elements.

Modifying the above lemma to allow the curve lengths to vary gives the following lemma.

Lemma 9. *Given $\tilde{\gamma}(s) : [0, \tilde{l}] \to \mathbb{R}^2$, $\tilde{\gamma} \in C^1$, with tangent angle function $\tilde{\theta}$, define:*

$$\Gamma'_\delta(\tilde{\gamma}) = \left\{ \gamma : [0, \tilde{l}] \to \mathbb{R}^2 \ \mid \ \gamma \in C^1, \ |\gamma(0) - \tilde{\gamma}(0)| \le \tfrac{\delta^\xi}{2}, \ |\theta(s) - \tilde{\theta}(s)| \le \delta, \ l \in [\tilde{l} - \tfrac{\delta^\xi}{4}, \tilde{l}] \right\},$$

where l is arclength of γ, s is the arclength parameter for $\tilde{\gamma}$, and all other curves are parameterized to have constant speed $l/\tilde{l}$. There exists a $(\delta + \frac{\delta^\xi}{\lambda})$-cover for $(\Gamma'_\delta(\tilde{\gamma}), \rho)$ with at most $2^{\log 3 \left\lceil \frac{4L}{\delta^{\xi-1}} \right\rceil}$ elements.

We now have the necessary ingredients for constructing an ε-cover on $\mathcal{C}_K^L$, applying Lemma 9 together with Theorem 3.

Proposition 10. *There exists a $\left(\delta + \frac{\sqrt{\delta^3}}{\lambda}\right)$-cover for $(\mathcal{C}_K^L, \rho)$ with no more than:*

$$\left\lceil \frac{4L}{\sqrt{\delta^3}} \right\rceil 2^{\left\lceil \frac{KL}{\delta} \right\rceil + \log 3 \left\lceil \frac{4L}{\sqrt{\delta}} \right\rceil}$$

elements.

Proof. To construct the cover, partition the interval $[0, L]$ of possible arclength into subintervals of width $\frac{\sqrt{\delta^3}}{4}$, giving $\left\lceil \frac{4L}{\sqrt{\delta^3}} \right\rceil$ subintervals. Let l_δ be the right endpoint of any such subinterval. We will parameterize γ of length $l \in \left(l_\delta - \frac{\sqrt{\delta^3}}{4}, l_\delta\right]$ by $s \in [0, l_\delta]$ so that $\gamma(0) = 0$, $\theta(0) = 0$ and $\frac{d\gamma}{ds} = \frac{l}{l_\delta}$.

For all curves with lengths in a particular subinterval, $|d\theta/ds| \le \frac{l}{l_\delta} K \le K$, and so we may apply Theorem 3 to obtain an L^∞ δ-net for the angle functions with at most $2^{\left\lceil \frac{KL}{\delta} \right\rceil}$ elements θ_δ. Each θ_δ gives rise to a curve

$$\tilde{\gamma}_\delta(s) = \int_0^s \langle \cos \theta_\delta(t), \sin \theta_\delta(t) \rangle \ dt,$$

and its associated neighborhood of curves $\Gamma'_\delta(\tilde{\gamma}_\delta)$. Then by Lemma 9, we may construct a $(\delta + \frac{\delta^\xi}{\lambda})$-cover for each $(\Gamma'_\delta(\tilde{\gamma}), \rho)$ with at most $2^{\log 3 \left\lceil \frac{4L}{\sqrt{\delta^3}} \right\rceil}$ elements.

Applying this process within each length subinterval gives the result. ∎

Taking δ_1 so that $\varepsilon = \delta_1 + \sqrt{\delta_1^3}$, and applying Proposition 10 gives a $(\delta_1 + \frac{\sqrt{\delta_1^3}}{\lambda})$-cover for $(\mathcal{C}_K^L, \rho)$ with

$$K_{\delta_1} = \left\lceil \frac{4L}{\sqrt{\delta_1^3}} \right\rceil 2^{\left\lceil \frac{KL}{\delta_1} \right\rceil + \log 3 \left\lceil \frac{4L}{\sqrt{\delta_1}} \right\rceil}$$

elements, thus proving the following corollary, which gives out upper bound.

Corollary 11. $\mathcal{H}_\varepsilon(\mathcal{C}_K^L, \rho) \preceq \frac{KL}{\varepsilon}$.

4.1.2 Lower Bound

We construct a 2ε-separated set in $\mathcal{C}_K^L$. This is not a straightforward as it might appear, as we must guard against pairs of curves where $|\gamma_1(s) - \gamma_2(s)|$ is large but there exists and s' such that $|\gamma_1(s) - \gamma_2(s')|$ is small. To do so, we will restrict how far a curve can wander away from the horizontal axis.

As a first step, we count the number of realizations of an n-step symmetric random walk g satisfying $g(0) = g(n) = 0$ and $\int g = 0$, where g is piecewise linear with slope ± 1. This will allow us to piece together curves from functions that stay close to the horizontal axis, while maintaining differentiability of the curve at the joins.

We count the number of such walks using a probabilistic argument. Let $a(n) = \int_0^n g$, and for fixed n consider the random variable $z = \langle g(n), a(n)\rangle \ \Sigma^{-1} \ \langle g(n), a(n)\rangle^T$, where Σ is the covariance matrix for the random vector $\langle g(n), a(n)\rangle$. We will use level sets for z to find a lower bound for the number of walks g so that $\langle g(n), a(n)\rangle$ are "close enough" to the origin, then we will add steps to those walks to bring the endpoints and areas back to 0. Proofs of the next two lemmas are left to the Appendix.

Lemma 12. *The number of realizations of a symmetric n-step random walk g so that $g(0) = 0$, $|g'(n')| \leq 3\sqrt{n}$, and $|a(n')| \leq \frac{\sqrt{3}}{2}\sqrt{4n^3 - n}$ is at least $\frac{1}{3}2^n$.*

Lemma 13. *Given a realization g' of an n'-step random walk so that $g'(0) = 0$, $|g'(n')| \leq 3\sqrt{n}$ and $|a(n')| \leq \frac{\sqrt{3}}{2}\sqrt{4n^3 - n}$, there exists a realization g of an n-step random walk, coinciding with g' on the first n' steps, satisfying $g(0) = g(n) = 0$ and $\int_{[0,n]} g = 0$, where $n \leq n' + 3\sqrt{n'} + 8\sqrt{\frac{9n'}{2} + \frac{\sqrt{3}}{2}\sqrt{4n'^3 - n'}}$.*

We now construct an appropriate collection of functions. Recall the definition of $\mathcal{F}_1^I$:

$$\mathcal{F}_1^I(C) = \{f : I \to \mathbb{R} \ | \ f(a) = 0, \ |f(x) - f(x')| \leq C|x - x'|, \ \forall x, x' \in I\}.$$

Proposition 14. *Let $I = [a, b]$, and define:*

$$\mathcal{F}_{1,0}^I(C) = \left\{ f \in \mathcal{F}_1^I \ | \ f(b) = 0, \int_I f = 0 \right\}.$$

There exists a 2ε-separated set in $(\mathcal{F}_{1,0}^I, L^\infty)$ with $\mathcal{M}'_{2\varepsilon}$ elements, where $\mathcal{M}'_{2\varepsilon} \succeq 2^{\frac{C|b-a|}{\varepsilon}}$.

Proof. Set $n = \left[\frac{C|b-a|}{\varepsilon}\right]$, take $n' = n - 3\sqrt{n} - 2\sqrt{\frac{9n}{2} + \frac{\sqrt{3}}{2}\sqrt{4n^3 - n}}$ and $\Delta = \frac{C|b-a|}{n} \geq \varepsilon$. On the interval $[a, a + n'\Delta]$, Theorem 3 and Lemma 12 gives at least $\frac{1}{3}2^{n'}$ elements in a 2ε-separated set consisting of functions $\hat{f}$ which are piecewise linear with slopes $\pm C$ satisfying, for sufficiently large n:

$$\begin{aligned}
\hat{f}(a) &= 0, \\
\hat{f}(a + n'\Delta) &\leq \frac{3C|b-a|}{n}\sqrt{n'} \leq \frac{3C|b-a|}{n}\sqrt{n}, \\
\int_{[a, a+n'\Delta]} \hat{f} &\leq \frac{\sqrt{3}C|b-a|^2}{2n^2}\sqrt{4n'^3 - n'} \leq \frac{\sqrt{3}C|b-a|^2}{2n^2}\sqrt{4n^3 - n}.
\end{aligned}$$

Now apply Lemma 13 to extend $\hat{f}$, in fewer than $n - n'$ steps, to obtain a function with endpoint and integral values of zero. Extend $\hat{f}$ to the full interval $[a, b]$ by setting it to zero on the remainder of the interval, producing a function f with the desired properties.

As $\varepsilon \to 0$, $n \to \infty$, and $\lim_{n\to\infty} \frac{n}{n'} = 1$, giving the desired result.

□

We are now ready to estimate the lower bound.

Proposition 15. *There exists a 2ε-separated set for $(\mathcal{C}_K^L, \rho)$ with $\mathcal{M}_{2\varepsilon}\left(\mathcal{C}_K^L\right)$ elements, where $\mathcal{M}_{2\varepsilon}\left(\mathcal{C}_K^L\right) \succeq 2^{\left[\frac{KL}{\varepsilon}\right]}$.*

Proof. Choose δ so that $\varepsilon = \frac{\delta}{1+\frac{K^2\delta}{4}}$, giving $\lim_{\varepsilon\to 0} \frac{\delta}{\varepsilon} = 1$. For a fixed L', divide the interval $[0, L']$ into subintervals $I_k = [a_k, a_{k+1}]$ of width $\sqrt{\delta}$, giving $k \leq \left[\frac{L'}{\sqrt{\delta}}\right]$. Within each I_k, apply Proposition 14 with $C = K$ to obtain a collection of 2δ-separated functions $\{\frac{df_j^k}{dx}\}$, $j = 1, \ldots, n$, where $n \succeq \frac{1}{3} 2^{\frac{K\sqrt{\delta}}{\delta}}$. The functions satisfy, for $i \neq j$:

(i) $\frac{df_j^k}{dx}(a_k) = \frac{df_j^k}{dx}(a_{k+1}) = 0$

(ii) $\int_{I_k} \frac{df_j^k}{dx} = 0$

(iii) $\left|\left|\frac{df_i^k}{dx} - \frac{df_i^k}{dx}\right|\right|_\infty \geq 2\delta$

(iv) $\left|\left|\frac{df_j^k}{dx}\right|\right|_\infty \leq \frac{K\sqrt{\delta}}{2}$

Integrating, we obtain a collection of primitives $\{f_j^k\}$ with curvature functions bounded by K, and tangent angle functions $\{\theta_j^k\}$, where $\theta_j^k = \arctan \frac{df_j^k}{dx}$. These primitives satisfy the following properties for $i \neq j$:

(i) $f_j^k(a_k) = f_j^k(a_{k+1}) = 0$

(ii) $\theta_j^k(a_k) = \theta_j^k(a_{k+1}) = 0$

(iii) $\left|\left|\theta_i^k - \theta_j^k\right|\right|_\infty \geq \left|\left|\arctan \frac{df_i^k}{dx} - \arctan \frac{df_j^k}{dx}\right|\right|_\infty \geq \frac{1}{1+\frac{K^2\delta}{4}} \left|\left|\frac{df_i^k}{dx} - \frac{df_j^k}{dx}\right|\right|_\infty \geq \frac{2\delta}{1+\frac{K^2\delta}{4}} = 2\varepsilon.$

Construct functions $\{f_i\}$ on $[0, L']$ by concatenating sequences $\{f_i^k\}$. These will be C^1 and piecewise quadratic (not piecewise circular as in the construction of the upper bound). The number of such functions will be 2^m, where $m \succeq \left(\frac{L'}{\sqrt{\delta}}\right)\left(\frac{K\sqrt{\delta}}{\delta}\right) = \frac{KL'}{\delta}$, and each will the properties above. In particular, $\frac{df_i}{dx}(0) = \frac{df_i}{dx}(L') = 0$ and $f_i(0) = f_i(L') = 0$ for every i.

Each function f_i^k has length

$$
\begin{aligned}
L_i^k &= \int_{I_k} \sqrt{1 + f_i'^2}\, dx \\
&\leq \int_{I_k} \sqrt{1 + (K\tfrac{\sqrt{\delta}}{2})^2}\, dx \\
&= \sqrt{\delta} \cdot \sqrt{1 + \tfrac{K^2\delta}{4}}.
\end{aligned}
$$

This gives the length of f_i to be at most $\frac{L'}{\sqrt{\delta}}\sqrt{\delta} \cdot \sqrt{1 + \frac{K^2\delta}{4}} = L'\sqrt{1 + \frac{K^2\delta}{4}}$. Take:

$$
L' = \frac{L}{\sqrt{1 + \frac{K^2\delta}{4}}},
$$

to find $m \succeq \frac{KL}{\delta} = \frac{KL}{\varepsilon}$.

Finally, we demonstrate the 2ε-separation of these functions as curves. Recall from the definition of ρ that $\lambda < 1/K$. For $f_i,\ f_j,\ i \neq j$, we may assume there exists some subinterval $I = [x_0, x_0 + \frac{\delta}{K}]$ so that $\frac{df_i}{dx}|_I$ has slope $+K$ and $\frac{df_j}{dx}|_I$ has slope $-K$, giving $|\frac{df_i}{dx}(x_0 + \frac{\delta}{K}) - \frac{df_j}{dx}(x_0 + \frac{\delta}{K})| \geq 2\delta$. See Figure 4.

Figure 4: Separation of curves.

Consider the ρ-distance of f_j to the point $P = \left(\frac{\delta}{K}, f_i(\frac{\delta}{K})\right)$:

$$\begin{aligned}
\rho(f_1, f_2) &\geq \rho(P, f_j) \\
&\geq \min_x \frac{1}{\lambda}\left|x_0 + \frac{\delta}{K} - x\right| + \left|\theta_i(x_0 + \tfrac{\delta}{K}) - \theta_j(x)\right| \\
&= \min_x \frac{1}{\lambda}\left|x_0 + \frac{\delta}{K} - x\right| + \left|\arctan \tfrac{df_i}{dx}(x_0 + \tfrac{\delta}{K}) - \arctan \tfrac{df_j}{dx}(x)\right| \\
&\geq \min_x \frac{1}{\lambda}\left|x_0 + \frac{\delta}{K} - x\right| + \frac{1}{1 + \frac{K^2\delta}{4}}\left|\tfrac{df_i}{dx}(x_0 + \tfrac{\delta}{K}) - \tfrac{df_j}{dx}(x)\right|.
\end{aligned}$$

Since this last expression does not depend on location in the plane, we may take $x_0 = 0$, $f_i' = Kx$, and $f_j' = -Kx$, as depicted in Figure 4. Assume also that $x \in [0, \frac{\delta}{K}]$ (the argument is analogous for $x > \delta/K$), giving:

$$\rho(f_i, f_j) \geq \min_x \frac{1}{\lambda}\left|\frac{\delta}{K} - x\right| + \frac{1}{1 + \frac{K^2\delta}{4}}\left|\delta + Kx\right|.$$

Taking derivatives with respect to x, we see that since $\lambda < 1/K$, the minimum of the last expression occurs when $x = \delta/K$, giving $\rho(f_1, f_2) = |\theta_i(\frac{\delta}{K}) - \theta_j(\frac{\delta}{K})| \geq \frac{2\delta}{1+\frac{K^2\delta}{4}} = 2\varepsilon$ as desired.

□

Corollary 16. $\mathcal{H}_\varepsilon(\mathcal{C}_K^L, \rho) \succeq \frac{KL}{\varepsilon}$.

Corollaries 11 and 16 result in a tight estimate of $\mathcal{H}_\varepsilon(\mathcal{C}_K^L, \rho)$, which we present in Theorem 19.

4.2 Estimation of $H_\varepsilon(\mathcal{I}_K^L, \rho)$

Corollary 11 also gives an upper bound for $\mathcal{H}_\varepsilon(\mathcal{I}_K^L, \rho)$. Any search for a lower bound, however, encounters the obstacle of constructing a 2ε-separated set of curves which are closed, putting constraints on areas under the tangent angle functions. This can be shown to be related to an unsolved problem of partition functions (c.f. [25]).

Instead, we sacrifice tightness of the lower bound and force closure another way. Suppose we have a collection of functions, 2δ-separated in the metric ρ_{C^1}, defined on an interval of length L' with second derivative bounded by $\pm K$. We may select any two functions defined on an interval $[a, b]$ so that $f(a) = f(b)$ and use the graphs of these to join two halves of a circle of radius $1/K$. See Figure 5. For functions satisfying $f(0) = f(L') = \frac{df}{dx}(0) = \frac{df}{dx}(L') = 0$, and for appropriate choices of L' and δ, these curves will give the desired 2ε-separated set. In other words, we may build upon the construction in Proposition 15 for open curves to obtain a 2ε-separated set of closed curves.

Theorem 17. *There exists a 2ε-separated set for $(\mathcal{I}_K^L, \rho)$ with $\mathcal{M}_{2\varepsilon}\left(\mathcal{I}_K^L\right) = 2^{2m}$ elements where*

$$2m \succeq \frac{KL - 2\pi}{\varepsilon}$$

.

Proof. Select δ so that $\varepsilon = \frac{\delta}{\sqrt{1+\frac{K^2\delta}{4}}}$. We will use a slight modification of the collection $\{f_i\}$ constructed in the proof of Proposition 15 to construct a 2ε-separated set of closed curves. Taking any two (not necessarily distinct) functions f_i, f_j, we may join two halves of a circle of radius $1/K$ with the functions, forming a closed C^1 curve with bounded curvature and length. See Figure 5. The number of such curves is 2^{2m}, where $m \succeq \frac{KL'}{\delta}$. We have from the proof of Proposition 15 that this collection of functions is 2ε-separated as curves in the metric ρ.

Figure 5: Construction of closed curves for 2ε-separated set.

It remains to determine a suitable value for L'. Each function f_i has length $L'\sqrt{1+\frac{K^2\delta}{4}}$. Together, the two halves of the circle of radius $1/K$ add an additional $2\pi/K$ in length. To construct a curve of length no more than L, L' must satisfy:

$$2L'\sqrt{1+\frac{K^2\delta}{4}} + \frac{2\pi}{K} \leq L,$$

which gives $2L' \leq \frac{L-\frac{2\pi}{K}}{\sqrt{1+\frac{K^2\delta}{4}}}$. Taking $L' = \frac{L-\frac{2\pi}{K}}{\sqrt{1+\frac{K^2\delta}{4}}}$, we have $m \succeq \frac{KL-2\pi}{\delta} \sim \frac{KL-2\pi}{\varepsilon}$, as claimed. □

Corollary 18. $\mathcal{H}_\varepsilon(\mathcal{I}_K^L, \rho) \succeq \frac{KL-2\pi}{\varepsilon}$.

4.3 Entropy Theorem

Together, Corollaries 11, 16, and 18 prove the following theorem. Although we were not able to prove it, we believe Theorem 19(b) should be $\mathcal{H}_\varepsilon(\mathcal{I}_K^L, \rho) \sim \frac{KL}{\varepsilon}$.

Theorem 19.

(a) $\mathcal{H}_\varepsilon(\mathcal{C}_K^L, \rho) \sim \frac{KL}{\varepsilon}$.

(b) $\mathcal{H}_\varepsilon(\mathcal{I}_K^L, \rho) \asymp \frac{1}{\varepsilon}$.

5 Variable Length Encoding: Non-compact Classes and Adaptive Codelengths

5.1 Minimum Description Length and Adaptive Coding

The ε-entropy gives the minimum number of bits required to represent elements in a compact space with a fixed code length. To relax the compactness constraint, we will allow a variable code length, thereby replacing the finite number of codewords of fixed length with a countable number of codewords of varying length in an adaptive coding scheme.

We will construct an adaptive coding scheme for functions in Lemma 20, which will give rise to two adaptive encodings for curves, one based on the boundary curve and the other on the medial axis. We will apply the Rissanen's MDL principle in Section 7 to determine which of those two encodings is most efficient.

5.1.1 Adaptive Coding for Lipschitz Functions

We carry out an adaptive coding scheme for Lipschitz functions, again motivated by Theorem 3.

Theorem 20. *For every $\varepsilon > 0$, there exists a countable codebook $F_\varepsilon = \{f_1, f_2, \ldots\}$, depending only on ε, with the following property. For every Lipschitz function f defined on $[a, b]$ so that $f(a) = 0$ and $f'(x)$ is continuous a.e., there are constants $C(f, \delta)$ such that for all δ, there is a codeword $f_n \in F_\varepsilon$ such that $||f - f_n||_\infty \leq \varepsilon$ and f_n has description length:*

$$L(f_n) \leq \left\lceil \frac{\int |f'| + \delta}{\varepsilon} \right\rceil + C(f, \delta).$$

Proof. Since f' is continuous a.e. and bounded, f' is Riemann integrable [7]. Therefore, for any δ, there exists a step function g taking on rational values, with a finite number of jumps at rational points $\{x'_j\}$, so that $|f'| \leq g$ and $\int g \leq \int |f'| + \delta$. This means that on each subinterval $I_j = [x_j, x_{j+1})$ where g is constant, f is Lipschitz with constant $g(x_j)$. Denote the number of jumps by m. Note that $g \equiv C$ gives the result in Theorem 3.

Using g, we may determine a variably spaced finite number of points $\{x_k\}$ so that for any k, $\int_{x_k}^{x_k+1} g \leq \varepsilon$. In particular, on each subinterval I_j, select the points spaced $\frac{\varepsilon}{g(x'_j)}$ apart. There will be at most:

$$\left\lceil \frac{g(x'_j)|I_j|}{\varepsilon} \right\rceil + 1 = \left\lceil \frac{\int_{I_j} g(x)}{\varepsilon} \right\rceil + 1$$

such points. Take $\{x_k\}$ to be the collection of $\{x'_j\}$ together with these equally spaced points. Note that when $g \equiv C$, this gives $x_{k+1} - x_k = \frac{\varepsilon}{C}$ as in Theorem 3.

We now construct an approximation f_n for f. We claim there exists a piecewise linear function ϕ_n, with slope $\pm g(x_k)$ on the interval $J_k = [x_k, x_{k+1})$, vanishing at a, so that $f \subset K(\phi_n)$, where $K(\phi_n)$ is defined as in the proof to Theorem 3. On J_1, take $\phi_n(x) = g(0)x$. Certainly, since f is Lipschitz with constant $g(0)$ on I_1, $f \subset K(\phi_n)$. Inductively, assume ϕ_n has been constructed so that $f \subset K(\phi_n)$ for $x \leq x_k$. We wish to define ϕ_n for I_k so that f remains in $K(\phi_n)$. Since f is Lipschitz with constant $g(x_k)$ on I_k and $f \subset K(\phi_n)$ for $x \leq x_k$, one of the following is true:

(a) $f(x_{k+1}) \in [\phi_n(x_k) - g(x_k)(x - x_k), \phi_n(x_k) + g(x_k)(x - x_k)] \subset [\phi_n(x_k) - \varepsilon, \phi_n(x_k) + \varepsilon]$,

(b) $$f(x_{k+1}) \in [\phi_n(x_k) - g(x_k)(x - x_k) - 2\varepsilon, \phi_n(x_k) + g(x_k)(x - x_k) - 2\varepsilon] \\ \subset [\phi_n(x_k) - 3\varepsilon, \phi_n(x_k) - \varepsilon].$$

If (a) is true, then defining ϕ_n to have positive slope on I_k gives $f \subset K(\phi_n)$. If (b) is true, then defining ϕ_n to have negative slope on I_k gives the desired result. See Figure 6. And so we have constructed a ϕ_n so that $f \subset K(\phi_n)$. Taking f_n to be the center of the corridor $K(\phi_n)$, we have $||f - f_n||_\infty \leq \varepsilon$.

Figure 6: Possible range for functions in a particular corridor.

Encode f by encoding f_n, or equivalently, ϕ_n. To do so requires describing g, which in turn requires describing the collection of points $\{x_k\}$ as outlined above. We must also describe the sequence of signs $\pm$ to assign to the slopes $g(x'_j)$ at each of the points $\{x_k\}$. Since g has rational jumps at rational values, encoding g requires a fixed and finite number of bits depending only on f and δ, yielding the constant $C(f, \delta)$. Describing a sign requires a single bit. As this must be done at each of the $\{x_k\}$, we see encoding the sequence of signs requires at most:

$$\sum_j \left(\left\lceil \frac{\int_{I_j} g}{\varepsilon} \right\rceil + 1 \right) + m \leq \left\lceil \frac{\int_{[a,b]} g}{\varepsilon} \right\rceil + 2m$$

bits. Then, absorbing $2m$ into $C(f, \delta)$, the total number of bits required to describe f_n satisfies:

$$L(f_n) \leq \frac{\int g}{\varepsilon} + C(f, \delta) \leq \frac{\int |f'| + \delta}{\varepsilon} + C(f, \delta),$$

as claimed.

□

5.2 Adaptive Code Lengths for Curves

In the compact setting, the leading term in the bit rate comes from approximating the tangent angle function; correcting for location requires only lower order terms. The same argument applies in the adaptive setting. Lemma 20 describes a method for adaptively encoding functions in a non-compact space by adaptively approximating each function's derivative, and we may use it to encode curves by applying it to the associated tangent angle function. Doing so gives an encoding with leading term for the bit rate of:

$$\frac{\int |\kappa_\gamma| \, ds + \delta}{\varepsilon},$$

for some fixed δ.

6 Medial Axis

Before we describe our encoding scheme for the medial axis representation and return to the discussion of efficient representation, we introduce the medial axis construction and derive some important properties.

6.1 Medial axis properties

In 1970, Blum introduced an elegant description for closed planar regions that captures local symmetries, preserving geometric information about the region's boundary in its own geometry. The **medial axis pair** of a closed region in $\mathbb{R}^2$ consists of $\mathbf{m}(t)$, the closure of the locus of centers of maximal bitangent circles contained within the region, and $r(t)$, the function of associated radii. The axis curve $\mathbf{m}$ is a subset of the symmetry set of the region, and can be thought of as the region's "skeleton," where r gives the length of "ribs." One of the appealing aspects of the medial axis from a shape modeling perspective is that this skeletal structure allows for a region to be decomposed into meaningful parts, allowing for parts-based comparisons between two shapes (*c.f.* [8, 24, 20, 29, 23, 30]).

We will mostly consider medial axes of regions bounded by a single simple, closed, C^2 curve. In this setting, many beautiful results may be proved. We list these here; proofs may be found in [8].

6.1.1 Geometry of the medial axis

In general, $\mathbf{m}$ will consist of several branches with a degree of smoothness determined by the smoothness of the boundary curve (c.f. Theorem 23), meeting at branch points. We will denote a branch contained in $\mathbf{m}$ by m. First, we introduce some notation; see Figure 7.

Figure 7: Notation and relationships for the medial axis.

Notation. *Refer to Figure 7. The following notations are applicable locally, within a branch of the medial curve for a region bounded by the closed curve γ. Let $m(t)$ be a local parameterization of m.*

- *Denote the tangent vector to $m(t)$ by $\mathbf{T}_m(t)$, which makes an angle $\theta_m(t)$ with the positive horizontal axis. Denote the normal vector to m by $\mathbf{N}_m(t)$ and the curvature by κ_m.*
- *Following $m(t)$ in the direction of its orientation, denote by $\gamma_+(t)$ the boundary point corresponding to $m(t)$ lying to the left of m and by $\gamma_-(t)$ the boundary point lying to the right.*
- *Parameterizing γ in the standard orientation, denote by $\mathbf{T}_-$ the tangent vector to γ_- with orientation inherited from $m(t)$, and by $\mathbf{T}_+$ the tangent vector to γ_+ with opposite orientation to the inherited one. Denote by $\theta_\pm$ the angles between the tangent vectors, $\mathbf{T}_\pm$, and the positive horizontal axis.*
- *With the standard orientation, denote by κ_+ the curvature at points on γ_+ and by κ_- the curvature at points on γ_-.*
- *Denote the smaller angles between $-\mathbf{T}_+$ and $\mathbf{T}_m$ by α, and between $-\mathbf{T}_+$ and $\mathbf{T}_-$ by β.*
- *Denote the smaller angle between $\mathbf{T}_m$ and $-\mathbf{N}_+$, the outward pointing normal to γ_+, by ϕ, and so $\phi = \alpha + \frac{\pi}{2}$.*
- *The symbols s, s_+, s_-, and v will always indicate the arclength parameter for γ, γ_+, γ_-, and m, respectively.*
- *The symbol $'$ will be reserved for derivatives with respect to an arclength parameter of m.*

The first result shows that the information contained in the medial axis pair is equivalent to that contained in the boundary curve, and gives explicit formulas for alternating between the two representations.

Theorem 21.

(a) Given a smooth, arclength parameterized medial axis pair $(m(v), r(v))$ for a region with smooth boundary, the pieces of the boundary curve corresponding to v are given by:

$$\gamma_\pm(v) = \left(m + rr'\mathbf{T}_m \mp r\sqrt{1-r'^2}\mathbf{N}_m\right)(v). \tag{1}$$

(b) Given associated boundary points $\gamma_\pm(s_\pm)$, the medial axis pair corresponding to $s_\pm$ is given by:

$$\begin{aligned} r(s_\pm) &= -\frac{(\gamma_+ - \gamma_-)\cdot(\mathbf{N}_+ - \mathbf{N}_-)}{(\mathbf{N}_+ - \mathbf{N}_-)\cdot(\mathbf{N}_+ - \mathbf{N}_-)}, \\ m(s_\pm) &= \gamma_\pm(s_\pm) + r(s_\pm)\mathbf{N}_+, \end{aligned}$$

for $\gamma_\pm$ oriented to have inward pointing normals.

We next describe the relationships between the geometry of the boundary curve and the geometry of the medial axis.

Theorem 22. *For a smooth branch of the medial curve, the following relationships hold:*

1. $r' = \sin\alpha = -\cos\phi$.

2. $\mathbf{T}_m = \pm\sqrt{1-r'^2}\mathbf{T}_\pm + r'\mathbf{N}_\pm$.

3. $\mathbf{N}_m = -r'\mathbf{T}_\pm \pm \sqrt{1-r'^2}\mathbf{N}_\pm$.

4. $\mathbf{T}_\pm = \mp\sqrt{1-r'^2}\mathbf{T}_m + r'\mathbf{N}_m = \mp\sin\phi\mathbf{T}_m + \cos\phi\mathbf{N}_m$.

5. $\mathbf{N}_\pm = r'\mathbf{T}_m \mp \sqrt{1-r'^2}\mathbf{N}_m = -\cos\phi \mp \sin\phi\mathbf{N}_m$.

6. $\kappa_\pm = \frac{\pm\kappa_m\sqrt{1-r'^2}-r''}{(1-r'^2)\pm r\kappa_m\sqrt{1-r'^2}-rr''}$.

7. $\pm\kappa_m = \frac{\kappa_\pm\sqrt{1-r'^2}}{1-r\kappa_\pm} + \frac{r''}{\sqrt{1-r'^2}} = \pm\frac{1}{2}\sin\left(\frac{\theta_+-\theta_-}{2}\right)\left(\frac{\kappa_-}{1-r\kappa_-} - \frac{\kappa_+}{1-r\kappa_+}\right)$.

8. $\alpha = \beta/2$ *(and so* ϕ *is also the angle between* $\mathbf{T}_m$ *and the outward pointing normal at* γ_-*).*

9. $\phi' = -\frac{1}{2}\sin\left(\frac{\theta_+-\theta_-}{2}\right)\left(\frac{\kappa_-}{1-r\kappa_-} + \frac{\kappa_+}{1-r\kappa_+}\right)$.

With some work, these geometric relationships give rise to the following smoothness relationships:

Theorem 23. *Let v and s be arclength parameters for m and γ, respectively.*

(a) If $(m(v), r(v))$ are a C^p, $p \geq 2$, interior portion of the medial axis pair for $\gamma_\pm(s)$ satisfying $r > 0$, $|r'| < 1$ and $|\kappa_m| < \frac{1-r'^2-rr''}{r(1-r'^2)}$, then $\gamma_\pm(s_\pm)$ are also C^p.

(b) Conversely, if $\gamma_\pm(s_\pm)$ are portions of a C^p curve γ, $p \geq 2$, corresponding via the medial axis so that:

$$1 + \kappa_\pm \frac{(\gamma_+ - \gamma_-)\dot{(}\mathbf{N}_+ - \mathbf{N}_-)}{(\mathbf{N}_+ - \mathbf{N}_-)\dot{(}\mathbf{N}_+ - \mathbf{N}_-)} > 0,$$

then the associated interior portion of the medial axis is C^p.

Given an axis curve $\mathbf{m}$, not every function defined on $\mathbf{m}$ can be a radius function. Theorem 24 gives constraints on r.

Theorem 24. *The smooth pair (m, r) is locally the medial axis of a smooth boundary curve if and only if*

1. $|r'| < 1$

2. $|\kappa_m| < \frac{1-r'^2-rr''}{r\sqrt{1-r'^2}}$.

Following a branch of a medial curve, the two boundary curves $\gamma_\pm$ are described simultaneously. If each is parameterized by its own arclength, $s_\pm$, quantities of interest are the relative velocities of traversal of those boundary pieces to each other and of each boundary piece to the velocity of the medial axis.

Theorem 25.

(a) Two boundary curves $\gamma_\pm(s_\pm)$ correspond to each other via a medial point if and only if the function $s_+(s_-)$ satisfies the ODE:

$$\frac{ds_-}{ds_+} = \frac{2\cos^2(\beta/2) + \kappa_+(\gamma_+ - \gamma_-)\cdot\mathbf{N}_+}{2\cos^2(\beta/2) - \kappa_-(\gamma_+ - \gamma_-)\cdot\mathbf{N}_-}$$

with initial condition $s_-(s_+^0) = s_-^0$ for $\gamma(s_-^0)$ and $\gamma(s_+^0)$ corresponding via the medial axis.

(b) The velocity of boundary points $\gamma_\pm$ corresponding to a medial axis point $m(v)$ is given by:

$$\frac{ds_\pm}{dv} = \frac{\mp\sqrt{1-r'^2}}{1-r\kappa_\pm}.$$

Corresponding boundary points must also fulfill a geometric condition:

Theorem 26. *The two points $\gamma_\pm$ with tangent vectors $\mathbf{T}_\pm$ are related by a bitangent circle, thereby corresponding to the same symmetry set point, if and only if:*

$$(\gamma_+ - \gamma_-) \cdot (\mathbf{T}_+ - \mathbf{T}_-) = 0.$$

We end our review of geometric properties of the axis with results about how smooth pieces of a medial curve can come together at a branch point. By a generic curve, we mean a curve for which small perturbations do not alter the structure of $\mathbf{m}$.

Theorem 27. *For a region bounded by generic, smooth curves $\gamma_\pm$, any point of the medial curve corresponds to a medial circle with one of the following types of tangency:*

1. *a fourth order osculating circle, in which case the point is an endpoint of a medial curve where the medial circle has fourth order contact with the boundary curve.*
2. *a bitangent circle, in which case the point is in the interior of a smooth part of the medial curve.*
3. *a tri-tangent circle, in which case the point is a branch point of the medial curve where smooth medial branches come together.*

6.1.2 Constraints on the Boundary Curve

It is well-known that the boundary curve is not a stable representation of the medial axis pair, as small perturbations in the boundary can cause drastic changes in the structure of $\mathbf{m}$. The reverse, however, is not true. The medial axis provides a very stable representation of the boundary curve, as we demonstrate in the following two results. The first is a geometric argument bounding the region in the plane where a boundary curve can go, given sufficiently well-sampled medial data. The second is an analytic analogue of the geometric result.

Suppose we are given medial points $m_i = m(v_i)$, with corresponding radii r_i and boundary points $\gamma_i^\pm$, $i = 1, 2$, where $|v_2 - v_1| \le \delta$ for $\delta < \min(r_1, r_2)$. Denote by C_i the medial circles centered at m_i of radius r_i. Since the arclength of the medial curve joining m_1 and m_2 is at most δ, $\delta < r_i$, we have $C_1 \bigcap C_2 \neq \emptyset$.

Figure 8: Possible regions for R^+ and R^-, containing boundary points.

We wish to determine the regions in the plane $R^\pm$ within which the boundary curves $\gamma_i^\pm(v)$ can lie for $v \in (v_1, v_2)$. We claim R^+ (resp. R^-) is the region outside $C_1 \bigcup C_2$ but inside the circle C^+ (resp. C^-) passing through the points γ_i^+ (resp. γ_i^-) as depicted in Figure 8. The construction of these regions is described below.

First, note that possible locations for medial centers must lie within the solid ellipse defined by $d((x,y), m_1) + d((x,y), m_2) \le \delta$. In addition, the medial curve cannot cross the normal lines joining m_i to the $\gamma_i^\pm$; otherwise, there would be a point p of the medial curve on the normal line with an associated circle C_p of radius $r_p \le \min d(p, \gamma_i^\pm)$, thereby violating the property that medial circles lie entirely within the boundary curve. We may therefore restrict the location of medial centers to lie between the normal lines. Denote this subset of the ellipse by E.

Now consider possible radii for each $p \in E$. The maximum possible radius at p, r_p, will be determined by the distance between p and the nearest boundary point, $\gamma_p \in \{\gamma_i^{\pm}\}$. This creates a natural partition of E into four regions $E_i^{\pm}$ consisting of those points at least as close to the corresponding boundary point as to all other boundary points. Furthermore, the boundaries of these regions will be given by the lines of equidistance between pairs of boundary points. Denote by $l^{\pm}$ the lines defined by $d((x,y),\gamma_1^{\pm}) = d((x,y),\gamma_2^{\pm})$.

The line l^+ enters E at a point p^+, and l^- enters E at a point p^-. Set $r^{\pm} = d(p^{\pm},\gamma_i^{\pm})$, the maximum radius possible at $p^{\pm}$. We define $C^{\pm}$ to be the circles of radius $r^{\pm}$ centered at $p^{\pm}$.

In order to prove the regions $R^{\pm}$ are maximal, we require the following lemma, proved in the Appendix.

Lemma 28. *Fix $x_0 \geq 0$ and $c \leq 0$, and consider all circles passing through the origin whose centers (a,b) satisfy:*

1. *$d((a,b),(0,0)) \leq d((a,b),(x_0,0))$,*
2. *$0 \leq a$, $b \leq c$.*

By construction, the radius associated to each center, $r_{a,b} = \sqrt{a^2+b^2}$. Define a function associated to each circle, $g_{a,b}(x)$, whose graph is the part of the circle lying above the x-axis. Then $g_{x_0/2,c}(x) \geq g_{a,b}(x)$ for $x \in [0, x_0/2]$.

Returning to the original problem, we show that for $p \in E$, the largest medial circle centered at p is contained in the region bounded by C_1, C_2, C^+ and C^-.

Choose $p \in E$. WLOG, assume $p \in E_1^+$, and set $\gamma_1^+ = (0,0)$, $\gamma_2^+ = (x_0/2, 0)$, $p^+ = (x_0/2, c)$, which makes l^+ correspond to the line $x = x_0/2$. Note that E_1^+ lies entirely in the valid region for centers (a,b) in the lemma, and therefore p^+ is the center of the circle corresponding to the maximum g of the lemma. Therefore, C^+ gives the arc of the highest circle above the line joining γ_1^+ and γ_2^+. The maximality of C^+, together with the constraints on the maximum radii, $\{r_p\}$, completes the proof of the following Theorem:

Theorem 29. *Let $m_1 = m(v_1)$, $m_2 = m(v_2)$ for $|v_1 - v_2| \leq \delta < min\{r_1, r_2\}$ be two points on a medial branch parameterized by arclength, with radii r_i, associated to boundary points, $\gamma_i^{\pm}$, $i = 1,2$. Define C^+ to be the circle centered at p^+ of radius $r^+ = |\gamma_1^+ - p^+|$ and C^- to be the circle centered at p^- of radius $r^- = |\gamma_1^- - p^-|$, where $p^{\pm}$ is the point of intersection of the line of equidistance between $\gamma_1^{\pm}$ and $\gamma_2^{\pm}$ and the ellipse of points whose distances to m_1 and m_2 sum to δ. Define $R^{\pm} = \operatorname{cl}\left(C^{\pm} \bigcup \left(C_1^{\pm} \bigcup C_2^{\pm}\right)^c\right)$. Then for $v \in [v_1, v_2]$, $\gamma_+(v) \subset R^+$ and $\gamma_-(v) \subset R^-$.*

Returning to the metric ρ, we present the analytic analogue of Theorem 29. Here $\bar{R}$ is the maximum value for r; its value depends on Ω.

Proposition 30. *Suppose $(m_i(v), r_i(v))$, $v \in [0,l]$, are C^1 medial branches for γ_i, a boundary curve with tangent angle functions θ_i, and inward-pointing normal vectors $\mathbf{N}_i$, $i = 1,2$. Then:*

$$\rho(\gamma_1,\gamma_2) \leq \sup_v \tfrac{1}{\lambda}\left(|m_1 - m_2| + |r_1 - r_2| + \bar{R}\left(|\theta_{m_1} - \theta_{m_2}| + |\phi_1 - \phi_2|\right)\right) + |\theta_{m_1} - \theta_{m_2}| + |\phi_1 - \phi_2|.$$

Proof.

$$
\begin{aligned}
\rho(\gamma_1, \gamma_2) &\leq \sup_v \tfrac{1}{\lambda} \left|\gamma_1(v) - \gamma_2(v)\right| + \left|\theta_1(v) - \theta_2(v)\right| \\
&= \sup_v \tfrac{1}{\lambda} \left(\left|m_1 + r_1\mathbf{N}_{\gamma_1} - m_2 - r_2\mathbf{N}_{\gamma_2}\right|\right) + \left|\theta_{m_1} + \phi_1 - \theta_{m_2} - \phi_2\right| \\
&\leq \sup_v \tfrac{1}{\lambda} \left(|m_1 - m_2| + \left|r_1\mathbf{N}_{\gamma_1} - r_2\mathbf{N}_{\gamma_2}\right|\right) + |\theta_{m_1} - \theta_{m_2}| + |\phi_1 - \phi_2| \\
&\leq \sup_v \tfrac{1}{\lambda} \left(|m_1 - m_2| + |r_1 - r_2| + \bar{R}\left|\mathbf{N}_{\gamma_1} - \mathbf{N}_{\gamma_2}\right|\right) + \left(|\theta_{m_1} - \theta_{m_2}| + |\phi_1 - \phi_2|\right) \\
&\leq \sup_v \tfrac{1}{\lambda} \left(|m_1 - m_2| + |r_1 - r_2| + \bar{R}\left(|\theta_{m_1} - \theta_{m_2}| + |\phi_1 - \phi_2|\right)\right) \\
&\qquad + |\theta_{m_1} - \theta_{m_2}| + |\phi_1 - \phi_2|
\end{aligned}
$$

□

6.1.3 Counting Branches

For the medial axis to be acceptable as a shape descriptor, we need to make sure that reasonably simple curves produce axes with a reasonable number of branches.

Lemma 31. *Let Ω be a bounded, simply connected region of the plane, and $(\mathbf{m}, r)$ the medial axis pair for a generic, twice-differentiable, simple closed curve $\gamma \subset \Omega$. Then if γ has N local maxima of curvature, the number of branches in $\mathbf{m}$ is at most $M = 2N - 3$.*

Proof. By Theorem 27, the medial axis of a generic curve consists of three types of points: points where the medial circle osculates at the endpoint of an axis branch, tritangent points where three medial branches come together, and bitangent points in the interior of a medial branch.

If the medial circle osculates at the endpoint of a branch, γ must have a local maximum of curvature at the point of osculation. On the other hand, every local maximum of curvature does not necessarily correspond to an osculating medial circle. Therefore, the number of endpoint branches is at most N.

Because γ is a simple, closed, C^2 curve, $\mathbf{m}$ is connected. As the deformation retract of the boundary of a contractible space, it is also contractible. Therefore, the graph of $\mathbf{m}$ is a tree, where edges correspond to bitangent circles and vertices correspond to tritangent circles. Since vertices will only occur at tritangent points, a simple counting argument gives that the number of vertices for the medial graph with k endpoints is $k - 2$, which gives the number of edges as $2k - 3$. Hence the number of branches M in $\mathbf{m}$ satisfies $M \leq 2N - 3$.

□

7 Adaptive Coding and Optimal Representation

At last we return to the question of whether the region-based medial axis model or the boundary curve is a more efficient representation for a particular shape. In the compact setting, ε-entropy provides a benchmark for optimal description length, and so efficiency of fixed length encodings for elements of compact spaces can be decisively determined by comparison to the ε-entropy of the space, yielding the boundary curve as the clear winner. No such benchmark exists in the non-compact setting, but we can make reasonable comparisons.

To compare efficiency of adaptive encodings, we turn to Rissanen's Minimum Description Length (MDL) principle [18]. Thanks to Shannon [22], it is well known that given a probability distribution P on a space $\mathcal{X}$, the best compression scheme for elements $x \in \mathcal{X}$ will assign the shortest code lengths to the most probable elements. Code lengths, $\mathcal{L}_P$, are therefore completely defined by the probability distribution: $\mathcal{L}_P(x) = -\log_2 P(x)$. Conversely, every coding scheme C gives rise to a sub-probability distribution on $\mathcal{X}$, where probabilities are defined by the code lengths: $P_C(x) = 2^{-L(x)}$. The better the compression scheme, the closer to a true probability distribution P_C will be.

Behind this formulation is the idea that the best compression scheme will exploit regularities in the space, thereby resulting in the smallest expected code length. This gives a method for evaluating coding schemes with or without a probabilistic framework: the encoding resulting in the shortest code length for the largest proportion of the space is the one best capturing the salient aspects of the space.

While the boundary curve gives an optimal fixed-length bit rate, moving to a variable-length adaptive code allows the medial axis to shine. We derive a precise characterization of curves for which the medial axis gives a shorter adaptive codelength.

In both fixed- and adaptive-length encoding, the high-order term in the bit length comes from encoding the tangent angle function. This observation allows us to exploit the way in which the medial axis encodes the geometry of the boundary curve. In particular, since $\theta_\pm = \theta_m \pm \phi + \pi/2$, to construct an ε-approximation to θ_γ requires approximations to θ_m and ϕ with error summing to ε. For $0 < \eta < \varepsilon$, if we η-encode θ_m and $(\varepsilon - \eta)$-encode ϕ, we obtain an encoding for θ_γ with leading bit length term:

$$\frac{\int |\theta_\gamma|\, dv + \delta_1}{\eta} + \frac{\int |\phi|\, dv + \delta_2}{\varepsilon - \eta}.$$

Note that this also explains why the medial axis does poorly in the fixed length setting: to construct a uniform covering with ε-balls, both θ_m and ϕ must be accurate within $\varepsilon/2$ to guarantee θ_γ has ε-accuracy. For most curves, such a high level of accuracy in both angle functions is overkill.

To relate the number of bits in this medial axis encoding to the number of bits in a boundary encoding, we require the following lemma:

Lemma 32. *Let m be a branch of a medial curve of length l for $\gamma \in \mathcal{E}$. Let s be an arclength parameter for γ so that $\gamma(s)|_{s\in D} = \gamma_+ \cup \gamma_-$. Then:*

$$\int_D |\kappa|\, ds = \int_{[0,l]} |\kappa_m| + |\phi'| + ||\kappa_m| - |\phi'||\ dv.$$

Proof. Recall from Theorem 22 that within a branch of a medial curve, $\mathbf{N}_\pm = -\cos\phi \mathbf{T}_m \mp \sin\phi \mathbf{N}_m$, and so:

$$\begin{aligned}
\tfrac{d\mathbf{N}_\pm}{dv} &= \phi' \sin\phi \mathbf{T}_m - \kappa_m \cos\phi \mathbf{N}_m \mp \phi' \cos\phi \mathbf{N}_m \pm \sin\phi \kappa_m \mathbf{T}_m \\
&= (-\kappa_m - \phi)\,(\mp \sin\phi \mathbf{T}_m) + (-\kappa_m - \phi) \cos\phi \mathbf{N}_m \\
&= (\kappa_m \mp \phi')\, \mathbf{T}_\pm.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_D |\kappa_\gamma|\, ds &= \int_{[0,l]} |\kappa_+|\, ds_+ \int_{[0,l]} |\kappa_-|\, ds_- \\
&= \int |\tfrac{d\mathbf{N}_+}{ds_+}||\tfrac{ds_+}{dv}|\, dv + \int |\tfrac{d\mathbf{N}_-}{ds_-}||\tfrac{ds_-}{dv}|\, dv \\
&= \int |\tfrac{d\mathbf{N}_+}{dv}||\tfrac{dv}{ds_+}||\tfrac{ds_+}{dv}|\, dv + \int |\tfrac{d\mathbf{N}_-}{dv}||\tfrac{dv}{ds_-}||\tfrac{ds_-}{dv}|\, dv \\
&= \int |\kappa_m - \phi'|\, dv + \int |\kappa_m + \phi'|\, dv \\
&= \int_{[0,l]} |\kappa_m| + |\phi'| + ||\kappa_m| - |\phi'||\ dv.
\end{aligned}$$

□

Selecting the optimal value for the error tolerance η within a fixed region of a branch of $\mathbf{m}$ depends on the behavior of the medial axis there. The optimal value for η will determine the bit length of a medial encoding for that region which will in turn determine whether or not the medial axis outperforms the boundary curve there. In other words, we wish to minimize the expression:

$$\frac{\int_{[0,l]} |\kappa_m| \, dv}{\eta} + \frac{\int_{[0,l]} |\phi'| \, dv}{\varepsilon - \eta}$$

with respect to η for $0 \leq \eta \leq \varepsilon$. Note that since the δ_i can be made arbitrarily small, we have dropped reference for ease of computation. Set $a = \int_{[0,l]} |\kappa_m| \, dv$ and $b = \int_{[0,l]} |\phi'| \, dv$ The minimization then gives $\eta = \varepsilon/2$ for $a = b$ and $\eta = \frac{\varepsilon}{b-a}\left(\sqrt{ab} - a\right)$ for $a \neq b$. We will now adaptively select regions of the medial curve and associated local optimal values for η to give the best bit rate for the medial axis within a particular branch.

Theorem 33. *Let m be an arclength parameterized medial branch defined on a closed interval I for $\gamma \in \mathcal{E}$ with corresponding boundary segments $\gamma_\pm$. Then encoding $\gamma_\pm$ via the medial axis is more efficient than directly adaptively encoding θ_γ whenever I can be partitioned into a finite number of subintervals I_j where for each j:*

$$\begin{aligned} \frac{\sup_{I_j} |\kappa_m|}{\sup_{I_j} |\phi'|} &> 2 + \sqrt{3} \text{ or} \\ \frac{\sup_{I_j} |\phi'|}{\sup_{I_j} |\kappa_m|} &> 2 + \sqrt{3}. \end{aligned}$$

Proof. Following the proof of Theorem 20, select g_κ and δ_1 satisfying $|\kappa_m| \leq g_\kappa$ and $\int g_\kappa \leq \int |\kappa_m| + \delta_1$, and select g_ϕ and δ_2 satisfying $|\phi'| \leq g_\phi$ and $\int g_\phi \leq \int |\phi'| + \delta_2$. Construct an encoding scheme as follows. Partition I, the domain of m, into maximal subdomains $\{I_i\}$ on which both g_κ and g_ϕ are constant. Since both functions are piecewise constant with a finite number of jumps, the number of such subdomains will be finite. On each subdomain, compute the minimizing η_i. Then the medial axis will be more efficient when:

$$g_\kappa + g_\phi + 2\sqrt{g_\kappa g_{\phi'}} < 2\max\{g_\kappa, g_{\phi'}\}.$$

For $g_\kappa \geq g_\phi$, this gives:

$$\frac{g_\kappa}{g_{\phi'}} > 2 + \sqrt{3},$$

otherwise take the reciprocal of the left side of the inequality.

Recalling the construction of the functions g_κ and g_ϕ, the result is proved.

□

An interpretation of Theorem 33 is that the medial axis decouples the curvature of the boundary curve into the portion coming from the curvature of the local axis of symmetry, i.e., the medial curve, and the portion coming from variation around that axis. When the boundary curvature comes primarily from one source or the other, the medial axis is more efficient. When the curvature of the boundary relies heavily on both sources (the most extreme case of which is when $a = b$), the boundary curve is more efficient.

This result partitions shapes silhouettes into two classes: those that are best modeled by the boundary (edge-based model) and those that are best modeled by a skeleton (region-based model). Figure 9 gives an example of each, with the leading term in the associated bit rates indicated for $\varepsilon = 0.01$. A forthcoming paper will apply this partitioning criterion to large databases of shape contours.

Figure 9: Two shapes with bit rates for medial axis and boundary encodings. The fish is better modeled by the medial axis while the rectangle prefers the boundary model.

Acknowledgements

Untold thanks is owed to my thesis advisor, David Mumford, and mentor, Jill Pipher, for their mathematical support of this work. Thanks is also due to the Brown University Department of Mathematics for equally necessary financial support.

A Proof of Lemma 8

Proof. Construct a cover by taking a collection of curves with tangent angle equal to $\tilde{\theta}$ and with jump discontinuities in suitable directions at suitable intervals. To see which jumps and which intervals, take $\gamma \in \Gamma_\delta$, and let γ' be a curve satisfying $|\gamma(0) - \gamma'(0)| \leq \frac{\delta^\xi}{2}$, $\theta'(s) = \tilde{\theta}(s)$, so that for some Δ and n, γ' jumps by $\frac{\delta^\xi}{4} + \frac{\delta\Delta}{2}$ in one of n specified directions every Δ in arclength. See Figure 10. We must determine values for Δ and n so that $\rho(\gamma, \gamma') \leq \delta + \frac{\delta^\xi}{\lambda}$. By hypothesis, $|\theta(s) - \theta'(s)| \leq \delta$, so it remains to show $|\gamma(s) - \gamma'(s)| \leq \delta^\xi$.

Figure 10: Correction of approximation curve γ'.

Assume $|\gamma(a_k) - \tilde{\gamma}(a - k)| \leq \frac{\delta^\xi}{2}$ for $a_k = k\Delta$, $k < \left[\frac{L}{\Delta}\right]$. Then:

$$\begin{aligned} |\gamma(a_k + \Delta) - \gamma'(a_k + \Delta)| &\leq \int_{a_k}^{a_k+\Delta} 2\left|\sin\left(\frac{\theta - \theta'}{2}\right)\right| dt + \frac{\delta^\xi}{2} \\ &\leq \delta\Delta + \tfrac{\delta^\xi}{2}. \end{aligned}$$

To ensure $|\gamma(s)-\gamma'(s)| \leq \delta^\xi$ for $s \in [a_k, a_k+\Delta]$, $\Delta \leq \frac{\delta^{\xi-1}}{2}$. If, for every choice of k, $|\gamma(a_k)-\gamma'(a_k)| \leq \frac{\delta^\xi}{2}$, then $|\gamma(s) - \gamma'(s)| \leq \delta^\xi$ for any s. We set about constructing such a γ'.

Figure 11: Distance after correction.

Suppose at the point $s = a_k + \Delta$, γ' jumps to the point O in Figure 11. Take the circle of radius $r = \delta\Delta + \frac{\delta^\xi}{2}$ centered at $P = \lim_{s \to a_k+\Delta} \gamma'(s)$. From P emerge radial lines in n directions, for some n, spaced so that each line makes an angle of $\frac{2\pi}{n}$ with the lines on either side of it. On the line closest to $\gamma(a_k + \Delta)$, γ' jumps out half the length of the spoke to O, which is at a distance of $\frac{r}{2} = \frac{\delta^\xi}{4} + \frac{\delta\Delta}{2}$ from P. The maximum for $|\gamma(a_k+\Delta)-\gamma'(a_k+\Delta)|$ occurs when $\gamma(a_k+\Delta)$ corresponds to the point Q on the radial line making an angle of $\frac{\pi}{2n}$ with the radial line containing O, giving a maximum value of $\sqrt{\frac{5}{4} - \cos\left(\frac{\pi}{n}\right)}\left(\delta\Delta + \frac{\delta^\xi}{2}\right)$. Choosing $n = 3$ and $\Delta = \frac{\delta^{\xi-1}}{2}$ produces a γ' so that $\rho(\gamma, \gamma') \leq \delta + \frac{\delta^\xi}{2}$ as desired.

Now construct the cover. For $\Delta = \frac{\gamma^{\xi-1}}{2}$ and $n = 3$ as above, divide the interval $[0, L]$ into subintervals of width Δ (except perhaps the last subinterval which might be shorter), giving subintervals $J_k = [a_k, a_{k+1}]$, $k = 1, \ldots, m$, where $m = [\frac{L}{\Delta}]$. We will construct piecewise C^1 curves that are allowed a jump discontinuity at each $s = a_{k+1}$ for $k < m$. Within each J_k, these curves will have tangent angle functions equal to $\tilde{\theta}$, and at $s = a_{k+1}$ each curve will jump a distance of $\frac{\delta^\xi}{4} + \frac{\delta\Delta}{2}$ in one of n directions, for some n. In other words, each of these curves satisfies the same properties as

γ', and we therefore take $n = 3$ and $\Delta = \frac{\delta^{\xi-1}}{2}$. We demonstrate that these are suitable curves for the centers of the covering.

Define $d_{i_k} = (\cos\psi_{i_k}, \sin\psi_{i_k})$, where for each $k = 1, \ldots, m$, $i_k \in \{1, \ldots, 3\}$, and ψ_{i_k} is an angle so that $\tilde{\theta}(a_k) \neq \psi_{i_k}$ for any k, and $|\psi_{i_k} - \psi_{i'_k}| = 2\pi/3$ for $i'_k = i_k \pm 1$. Then for $(i) = \{i_1, \ldots, i_m\}$, consider the collection of curves $\tilde{\gamma}^{(i)}(s) = \tilde{\gamma}(s) + \sum_{j=0}^{k} \frac{\delta^\xi}{2} d_{i_j}$ for $s \in [a_k, a_{k+1})$.

Associate to each $\tilde{\gamma}^{(i)}$ a curve $\gamma \in \Gamma_\delta$ by sequentially determining at each a_k the d_{i_k} that will minimize $|\tilde{\gamma}^{(i)}(a_k) - \gamma(a_k)|$. By the above argument for γ', for each (i), the set of associated curves is at most $\delta + \frac{\delta^\xi}{\lambda}$ away from $\tilde{\gamma}^{(i)}$ in the metric ρ, and therefore defines a ball of diameter no more than $2(\delta + \frac{\delta^\xi}{\lambda})$.

The number of curves $\tilde{\gamma}^{(i)}$ is $2^{\log 3 \left[\frac{2L}{\delta^{\xi-1}}\right]}$: for each k, $\tilde{\gamma}^{(i)}$ can jump in 3 possible directions, and $k \leq \left[\frac{2L}{\delta^{\xi-1}}\right]$.

□

B Proof of Lemma 9

Proof. For $\gamma \in \Gamma'_\delta$ of length l, taking γ' as in the proof of Lemma 8, of length $\tilde{l}$:

$$\begin{aligned}
|\gamma'(a_k + \Delta) - \gamma(a_k + \Delta)| &= \left| \int_{a_k}^{a_k+\Delta} \left\langle \cos\tilde{\theta}, \sin\tilde{\theta} \right\rangle - \frac{l}{\tilde{l}} \left\langle \cos\theta, \sin\theta \right\rangle dt \right| + \frac{\delta^\xi}{2} \\
&\leq \left| \int_{a_k}^{a_k+\Delta} \left\langle \cos\tilde{\theta}, \sin\tilde{\theta} \right\rangle - \frac{4\tilde{l} - \delta^\xi}{4\tilde{l}} \left\langle \cos\theta, \sin\theta \right\rangle dt \right| + \frac{\delta^\xi}{2} \\
&\leq \int_{a_k}^{a_k+\Delta} \left| \left\langle \cos\tilde{\theta}, \sin\tilde{\theta} \right\rangle - \left\langle \cos\theta, \sin\theta \right\rangle \right| dt + \frac{\delta^\xi}{4\tilde{l}} \left| \int_{a}^{a+\Delta} \left\langle \cos\theta, \sin\theta \right\rangle dt \right| \\
&\quad + \frac{\delta^\xi}{2} \\
&\leq \int_{a_k}^{a_k+\Delta} \left| 2\sin\left(\frac{\tilde{\theta}(t) - \theta(t)}{2} \right) \right| dt + \frac{\delta^\xi}{4\tilde{l}}\Delta + \frac{\delta^\xi}{2} \\
&\leq (\delta + \frac{\delta^\xi}{4\tilde{l}})\Delta + \frac{\delta^\xi}{2} \\
&\leq \delta\Delta + \frac{\delta^\xi}{4} + \frac{\delta^\xi}{2}.
\end{aligned}$$

Adjusting the argument from Lemma 8, we may take $n = 3$ as before, and set $\Delta = \frac{\delta^{\xi-1}}{4}$, giving a covering with at most $2^{\log 3 \left[\frac{4L}{\delta^{\xi-1}}\right]}$ as claimed.

□

C Proof of Lemma 12

Proof. Let $\{\epsilon_i\}_1^n$ be a collection of independent random variables taking on the values ± 1 with equal probability. Then a symmetric n-step random walk g has the property that $g(n) = \sum_1^n \epsilon_i$ and $a(n) = \sum_1^n (n - i + \frac{1}{2})\epsilon_i$, which leads to a simple computation for Σ: $\mathrm{Var}(g(n)) = n$, $\mathrm{Var}(a(n)) = \frac{n^3}{3} - \frac{n}{12}$, and $\mathrm{Cov}(g(n), a(n)) = \frac{n^2}{2}$. This gives:

$$\begin{aligned} z &= \frac{(4n^2-1)g(n)^2 - 12ng(n)a(n) + 12a^2}{n^3-n}, \\ \mathrm{E}(z) &= 2. \end{aligned}$$

Then by the Markov inequality, $P(z \geq \alpha) \leq \frac{\mathrm{E}(z)}{\alpha} = \frac{2}{\alpha}$, which gives $P(z \leq \alpha) \geq 1 - 2/\alpha$. Taking $\alpha = 3$, we have $P(z \leq \alpha) \geq 1/3$, which gives the number of random walks g resulting in $z \leq 3$ to be at least $\frac{1}{3}2^n$.

If $z \leq 3$, then the point $(g(n), a(n))$ is inside the ellipse

$$\frac{(4n^2-1)g(n)^2 - 12ng(n)a(n) + 12a^2}{n^3-n} = 3,$$

which gives $|g(n)| \leq 3\sqrt{n}$ and $|a(n)| \leq \frac{\sqrt{3}}{2}\sqrt{4n^3-n}$ as desired. This may be seen by finding where the tangent lines to the ellipse are horizontal and vertical.

□

D Proof of Lemma 13

Proof. WLOG, assume $g'(n') \geq 0$ and $a(n') \geq 0$. Clearly, we may extend g' to return to the origin in $g'(n')$ steps, which is bounded above by $3\sqrt{n}$. This correction adds at most $\frac{9n}{2}$ in positive area, giving

$$a(n' + g'(n')) \leq \frac{9n'}{2} + \frac{\sqrt{3}}{2}\sqrt{4n'^3 - n'}.$$

We will correct the area by adding on stepped pieces beginning and ending on the horizontal axis, forming triangles with base of width b_i, height $b_i/2$, and area $\frac{b_i^2}{4}$, $b_i \in \mathbb{Z}^+$. This gives $4a(n'+g(n')) = \sum_i b_i^2$, obtainable in $\sum_i b_i$ steps for some i. Since any integer may be expressed as a sum of four squares [21]: $4a(n' + g(n')) = b_1^2 + b_2^2 + b_3^2 + b_4^2$, where each $b_i \leq \left\lceil \sqrt{4a(n' + g(n'))} \right\rceil$. Therefore, we find the number of steps required to recover the area is:

$$\begin{aligned} \sum_{i=1}^{4} b_i &\leq 4 \cdot 2 \left\lceil \sqrt{a(n' + g(n'))} \right\rceil \\ &\leq 8\sqrt{\frac{9n'}{2} + \frac{\sqrt{3}}{2}\sqrt{4n'^3 - n'}}. \end{aligned}$$

This means that obtaining g such that $g(n) = 0$ and $\int g = 0$ requires at most $\frac{\sqrt{3}}{2}\sqrt{n'} + 8\sqrt{\frac{9n'}{2} + \frac{\sqrt{3}}{2}\sqrt{4n'^3 - n'}}$ steps which completes the proof of this lemma.

□

E Proof of Lemma 28

Proof. Choose any center (a, b) satisfying the hypotheses of the lemma. If $a \neq x_0/2$, then the circle centered at (a, b) will be completely contained in the circle centered at $(x_0/2, bx_0/2a)$, the point of intersection of $x = x_0/2$ and the line through the origin and (a, b). Therefore $g_{a,b}(x) < g_{x_0/2, bx_0/2a}(x)$, and we need only consider points on the line $x = x_0/2$ for $b \leq c$, resulting in circles containing the point $(x_0, 0)$ as well as the origin. See Figure 12.

Figure 12: Maximal circle.

Define $\theta_{a,b}(x)$ to be the angle between the x-axis and the tangent vector to $g_{a,b}(x)$. Then $b_1 \leq b_2 \Rightarrow \theta_{a,b_1}(x) \leq \theta_{a,b_2}(x)$, and so $\theta_{x_0/2,b}(x) \leq \theta_{x_0/2,c}(x)$ for $x < \frac{x_0}{2}$. This means $\frac{dg_{x_0/2,b}}{dx}(x) \leq \frac{dg_{x_0/2,c}}{dx}(x)$, which, together with $0 = g_{x_0/2,c}(0) = g_{x_0/2,b}(x)$, gives $g_{x_0/2,c}(0) \leq g_{x_0/2,b}(x)$ for every x in their domain.

□

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