THE SINGULAR SETS OF AREA MINIMIZING RECTIFIABLE CURRENTS WITH CODIMENSION ONE AND OF AREA MINIMIZING FLAT CHAINS MODULO TWO WITH ARBITRARY CODIMENSION[1]

BY HERBERT FEDERER

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1. When describing the interior structure of an area minimizing m dimensional locally rectifiable current T in $\mathbf{R}^{m+1}$, one calls a point $x \in \operatorname{spt} T \sim \operatorname{spt} \partial T$ regular or singular according to whether or not x has a neighborhood V such that $V \cap \operatorname{spt} T$ is a smooth m dimensional submanifold of $\mathbf{R}^{m+1}$. As a result of the efforts of many geometers it is known that there exist no singular points in case $m \leqq 6$; a detailed exposition of this theory may be found in [3, Chapter 5]. Recently it was proved in [2] that

$$Z = \partial(\mathbf{E}^8 \llcorner \mathbf{R}^8 \cap \{x : x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\})$$

is a 7 dimensional area minimizing current in $\mathbf{R}^8$ with the singular point 0. This implies that, for $m > 7$, $\mathbf{E}^{m-7} \times Z$ is an m dimensional area minimizing current in $\mathbf{R}^{m-7} \times \mathbf{R}^8 \simeq \mathbf{R}^{m+1}$ with the $m-7$ dimensional singular set $\mathbf{R}^{m-7} \times \{0\}$. Here we will show (Theorem 1) that the Hausdorff dimension of the singular set of an m dimensional area minimizing rectifiable current in $\mathbf{R}^{m+1}$ never exceeds $m-7$.

Our method also yields the result (Theorem 2) that the Hausdorff dimension of the singular set of an m dimensional area minimizing flat chain modulo 2 in $\mathbf{R}^{m+p}$ never exceeds $m-2$, for arbitrary codimension p.

2. We use the terminology of [3]. Given any positive integer m we choose Υ according to [3, 5.4.7] with $n = m+1$ and let

$$\omega(T) = \{x : \Theta^m(\|T\|, x) \geqq \Upsilon\} \quad \text{for } T \in \mathcal{R}_m^{\text{loc}}(\mathbf{R}^{m+1}).$$

Whenever $0 \leqq k \in \mathbf{R}$ and $A \subset \mathbf{R}^{m+1}$ we define $\phi_\infty^k(A)$ as the infimum of the set of numbers $\sum_{B \in G} \alpha(k) 2^{-k} (\operatorname{diam} B)^k$ corresponding to all countable open coverings G of A. We see from [3, 2.10.2] that

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$$\phi_\infty^k(A) = 0 \quad \text{if and only if } \mathcal{H}^k(A) = 0,$$

and from [3, 2.10.19(2)] that

$$\Theta^{*k}(\phi_\infty^k \llcorner A, x) \geqq 2^{-k} \quad \text{for } \mathcal{H}^k \text{ almost all } x \text{ in } A.$$

LEMMA 1. *If $Q_i \in \mathcal{R}_m^{\text{loc}}(R^{m+1})$ and Q_i is absolutely area minimizing with respect to R^{m+1} for each positive integer i,*

$$Q_i \to Q \quad in\ \mathcal{F}_m^{\text{loc}}(R^{m+1}) \quad as\ i \to \infty,$$

and K is a compact subset of $R^{m+1} \sim \text{Clos} \bigcup_{i=1}^\infty \text{spt}\, \partial Q_i$, then

$$\phi_\infty^k[\omega(Q) \cap K] \geqq \limsup_{i\to\infty} \phi_\infty^k[\omega(Q_i) \cap K].$$

PROOF. We observe that if V is any open set containing $\omega(Q) \cap K$, then V contains $\omega(Q_i) \cap K$ for all sufficiently large integers i. Otherwise we could choose a subsequence of points $x_i \in \omega(Q_i) \cap K \sim V$ converging to point $x \in K \sim V$. Since

$$d = \text{dist}\left(K, \bigcup_{i=1}^\infty \text{spt}\, \partial Q_i\right) > 0,$$

we would find whenever $d > r > s > 0$ that $s^{-m}\|Q_i\| U(x_i, s) \geqq \alpha(m)\Upsilon$ according to [3, 5.4.3(3)], with $B(x_i, s) \subset U(x, r)$ for large i, hence

$$\|Q\| U(x, r) \geqq \limsup_{i\to\infty} \|Q_i\| U(x_i, s) \geqq s^m \alpha(m)\Upsilon$$

by [3, 5.4.2]. Thus $\|Q\| U(x, r) \geqq r^m \alpha(m)\Upsilon$ for $0 < r < \delta$, and we could infer that $x \in \omega(Q) \cap (K \sim V) = \emptyset$.

LEMMA 2. *If $T \in \mathcal{R}_m^{\text{loc}}(R^{m+1})$, T is absolutely area minimizing with respect to R^{m+1}, $a \in \text{spt}\, T \sim \text{spt}\, \partial T$ and $\Theta^{*k}[\phi_\infty^k \llcorner \omega(T), a] > 0$, then there exists an oriented tangent cone Q of T at a such that $\mathcal{H}^k[\omega(Q)] > 0$.*

PROOF. Assuming $\Theta^{*k}[\phi_\infty^k \llcorner \omega(T), a] > 2^k c > 0$ and recalling the proof of [3, 5.4.3], in particular the argument on pages 624 and 625, we choose ρ_i and β_i for each positive integer i so that

$$0 < 2\rho_i < i^{-1}\sigma_i, \qquad \phi_\infty^k[\omega(T) \cap B(a, \rho_i)] > \alpha(k)\rho_i^k 2^k c,$$

$$\beta_i^{-1} \in G_i, \qquad \rho_i \leqq (1 - i2^{-i})2\rho_i < \beta_i^{-1} < 2\rho_i.$$

Then $\phi_\infty^k[\omega(T) \cap B(a, \beta_i^{-1})] > \alpha(k)\beta_i^{-k} c$ and the corresponding currents $Q_i = (\boldsymbol{\mu}_{\beta_i} \circ \boldsymbol{\tau}_{-a})_\# T$ satisfy the condition $\phi_\infty^k[\omega(Q_i) \cap B(0, 1)] > \alpha(k)c$. A subsequence of $Q_1, Q_2, Q_3, \cdots$ converges in $\mathcal{F}_m^{\text{loc}}(R^{m+1})$ to an oriented tangent cone Q of T at a, for which $\phi_\infty^k[\omega(Q)\cap \boldsymbol{B}(0, 1)] \geqq \boldsymbol{\alpha}(k)c$ according to Lemma 1.

THEOREM 1. *If $T\in \mathcal{R}_m^{\mathrm{loc}}(\boldsymbol{R}^{m+1})$, $m\geqq 7$ and T is absolutely area minimizing with respect to $\boldsymbol{R}^{m+1}$, then there exists an open set V such that $V\cap \operatorname{spt} T$ is an m dimensional submanifold of class ∞ of $\boldsymbol{R}^{m+1}$ and*

$$\mathcal{H}^k[\boldsymbol{R}^{m+1} \sim (V \cup \operatorname{spt} \partial T)] = 0 \quad \textit{whenever } m - 7 < k \in \boldsymbol{R}.$$

PROOF. We use induction with respect to m. First we will prove the following statement:

If M is an $\mathcal{L}^{m+1}$ measurable set, U is an open subset of $\boldsymbol{R}^{m+1}$,

$$S = [\partial(\boldsymbol{E}^{m+1} \llcorner M)] \llcorner U \in \mathcal{R}_m(\boldsymbol{R}^{m+1})$$

and S is absolutely area minimizing with respect to $\boldsymbol{R}^{m+1}$, then there exist an open set W such that $W\cap \operatorname{spt} S$ is an m dimensional submanifold of class ∞ of $\boldsymbol{R}^{m+1}$ and

$$\mathcal{H}^k(U \sim W) = 0 \quad \textit{whenever } m - 7 < k \in \boldsymbol{R}.$$

In view of [3, 5.4.7] it suffices to show that

$$\mathcal{H}^k[U \cap \omega(S)] = 0 \quad \text{whenever } m - 7 < k \in \boldsymbol{R}.$$

Assuming the contrary we choose $k > m-7$ and $a\in U\cap \omega(S)$ so that $\Theta^{*k}[\phi_\infty^k \llcorner \omega(S), a] > 0$, apply Lemma 2 to obtain an oriented tangent cone C of S at a with $\mathcal{H}^k[\omega(C)] > 0$, and infer from [3, 5.4.3(5), (8)] that C is absolutely area minimizing with respect to $\boldsymbol{R}^{m+1}$ and $C = \partial(\boldsymbol{E}^{m+1} \llcorner N)$ for some $\mathcal{L}^{m+1}$ measurable set N. Since $\mathcal{H}^k\{0\} = 0$ we can choose $b\in\omega(C)\sim\{0\}$ so that $\Theta^{*k}[\phi_\infty^k \llcorner \omega(C), b] > 0$, and repeat the procedure to construct an oriented tangent cone D of C at b such that $\mathcal{H}^k[\omega(D)] > 0$, D is absolutely area minimizing with respect to $\boldsymbol{R}^{m+1}$ and $D = \partial(\boldsymbol{E}^{m+1} \llcorner P)$ for some $\mathcal{L}^{m+1}$ measurable set P. We infer from [3, 4.3.16] that D is a cylinder with direction $b/|b|$, from [3, 4.3.15] that there exist an isometry H mapping $\boldsymbol{R}\times \boldsymbol{R}^m$ onto $\boldsymbol{R}^{m+1}$ and a current $Q\in \mathcal{R}_{m-1}^{\mathrm{loc}}(\boldsymbol{R}^m)$ with $D = H_\#(\boldsymbol{E}^1\times Q)$, and from [3, 5.4.8] that Q is absolutely $m-1$ area minimizing with respect to $\boldsymbol{R}^m$. We note that $\partial Q = 0$ because $\partial D = 0$. In case $m\geqq 8$ we inductively obtain an open subset Y of $\boldsymbol{R}^m$ such that $Y\cap \operatorname{spt} Q$ is an $m-1$ dimensional submanifold of class ∞ of $\boldsymbol{R}^m$ and $\mathcal{H}^{k-1}(\boldsymbol{R}^m \sim Y) = 0$. In case $m = 7$ we know from [3, 5.4.15] that spt Q is a 6 dimensional submanifold of class ∞ of $\boldsymbol{R}^7$, and we take $Y = \boldsymbol{R}^7$. In both cases $H(\boldsymbol{R}\times Y)\cap \operatorname{spt} D$ is an m dimensional submanifold of class ∞ of $\boldsymbol{R}^{m+1}$ and

$$\mathcal{H}^k[\boldsymbol{R}^{m+1} \sim H(\boldsymbol{R} \times Y)] = \mathcal{H}^k[\boldsymbol{R} \times (\boldsymbol{R}^m \sim Y)] = 0$$

by [**3**, 2.10.45]. Since $D=\partial(\mathbf{E}^{m+1} \llcorner P)$ we see that

$$\Theta^m(\|D\|, x) = 1 \quad \text{for } x \in H(\mathbf{R} \times Y) \cap \operatorname{spt} D,$$

hence $\omega(D) \subset \operatorname{spt} D \sim H(\mathbf{R} \times Y)$ and $\mathcal{H}^k[\omega(D)]=0$, which is inconsistent with our previous assertion that $\mathcal{H}^k[\omega(D)]>0$.

To deduce the conclusion of the theorem from the statement verified above we suppose $a \in \mathbf{R}^{m+1} \sim \operatorname{spt} \partial T$ and proceed as in [**3**, 5.3.18] to find a positive number ρ and a representation

$$T \llcorner U(a, \rho) = \sum_{i \in \mathbf{Z}} S_i \quad \text{with} \quad \|T\| \llcorner U(a, \rho) = \sum_{i \in \mathbf{Z}} \|S_i\|,$$

where $S_i = [\partial(\mathbf{E}^{m+1} \llcorner M_i)] \llcorner U(a, \rho)$ for certain $\mathcal{L}^{m+1}$ measurable sets M_i such that $M_i \subset M_{i-1}$; moreover $\{i: b \in \operatorname{spt} S_i\}$ is finite whenever $b \in U(a, \rho)$. For each integer i we choose an open set W_i such that $W_i \cap \operatorname{spt} S_i$ is an m dimensional submanifold of class ∞ of $\mathbf{R}^{m+1}$ and

$$\mathcal{H}^k[U(a, \rho) \sim W_i] = 0 \quad \text{whenever} \quad m - 7 < k \in \mathbf{R}.$$

We conclude that $B = U(a, \rho) \sim \bigcup_{i \in \mathbf{Z}} (\operatorname{spt} S_i \sim W_i)$ is open,

$$U(a, \rho) \sim B \subset \bigcup_{i \in \mathbf{Z}} [U(a, \rho) \sim W_i],$$

$$\mathcal{H}^k[U(a, \rho) \sim B] = 0 \quad \text{whenever} \quad m - 7 < k \in \mathbf{R},$$

$$B \cap \operatorname{spt} T = \bigcup_{i \in \mathbf{Z}} B \cap \operatorname{spt} S_i = \bigcup_{i \in \mathbf{Z}} B \cap W_i \cap \operatorname{spt} S_i,$$

and $B \cap \operatorname{spt} T$ is an m dimensional submanifold of class ∞ of $\mathbf{R}^{m+1}$ because for each $b \in B \cap \operatorname{spt} T$ one can reason as in [**3**, 5.4.15, p. 646] with a replaced by b to see that $\operatorname{Tan}(\operatorname{spt} T, b)$ is an m dimensional vector space, hence infer from [**3**, 5.3.18] that b is a regular point for T.

It is not yet known whether the conclusion of Theorem 1 could be sharpened so as to require that $\mathcal{H}^{m-7}(K \sim V) < \infty$ for every compact subset K of $\operatorname{spt} T \sim \operatorname{spt} \partial T$; in case $m=7$ this holds according to [**3**, 5.4.16].

3. Next we discuss area minimizing m dimensional chains with arbitrary codimension p in $\mathbf{R}^{m+p}$. When $p>1$ the singular set can have dimension $m-2$, as illustrated in [**3**, 5.4.19] by the example of holomorphic chains. It follows from [**3**, 5.3.16] that the singular set of an area minimizing m dimensional rectifiable current T is nowhere dense in spt T, but the largest possible value of the dimension of the singular set is not yet known in case $p>1$ and $m>1$.

The situation becomes much simpler when $\mathbf{Z}$ is replaced as coefficient group by the cyclic group Z_2 of order 2. Reducing modulo 2 in the context of geometric measure theory as explained in [3, 4.2.26], one can modify the proof of Theorem 1 to obtain the following proposition:

THEOREM 2. *If* $T\in\mathcal{R}_m(R^{m+p})$ *and* T *is homologically area minimizing modulo* 2 *with respect to* R^{m+p}, *which means that* $M(T+\partial S+2R)\geq M(T)$ *whenever* $S\in\mathcal{R}_{m+1}(R^{m+p})$ *and* $R\in\mathcal{R}_m(R^{m+p})$, *then there exists an open set* V *such that* $V\cap\operatorname{spt} T$ *is an* m *dimensional submanifold of class* ∞ *of* R^{m+p} *and*

$$\mathcal{H}^k[R^{m+p}\sim(V\cup\operatorname{spt}^2\partial T)]=0\quad\textit{whenever}\quad\sup\{m-2,0\}<k\in R.$$

In fact the extension of our two lemmas from R^{m+1} to R^{m+p} is trivial, the present current T is representative modulo 2, hence $\omega(T)\sim\operatorname{spt}^2\partial T$ equals the singular subset of $\operatorname{spt} T\sim\operatorname{spt}^2\partial T$, and the induction now starts with the case $m=1$ where the singular set is known to be empty.

For $m=2$ it was found in [1, Theorem 3(1)] that the singular set is isolated and $\operatorname{spt} T\sim\operatorname{spt}^2\partial T$ is the image of an immersion of a 2 dimension manifold in R^{2+p}. However, for $m>2$ it is not yet known whether one could sharpen the conclusion of Theorem 2 so as to require that $\mathcal{H}^{m-2}(K\sim V)<\infty$ for every compact subset K of $\operatorname{spt} T\sim\operatorname{spt}^2\partial T$.

Recalling [3, 5.4.4] one sees that *Theorem* 2 *remains valid with* R^{m+p} *replaced by any* $m+p$ *dimensional Riemmanian manifold of class* ∞.

For the study of interior regularity of solutions of the problem of least area, use of m dimensional flat chains modulo 2 is substantially equivalent to use of sets with finite m dimensional Hausdorff measure as employed in Reifenberg's approach presented in [4, Chapter 10], provided $G=Z_2$ and L is cyclic (see [4, p. 411]). Then our method shows that the Hausdorff dimension of the singular set of Reifenberg's solution of the m dimensional Plateau problem does not exceed $m-2$.

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BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND 02912