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MEAN OSCILLATION AND BESOV SPACES

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ABSTRACT. The homogeneous Besov-Lipschitz spaces, usually defined by difference operators or Fourier transform, are studied in terms of mean oscillation, and several equivalent characterisations are given.

1. **Definitions and main results**. The purpose of this paper is to show how mean oscillation characterisations similar to those already known ([1], [6]) for Lipschitz spaces Λ_α can be given for their L^p counterparts, the Besov spaces $B^\alpha_{p,q}$. We recall their definition: for each positive integer k, the k-th order L^p modulus of continuity of a function f is defined as

$$\omega_{p,k}(f,t) = \sup\{\|\Delta_h^k f\|_p : |h| \leqq t\}, \qquad t > 0,$$

where Δ_h denotes the difference operator $\Delta_h f(x) = f(x+h) - f(x)$, $x, h \in \mathbb{R}^n$; then, the (homogeneous) Besov space $B^{\alpha,k}_{p,q}$, $\alpha > 0$, $k > [\alpha]$, $1 \leqq p, q \leqq \infty$, is ([10]) the space of those functions f (or, more precisely, classes of functions modulo $\mathcal{P}_{k-1}$, the polynomials of degree $\leqq k-1$) such that

$$\|f\|_{B^{\alpha,k}_{p,q}} = \left(\int_0^\infty (t^{-\alpha}\omega_{p,k}(f,t))^q t^{-1}\,dt\right)^{1/q} < \infty$$

(equivalent definitions ([8]) can be given in terms of Fourier transform). In particular, if $p = q = \infty$, we obtain the classical Lipschitz spaces Λ_α if α is not an integer and the Zygmund classes for α integral.

Shortly after the introduction of BMO by John and Nirenberg, mean oscillation characterisations of Λ_α were given by Meyers ([6]) for $0 < \alpha < 1$ and by Campanato ([1]) for a general α. It is therefore natural to ask whether the same is true for the $B^{\alpha,k}_{p,q}$ and an affirmative answer is obtained as follows: let f be a locally integrable function and Q a cube in $\mathbb{R}^n$; by $P^k_Q f$ we denote the unique polynomial in $\mathcal{P}_k$ such that

$$\int_Q (f - P^k_Q f)x^\alpha\,dx = 0$$

for all $\alpha \in \mathbb{N}^n$, $0 \leqq |\alpha| \leqq k$ (e.g., if $k = 0$, $P^0_Q f = f_Q = |Q|^{-1}\int_Q f$). We will write $P_Q f$ and even P_Q if there is no chance of confusion. The following operator gives a pointwise measure of the mean oscillation of f

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$$\Omega_f^{1,k}(x,t) = \Omega_f^k(x,t) = \sup\left\{|Q|^{-1}\int_Q |f - P_Q^k f|\, dz\colon x \in Q,\ |Q| = t^n\right\}$$

and we have

THEOREM 1. *$f \in B_{p,q}^{\alpha,k}$ if and only if*

$$\|f\|_{\alpha,k;p,q} = \left(\int_0^\infty (t^{-\alpha}\|\Omega_f^{k-1}(\cdot,t)\|_p)^q\, t^{-1}\, dt\right)^{1/q} < \infty, \tag{1}$$

in which case $\|f\|_{\alpha,k;p,q} \sim \|f\|_{B_{p,q}^{\alpha,k}}$ ($A \sim B$ means there is an absolute constant C such that $C^{-1}A < B < CA$).

Mean oscillation characterisations are well suited to obtain estimates for singular integrals and also to generalise these spaces to domains whose geometry makes difficult the use of the classical definitions (see [5] and [7]). In any case, they allow great flexibility in the definition of $B_{p,q}^{\alpha,k}$; for instance, in view of John-Nirenberg's inequality a natural question is for which values of r the following variants of the operator Ω_f^1

$$\Omega_f^{r,k}(x,t) = \sup\left\{\left(|Q|^{-1}\int_Q |f - P_Q^k|^r\right)^{1/r}\colon\ x \in Q,\ \ |Q| = t^n\right\}$$

can be used to define $B_{p,q}^{\alpha,k}$. The precise answer is

THEOREM 2. (*i*) *If $\alpha \leqq n/p$, $1/q = 1/p - \alpha/n$ and $r < q$, replacing in* (1) *Ω_f^1 by Ω_f^r we obtain a norm equivalent to $\|f\|_{\alpha,k;p,q}$.* (*ii*) *If $\alpha > n/p$, Ω_f^∞ can be used in just the same way.*

Furthermore, it is well known that for $k > [\alpha]$ the spaces $B_{p,q}^{\alpha,k}$ are all equivalent modulo $\mathcal{P}_k$; this fact is also an immediate consequence of the following mean oscillation version of Marchaud's inequality:

THEOREM 3. *Given positive integers $k' > k > [\alpha]$, if $f \in B_{p,q}^{\alpha,k'}$ there exists a polynomial $R \in \mathcal{P}_{k'-1}$ such that*

$$\|\Omega_{f-R}^{1,k-1}(\cdot,t)\|_p \leqq Ct^{k'}\int_t^\infty \|\Omega_f^{1,k'-1}(\cdot,s)\|_p\, s^{-k'-1}\, ds$$

The rest of the paper is devoted to the proofs of these results but before giving them we should like to point out that (1) can be shown to be equivalent with the norm used by Ricci and Taibleson ([13]) to define MO spaces; in these sense, theorem 1 gives, for suitable indices, an identification between MO and Besov spaces, a fact proved in $\mathbb{R}$ by Ricci and Taibleson and by Greenwald ([2]) in $\mathbb{R}^n$. Their arguments, however, make extensive use of the structure theory of Besov spaces and are somewhat less direct than those given here.

2. **Proof of theorem 1**. Recalling the definition of $P_Q^{k-1}f$, an easy homogeneity argument gives

$$ess\sup_Q |P_Q^{k-1}f| \leqq C|Q|^{-1}\int_Q |f|\, dx \tag{2}$$

for any cube Q and $k \geqq 1$; it then follows that, since $P_Q^k(f+R) = P_Q^k f + R$ for any polynomial R of degree $\leqq k$,

$$(3) \qquad |Q|^{-1} \int_Q |f - P_Q^k f| \leq C|Q|^{-1} \int_Q |f - R|,$$

and that if $Q \subset Q'$,

$$(4) \qquad |Q|^{-1} \int_Q |f - P_Q^k f| \leq C(|Q'|/|Q|)|Q'|^{-1} \int_{Q'} |f - P_{Q'}^k f|.$$

We have now

LEMMA 1. *If Q has side length t, then, for a.e. $x \in Q$*

$$|f(x) - P_Q^{k-1}(x)| \leqq C \int_0^t \Omega_f^{k-1}(x,s)s^{-1}\,\mathrm{d}s$$

PROOF. By Lebesgue's differentiation theorem,

$$|f(x) - P_Q^{k-1}(x)| = \lim |Q'|^{-1} \int_{Q'} |f - P_Q^{k-1}|\,\mathrm{d}z \quad \text{as} \quad Q' \to \{x\}$$

a.e. in Q. Let $\{Q_n\}$ be a sequence of cubes tending to x and such that $Q = Q_0$, $Q_i \subset Q_{i+1}$ and $|Q_i| = 2^n|Q_{i+1}|$; by (4)

$$\begin{aligned} |Q_i|^{-1} \int_{Q_i} |f - P_Q^{k-1}| &\leqq |Q_i|^{-1} \int_{Q_i} |f - P_{Q_i}^{k-1}| + \sum_{j=1}^{i} |Q_i|^{-1} \int_{Q_i} |P_{Q_j}^{k-1} - P_{Q_{j-1}}^{k-1}| \\ &\leqq C \sum_{j=0}^{i} |Q_j|^{-1} \int_{Q_j} |f - P_{Q_j}^{k-1}| \leqq C \sum_{j=0}^{i} \Omega_f^{k-1}(x, 2^{-j}t) \\ &\leqq C \int_0^t \Omega_f^{k-1}(x,s)s^{-1}\,\mathrm{d}s \end{aligned}$$

and the lemma follows.

Fix now x, h and let Q be the cube with centre x and side length $2k|h|$. Since Δ^k annihilates $\mathcal{P}_{k-1}$, the estimate

$$|\Delta_h^k f(x)| = |\Delta_h^k(f - P_Q^{k-1})(x)| \leqq C \int_0^{2k|h|} \sum_{j=0}^{k} \Omega_f^{k-1}(x + jh, s)s^{-1}\,\mathrm{d}s$$

holds a.e. in $\mathbb{R}^n$ and it follows that

$$\|\Delta_h^k f\|_p \leqq C \int_0^{2k|h|} \|\Omega_f^{k-1}(\cdot, s)\|_p\, s^{-1}\,\mathrm{d}s;$$

therefore

$$\omega_{p,k}(f,t) \leqq C \int_0^{2kt} \|\Omega_f^{k-1}(\cdot,s)\|_p s^{-1}\,\mathrm{d}s$$

which, together with Hardy's inequality implies $\|f\|_{B_{p,q}^{\alpha,k}} \leqq C\|f\|_{\alpha,k;p,q}$.

Conversely, if $f \in B^{\alpha,k}_{p,q}$, f is then locally integrable and for each $t > 0$ we can construct (see [4]) functions $g_t \in L^p$ and h_t in L^1 with weak derivatives $D^\beta h_t \in L^p$, $|\beta| = k$, such that $f = g_t + h_t$ and

$$\|g_t\|_p \leqq C\omega_{p,k}(f,t),\ t^k|h_t|_{p,k} = t^k \sum_{|\beta|=k} \|D^\beta h_t\|_p \leqq C\omega_{p,k}(f,t).$$

It now follows from (4) that

$$\|\Omega^{k-1}_{g_t}(\cdot,t)\|_p \leqq C\|g_t\|_p \leqq C\omega_{p,k}(f,t);$$

also if $\phi \in C^\infty_0(\mathbb{R}^n)$, $\int \phi \mathrm{d}x = 1$ and $\phi_\lambda(x) = \lambda^{-n}\phi(x/\lambda)$, we can estimate the mean oscillation of the smooth function $h_t * \phi_\lambda$ using its $k-1$ order Taylor expansion and we obtain that $\|\Omega^{k-1}_{h_t * \phi_\lambda}(\cdot,t)\|_p \leqq C\cdot t^k|h_t * \phi_\lambda|_{p,k}$; hence by Fatou's lemma, $\|\Omega^{k-1}_{h_t}(\cdot,t)\|_p \leqq C\omega_{p,k}(f,t)$, and therefore,

$$\|\Omega^{k-1}_{g_t}(\cdot,t)\|_p \leqq C(\|\Omega^{k-1}_{g_t}(\cdot,t)\|_p + \|\Omega^{k-1}_{h_t}(\cdot,t)\|_p) \leqq C\omega_{p,k}(f,t),$$

which implies $\|f\|_{\alpha,k;p,q} \leqq C\|f\|_{B^{\alpha,k}_{p,q}}$.

3. **Proof of theorem 2**. By Hardy's inequality theorem 2 follows from

LEMMA 2. *(i) If* $\alpha \leqq n/p$, $0 < \alpha' < \alpha'' < \alpha$ *and* $1/r = 1/p - \alpha''/n$,

$$\|\Omega^r_f(\cdot,t)\|_p \leqq Ct^{\alpha'} \int_0^t \|\Omega^1_f(\cdot,4s)\|_p\, s^{-\alpha'-1}\, \mathrm{d}s$$

(ii) If $\alpha > n/p$,

$$\|\Omega^\infty_f(\cdot,t)\|_p \leqq Ct^{n/p} \int_0^t \|\Omega^1_f(\cdot,4s)\|_p\, s^{-n/p-1}\, \mathrm{d}s$$

PROOF. (*i*) Fix $t > 0$ and let Q be a cube with side length t; we have from lemma 1

$$|f(y) - P_Q f(y)| \leqq C \int_0^t \Omega^1_f(y,s)s^{-1}\, \mathrm{d}s \tag{5}$$

a.e. in Q. Writing $Q^* = Q(y,2s)$ we obtain by (4)

$$\Omega^1_f(y,s) \leqq C|Q^*|^{-1} \int_{Q^*} |f - P_{Q^*}|\, \mathrm{d}u \leqq C \inf_{Q^*} \Omega^1_f(z,2s)$$

$$\leqq Cs^{-n} \int_{Q^*} \Omega^1_f(z,2s)\, \mathrm{d}z;$$

therefore, if Q_0 is the cube with centre 0 and side length t,

$$\Omega^1_f(y,s) \leqq Cs^{-\alpha'} \int_{Q^*} \frac{\Omega^1_f(z,2s)}{|y-z|^{n-\alpha'}}\, \mathrm{d}z$$

$$\leqq Cs^{-\alpha'}(\Omega^1_f(\cdot,2s)\chi_{2Q}) * \left(\frac{\chi_{2Q_0}}{|\cdot|^{n-\alpha'}}\right)(y)$$

Inserting this estimate in (5) and using Minkowski's and Young's inequalities, we have for $1/r = 1/p + (n - \alpha'')/n - 1$

$$(|Q|^{-1}\int_Q |f-P_Q|^r)^{1/r} \leqq Ct^{-n/r}\int_0^t \|\Omega_f^1(\cdot,2s)\chi_{2Q*}(|\cdot|^{\alpha'-n}\chi_{2Q_0})\|_r s^{-\alpha'-1}\,ds$$
$$\leqq Ct^{-n/r+\alpha'-\alpha''}\int_0^t\left(\int_{2Q}\Omega_f^1(z,2s)^p\,dz\right)^{1/p}s^{-\alpha'-1}\,ds$$
$$= Ct^{\alpha'-n/p}\int_0^t\left(\int_{2Q}\Omega_f^1(z,2s)^p\,dz\right)^{1/p}s^{-\alpha'-1}\,ds.$$

If $\{Q_i\}$ is now a disjoint family of cubes with side length t and such that $\mathbb{R}^n = U Q_i$, it follows that for a general $x \in \mathbb{R}^n$

$$\Omega_f^r(x,t) \leqq C\sum_i\left(|2Q_i|^{-1}\int_{2Q_i}|f-P_{2Q_i}|^r\right)^{1/r}\chi_{Q_i}(x)$$
$$\leqq Ct^{\alpha'-n/p}\int_0^t\left|\sum_i\left(\int_{4Q_i}\Omega_f^1(z,4s)^p\,dz\right)^{1/p}\chi_{Q_i}(x)\right|s^{-\alpha'-1}\,ds$$

and again by Minkowski's inequality,

$$\|\Omega_f^r(\cdot,t)\|_p \leqq Ct^{\alpha'-n/p}\int_0^t\left\|\sum_i\left(\int_{4Q_i}\Omega_f^1(z,4s)^p\,dz\right)^{1/p}\chi_{Q_i}\right\|_p s^{-\alpha'-1}\,ds$$
$$\leqq Ct^{\alpha'-n/p}\int_0^t\left|\left(\sum_i\int_{4Q_i}\Omega_f^1(z,4s)^p\,dz\right)t^n\right|^{1/p}s^{-\alpha'-1}\,ds$$
$$\leqq Ct^{\alpha'}\int_0^t\|\Omega_f^1(\cdot,4s)\|_p s^{-\alpha'-1}\,ds.$$

(*ii*) We prove first that f is locally bounded. Fix a cube Q, $|Q| = t^n$; we have for a.e. $y \in Q$

$$|f(y)-P_Q(y)| \leqq C\int_0^t\Omega_f^1(y,s)s^{-1}\,ds$$
$$\leqq C\int_0^t\left(s^{-n}\int_{2Q_s(y)}\Omega_f^1(z,2s)\,dz\right)s^{-1}\,ds$$
$$\leqq C\int_0^t s^{-n/p}\left(\int_{2Q_s(y)}\Omega_f^1(z,2s)^p\,dz\right)^{1/p}s^{-1}\,ds$$
$$\leqq C\left(\int_0^t s^{(\alpha-n/p)q'-1}\,ds\right)^{1/q'}\left(\int_0^\infty(s^{-\alpha}\|\Omega_f^1(\cdot,2s)\|_p)^{q_s-1}\,ds\right)^{1/q}$$
$$= Ct^{\alpha-n/p}\|f\|_{\alpha;p,q}.$$

Now, if $y_0 \in Q$ is such that $\|(f-P_Q)\chi_Q\|_\infty \leqq 2|f(y_0)-P_Q(y_0)|$, then

$$\|(f-P_Q)\chi_Q\|_\infty \leqq C\int_0^t s^{-n}\left(\int_{2Q_s(y_0)}\Omega_f^1(z,2s)\,dz\right)s^{-1}\,ds$$
$$\leqq C\int_0^t s^{-n/p}\left(\int_{2Q}\Omega_f^1(z,2s)^P\,dz\right)^{1/P}s^{-1}\,ds;$$

and writing as before $\mathbb{R}^n = UQ_i$, Q_i disjoint, $|Q_i| = t^n$,

$$\Omega_f^\infty(x,t) \leqq \sum_i \|f - P_{2Q_i})\chi_{2Q_i}\|_\infty \chi_{Q_i}(x)$$

$$\leqq C \int_0^t \left| \sum_i \left(\int_{4Q_i} \Omega_f^1(z,4s)^p \, \mathrm{d}z \right)^{1/p} \chi_{Q_i}(x) \right| s^{-n/p-1} \, \mathrm{d}s;$$

therefore

$$\left\| \Omega_f^\infty(\cdot,t) \right\|_p \leqq C \int_0^t \left\| \sum_i \left(\int_{4Q_i} \Omega_f^1(z,4s)^p \, \mathrm{d}z \right)^{1/p} \chi_{Q_i} \right\|_p s^{-n/p-1} \, \mathrm{d}s$$

$$\leqq Ct^{n/p} \int_0^t \|\Omega_f^1(\cdot,4s)\|_p s^{-n/p-1} \, \mathrm{d}s.$$

4. **Proof of theorem 3**. Our approach originates in the proofs for the Lipschitz case given in [1] or [3]. Clearly, it is enough to assume $k' = k + 1$ and we construct first the polynomial R. Let Q_0^ℓ be the cube with centre 0 and side length 2^ℓ and for $f \in B_{p,q}^{\alpha,k+1}$ write

$$P_{Q_0^\ell}^k f(x) = \sum_{|\beta| \leqq k} (a_{\beta,\ell}/\beta!) x^\beta;$$

if $|\beta| = k$, using lemma 2.1 in [1] we obtain

$$(6) \qquad |a_{\beta,j} - a_{\beta,j'}| \leqq \sum_{j+1}^{j'} |a_{\beta,i} - a_{\beta,i-1}| \leqq C \sum_{j+1}^{j'} 2^{-ik} |Q_0^i|^{-1} \int_{Q_0^i} |f - P_{Q_0^i}^k|,$$

which can be bounded by

$$C \sum_{j+1}^{\infty} 2^{-ik} |Q_0^i|^{-1} \int_{Q_0^i} \Omega_f^k(z,2^i) \, \mathrm{d}z \leqq C \sum_j^\infty 2^{-ik-in/p} \|\Omega_f^k(\cdot,2^i)\|_p$$

$$\leqq C\, 2^{-j(k+n/p-\alpha)} \|f\|_{\alpha,k+1;p,q}.$$

Thus, $\{a_{\beta,\ell}\}$ is a Cauchy sequence as $\ell \to \infty$ and we set $a_\beta = \lim a_{\beta,\ell}$, $|\beta| = k$. Let $R(x) = \sum_{|\beta|=k} (a_\beta/\beta!)x^\beta$ and $g = f - R$; we have now

$$(7) \qquad \Omega_g^{k-1}(x,t) \leqq Ct^k \int_t^\infty \Omega_f^k(x,s) s^{-k-1} \, \mathrm{d}s,$$

which obviously gives the desired result. To prove it, let $x \in Q$, $|Q| = t^n$ and let $Q = Q_0 \subset Q_1 \subset \ldots \subset Q_\lambda = Q_0^N$ be a sequence of cubes such that $|Q_i| = 2^n |Q_{i-1}|$; writing

$$P_{Q_i}^k f(y) = \sum_{|\beta|=k} (a_{\beta,i}/\beta!)(y-x)^\beta + S_i(y) = R_i(y) + S_i(y)$$

$$R(y) = \sum_{|\beta|=k} (a_\beta/\beta!)(y-x)^\beta + S'(y) = R'(y) + S'(y)$$

and using (3) we obtain

$$(8) \qquad |Q|^{-1}\int_Q |g - P_Q^{k-1}g|\,\mathrm{d}y \leqq C|Q|^{-1}\int_Q |g - (P_Q^k f - R_0) + S'|\,\mathrm{d}y$$

$$\leqq C\Big[\,|Q|^{-1}\int_Q |f - P_Q^k f|\,\mathrm{d}y + ess\,\sup_Q |R - R_0|\,\Big].$$

Since again by lemma 2.1 of [1]

$$ess\,\sup_Q |R_i - R_{i-1}| \leqq Ct^k \sum_{|\beta|=k} |a_{\beta,i} - a_{\beta,i-1}|$$

$$\leqq Ct^k(2^i t)^{-k}|Q_i|^{-1}\int_{Q_i} |f - P_{Q_i}^k|$$

and from (6)

$$ess_Q \sup |R'_\lambda - R'| \leqq Ct^k \sum_{|\beta|=k} |a_{\beta,N} - a_\beta| \leqq Ct^k \int_{2^N}^\infty \Omega_f^k(x,s)s^{-k-1}\,\mathrm{d}s$$

we arrive at (7) by estimating (8) by

$$Ct^k\left[\sum_0^\lambda (2^i t)^{-k}\Omega_f^k(x,2^i t) + \int_{2^N}^\infty \Omega_f^k(x,s)s^{-k-1}\,\mathrm{d}s\right] \leqq Ct^k\int_t^\infty \Omega_f^k(x,s)s^{-k-1}\,\mathrm{d}s.$$

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