SQUARE FUNCTIONS, NON-TANGENTIAL LIMITS AND HARMONIC MEASURE IN CO-DIMENSIONS LARGER THAN ONE

GUY DAVID, MAX ENGELSTEIN, AND SVITLANA MAYBORODA

ABSTRACT. In this paper, we characterize the rectifiability (both uniform and not) of an Ahlfors regular set, E, of arbitrary co-dimension by the behavior of a regularized distance function in the complement of that set. In particular, we establish a certain version of the Riesz transform characterization of rectifiability for lower-dimensional sets. We also uncover a special situation in which the regularized distance is itself a solution to a degenerate elliptic operator in the complement of E. This allows us to precisely compute the harmonic measure of those sets associated to this degenerate operator and prove that, in a sharp contrast with the usual setting of co-dimension one, a converse to the Dahlberg's theorem (see [Da] and [DFM2]) must be false on lower dimensional boundaries without additional assumptions.

RÉSUMÉ. On caractérise la rectifiabilité (uniforme ou non) d'un ensemble Ahlfors-régulier E, de codimension arbitraire, par le comportement d'une distance régularisée définie sur le complémentaire de E. En particulier on donne, pour les ensembles de dimension $< n-1$, une sorte de caractérisation des ensembles rectifiables par la transformée de Riesz. On découvre également uns situation particulière où la distance régularisée est elle-même solution dans le complémentaire de E d'une équation elliptique dégénérée. Ceci nous permet de calculer précisément la mesure harmonique associée à cette équation et de démontrer, à l'opposé de ce qui se passe en codimension 1, que la réciproque du théorème de Dahlberg (voir [Da] et [DFM2]) est fausse en codimension > 1 sans hypothèses supplémentaires.

CONTENTS

1. Introduction

The beginning of the XXI century has brought a series of long sought-after results en-lightening connections between the scale-invariant geometric, analytic, and PDE properties of sets. Among the most celebrated ones were the Riesz transform characterizations of uni-form rectifiability [DS2, To1, NTV] and full description of the sets for which the harmonic measure is absolutely continuous with respect to the Lebesgue measure, in terms of uniform rectifiability along with a certain topological condition [HM, AMT]. Both of these results rest on, and have been surrounded by, a plethora of important related advancements. We do not intend to review the related literature in the present work but point out that virtually the entire theory has been restricted to the $n-1$ dimensional boundaries of domains in $\mathbb{R}^n$. The question of a possible extension of these results to lower-dimensional sets has become one the central open problems in the subject ever since.

In [DFM3], the first and third author (together with Joseph Feneuil) introduced a reg-ularized distance function $D_{\alpha,\mu}$ (see (1.3) and (1.5) below) as a tool in a long program to characterize uniformly rectifiable sets of co-dimension greater than one by the behavior of certain (degenerate) elliptic operators in the complement of that set. In this paper we pro-vide necessary and sufficient conditions for both the rectifiability and uniform rectifiability (see Definition 1.1) of a d-Ahlfors regular set, E, in terms of the oscillation of $|\nabla D_{\alpha,\mu}|$ in the complement of E. These results are new even in the classical context of co-dimension one. However, most notably, for lower dimensional sets, and for special values of involved parameters, they provide an unexpected version of the Riesz transform characterization - we will discuss the details after appropriate definitions.

We also discover a surprising situation in which the distance function itself is a solution to the degenerate elliptic operator introduced by [DFM3]. This allows us to compute the Green function *explicitly* and compare the associated harmonic measure (see (1.11) below) to the Hausdorff measure *no matter how irregular E is* – a situation unheard of in co-dimension one. In particular, as we mentioned above, recently, as a culmination of a long line of research starting with the work of F. and M. Riesz [RR], the combined results of Hofmann, Martell [HM] and Azzam, Mourgoglou, Tolsa [AMT] characterized (under mild topological assumptions) the uniform rectifiability of a co-dimension 1 set E by the regularity of the harmonic measure supported on E. Our result shows that the analogous characterization fails brutally in the situation described above.

In order to more precisely state our results (and the analogous work in co-dimension one), let us introduce some notation and notions. We are given an Ahlfors regular set $E \subset \mathbb{R}^n$, of any dimension $d < n$, and an Ahlfors regular measure μ on E. Recall, a measure μ is

d-Ahlfors regular if

$$(1.1) \qquad C^{-1}r^d \le \mu(B(Q,r)) \le Cr^d \ \ \forall Q \in E, \ \forall r > 0.$$

A set, E, is d-Ahlfors regular if $\mathcal{H}^d|_E$ is a d-Ahlfors regular measure, or equivalently if it supports some d-Ahlfors regular measure.

The most salient class of regularity for us is uniform rectifiability. Recall that a set $E \subset \mathbb{R}^n$ is d-rectifiable if

$$(1.2) \qquad \text{there exist countably many Lipschitz functions}$$
$$f_i : \mathbb{R}^d \to \mathbb{R}^n \ \ \text{such that} \ \mathcal{H}^d\left(E \backslash \bigcup_i f_i(\mathbb{R}^d)\right) = 0.$$

Uniform rectifiability is a quantitative version of this (cf. [DS1]):

Definition 1.1. *Let d be an integer and E be a d-Ahlfors regular set. We say that E is d-uniformly rectifiable (with constants $\theta > 0, L > 0$), if it has big pieces of Lipschitz images. That is, if for any $x \in E, r > 0$ there exists an $f : \mathbb{R}^d \to \mathbb{R}^n$ which is L-Lipschitz and such that*

$$\mathcal{H}^d(E \cap B(x,r) \cap f(\mathbb{R}^d \cap B(0,r))) \ge \theta r^d.$$

The definition above is just one of several (equivalent) definitions of uniform rectifiability, which we chose because we like its geometric flavor. Through the paper we will introduce as needed (and use) other characterizations of uniform rectifiability (e.g. using β or α numbers). One of the goals of this paper is to provide another characterization of uniform rectifiability using the regularized distance to E.

1.1. **Regularized Distances and (Uniform) Rectifiability.**
As above let E be a d-Ahlfors regular set and μ a d-Ahlfors regular measure whose support is E. In some cases, μ will be the restriction to E of the d-dimensional Hausdorff measure $\mathcal{H}^d$, but not always.

For each $\alpha > 0$ (often fixed in the argument), we define first a function, $R \equiv R_{\alpha,\mu}$, on $\Omega = \mathbb{R}^n \setminus E$ by

$$(1.3) \qquad R(x) = R_{\alpha,\mu}(x) = \int_{y \in E} |x - y|^{-d-\alpha} d\mu(y),$$

where the convergence comes from the Ahlfors regularity of μ. In fact, a simple estimate with dyadic annuli shows that

$$(1.4) \qquad C^{-1}\delta(x)^{-\alpha} \le R(x) \le C\delta(x)^{-\alpha} \ \ \text{for } x \in \mathbb{R}^n \backslash E,$$

where we set $\delta(x) = \text{dist}(x, E)$ and C depends on μ and α. After this, we define $D = D_{\alpha,\mu}$ by

$$(1.5) \qquad D(x) \equiv D_{\alpha,\mu}(x) = R_{\alpha,\mu}(x)^{-1/\alpha}(x) \ \ \text{for } x \in \mathbb{R}^n \backslash E.$$

These distances were first introduced by [DFM3] to study degenerate elliptic PDE, but in the first part of our paper we are more concerned with the analytic properties of ∇D_α, which

we think of as analogous to the Riesz transform (though it is regularized by the presence of α in the kernel). Indeed,

$$(1.6) \quad \nabla D_\alpha(x) = -\frac{1}{\alpha} \left(\int_{y \in E} |x-y|^{-d-\alpha} d\mu(y) \right)^{-\frac{1}{\alpha}-1} \int_{y \in E} \nabla_x(|x-y|^{-d-\alpha}) \, d\mu(y)$$

$$= \frac{d+\alpha}{\alpha} \left(\int_{y \in E} |x-y|^{-d-\alpha} d\mu(y) \right)^{-\frac{1}{\alpha}-1} \int_{y \in E} \frac{x-y}{|x-y|^{d+\alpha+2}} \, d\mu(y),$$

for every $x \in \Omega$. Setting formally $\alpha = -1$ above and properly re-interpreting the integrals would transform the latter term into the classical Riesz transform. However, our α is always a positive number, so that the resultant expression, while analogous, is actually a quite surprising extension of the concept of the Riesz transform. One of the main discoveries of this paper is that ∇D_α carries rich geometric information, similar to the original Riesz tranform, for sets of *arbitrary* dimension (not necessarily $n-1$).

To measure the oscillation of $\nabla D_{\alpha,\mu}$ in a scale invariant way we define

$$(1.7) \qquad F(x) \equiv F_{\mu,\alpha}(x) = \delta(x) \big| \nabla(|\nabla D|^2)(x) \big| = \delta(x) \left(\sum_{k=1}^n \left| \frac{\partial}{\partial x_k}(|\nabla D|^2)(x) \right|^2 \right)^{1/2}$$

for $x \in \Omega \equiv \mathbb{R}^n \backslash E$. This is a dimensionless quantity, or rather, in crude terms, it is easy to see that D_α is Lipschitz and $|\nabla(|\nabla D|^2)(x)|$ is bounded by $\delta(x)^{-1}$. We say that $D_{\alpha,\mu}$ satisfies the "usual square function estimates" (USFE for short) when

$$(1.8) \qquad\qquad F(x)^2 \delta(x)^{-n+d} dx \text{ is a Carleson measure on } \Omega.$$

Let us recall the definition of a Carleson measure (which is intimately linked to uniform rectifiability and will be used several times), first on $E \times \mathbb{R}_+$ (the standard case) and then on Ω (as needed above).

Definition 1.2. *We say that $\nu(x,r)$ is a Carleson measure on $E \times \mathbb{R}_+$ if there exists a $C > 0$ such that for every $X \in E$ and $R > 0$ we have that*

$$(1.9) \qquad\qquad \nu(B(X,R) \cap E \times [0,R]) \le CR^d.$$

Similarly, $\mathcal{G}$ is a Carleson subset of $E \times (0,+\infty)$ if the measure $\mathbf{1}_\mathcal{G}(x,r) \frac{d\mathcal{H}^d(x) dr}{r}$ is a Carleson measure on $E \times \mathbb{R}_+$.

Definition 1.3. *We say that $\lambda(x)$ is a Carleson measure on Ω if there exists a $C \ge 0$ such that*

$$\lambda(\Omega \cap B(X,R)) \le CR^d$$

for all $X \in E$ and $R > 0$.
Similarly, $Z \subset \Omega$ is a Carleson set if $\mathbf{1}_Z(x)\delta(x)^{-n+d} dx$ is a Carleson measure on Ω.

Notice that for the moment we use the definition of F that is the simplest for us to use; taking $F(x) = \big| \nabla(|\nabla D|)(x) \big|$ instead will give the same results; see e.g. Corollary 2.2. The results of Sections 2 and 4 characterize uniform rectifiability through the USFE; in particular we show

Theorem 1.4. *A d-Ahlfors regular set in $\mathbb{R}^n$ equipped with a d-Ahlfors regular measure μ, $d < n$, is uniformly rectifiable if and only if $F_{\alpha,\mu}$ satisfies the USFE for some $\alpha > 0$.*

In fact a slightly weaker condition than the USFE, which we call "weak USFE", implies that d is integer and E is uniformly rectifiable. We note that this characterization is new even in co-dimension one and in some instances may be easier to check than previous conditions involving square functions (due to David and Semmes [DS1]) and singular integrals [DS2, NTV, To1]. Indeed, while it is often hard to check the L^2 boundedness of singular operators, given a set E, $F_{\alpha,\mu}$ can be computed fairly explicitly. We also remark the parallel with the "classical" USFE, involving a Carleson measure condition similar to the above for the second derivative of the Newtonian kernel, that is, the gradient of the kernel of the classical Riesz transform [DS2]. Just as ∇D in (1.6) resembles the Riesz transform only formally, our expressions here with $\alpha > 0$ are, of course, different, both intuitively (we really think of them as derivatives of a regularized distance) and factually (these are not classical singular integrals). Most importantly, the results here apply to the lower-dimensional setting while the (obvious extension of) the classical USFE is known to fail for sets of dimension lower than $n-1$ [DS2], p. 267, and no lower-dimensional analogue of this characterization has been known thus far.

We would like to highlight a crucial component of the proof that the USFE implies uniformly rectifiability: Corollary 3.2. There we show that if $|\nabla D_{\alpha,\mu}|$ is constant, then E must be a d-affine space and μ a constant multiple of $\mathcal{H}^d|_E$. This corollary follows from Theorem 3.1, which is possibly of wider interest, and states that if the distance to a set, E, is a C^1 function, the set E must be convex. As we were writing, we learned that this line of inquiry is related to results in convex analysis (see Section 3 for details).

We are also interested in the existence of non-tangential limits for $|\nabla D_\alpha|$. That is, the limit of $|\nabla D_\alpha(x)|$ as x approaches a point $Q \in E$ without getting "too close" to E (see (5.3) for the definition of a non-tangential region and Definition 5.2 for the definition of non-tangential convergence). Non-tangential limits are an important concept in harmonic analysis (e.g. the classical Fatou's theorem). In Section 5, we prove

Theorem 1.5. *A d-Ahlfors regular set in $\mathbb{R}^n$ equipped with a d-Ahlfors regular measure μ, $d < n$, is rectifiable if and only if the non-tangential limit of $|\nabla D_{\alpha,\mu}|$ exists at μ-almost every point in E (for cones of every aperture).*

Continuing to think of $\nabla D_{\alpha,\mu}$ as a slightly smoother version of Riesz transform, these results are in the vein of Tolsa's, [To1], who shows that the existence of principle values of the Riesz transform is equivalent to rectifiability. We remark that the L^2 boundedness of the operator

$$\frac{d+\alpha}{\alpha} \left(\int_{y \in E} |x-y|^{-d-\alpha} d\mu(y) \right)^{-\frac{1}{\alpha}-1} \int_{y \in E} \frac{x-y}{|x-y|^{d+\alpha+2}} \, f(y) \, d\mu(y),$$

naturally associated to $\nabla D_{\alpha,\mu}$, is valid on all Ahlfors regular sets, using a simple domination by the Hardy-Littlewood maximal function. Thus, the celebrated result of [NTV] that says that for an Ahlfors regular set E of co-dimension 1, E is uniformly rectifiable if and only if the Riesz kernel defines a bounded operator on $L^2(E, \mathcal{H}^{n-1})$ trivially fails in our case.

Yet, much as in [To1], the existence of the limits characterizes rectifiability, albeit these are different, non-tangential limits.

1.2. **Harmonic Measure in co-dimension one and greater.** Associated to these distances is the degenerate elliptic PDE:

$$(1.10) \qquad L_{\alpha,\mu}u \equiv -\mathrm{div}\left(\frac{1}{D_{\alpha,\mu}^{n-d-1}}\nabla u\right) = 0.$$

In [DFM2], elliptic estimates and some potential theory were established for solutions of $L_{\alpha,\mu}$ in the complement of E. Most saliently for our purposes, it was shown that a maximal principle holds and that the Dirichlet problem could be solved for continuous data. Thus, for $X \in \Omega \equiv \mathbb{R}^n \backslash E$, we can define a harmonic measure $\omega^X \equiv \omega_{\alpha,\mu}^X$ as the measure given by the Riesz representation theorem with the property that if u_f is the unique solution to

$$L_{\alpha,\mu}u_f = 0, \ \text{ in } \Omega,$$
$$u_f = f, \ \text{ in } E,$$

then

$$(1.11) \qquad u_f(X) = \int_E f(Q)d\omega_{\alpha,\mu}^X(Q).$$

The distances, $D_{\alpha,\mu}$, were introduced by [DFM3], as a smooth replacement for $\delta(x)$; this smoothness was essential to the proof of the co-dimension greater than one analogue of Dahlberg's theorem in [DFM3]. Recall that for the Laplacian in co-dimension one, Dahlberg proved that for Lipschitz domains the harmonic measure is quantitatively absolutely continuous (precisely, an A_∞-weight) with respect to surface measure; see [Da]. In [DFM3], the authors proved that in co-dimension greater than one, $\omega_{\alpha,\mu} \in A_\infty(d\mathcal{H}^d|_E)$ when E is a graph with small Lipschitz constant. In work in progress, [DM], the first and third author are looking to extend this result to all uniformly rectifiable sets E.

As mentioned above, the analogue of this program in co-dimension one has been a question of central interest for years, in particular, because the behavior of harmonic measure supported on a set E has important consequences for the solutions of the Dirichlet problem in the complement of that set (see, e.g. [FKP] and for recent results in higher co-dimension, [MZ]). Recently, Hofmann and Martell [HM] showed that under weak topological conditions (which are vacuously satisfied in higher co-dimension) the $n-1$ uniform rectifiability of a set E implies that the harmonic measure of $\mathbb{R}^n \backslash E$ is quantitatively absolutely continuous with respect to $\mathcal{H}^{n-1}|_E$ (they use a condition known as weak-A_∞, which takes into account the pole of the harmonic measure and is natural due to the potentially nasty topology of $\mathbb{R}^n \backslash E$). The aforementioned work of [DM] should be seen as a generalization of their work to higher co-dimension.

More impressively, the conditions of [HM] are necessary and sufficient, as recent work of Azzam, Mourgoglou, and Tolsa, [AMT], shows that if the harmonic measure in the complement of a $(n-1)$-Ahlfors regular set E is in weak A_∞, then it must be that E is uniformly rectifiable and $\mathbb{R}^n \backslash E$ satisfies the aforementioned topological assumption.

Our initial goal was to connect the USFE and the existence of non-tangential limits with the behavior of the harmonic measure, $\omega_{\alpha,\mu}$, with the hopes of proving a higher co-dimension version of [AMT] and therefore characterizing uniform rectifiability by the behavior of $\omega_{\alpha,\mu}$.

However, as mentioned above, such a result turns out to be completely false in some cases. There is a specific value of α, described in Section 6, where $D_{\alpha,\mu}$ itself is a solution of $L_{\alpha,\mu}u = 0$. In this scenario, we can explicitly compute $\omega_{\alpha,\mu}$ by showing that $D_{\alpha,\mu}$ is, in fact, the Green function with pole at infinity (see Definition 6.2 and Corollary 6.8).

More precisely, we show that $\omega_{\alpha,\mu}$ is proportional to $\sigma = \mathcal{H}^d|_E$ for *any Ahlfors regular set* E, including purely unrectifiable ones (Theorem 6.7). Thus, in the case of "magic-α" the converse to [DM] (hence the higher co-dimension generalization of [AMT]) fails in the most spectacular way possible. We also show that for any rectifiable set E, the harmonic measure for magic α is a constant multiple of σ (see Corollary 6.10). This is surprising for two reasons. First, for the Laplacian in co-dimension one, under mild topological assumptions, the only set for which $\omega = \sigma$ is the half-space ([KT2]). Second, there are very few situations in co-dimension one in which the Poisson kernel, $\frac{d\omega}{d\sigma}$, or the Green function can be precisely computed. Essentially only in the presence of lots of symmetry (e.g. the ball) or where there is an explicit conformal transformation from the ball (e.g. polygonal domains in $\mathbb{R}^2$) are the Poisson and Green kernels known. Here, for magic α, we are able to compute the Green function with pole at infinity *for any Ahlfors regular set* and the Poisson kernel $\frac{d\omega_\alpha}{d\sigma}$ *for any rectifiable set* E.

We now expect that the situation for magic α is really exceptional, and hope to make precise how this is so in future investigations.

2. The square function estimate for uniformly rectifiable sets

In this section we prove the direct results concerning the USFE.

Theorem 2.1. *Let $E \subset \mathbb{R}^n$ be uniformly rectifiable set of dimension $d < n$, and let μ be a d-Ahlfors regular measure whose support is E. Then for each $\beta > 0$, $D_{\beta,\mu}$ satisfies the USFE. That is, if $D_{\beta,\mu}$ is as in (1.3) and (1.5), and F is defined as in (1.7), then (1.8) holds.*

See Corollary 2.2 concerning another, roughly equivalent, function $\widetilde{F}$. Notice that we do not assume that $d < n - 1$ here, but we do not talk about the possible relations with the operator L either.

The main outline of the proof is as follows; it is clear that $F_{\beta,\nu} = 0$ when ν is a multiple of Hausdorff measure restricted to a plane (we call these measures flat; see (2.1), below). The key estimate, (2.4), makes this quantitative: the size of $F_{\beta,\mu}$ can be estimated by the distance between μ and a well chosen flat measure (this distance is measured by the α numbers, see (2.3)). Tolsa's characterization of uniform rectifiability using α numbers finishes the proof.

Proof. Fix $\beta > 0$ and μ as in the Theorem statement. From now on we will abuse notation and refer to $D_{\beta,\mu}$ as D (similarly for R and F). Occasionally, we will have to work with $D_{\beta,\nu}$ for some other measure ν. Here again we will suppress the dependence on β and just refer to D_ν.

Before we can introduce the key estimate alluded to above, we must introduce the Wasserstein distances and α-numbers. Let us denote by $\mathcal{F} = \mathcal{F}_d$ the set of flat measures;

(2.1)   a flat measure is a positive multiple of the Lebesgue measure on an affine d-plane.

We are interested in Wasserstein distances, which we define as follows. Given two positive measures μ and ν and a ball $B(x,r)$, we define $\mathcal{D}_{x,r}(\mu, \nu)$ by

$$(2.2) \qquad \mathcal{D}_{x,r}(\mu, \nu) = r^{-d-1} \sup_{f \in \Lambda(x,r)} \left| \int_{B(x,r)} f(d\mu - d\nu) \right|,$$

where we denote by $\Lambda(x,r)$ the set of functions f that are 1-Lipschitz on $\mathbb{R}^n$ and vanish on $\mathbb{R}^n \setminus B(x,r)$. Notice that the normalization is such that $\mathcal{D}_{x,r}(\mu, \nu) \leq C$ when μ and ν are Ahlfors regular, with a constant C that does not depend on x or r. Let us note that we will not require μ, ν to be probability measures, so it is misleading to say that $\mathcal{D}_{x,r}$ is a "distance".

We are especially interested in the numbers

$$(2.3) \qquad \alpha(x,r) = \inf_{\nu \in \mathcal{F}} \mathcal{D}_{x,r}(\mu, \nu),$$

where $x \in \mathbb{R}^n$ and $r > 0$ are such that $B(x,r)$ meets E. These "α-numbers" measure the local Wasserstein distances from μ to flat measures. In the context of quantitative rectifiability, these numbers were introduced and widely used by X. Tolsa (see, e.g. [To]), to create a theory for measures that is analogous to P. Jones' β-numbers. We will use the fact that, see [To], for any uniformly rectifiable set E has Carleson measure estimates on the $\alpha(x,r)^2$ (see (2.5)).

We can now introduce the key estimate: fix $x \in \Omega$ and set $r_0 = \delta(x)$ and for $k \geq 0$ let $r_k \equiv 2^k r_0$. We will prove that for $1 \leq i \leq n$,

$$
\begin{aligned}
(2.4) \qquad \left| \partial_i \left(|\nabla D(x)|^2 \right) \right| &= \left| \partial_i \left(|\nabla D(x)|^2 \right) - \partial_i \left(|\nabla D_\nu(x)|^2 \right) \right| \\
&\leq C \delta(x)^{-1} \sum_{l \geq 0} 2^{-(\beta+1)l} \alpha(y, 2^{l+6} r_0),
\end{aligned}
$$

for $y \in E \cap B(x, 16\delta(x))$, and where ν a correctly chosen flat measure. The first equality comes from the easy fact (proven below) that for any flat measure ν, $|\nabla D_\nu(x)|^2$ is constant.

Before we prove (2.4), let us see how the estimate implies the final result. Theorem 1.2 in [To] says (amongst other things) that when E is a d-dimensional uniformly rectifiable set and μ is an Ahlfors regular measure whose support is E,

$$(2.5) \qquad \alpha(x,r)^2 \frac{d\mu(x)dr}{r} \quad \text{is a Carleson measure on } E \times \mathbb{R}_+,$$

which means that for $X \in E$ and $R > 0$,

$$(2.6) \qquad \int_{x \in E \cap B(X,R)} \int_{r=0}^R \alpha(x,r)^2 \frac{d\mu(x)dr}{r} \leq CR^d.$$

To be precise, in [To], the estimate (2.6) is not written in terms of the numbers $\alpha(x,r)^2$, but numbers $\alpha(Q)$ indexed by dyadic pseudocubes $Q \subset E$; however the $\alpha(Q)$ and the $\alpha(x,r)$ mutually dominate each other for comparable values of $l(Q)$ and r, and it is a standard

argument based on Fubini's theorem to go from the condition of [To] to (2.6). We skip the computation because it is both easy and done in Lemma 5.9 in [DFM3].

Notice that the function $\alpha(x, r)$ depends both on E and μ, and (2.6) contains information both on the geometry of E (the fact that it is close to a d-plane in most balls) and on the distribution of μ inside E. In fact, Tolsa's result is already significant when $E = \mathbb{R}^d$ and $d\mu = f d\lambda$ for some function f such that $C^{-1} \leq f \leq C$.

Now we claim that the Carleson measure estimate (1.8) follows from (2.6) and the key inequality (2.4); the verification is fairly easy: we have an integral over $\Omega \cap B(X, R)$ to compute, we use the estimate (2.4) to replace F by α-numbers, average this estimate over $y \in E \cap B(x, 16\delta(x))$, and thus get a multiple integral to estimate. We use Fubini, integrate over a piece of Ω, and find ourselves with an integral that is bounded by the one in (2.6). We skip the computation because we claim that the same one is done in Lemma 5.89 of [DFM3]. This completes our proof of Theorem 2.1 up to establishing (2.4).

Proving (2.4): To prove (2.4), recall $F = F_{\beta, \mu}$ from (1.7) and $R = R_{\beta, \mu}$ from (1.3); we will need to compute their derivatives. Set $h(z) = |z|^{-d-\beta}$; this is a smooth function on $\mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\}$, and we denote by $\nabla^j h$ its iterated gradient (i.e. the collection of all its derivatives of order j). Notice that R is smooth on Ω, and

$$(2.7) \qquad \nabla^j R(x) = \int_E \nabla^j h(x - y) d\mu(y).$$

Next, D is smooth on Ω, and

$$(2.8) \qquad \nabla D(x) = -\frac{1}{\beta} R(x)^{-\frac{1}{\beta}-1} \nabla R(x)$$

and

$$(2.9) \qquad |\nabla D(x)|^2 = \frac{1}{\beta^2} R(x)^{-\frac{2}{\beta}-2} |\nabla R(x)|^2 = \frac{1}{\beta^2} R(x)^{-\frac{2}{\beta}-2} \sum_j |\partial_j R(x)|^2$$

and then

$$(2.10) \qquad \begin{aligned} \partial_i \big(|\nabla D(x)|^2 \big) &= -\frac{1}{\beta^2} \frac{2 + 2\beta}{\beta} R(x)^{-\frac{2}{\beta}-3} \partial_i R(x) \sum_j |\partial_j R(x)|^2 \\ &\quad + \frac{2}{\beta^2} R(x)^{-\frac{2}{\beta}-2} \sum_j \partial_j R(x) \partial_i \partial_j R(x). \end{aligned}$$

The precise structure of (2.10) is not so important; it suffices that we can compute (and bound) the errors we get from modifying the measure μ more or less explicitly.

The computations above are simpler when $F = F_\nu$ for a flat ν; let $\nu = \lambda \mathcal{H}^d_{|P}$ for some $\lambda > 0$. In this case,

$$(2.11) \qquad R_\nu(x) = c_1 \lambda \delta_P^{-\beta}(x),$$

where $c_1 > 0$ is a constant that depends on d and α, and $\delta_P(x) \equiv \text{dist}(x, P)$. In this case (1.5) yields

$$(2.12) \qquad D_\nu(x) = c_2 \lambda^{-1/\beta} \delta_P(x),$$

with $c_2 = c_1^{-1/\beta}$ It follows that $|\nabla D_\nu|^2 = c_2^2 \lambda^{-2/\beta}$, and $\partial_i\big(|\nabla D_\nu(x)|^2\big) = 0$ for $1 \le i \le n$.

Our next goal is to estimate the differences, $|\nabla^j R(x) - \nabla^j R_\nu(x)|$ where $j \ge 0$ and ν is a well chosen flat measure. In turn, these will allow us to estimate the difference, $\partial_i\big(|\nabla D(x)|^2\big) - \partial_i\big(|\nabla D_\nu(x)|^2\big)$. Given the complexity of (2.10), we expect lots of terms, but they will all involve differences of the form $|\nabla^j R(x) - \nabla^j R_\nu(x)|$. We will start with the simplest case, $j = 0$.

Recall that $x \in \Omega$, $r_0 = \delta(x)$ and, for $k \ge 0$, $r_k = 2^k r_0$. Let φ be a (fixed) smooth bump function such that $0 \le \varphi \le 1$ on $\mathbb{R}^n$, φ is radial, $\varphi = 1$ on $B(0, 8r_0)$, $\varphi = 0$ on $\mathbb{R}^n \setminus B(0, 16r_0)$; then let $\varphi_0 = \varphi$ and, for $k \ge 1$, $\varphi_k(x) = \varphi(2^{-k}x) - \varphi(2^{-k+1}x)$ (so that φ_k is supported on $A_k \equiv \overline{B}(0, 2^{k+4}r_0) \setminus B(0, 2^{k+2}r_0)$, where $A_0 = \overline{B}(0, 16r_0)$). Note that $\sum_{k \ge 0} \varphi_k = 1$.

Next we choose a flat measure ν_k which is nearly optimal for the definition of $\alpha(x, 32r_k)$. That is, we choose $\nu_k = \lambda_k \mathcal{H}^d_{|P_k}$ for some $\lambda_k > 0$ and some affine d-plane P_k, so that, in particular,

$$(2.13) \qquad\qquad \mathcal{D}_{x, 32r_k}(\mu, \nu_k) \le C\alpha(x, 32r_k),$$

where C depends on n, d, and the Ahlfors regularity constant for μ. As we shall see, at points and scales where μ is not well approximated by flat measures, rather than choosing the measure ν_k which minimizes the right hand side of (2.13), we will prefer to make sure that we keep some control on λ_k and the support of ν_k.

To pick the ν_k, let $c > 0$ be small, to be chosen soon. If $\alpha(x, 32r_k) \le c$, let us just pick ν_k so that $\mathcal{D}_{x, 32r_k}(\mu, \nu_k) = \alpha(x, 32r_k)$; to see that such a minimizer exists (and is nice), recall that $\mu(B(x, 2r_k)) \ge C^{-1}r_k^d$, by Ahlfors regularity; if c is small enough, depending only on n, d, and the Ahlfors regularity constant for μ, then any flat measure η such that $\mathcal{D}_{x, 32r_k}(\mu, \eta) \le 2\alpha(x, 32r_k)$ must be such that $\eta(B(x, 2r_k)) \ge (2C)^{-1}r_k^d$ (test (2.2) on a bump function centered at x). Hence (writing $\eta = \lambda \mathcal{H}^d_{|P}$ as above)

$$(2.14) \qquad\qquad P \cap B(x, 2r_k) \ne \emptyset \quad \text{and} \quad C^{-1} \le \lambda \le C.$$

It is now easy to find a minimizing $\nu_k = \lambda_k \mathcal{H}^d_{|P_k}$, where (2.14) holds for P_k, λ_k. In the remaining case where $\alpha(x, 32r_k) \ge c$, we do not complicate our life, and pick $\nu_k = \mathcal{H}^d_{|P_k}$, where P_k is any d-plane through $B(x, 2r_k)$. Then (2.13) and (2.14) hold trivially.

Set $\nu = \nu_0$ and $P = P_0$. By the translation invariance of our problem, we may assume that the origin lies in $P \cap B(x, 2r_0)$ (this will simplify our notation, because this way we don't need to translate our bump functions φ_k).

Recall that $R_\nu(x) = c_1 \lambda_0 \delta_P^{-\beta}(x)$ by (2.11). Recall, furthermore, the notation in (2.7). We want to estimate the difference

$$(2.15) \qquad\qquad |R(x) - R_\nu(x)| = \left| \sum_{k \ge 0} \int_{A_k} \varphi_k(y) h(x - y)\big(d\mu - d\nu\big)(y) \right|$$

(by (1.3), $\sum_k \varphi_k = 1$ and supp $\varphi_k \subset A_k$).

Notice that $\varphi_k(y)h(x - y)$ is Lipschitz, with a constant at most $Cr_k^{-d-\beta-1}$, and it vanishes outside of $B(0, 2^{k+4}r_0) \subset B(x, 2^{k+5}r_0)$; thus by (2.2) (applied with $C^{-1}r_k^{d+\beta+1}\varphi_k(\cdot)h(x - \cdot) \in$

$\Lambda(x, 2^{k+5}r_0))$

$$(2.16) \qquad \left| \int_{A_k} \varphi_k(y) h(x-y) (d\mu - d\nu)(y) \right| \leq C r_k^{-\beta} \mathcal{D}_{x, 2^{k+5}r_0}(\mu, \nu).$$

For $k = 0$, $\mathcal{D}_{x, 2^{k+5}r_0}(\mu, \nu) = \mathcal{D}_{x, 32r_0}(\mu, \nu) = \mathcal{D}_{x, 32r_0}(\mu, \nu_0)$. For $k \geq 1$, we use intermediate measures. We start with

(2.17)

$$\mathcal{D}_{x, 2^{k+5}r_0}(\mu, \nu) \leq \mathcal{D}_{x, 2^{k+5}r_0}(\mu, \nu_k) + \mathcal{D}_{x, 2^{k+5}r_0}(\nu_k, \nu_0) \leq \mathcal{D}_{x, 2^{k+5}r_0}(\mu, \nu_k) + \sum_{l=1}^{k} \mathcal{D}_{x, 2^{k+5}r_0}(\nu_l, \nu_{l-1}),$$

where the triangle inequality comes directly from the definition (2.2). We claim that

$$(2.18) \qquad \mathcal{D}_{x, 2^{k+5}r_0}(\nu_l, \nu_{l-1}) \leq C\alpha(x, 2^{l+5}r_0),$$

because both measures ν_l and ν_{l-1} approximate μ well in $B(x, 2^{l+5}r_0)$. The general idea is that since the two measures are flat measures associated to planes that pass near x (i.e. (2.14) holds), a good control on $B(x, 2^{l+5}r_0)$ implies a good control on $B(x, 2^{k+5}r_0)$. The proof is almost the same as for equation (6.78) in [DFM3], so we leave it.

We return to (2.15): use (2.13), (2.16), (2.17), and (2.18) to obtain

$$(2.19) \qquad |R(x) - R_\nu(x)| \leq C \sum_{k \geq 0} r_k^{-\beta} \sum_{0 \leq l \leq k} \alpha(x, 2^{l+5}r_0) \leq C \sum_{l \geq 0} r_l^{-\beta} \alpha(x, 2^{l+5}r_0).$$

Again the computations are the same as [DFM3]. Notice that $\alpha(x, 2^{l+5}r_0) \leq 2^{d+1}\alpha(y, 2^{l+6}r_0)$ for every $y \in B(x, 16r_0)$, just by (2.2), (2.3), and because $B(x, 2^{l+5}r_0) \subset B(y, 2^{l+6}r_0)$ and hence $\Lambda(x, 2^{l+5}r_0) \subset \Lambda(y, 2^{l+6}r_0)$ (recall, from (2.2), that $\Lambda(x, r)$ is the set of functions f that are 1-Lipschitz on $\mathbb{R}^n$ and vanish on $\mathbb{R}^n \setminus B(x, r)$). Thus

$$|R(x) - R_\nu(x)| \leq C \sum_{l \geq 0} r_l^{-\beta} \alpha(y, 2^{l+6}r_0) = C r_0^{-\beta} \sum_{l \geq 0} 2^{-\beta l} \alpha(y, 2^{l+6}r_0)$$

(2.20)

$$= C\delta(x)^{-\beta} \sum_{l \geq 0} 2^{-\beta l} \alpha(y, 2^{l+6}r_0)$$

for $y \in B(x, 16r_0)$.

This was our estimate for R, but we have a similar estimate for the iterated derivatives of R. That is, we start from (2.7) instead of (1.3), and observe that we can compute as above, with an extra $|x - y|^{-j}$, which transforms into an extra $r_k^{-j} \leq C\delta(x)^{-j}$ in the estimates below. This yields

$$|\nabla^j R(x) - \nabla^j R_\nu(x)| = \left| \int_E \nabla^j h(x-y)(d\mu - d\nu)(y) \right|$$

(2.21)

$$\leq C\delta(x)^{-\beta-j} \sum_{l \geq 0} 2^{-(\beta+j)l} \alpha(y, 2^{l+6}r_0)$$

for $y \in B(x, 16\delta(x))$. Observe also that a direct estimate with (2.7) yields

$$(2.22) \qquad |\nabla^j R(x)| \leq C\delta(x)^{-\beta-j}.$$

Let us check that if we pick P_0, our initial plane, correctly we have a similar estimate for $R_{\nu,\beta}$, i.e.,

$$(2.23) \qquad\qquad |\nabla^j R_\nu(x)| \le C\delta(x)^{-\beta-j}.$$

Recall from the discussion below (2.13), we choose ν_0 such that $\mathcal{D}_{x,32r_0}(\mu,\nu_k) = \alpha(x, 32r_0)$ when $\alpha(x, 32r_k r) \le c$. In this regime, we claim that, perhaps by choosing c a little bit smaller, the following inequality holds:

$$(2.24) \qquad\qquad \operatorname{dist}(y, E) \le 10^{-1}r_0 \ \text{ for } y \in P_0 \cap B(x, 16r_0).$$

Otherwise, pick $y \in P_0 \cap B(x, 16r_0)$, at distance at least $10^{-1}r_0$ from E, choose a Lipschitz bump function f, supported in $B(y, 2 \cdot 10^{-2}r_0)$ so that $f = 10^{-2}r_0$ on $B(y, 10^{-2}r_0)$ and f is 1-Lipschitz. Then (2.2) yields $\int f(d\mu - d\nu_0) \le cr_0^{d+1}$, while $\int f d\mu = 0$ (because E does not meet $B(y, 2 \cdot 10^{-2}r_0)$) and $\int f d\nu \ge 10^{-2}r_0\nu(B(y, \tau r_0)) \ge C^{-1}r_0^{d+1}$ by (2.14) and because $y \in P_0$. If we take c small enough, we get a contradiction that proves (2.24). We deduce from this that

$$(2.25) \qquad\qquad |y - x| \ge \frac{r_0}{2} \ \text{ for } y \in P_0,$$

because either $y \in B(x, 16r_0)$ and we use the fact that $r_0 = \delta(x) = \operatorname{dist}(y, E)$, or else $|y - x| \ge 16r_0$ anyway.

When $\alpha(x, 32r_0) \ge c$, we decided to pick any d-plane through $B(x, 2r_0)$, and we simply make sure that (2.25) holds when we do this.

Once we have (2.25), (2.23) easily follows from (2.14) and the usual computations.

We may now return to our original formula, (2.10). It says that $\partial_i\big(|\nabla D(x)|^2\big)$ is a sum of $2n$ terms, and we claim that because of (2.22), each of these terms is bounded from above by $C\delta(x)^{-1}$.

Indeed, if we did not have any derivatives, we would simply get $CR(x)^{-\frac{2}{\beta}} \le CD^2 \le C\delta(x)^2$ by (1.5). But we have three additional derivatives, which give an extra $\delta(x)^{-3}$. Altogether, the brutal estimate is $|\partial_i\big(|\nabla D(x)|^2\big)| \le C\delta(x)^{-1}$. By (2.23), we would have the same estimate when we replace R with $R_{\nu,\beta}$ in some places. Now we need to estimate $\partial_i\big(|\nabla D(x)|^2\big) - \partial_i\big(|\nabla D_{\nu,\beta}(x)|^2\big)$, which is a sum of terms like the above, except that now one of the terms of each product is replaced with the corresponding difference involving $|\nabla^j R(x) - \nabla^j R_{\nu,\beta}(x)|$. We use (2.21) for this difference (which allows us to multiply the estimate by a sum of α-numbers), keep the same estimates for the rest of each product, sum everything up, and get that

$$\big|\partial_i\big(|\nabla D(x)|^2\big)\big| = \big|\partial_i\big(|\nabla D(x)|^2\big) - \partial_i\big(|\nabla D_{\nu,\beta}(x)|^2\big)\big|$$
$$\le C\delta(x)^{-1}\sum_{l\ge 0} 2^{-(\beta+1)l}\alpha(y, 2^{l+6}r_0),$$

for $y \in B(x, 16\delta(x))$. This is (2.4), the desired result. $\qquad\qquad\square$

The attentive reader may ask why we raise $|\nabla D|$ to the second power in the definition of F (see (1.7)). Indeed, this is done mostly for aesthetic reasons (mainly so that (2.10) doesn't look so nasty). In the following Corollary we show that our result still holds if F is replaced by $\tilde{F}$ (which is the same except we do not square $|\nabla D|$).

Corollary 2.2. *Theorem 2.1 is still valid when we replace $F(x)$ with*

$$(2.26) \qquad \widetilde{F}(x) = \delta(x)\big|\nabla(|\nabla D|)(x)\big|.$$

Proof. Noting that $|\nabla D| = (|\nabla D|^2)^{1/2}$ we see that $\widetilde{F}(x) = \frac{1}{2}F(x)|\nabla D|^{-1}$, at least for x such that $\nabla D(x) \neq 0$. Let $C_1 \geq 0$ be large, to be chosen soon (depending on n, d, and the Ahlfors regularity constant for μ), and set $Z = \{x \in \Omega \,;\, |\nabla D| \leq C_1^{-1}\}$. It is enough to control $\widetilde{F}(x)\mathbf{1}_{\mathbf{Z}}(\mathbf{x})$, because we can use Theorem 2.1 for the rest of $\widetilde{F}$.

Even on Z, $\widetilde{F}$ is not as large as one may fear; for $1 \leq i \leq n$,

$$(2.27) \qquad \begin{aligned} \left|\partial_i(|\nabla D|)(x)\right| &= \left|\partial_i(\sqrt{|\nabla D|^2})(x)\right| = \frac{1}{2}\left|\frac{\partial_i(|\nabla D|^2)(x)}{|\nabla D(x)|}\right| \\ &= \left|\frac{\partial_i\nabla D(x)\cdot\nabla D(x)}{|\nabla D(x)|}\right| \leq \left|\nabla^2 D(x)\right| \leq C\delta(x)^{-1} \end{aligned}$$

by brutal computations, and at the end, (2.8) and (2.22). In particular, this implies that $\widetilde{F}$ is bounded uniformly on Z:

$$(2.28) \qquad |\widetilde{F}(x)| = \delta(x)|\nabla|\nabla D(x)|| \leq C, \ \ \forall x \in Z.$$

In addition, we claim that Z itself is not large. Indeed, let $x \in Z$ be given; recall the notation used in the proof of Theorem 2.1, specifically that ν is a well chosen flat measure so that $D_{\nu,\beta}(z) = c_2\lambda_0^{-1/\beta}\delta_P(z)$. Hence, by (2.14), $|\nabla D_{\nu,\beta}(x)| \geq C^{-1}$ and, by (2.8) (and (2.25)),

$$(2.29) \qquad |\nabla R_{\nu,\beta}(x)| \geq C^{-1}R_{\nu,\beta}(x)^{\frac{1}{\beta}+1} \geq C^{-1}\delta(x)^{-1-\beta}.$$

On the other hand, $|\nabla D| \leq C_1^{-1}$ by definition of Z hence, by (2.8) again,

$$(2.30) \qquad |\nabla R(x)| \leq CC_1^{-1}R(x)^{\frac{1}{\beta}+1} \leq CC_1^{-1}\delta(x)^{-1-\beta}.$$

If we choose C_1 large enough, we deduce from the two that

$$(2.31) \qquad |\nabla R_{\nu,\beta}(x) - \nabla R(x)| \geq c\delta(x)^{-1-\beta}$$

for some $c > 0$. Then by (2.21) (with $j = 1$),

$$(2.32) \qquad \sum_{l\geq 0} 2^{-(\beta+j)l}\alpha(y, 2^{l+6}r_0) \geq C^{-1} \ \ \text{for } y \in B(x, 16\delta(x)).$$

But we have seen earlier that the work of Tolsa [To] gives a Carleson estimate on the square of the sum of (2.32), which implies, by Chebyshev (and the same computations using Fubini that lead from (2.6) to (1.8); see the two paragraphs after (2.6)) that Z is a Carleson set. That is, there is a constant $C \geq 0$ such that for $X \in E$ and $R > 0$,

$$(2.33) \qquad \int_{B(X,R)\cap Z} \frac{d\mu(x)}{\delta(x)^{n-d}} \leq CR^d.$$

This immediately leads to a Carleson bound on $\widetilde{F}|_Z$,

$$(2.34) \qquad \int_{B(X,R)\cap Z} |\widetilde{F}(x)|^2\frac{d\mu(x)}{\delta(x)^{n-d}} \overset{(2.28)}{=} \int_{B(X,R)\cap Z} C^2\frac{d\mu(x)}{\delta(x)^{n-d}} \leq CR^d.$$

This completes the proof of Corollary 2.2.   □

3. E IS FLAT WHEN $|\nabla D|$ IS CONSTANT ON Ω.

To prove the converse to Theorem 2.1 (and later in Section 5 to study non-tangential limits), we first prove the "limiting result": if F vanishes, i.e., if $|\nabla D_{\beta,\mu}|$ is constant, then μ must be supported on a plane.

More precisely, we show in this section that in this case $D_{\beta,\mu}$ is a multiple of δ, E is a d-plane and μ is a multiple of $\mathcal{H}^d|_E$.

We learned while writing the paper that a subset, C, of a Banach space, X, with the property that for every $x \in X$ there is a unique closest point $c \in C$ to x is called a **Chebyshev set**. Chebyshev sets are well studied (see the survey, [Bo]), and it is an old theorem, attributed to Bunt, that every Chebyshev set in Euclidean space is convex (Theorem 13 in [Bo]). Invoking this result would allow us end the proof of Theorem 3.1 after (3.5). However, we include the whole argument for the sake of completeness. As an aside, it is apparently an interesting open question as to whether every Chebyshev set in a Hilbert space is convex.

Theorem 3.1. *Let E be a closed set in $\mathbb{R}^n$, and let D be a continuous nonnegative function on $\mathbb{R}^n$, which vanishes on E, is of class C^1 on $\Omega = \mathbb{R}^n \setminus E$, and such that $|\nabla D|$ is positive and constant on every connected component of Ω. Then E is convex. If, in addition, $|\nabla D| = 1$ on Ω, then $D(x) = \text{dist}(x, E)$ for $x \in \mathbb{R}^n$.*

Theorem 3.1 is stated as is so that we may apply it easily in the proof of Corollary 3.2. However, the discerning reader will notice that the theorem is really the combination of two separate facts: first that a C^1 function vanishing on E with constant derivative on a connected component of $\mathbb{R}^n \setminus E$ is a constant multiple of $\text{dist}(x, E)$ on that component. Second, the fact, mentioned in the introduction, that if $\text{dist}(x, E)$ is C^1 in $\mathbb{R}^n \setminus E$, then E is convex.

Note that the function D in Theorem 3.1 is not necessarily of the form $D_{\alpha,\mu}$ defined in (1.3). However, we will eventually apply the theorem to exactly those functions.

Proof. We start with the assumption that $|\nabla D| = 1$ on some connected component Ω_0 of Ω. Observe first that $D(x) > 0$ on Ω_0, because of our assumption that $|\nabla D| \neq 0$. We may of course assume that $\Omega_0 \neq \emptyset$, and in our main case when E is Ahlfors regular of dimension $d < n - 1$, we even know that Ω is connected and dense in $\mathbb{R}^n$, so $\Omega_0 = \Omega$.

Set $v(x) = \nabla D(x)$ on Ω_0; this is a C^0 vector field that does not vanish, and we can use it to define a flow. That is, given $x \in \Omega_0$, we can define $\varphi(x, \cdot)$ to be the solution of the equation $\frac{\partial \varphi(x,t)}{dt} = -v(\varphi(x, t))$ such that $\varphi(x, 0) = x$, which is defined on a maximal (open) interval $I(x)$. By the chain rule (and $|\nabla D| \equiv 1$), $\partial_t D(\varphi(x, t)) = -1$ for $t \in I(x)$. Integrating this, we note that $D(\varphi(x, t)) = D(x) - t$ for $t \in I(x)$.

This solution can be extended as long as $\varphi(x, t)$ stays in Ω (or equivalently Ω_0), which means at least as long as $D(\varphi(x, t)) > 0$. So I contains $[0, D(x))$, and $\lim_{t \uparrow D(x)} D(\varphi(x, t)) = 0 \Rightarrow \varphi(x, D(x)) \in E$. To be precise; while the flow cannot be extended naturally to time $t = D(x)$, the limit $p(x) \equiv \lim_{t \uparrow D(x)} \varphi(x, t)$ exists and $p(x) \in E$.

Since δ and $\varphi(x, \cdot)$ are 1-Lipschitz, $t \mapsto \delta(\varphi(x, t))$ is 1-Lipschitz, and

(3.1) $\delta(x) = \delta(\varphi(x, 0)) \leq \delta(\varphi(x, D(x))) + D(x) = D(x)$.

On the other hand, if $p_{\delta(x)}$ is a point of E that minimizes the distance to x, then the bound on the gradient of D implies that

$$D(x) = |D(p_{\delta(x)}) - D(x)| \leq |p_{\delta(x)} - x| = \delta(x),$$

where the first equality follows from the continuity of D at $p_{\delta(x)}$. That is, not only did we prove that

(3.2)
$$\delta(x) = D(x) \text{ for } x \in \Omega_0,$$

but we also learned that the flow follows straight lines. More precisely, setting

(3.3)
$$\Gamma_x = \{\varphi(x,t); 0 \leq t < D(x)\},$$

we know that the length of Γ_x is $D(x) = \delta(x)$, and since $|p(x) - x| \geq \delta(x)$ by definition of $\delta(x)$, the fact that Γ_x goes from x to $p(x)$ and has a length $D(x)$ implies that it is the line segment $[x, p(x))$.

Let us pause to point out that we have already proven the second conclusion of the theorem; that if $|\nabla D(x)| \equiv 1$ then $D(x) = \text{dist}(x, E)$. To prove the first part of the theorem it will suffice to show that

(3.4)
$$p \text{ is 1-Lipschitz on } \overline{\Omega}_0.$$

Indeed, if we know (3.4), let us assume for the sake of contradiction that E is not convex. In particular, there are points $a, b \in E$ be given, with $b \neq a$ such that there exists an $x \in (a, b)$ (the open line segment between a and b) that does not lie in E. Let Ω_0 denote the connected component of Ω that contains x, and define p on $\overline{\Omega}_0$, as above (if $|\nabla D| = c \neq 1$ on Ω_0 we can always consider D/c without losing generality). Denote by I_0 the connected component of $(a, b) \cap \Omega$ that contains x; this is an interval $(a', b') \subset (a, b)$, which is contained in Ω_0 (because it is connected and contained in Ω), and $a', b' \in E$. By (3.4) the length of the arc $p(I_0)$ is at most $|b' - a'|$, and since p is the identity on E we get that $p(I_0) = (a', b')$. In particular, $x \in p(I_0) \subset E$, which is a contradiction.

For the remainder of the proof we study p, aiming towards (3.4). We first check that $p(x)$ is the unique closest point in E to x.

Observe that $\nabla D(x) = -\frac{p(x)-x}{|p(x)-x|}$ for $x \in \Omega_0$. This is, for instance, because Γ_x has a tangent at x that points in the direction of $-v(x) = -\nabla D(x)$. But Γ_x also points in the direction of $p(x) - x$, since $\Gamma_x = [x, p(x))$. Let us deduce from this that

(3.5)
$$|y - x| > \delta(x) \text{ for } y \in E \setminus \{p(x)\},$$

i.e., that $p(x)$ is the only point of E that realizes the distance to x. Indeed, let $y \in E$ be such that $|y - x| = \delta(x)$, and observe that along $[y, x]$ the function $D(\xi)$ goes from 0 to $\delta(x)$; since $D(x)$ is 1-Lipschitz and the length of the segment is $\delta(x)$, we see that $\langle \nabla D(\xi), \frac{y-x}{|y-x|} \rangle = -1$, for all $\xi \in [x, y)$. Therefore, $\frac{y-x}{|y-x|}$ also points in the direction of $-v(x)$. We conclude that $y - x$ and $p(x) - x$ are two vectors which point in the same direction and have the same length, hence $y = p(x)$. The claim, (3.5), follows.

Let us extend p to $\overline{\Omega}_0$ by setting $p(x) = x$ when $x \in E$. We claim that p is continuous on $\overline{\Omega}_0$. Indeed, if $\{x_k\}$ in $\overline{\Omega}_0$ converges to x, the sequence $\{p(x_k)\}$ is bounded (because $|p(x_k) - x_k| = \delta(x_k)$), and it is easy to see that any point of accumulation, y, of this sequence

is such that $|y - x| = \lim_{k \to +\infty} |p(x_k) - x_k| = \delta(x)$, hence is equal to $p(x)$. That is, $\{p(x_k)\}$ converges to $p(x)$, as needed for the continuity of p.

To prove the higher regularity of p, we start by showing that if $L_+(p(x), x)$ is the closed half line that starts from $p(x)$ and contains x, then

$$(3.6) \qquad L_+(p(x), x) \subset \Omega_0 \cup \{p(x)\} \text{ and } p(y) = p(x) \text{ for } y \in L_+(p(x), x).$$

Note that if $y \in [p(x), x]$ then (3.6) is immediate by the uniqueness of C^0 vector flows; $y = \varphi(x, t)$ for some $t \in (0, \delta(x)]$ and therefore $\varphi(y, s) = \varphi(x, t + s)$ for all $s \in [0, \delta(y)]$. To prove (3.6) for $y \notin [p(x), x]$ we must reverse the flow. For $x \in \Omega_0$, we define $\varphi_+(x, \cdot)$ to be reverse flow of φ; that is $\frac{\partial \varphi_+(x,t)}{dt} = v(\varphi_+(x, t))$, with the initial value $\varphi_+(x, 0) = x$. This function is defined on an interval $I_+ \subset [0, +\infty)$, and since we can check as we did for φ above that $D(\varphi_+(x, t)) = D(x) + t \geq D(x) > 0$ for $t \in I(x)$, and that we can extend the solution as long as $\varphi_+(x, t) \in \Omega_0$, it follows that $I(x) = [0, +\infty)$.

Let $x \in \Omega_0$ and $t_0 > 0$ be given, and set $y = \varphi_+(x, t_0)$. Notice that $\varphi_+(x, t_0 - t) = \varphi(y, t)$ for $0 \leq t \leq t_0$, because φ_+, φ come from reverse flows. This implies that $x \in \Gamma_y$ (recall the notation from (3.3)). But we know that Γ_y is a straight line from y to $p(y)$. From this, $p(y) = p(x)$ immediately follows; indeed, if $p(y) \in (p(x), x]$ then $\delta(x) \leq |p(y) - x| < |p(x) - x|$ a contradiction. Similarly if $p(x) \in (p(y), y]$, thus (3.6) follows.

For $x \in \Omega_0$, denote by $P(x)$ the hyperplane through $p(x)$ which is orthogonal to $x - p(x)$. Then let $H(x)$ denote the half space on the other side of $P(x)$. That is, set

$$(3.7) \qquad H(x) = \big\{ z \in \mathbb{R}^n \,;\, \langle z, x - p(x) \rangle \leq \langle p(x), x - p(x) \rangle \big\}.$$

We claim that $E \subset H(x)$. To check this, we may assume that $p(x) = 0$ and $x = \lambda e_n$, where e_n is the last element of the canonical basis. By the discussion above, $p(te_n) = 0$ for every $t > 0$, and this means that $t = \text{dist}(te_n, 0) \leq \text{dist}(te_n, z)$ for every $z \in E$. Write $z = ae_n + v$, with $v \perp e_n$; then $\text{dist}(te_n, z)^2 = |(t - a)e_n - v|^2 = (t - a)^2 + |v|^2$ and we get that $t^2 \leq (t - a)^2 + |v|^2$. We let t tend to $+\infty$ and get that $a \leq 0$, hence $\langle z, x - p(x) \rangle = \langle z, x \rangle = a\lambda \leq 0 = \langle p(x), x - p(x) \rangle$, which means that $z \in H(x)$, as needed.

We now turn to (3.4). Let $x, y \in \overline{\Omega}_0$ be given; we want to prove that

$$(3.8) \qquad |p(x) - p(y)| \leq |x - y|.$$

Without loss of generality, we may assume that $p(x) = 0$ and $x = \lambda e_n$ for some $\lambda \geq 0$. We have two inequalities that we can use, the fact that $p(y) \in H(x)$ (because $p(y) \in E$), which says that

$$(3.9) \qquad \langle p(y), x \rangle \leq 0,$$

and similarly the fact that $0 = p(x) \in H(y)$, i.e., $0 = \langle p(x), y - p(y) \rangle \leq \langle p(y), y - p(y) \rangle$, or equivalently

$$(3.10) \qquad \langle p(y), y \rangle \geq |p(y)|^2.$$

Write $y = \mu e_n + y_0$ for some $y_0 \in e_n^\perp$, and first assume that $\mu \leq 0$. Replacing x with 0 diminishes $|x - y|$ but does not change $|p(x) - p(y)|$; thus it is enough to prove (3.8) for $x = 0$. That is, we just need to show that $|p(y)| \leq |y|$, which follows from (3.10) and Cauchy-Schwarz.

So we may assume that $\mu > 0$. Replacing x with μe_n diminishes $|x - y|$ but does not change $|p(x) - p(y)|$, so as before we may assume that $\lambda = \mu$. That is, $y = \lambda e_n + y_0$. Now write $p(y) = a e_n + b y_0 + z$, with $a, b \in \mathbb{R}$ and $z \in e_n^\perp \cap y_0^\perp$. Then $a \leq 0$ by (3.9) and because $\lambda > 0$, and (3.10) yields

$$(3.11) \qquad \lambda a + b|y_0|^2 = \langle y, p(y) \rangle \geq |p(y)|^2 = a^2 + b^2|y_0|^2 + |z|^2.$$

Now

$$(3.12) \qquad |p(x) - p(y)|^2 = |p(y)|^2 \leq \lambda a + b|y_0|^2$$

and for (3.8) we just need to know that $\lambda a + b|y_0|^2 \leq |x - y|^2 = |y_0|^2$. Since $a \leq 0$, we just need to check that $b \leq 1$ or $y_0 = 0$. We return to (3.11), which says that

$$(3.13) \qquad b(b-1)|y_0|^2 \leq \lambda a - a^2 - |z|^2.$$

The right-hand side is non positive, so $y_0 = 0$ or else $b(b-1) \leq 0$; this last case is impossible if $b > 1$, so finally (3.8) holds and p is 1-Lipschitz. $\qquad\square$

Notice that there is a (less interesting) converse. If E is convex and $D(x) = \operatorname{dist}(x, E)$, then D is 1-Lipschitz, the point $p(x) \in E$ such that $|x - p(x)| = D(x)$ is unique, and it is not so hard to check that $\nabla D(x) = -\frac{p(x)-x}{|p(x)-x|}$ and so $|\nabla D(x)| \equiv 1$.

We now apply Theorem 3.1 to the situation where $D = D_{\alpha,\mu}$ is defined by (1.3).

Corollary 3.2. *Let μ be a d-dimensional Ahlfors regular measure supported on the closed set $E \subset \mathbb{R}^n$. Suppose that for some $\alpha > 0$, the function $D_{\alpha,\mu}$ defined by (1.3) and (1.5) is such that on $\Omega = \mathbb{R}^n \setminus E$, $|\nabla D_{\alpha,\mu}|$ is locally constant and positive. Then d is an integer, E is a d-plane, and the density of μ with respect to $\mathcal{H}^d_{|E}$ is constant. If $d < n - 1$, there is a constant $c > 0$ such that $D_{\alpha,\mu}(x) = c \operatorname{dist}(x, E)$ for $x \in \Omega$; if $d = n - 1$, we may have two different formulas like this for the two components of Ω.*

For this result we only need to know that $d < n$. However, when we relate this result to the square function estimates below we may need $d < n - 1$.

Proof. Let μ, E, and α satisfy the assumptions and let $D = D_{\alpha,\mu}$. We observed earlier that D is smooth on Ω, and by (1.4) it is equivalent to δ on Ω, hence has a continuous extension to $\mathbb{R}^n$ such that $D(x) = 0$ on E. Then $c^{-1}D$ satisfies the assumptions of Theorem 3.1, hence E is convex and D is a constant multiple of δ on each of the connected components of Ω.

Next we check the geometric fact that if $d < n$ and E is a convex Ahlfors regular set of dimension d, then d is an integer and E is a subset of an affine d-space.

Denote by m the smallest integer greater than or equal to to d. That is, $m = d$ if d is an integer, and $m = [d] + 1$ otherwise. First we check that $m = d$ (and d is an integer). Suppose that $0 \in E$. It is easy to find $m + 1$ independent points in E, i.e., points $x_0, \cdots x_m \in E$ that are not contained in any $(m-1)$-plane (a d-Ahlfors regular set cannot be a subset of a $(m-1)$-dimensional plane since $m - 1 < d$).

Since E is convex, it contains the convex hull of the $m + 1$ points above, and in particular it contains an m-disk Δ. This forces $d \geq m$, and hence $d = m$. In addition, let P denote the affine d-plane that contains Δ; notice that $E \subset P$, because otherwise E contains a $(d+1)$-disk (by convexity again), and cannot be Ahlfors regular of dimension d.

We want to show now that E is all of P. We will show a slightly more general statement, that if $d < n$ and E is a convex, d-Ahlfors regular subset of a d-plane $P \subset \mathbb{R}^n$, and if $\delta(x) = \operatorname{dist}(x, E)$ is of class C^2 on all of $\mathbb{R}^n \setminus P$, then $E = P$. Since in the present situation $D = c\delta$ and $D \in C^\infty(\mathbb{R}^n \setminus E)$, we will conclude that $E = P$.

To see this, assume that $E \neq P$ and, without loss of generality, that 0 is a boundary point of E, considered as a subset of P. That is, $0 \in E$ (because E is closed) and every ball around 0 contains a point in $P \setminus E$. Let C be the set of points e such that $\lambda e \in E$ for some $\lambda > 0$. Since E is convex and contains 0, C is also the set of points e such that $\lambda e \in E$ for $\lambda > 0$ small, and then C is a convex cone. Next let e_1 lie in the interior of C; such a point exists because E contains an m-disk Δ (as above), and $-e_1 \notin C$ because otherwise 0 would be an interior point of E.

Let e_2 be a unit direction which is normal to P; we claim that δ is not C^2 in the direction e_1 at the point e_2. For small $\epsilon > 0$, we have $\epsilon e_1 \in E$; this is the definition of C. Thus $\delta(\epsilon e_1 + e_2) = 1$ for all small enough positive $\epsilon > 0$, and consequently, $\partial_{e_1}\delta(\epsilon e_1 + e_2) = \partial^2_{e_1 e_1}\delta(\epsilon e_1 + e_2) = 0$. Other the other hand, the fact that $-e_1 \notin \overline{C}$ implies that there exists some $\theta > 0$ such that $e_1 \cdot x > -(1-\theta)\|x\|$ for all $x \in E$ (θ depends on the distance between $-e_1$ and $\overline{C}$). Let x_ϵ be the closest point in E to the point $e_2 - \epsilon e_1$; then

$$\delta^2(e_2 - \epsilon e_1) \equiv \|x_\epsilon - e_2 + \epsilon e_1\|^2 = 1 + \epsilon^2 + \|x_\epsilon\|^2 + 2\epsilon \langle e_1, x_\epsilon \rangle \geq 1 + (\|x_\epsilon\| - \epsilon)^2 + 2\epsilon\theta\|x_\epsilon\|.$$

After analyzing two cases, depending on the relative size of $\|x_\epsilon\|$ and $\varepsilon/2$, we find that

$$\delta(e_2 - \epsilon e_1) \geq 1 + c\epsilon^2.$$

Let $M_\epsilon = \sup_{t \in [0,\epsilon]} |\partial^2_{e_1 e_1}\delta(e_2 - te_1)|$. If we assume that $\partial_{e_1}\delta$ is continuous at e_2 (i.e. that $\partial_{e_1}\delta(e_2) = 0$) then by the Taylor remainder theorem:

$$1 + c\epsilon^2 \leq \delta(e_2 - \epsilon e_1) \leq 1 + M_\epsilon \epsilon^2.$$

This implies that $\lim_{\epsilon \downarrow 0} M_\epsilon > c$ which in turn implies that δ is not C^2 at the point e_2. This contradicts the initial assumption that $E \neq P$.

We are left to prove the final claim; that μ must be a constant times $\mathcal{H}^d|_P$. Assume without losing generality that $0 \in P$ is a point of density for μ, with density $c > 0$ (clearly everything is invariant under translation). For $r_k \downarrow 0$ define the measure μ_k supported on P by $\mu_k(S) \equiv \frac{\mu(r_k S)}{r_k^d}$. Note that μ_k is still a d-Ahlfors regular measure supported on P. It is then easy to see that $\mu_k \rightharpoonup c\mathcal{H}^d|_P$ weakly as measures. By changing coordinates, $y = r_k z$, it is also clear that

$$D_{\mu,\alpha}(r_k x) = \left(\int_P \frac{d\mu(y)}{|r_k x - y|^{d+\alpha}} \right)^{-1/\alpha} = \left(\frac{1}{r_k^\alpha} \int_P \frac{d\mu(r_k z)}{r_k^d |x - z|^{d+\alpha}} \right)^{-1/\alpha} = r_k D_{\mu_k,\alpha}(x), \forall x \in \mathbb{R}^n.$$

Since $D_{\mu,\alpha}(r_k x) = \delta(r_k x) = r_k \delta(x)$ it follows that $D_{\mu_k,\alpha}(x) = \delta(x)$. Letting $k \to \infty$ and using that $\mu_k \rightharpoonup c\mathcal{H}^d|_P$ we get that $D_{c\mathcal{H}^d|_P,\alpha}(x) = \delta(x)$. However, by (2.12) there is only one $c > 0$ for which $D_{c\mathcal{H}^d|_P,\alpha}(x) = \delta(x)$. This implies that the density of μ with respect to $\mathcal{H}^d|_P$ is the same at all points of density in P. Therefore, $\mu = c\mathcal{H}^d|_P$ and we are done. □

Readers familiar with the concept of tangent measure will note that we essentially analyzed the tangent measures of μ at x_0 to obtain that μ has constant density with respect to $\mathcal{H}^d|_P$. This analysis was particularly easy in the case above, i.e. when μ is an Ahlfors regular

measure whose support is a d-plane. Later, in Section 5, we will need to understand the behavior of $D_{\alpha,\mu}(x)$ as $x \to E$ for more complicated sets E. In that section we will treat the concepts of tangent measure and blowup with more care and comprehensiveness.

4. A weak USFE implies the uniform rectifiability of E

In this section we use "endpoint result" of Section 3 to prove a (*a priori* slightly stronger) converse to Theorem 2.1 and Corollary 2.2. Let us note that throughout this section d is not assumed to be an integer (but will be forced to be so *a posteriori*).

Theorem 4.1. *Let $n \geq 1$ be an integer, and let $0 < d < n$ be given. Let μ be a d-dimensional Ahlfors regular measure supported on the closed set $E \subset \mathbb{R}^n$. Let $\alpha > 0$ be given, define $R = R_{\mu,\alpha}$, $D = D_{\mu,\alpha}$, $F = F_{\mu,\alpha}$, and $\widetilde{F} = \widetilde{F}_{\mu,\alpha}$ by (1.3), (1.5), (1.7), and (2.26). For $\varepsilon > 0$, set*

$$(4.1) \qquad Z(\varepsilon) = \big\{ x \in \Omega \,;\, F(x) > \varepsilon \big\} \quad and \quad \widetilde{Z}(\varepsilon) = \big\{ x \in \Omega \,;\, \widetilde{F}(x) > \varepsilon \big\}.$$

If for every $\varepsilon > 0$ $Z(\varepsilon)$ or $\widetilde{Z}(\varepsilon)$ is a Carleson set (see Definition 1.3), then d is an integer and E is uniformly rectifiable.

Notice that the USFE (applied to either F or $\widetilde{F}$) implies the Carleson condition on $Z(\varepsilon)$, respectively $\widetilde{Z}(\varepsilon)$, in the statement, by Chebyshev; thus we will refer to the condition that $Z(\epsilon)$ (or $\widetilde{Z}(\epsilon)$) is a Carleson set as the *weak USFE*.

As is always the case with these types of results, what we will prove is that there is a constant $\varepsilon_0 > 0$, that depends on n, d, α, and the Ahlfors regularity constant for μ, such that if $Z(\varepsilon_0)$ or $\widetilde{Z}(\varepsilon_0)$ is a Carleson set, then d is an integer and E is uniformly rectifiable. But this is not such a useful difference anyway, since ε_0 comes from a compactness argument and cannot be computed.

Before beginning the proof of Theorem 4.1, let us first check that it is enough to prove the theorem for Z. Indeed, recall from (1.7) and (2.26) that $F(x) = \delta(x)\big|\nabla(|\nabla D|^2)(x)\big|$ and $\widetilde{F}(x) = \delta(x)\big|\nabla(|\nabla D|)(x)\big|$. Notice that $\partial_j(|\nabla D|^2) = 2|\nabla D|\partial_j(|\nabla D|)$, so $F \leq 2|\nabla D|\widetilde{F}$. By (2.8), (1.4), and (2.22)

$$(4.2) \qquad |\nabla D|(x) \leq CR(x)^{-\frac{1}{\alpha}-1}|\nabla R(x)| \leq C\delta(x)^{1+\alpha}|\nabla R(x)| \leq C.$$

So $F(x) \leq C\widetilde{F}(x)$. If $F(x) > \varepsilon$, then $\widetilde{F}(x) > \varepsilon/C$. That is, $Z(\varepsilon) \subset \widetilde{Z}(\varepsilon/C)$. If $\widetilde{Z}(\varepsilon/C)$ is a Carleson set, then $Z(\varepsilon)$ is a Carleson set, and if we know the result for Z, we can deduce the uniform rectifiability from this and get the result for $\widetilde{Z}$.

To prove the result for Z we will show that the weak USFE (i.e. the condition that $Z(\varepsilon)$ is a Carleson set) implies that d is an integer and E satisfies the condition known as the Bilateral Weak Geometric Lemma (BWGL). The BWGL property, along with Ahlfors regularity, characterizes uniform rectifiability and so this will complete the proof. Let us quickly recall what the BWGL is; for a more comprehensive introduction to this and other characterizations of uniform rectifiability see, e.g., [DS2].

Recall the local normalized Hausdorff distances, $d_{x,r}$ defined for $x \in \mathbb{R}^n$ and $r > 0$ by
(4.3)
$$d_{x,r}(E, F) = \frac{1}{r} \left(\sup \left\{ \operatorname{dist}(y, F) \,;\, y \in E \cap \overline{B}(x, r) \right\} + \sup \left\{ \operatorname{dist}(y, E) \,;\, y \in F \cap \overline{B}(x, r) \right\} \right),$$

where E, F are closed sets that meet $\overline{B}(x, r)$ (we will not need the other cases). Using this distance we can define a bilateral version of P. Jones' β numbers [Jo], which we denote $\beta_b(x, r)$, as

(4.4)
$$\beta_b(x, r) \equiv \inf_P d_{x,r}(E, P)$$

where the infimum is taken over all affine d-planes P that meet $\overline{B}(x, r)$. These numbers measure, in a two-sided way, how close the set E is to being flat at the point x and scale $r > 0$. We can now state the BWGL:

Definition 4.2. *Let $E \subset \mathbb{R}^n$ be a closed set and $\beta_b(x, r)$ be defined with respect to E as in (4.4). Then E satisfies the condition known as the Bilateral Weak Geometric Lemma (BWGL) if the set $\mathcal{R}(\tau)$ defined by*

(4.5)
$$\mathcal{R}(\tau) = \left\{ (x, r) \in E \times (0, +\infty) \,;\, \beta_b(x, r) \geq \tau \right\},$$

is a Carleson subset of $E \times (0, +\infty)$ for all $\tau > 0$. Recall that any $\mathcal{G} \subset E \times (0, +\infty)$ is a Carleson subset of E if there exists a $C > 0$ such that for all $X \in \mathbb{R}^n$ and $R > 0$,

(4.6)
$$\int_{x \in E \cap B(X, R)} \int_{r \in (0, R]} \mathbf{1}_{\mathcal{G}}(x, r) \frac{d\mathcal{H}^d(x) dr}{r} \leq C R^d.$$

It is proved in [DS1] that if E is Ahlfors regular (of some integer dimension d) and satisfies the BWGL, then it is uniformly rectifiable. See also Theorem 1.76 in [DS2] for the statement. In fact, it is enough to show that $\mathcal{R}(\tau)$ is a Carleson set for a single $\tau > 0$, that depends on the dimensions and the Ahlfors regularity constant for E.

To show that the BWGL holds, we will first replace the $Z(\epsilon)$ with other similar sets $\mathcal{B}(\eta)$, which also satisfy a Carleson condition when the $Z(\epsilon)$ do, and which are more amenable to a later compactness argument that will invoke Corollary 3.2.

Lemma 4.3. *Let E and μ be given, with the assumptions above, and for $M \geq 1$ (a large constant, to be chosen later) and $x \in \Omega$, define a big (Whitney) neighborhood of x as*

(4.7)
$$W(x) = W_M(x) = \left\{ y \in \Omega \cap B(x, M\delta(x)) \,;\, \operatorname{dist}(y, E) \geq M^{-1}\delta(x) \right\}.$$

Define the bad set, $\mathcal{B}(\eta) = \mathcal{B}_M(\eta)$, by

(4.8)
$$\mathcal{B}_M(\eta) = \left\{ x \in \Omega \,;\, F(y) \geq \eta \text{ for some } y \in W_M(x) \right\}.$$

With this notation, if $Z(\varepsilon)$ is a Carleson set, then for each large enough M, $\mathcal{B}_M(3\varepsilon)$ is a Carleson set as well.

Thus, with our assumption that the weak USFE holds, each $\mathcal{B}_M(\eta)$ is a Carleson set.

Proof. This will be a relatively simple covering argument. Let $\tau \in (0, 1)$ be small, to be chosen soon (depending on ε). We define a very dense collection, H_τ, of points in Ω, which is a maximal subset of Ω with the property that $|x - y| \geq \tau \max\{\delta(x), \delta(y)\}$ when $x, y \in H_\tau$ are different.

The net H_τ is useful because F varies so slowly. Indeed, recalling the estimates below (2.25), $\delta(x)^{-1}F(x) = |\nabla(|\nabla D(x)|^2)| \leq C\delta(x)^{-1}$. The same argument, still based on the formula (2.10) and the estimate (2.22), yields

$$(4.9) \qquad |\nabla(\delta(x)^{-1}F)|(x) \leq C\delta(x)^{-2}.$$

Let us use this to check that if τ is small enough (depending on ε),

$$(4.10) \qquad |F(x) - F(x')| \leq \varepsilon \text{ for } x, x' \in \Omega \text{ such that } |x' - x| \leq 4\tau\delta(x).$$

First observe that $\delta(x') \geq \delta(x) - |x' - x| \geq (1 - 4\tau)\delta(x) \geq \delta(x)/2$ and in fact $\delta(x)/2 \leq \delta(z) \leq 2\delta(x)$ for $z \in [x, x']$. Then, setting $G(x) = \delta(x)^{-1}F(x)$ just for the sake of the computation,

$$|F(x) - F(x')| = |G(x)\delta(x) - G(x')\delta(x')| \leq \delta(x)|G(x) - G(x')| + G(x')|\delta(x) - \delta(x')|$$
$$(4.11) \qquad \leq C\delta(x)[|x' - x|\delta(x)^{-2}] + G(x')|x' - x| \leq C|x' - x|\delta(x)^{-1} \leq C\tau \leq \varepsilon,$$

where we used (4.9) and the fact that $G(x') = \delta(x')^{-1}F(x') \leq C\delta(x')^{-1}$ by the estimate above (4.9). So (4.10) holds.

Now let $x \in \mathcal{B}(3\varepsilon)$ be given. This means that we can find $y \in W_M(x)$ such that $F(y) \geq 3\varepsilon$.

By maximality of H_τ, we can find $z \in H_\tau$ such that $|z - y| \leq \tau \max\{\delta(z), \delta(y)\}$ (otherwise, add y to H_τ). If τ is small enough, the triangle inequality yields $\delta(y) \leq 2\delta(z)$ and so $|z - y| \leq 2\tau\delta(z)$.

If τ is small enough, (4.11) implies that $F(z) \geq 2\varepsilon$. In fact, this stays true for all $w \in B(z, \tau\delta(z))$. Notice also that since $y \in W_M(x)$, $(2M)^{-1}\delta(z) \leq \delta(x) \leq 2M\delta(z)$, and also $|x - z| < 2M\delta(x) \leq 4M^2\delta(z)$. In short, $x \in V(z)$, where

$$V(z) = \{x \in \Omega \cap B(z, 4M^2\delta(z)) \mid \delta(x) \geq (2M)^{-1}\delta(z)\}.$$

We are ready for the Carleson estimate. Recall (2.33) and set $d\sigma(x) = \delta(x)^{-n+d}dx$; we need to show that

$$(4.12) \qquad A(X, R) := \int_{x \in \Omega \cap B(X,R) \cap \mathcal{B}(3\varepsilon)} d\sigma(x) \leq CR^d$$

for $X \in E$ and $R > 0$. Let X and R be given. Observe that if $x \in \Omega \cap B(X, R) \cap \mathcal{B}(3\varepsilon)$, any point $z \in H_\tau$ constructed as above lies in $H_\tau \cap B(X, 3MR)$, because $|z - x| \leq 2M\delta(x)$ and $\delta(x) \leq |x - X| \leq R$. Furthermore, the argument above tells us that every $x \in \Omega \cap B(X, R) \cap \mathcal{B}(3\varepsilon)$ is in $V(z)$ for some $z \in H(\tau, X, M, R, \varepsilon) \equiv \{z \in H_\tau \cap B(X, 3MR) \mid F(z) \geq 2\varepsilon\}$. Thus

$$A(X, R) \leq \sum_{z \in H(\tau,X,M,R,\varepsilon)} \int_{x \in V(z)} d\sigma(x) \leq C \sum_{z \in H(\tau,X,M,R,\varepsilon)} \delta(z)^{-n+d}|V(z)|$$

$$\leq C \sum_{z \in H(\tau,X,M,R,\varepsilon)} \delta(z)^d \leq C \sum_{z \in H(\tau,X,M,R,\varepsilon)} \sigma(B(z, \tau\delta(z)/10))$$

$$(4.13) \qquad \leq C\sigma(Z(\varepsilon) \cap B(X, 4MR)) \leq CR^d,$$

by definition of σ (for the second inequality) and the fact that $V(z) \subset B(z, 4M^2\delta(z))$ (for the third one), then because $\sigma(B(z, \tau\delta(z)/10)) \geq C^{-1}\delta(z)^d$ and the balls $B(z, \tau\delta(z)/10)$ are disjoint by definition of H_τ, and finally (for the last line) since each $B(z, \tau\delta(z)/10)$ is contained in $Z(\epsilon) \cap B(X, 4MR)$, and by our Carleson estimate assumption on Z. The lemma follows. $\qquad\square$

As mentioned above, the set $\mathcal{B}_M(3\varepsilon)$ is defined in the right way to make it amenable to a compactness argument. In the following lemma we will show that if x is not in $\mathcal{B}_M(3\varepsilon)$ then the set E is relatively flat in a neighborhood of x of radius comparable to $\delta(x)$.

Lemma 4.4. *For each choice of $0 < d < n$, $\alpha > 0$, an Ahlfors regularity constant C_0, and constants $\eta > 0$ (small) and $N \geq 1$ (large), we can find $M \geq 1$ and $\varepsilon > 0$ such that if μ is Ahlfors regular (of dimension d, constant C_0, and support $E \subset \mathbb{R}^n$), and if $x \in \Omega \setminus \mathcal{B}(3\varepsilon)$, then d is an integer and there is a d-plane P such that $d_{x,N\delta(x)}(E, P) \leq \eta$.*

More explicitly, if d is not an integer, we can find M and ε (depending on d too) such that $\Omega \setminus \mathcal{B}_M(3\varepsilon)$ is empty.

Proof. We will prove this by compactness. That is, let $0 < d < n$, C_0, $\alpha > 0$, N, and $\eta > 0$ be given, and suppose that for each $k \geq 0$, there is a set, E_k, a d-Ahlfors regular measure, μ_k, with constant C_0 and whose support is E_k, which provide a counterexample with $M_k = 2^k$ and $\varepsilon_k = 2^{-k}$. That is, let F_k be defined as in (1.7) but adapted to E_k, μ_k and α. We assume that there are points $x_k \in \Omega_k \equiv \mathbb{R}^n \setminus E_k$, that do not lie in the corresponding bad set $\mathcal{B}_{M_k}^{E_k}(3\varepsilon_k)$, i.e.,

(4.14) $\qquad F_k(y) < 2^{-k}$ for all $y \in \Omega_k \cap B(x_k, 2^k \delta_{E_k}(x_k))$ with $\delta_{E_k}(y) \geq 2^{-k}\delta_{E_k}(x_k)$,

and yet for which the conclusion does not hold. That is, either d is not an integer, or d is an integer but there is no d-plane P_k such that $d_{x_k,N\delta(x_k)}(E_k, P_k) \leq \eta$. We want to reach a contradiction.

By translation and dilation invariance, we may assume that $x_k = 0$ and $\delta_{E_k}(x_k) = \mathrm{dist}(0, E_k) = 1$. We use the uniform Ahlfors regularity to replace $\{(E_k, \mu_k)\}$ with a subsequence for which μ_k converges (in the weak sense) to an Ahlfors regular measure μ_∞, and E_k converges (in the Hausdorff distance sense) to a closed set E_∞ (locally in $\mathbb{R}^n$). It is also easy to check that E_∞ is the support of μ_∞ and that μ_∞ is d-Ahlfors regular with a constant that depends only on C_0 and n. For more details (albeit in a slightly less general context) see the discussion before Lemma 5.1 below.

Additionally, $R_{\mu_k,\alpha} \to R_{\mu_\infty,\alpha}$ (uniformly on compact sets of $\Omega_\infty \equiv \mathbb{R}^n \setminus E_\infty$) and similarly for $D_{\mu_k,\alpha}$ and its derivatives. This follows from the weak convergence of the μ_k (actually a little work is necessary as $\partial^j h(x-y)$ is not compactly supported, but one can argue exactly as in Lemma 5.1 below). Because of this, and with hopefully obvious notation,

(4.15) $$F_\infty(y) = \lim_{k \to +\infty} F_k(y)$$

for every $y \in \Omega_\infty$.

Let $W_k \equiv W_{2^k}(0)$ be as in (4.7) but associated to the set E_k. Clearly, any $y \in \Omega_\infty$ lies in W_k for k large, and so, by assumption, $F_k(y) \leq 2^{-k}$ for k large. Taking limits, (4.15) implies that $F_\infty(y) = 0$ for $y \in \Omega_\infty$ and, by (1.7), $|\nabla D_\infty|$ is locally constant. If by bad luck $|\nabla D_\infty| = 0$ on some connected component $\Omega_0 \subset \Omega_\infty$, we also get that $D_\infty = 0$ on Ω_0 (because D_∞ vanishes on E); this is impossible, by the definition of D_∞ (cf. (1.4) and (1.5)). So $|\nabla D_\infty| \neq 0$ on Ω_0, and now Corollary 3.2 says that d is an integer and E_∞ is a d-plane.

Now recall that E_k converges to the d-plane E_∞; we thus get that for k large, $d_{0,N}(E_k, E_\infty) \leq \eta$, the desired contradiction. Lemma 4.4 follows. $\qquad\square$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. In view of Lemma 4.3, to prove Theorem 4.1, we just need to choose $\varepsilon = \varepsilon_\tau > 0$ such that

(4.16)   if $\mathcal{B}(3\varepsilon)$ is a Carleson set in Ω, then $\mathcal{R}(\tau)$ is a Carleson set in $E \times (0, +\infty)$.

Let us do this. For $(x, r) \in \mathcal{R}(\tau)$, we first use the Ahlfors regularity of E to choose $y \in \Omega \cap B(x, r/2)$ such that $\delta(y) \geq 2\kappa r$, where the constant $\kappa > 0$ depends on the dimensions and the Ahlfors regularity constant. The existence of y is standard; if we could not find it, we would be able to find $C_n \kappa^{-n}$ balls B_j of radius $\kappa r/2$, centered on $E \cap B(x, r/2)$, and that are disjoint. This would yield

$$C_n^{-1} \kappa^{-n} (\kappa r/2)^d \leq C \sum_j \mathcal{H}^d(E \cap B_j) \leq C\mathcal{H}^d(E \cap B(x, r)) \leq Cr^d,$$

a contradiction for κ small because $d < n$. Denote by $H(x, r)$ the ball $B(y, \kappa r)$. Then

(4.17) $|z - x| \leq r$ and $\delta(z) \geq \kappa r$ for $z \in H(x, r)$.

Take $N = 10\kappa^{-1}$; this way, $B(z, N\delta(z))$ contains $B(x, r)$ for $z \in H(x, r)$. Let $\eta > 0$, to be chosen soon in terms of τ, N, and choose $M = M_{N,\eta} > 0$ and $\varepsilon = \varepsilon_{N,\eta} > 0$ as in Lemma 4.4. That lemma says that if $z \in H(x, r) \setminus \mathcal{B}_M(3\varepsilon)$, then we can find a d-plane P such that $d_{z,N\delta(z)}(E, P) \leq \eta$. This also implies that $d_{x,r}(E, P) \leq r^{-1} N\delta(z) d_{z,N\delta(z)}(E, P) \leq N d_{z,N\delta(z)}(E, P) \leq N\eta$ (because $B(z, N\delta(z))$ contains $B(x, r)$ and by the definition (4.3)). We choose η so small that $N\eta < \tau$, and we get that $\beta(x, r) \leq \tau$. By contraposition, we proved that if $(x, r) \in \mathcal{R}(\tau)$, then every $z \in H(x, r)$ lies in $\mathcal{B}_M(3\varepsilon)$.

Return to the proof of (4.16) and assume that $\mathcal{B}_M(3\varepsilon)$ is a Carleson set in Ω. Let $X \in E$ and $R > 0$ be given, and denote by $A(X, R)$ the left-hand side of (4.6), with $\mathcal{G} = \mathcal{R}(\tau)$. Since $|H(x, r)| \geq C^{-1}r^n$, we see that

(4.18) $A(X, R) \leq C \int_{x \in E \cap B(X,R)} \int_{r \in (0, R]} \mathbf{1}_{\mathcal{R}(\tau)}(x, r) r^{-n} \left(\int \mathbf{1}_{H(x,r)}(z) dz \right) \frac{d\mathcal{H}^d(x) dr}{r}.$

Of course we apply Fubini and integrate in x and r first.

Notice that $z \in B(X, 2R) \cap \mathcal{B}_M(3\varepsilon)$ and $|x - z| \leq r \leq \kappa^{-1}\delta(z)$, so we get that

(4.19) $A(X, R) \leq \int_{B(X,2R) \cap \mathcal{B}_M(3\varepsilon)} h(z) dz,$

with

$$h(z) = \int_{x \in E \cap B(z, \kappa^{-1}\delta(z))} \int_{\delta(z) \leq r \leq R} \frac{d\mathcal{H}^d(x) dr}{r^{n+1}}$$

(because $r \geq |x - z| \geq \delta(z)$). The integral in r is at most $C\delta(z)^{-n}$. Then we integrate in x and get that $h(z) \leq C\delta(z)^{d-n}$. Finally,

(4.20) $A(X, R) \leq C \int_{B(X,2R) \cap \mathcal{B}_M(3\varepsilon)} \delta(z)^{d-n} dz \leq CR^d,$

by the assumption that $\mathcal{B}_M(3\varepsilon)$ is a Carleson set. $\qquad\square$

5. Blow-ups and Non-Tangential Limits of $|\nabla D_\beta|$

Throughout this section let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set with $d < n - 1$; this assumption is not strictly necessary for all our proofs but without it we must be a bit more careful as to questions of topology and anyways it is the only scenario in which we are interested. Let μ be a d-Ahlfors regular measure supported on E.

We are interested in the behavior of $\nabla D_{\beta,\mu}$ near E. One convenient tool for studying this is the blowup procedure.

For $Q \in E$, $S \subset E$, $y \in \Omega$, and $r_i \downarrow 0$ we can define

$$
\begin{aligned}
E_{i,Q} &\equiv \frac{E - Q}{r_i} \\
\mu_{i,Q}(S) &\equiv \frac{\mu(r_i S + Q)}{r_i^d} \\
D_{i,\beta,Q}(y) &\equiv \frac{D_{\beta,\mu}(r_i y + Q)}{r_i}.
\end{aligned}
$$

(5.1)

When the point Q is unimportant or clear from context we may abuse notation and refer simply to E_i, μ_i and $D_{i,\beta}$. Note that μ_i is still Ahlfors-regular (with the same constants as μ) and E_i is the support of μ_i. To explain the definition of D_i, let $y \in \mathbb{R}^n \setminus E_i$, which implies that $y = \frac{z-Q}{r_i}$ for some $z \in \mathbb{R}^n \setminus E$. Then we can calculate

$$
(5.2) \qquad D_{\beta,\mu_i}(y)^{-\beta} \equiv \int_{E_i} \frac{d\mu_i(x)}{|x - y|^{d+\beta}} \overset{w = r_i x + Q \in E}{=} \int_E \frac{d\mu(w)}{r_i^d |\frac{w-Q}{r_i} - \frac{z-Q}{r_i}|^{d+\beta}} = \left(\frac{D_{\beta,\mu}(z)}{r_i} \right)^{-\beta}.
$$

As the μ_i are uniformly Ahlfors regular, we know that, perhaps passing to a subsequence, we have $\mu_i \rightharpoonup \mu_\infty$. Since the μ_i are uniformly Ahlfors regular, μ_∞ is also Ahlfors regular and its support, E_∞, is the limit (in the Hausdorff distance sense) of the E_i. We want to show that R_i and D_i converge to R_∞ and D_∞.

Lemma 5.1. *Let E, μ be as above and $r_k \downarrow 0$ and $Q \in E$. With the notation and assumptions above, R_k, D_k, and their derivatives converge to R_∞, D_∞, and their derivatives, uniformly on every compact subset of $\Omega_\infty = \mathbb{R}^n \setminus E_\infty$.*

Proof. Consider $\nabla^j R_k(x) = \int_{E_k} \nabla^j h(x-y) d\mu_k(y)$, as in (2.7), and fix a compact set $K \subset \Omega_\infty$. Also let $\varepsilon > 0$ be given. It is easy to find a smooth cut-off function φ, supported in a large ball $B(0, R)$, but chosen close enough to $\chi_{B(0,R)}$ so that $\int |\nabla^j h(x-y)||1 - \varphi(y)| d\mu_k(y) \le \varepsilon$ for all k, including $+\infty$. The functions $\{\nabla^j h(x-y)\varphi(y)\}_{x \in K}$, are uniformly bounded in the Lipschitz norm with fixed support inside $\overline{B(0,R)}$. By Arzela-Ascoli this means that we can find a finite collection of continuous functions, $\{g_i\}$, supported in $B(0, 2R)$ such that for each $x \in K$ there is a g_i with $|g_i(y) - \nabla^j h(x-y)\varphi(y)| \le \varepsilon R^{-d}$ for all $y \in E_k \cap B(0, 2R)$ (for any k large enough). Then by Ahlfors-regularity $\int |g_i(y) - \nabla^j h(x-y)\varphi(y)| d\mu_k + \int |g_i(y) - \nabla^j h(x-y)\varphi(y)| d\mu_\infty \le C\varepsilon$ for all k. Since each $\int g_i d\mu_k$ converges to $\int g_i d\mu_\infty$, we see that for k large,

$$
\left| \int \nabla^j h(x-y)[d\mu_k - d\mu_\infty] \right| \le C\varepsilon;
$$

the conclusion (for $\nabla^j R_k(x)$) follows. The same estimates for $\nabla^j D$ follow as well, because on the compact set K we have uniform lower and upper bounds on the R_k. This proves the lemma. $\qquad\square$

Lemma 5.1 gives convergence on compact sets separated from E_∞. But we want to understand the convergence up the boundary. In order to do this it will be convenient to introduce "non-tangential access" regions, for reasons that we will make clear shortly. For $Q \in E, R > 0$ and $\eta \in (0,1)$ define:

$$(5.3) \qquad \Gamma_{R,\eta}(Q) = \left\{ x \in \Omega \cap B(Q,R); \ \mathrm{dist}(x,E) \geq \eta|x - Q| \right\}.$$

Associated to these non-tangential regions is the concept of a non-tangential limit.

Definition 5.2. *We say that f has a non-tangential limit, L, at $Q \in E$, if there is some $\eta \in (0,1)$ such that*

$$\lim_{R \downarrow 0} \sup_{x \in \Gamma_{R,\eta}(Q)} |f(x) - L| = 0.$$

We will denote this limit L by $\mathrm{n.t.lim}_{x \to Q} f(x)$, *or even* $\mathrm{n.t.lim}^{\eta}_{x \to Q} f(x)$ *to be explicit.*

Let E, Q, and $\{r_i\}$ be as for (5.1), and assume that the E_i converge to E_∞. Let $\Gamma^{\infty}_{R,\eta}(0)$ be defined as (5.3) but with respect to E_∞. Then, after a new sequence extraction, the sets $\frac{\Gamma_{Rr_i,\eta}(Q)-Q}{r_i}$ converge to a limit Γ, with $\Gamma^{\infty}_{R,\eta/2}(0) \supset \Gamma \supset \Gamma^{\infty}_{R,2\eta}(0)$. For the moment, we only know that D_i, R_i, and their derivatives, converge to D_∞, R_∞, and their derivatives, uniformly on compact subsets of $\mathbb{R}^n \setminus E_\infty$. If we want ∇D_i to converge to ∇D_∞ uniformly on compact subsets of $\mathbb{R}^n$, then it should at least converge uniformly on each $\Gamma^{\infty}_{R,\eta}(0)$, which roughly corresponds, after a change of variables, to ∇D having a non-tangential limit at Q (in fact, for every small η).

In the following two theorems we give a characterization of the existence of non-tangential limits of $|\nabla D|$ at μ-a.e. point $Q \in E$. It turns out that the existence of this limit is intimately linked to the tangent measures of μ at Q (and thus the rectifiability of μ). We will assume some basic familiarity with tangent measures here; for more background we suggest Chapter 17 of [Ma].

Theorem 5.3. *Let E be d-Ahlfors regular and d-rectifiable, μ be a d-Ahlfors regular measure supported on E and $\beta > 0$. Then for μ-almost every $Q \in E$, the limit $\mathrm{n.t.lim}^{\eta}_{x \to Q} |\nabla D_{\beta,\mu}(x)|$ exists for every $\eta > 0$.*

Proof. Notice first that it will be enough to show that for *each* $\eta > 0$, the non-tangential limit $\mathrm{n.t.lim}^{\eta}_{x \to Q} |\nabla D_{\beta,\mu}(x)|$ exists for μ-almost every $Q \in E$, because then the exceptional set of $Q \in E$ for which the limit fails to exist for *all* η is contained in the countable union of the exceptional sets for $\eta_i \equiv 2^{-i}$.

Let $x_i \in \Omega \equiv \mathbb{R}^n \setminus E$ be a sequence of points approaching $Q \in E$ non-tangentially (i.e. $x_i \in \Gamma_{R,\eta}(Q)$ for some $\eta \in (0,1), R > 0$ and $x_i \to Q$). Let $r_i = |x_i - Q|$ and define E_i, μ_i, D_i as in (5.1).

By Lemma 5.1 (perhaps passing to a subsequence) $E_i \to E_\infty$ and $\mu_i \rightharpoonup \mu_\infty$ which is a d-Ahlfors regular measure supported on E_∞. Furthermore $D_i \to D_\infty \equiv D_{\beta,\mu_\infty}$. This

convergence happens uniformly on compacta inside of Ω_∞ in the C^∞ topology. Note that $X_i \equiv \frac{x_i - Q}{r_i} \in \Omega_i \cap B(0,1)$. We also note (by the assumption that x_i is a non-tangential sequence), that $\mathrm{dist}(X_i, E_i) \geq \eta$.

Passing to a subsequence, we may assume that $X_i \to X_\infty$, and then $X_\infty \in \Omega_\infty$ because $\mathrm{dist}(X_\infty, E_\infty) \geq \eta$ (recall that $r_i = |x_i - Q|$). Then by (5.2),

$$(5.4) \qquad |\nabla D_\infty(X_\infty)| = \lim_i |\nabla D_i(X_i)| = \lim_i |\nabla D(x_i)|.$$

The reader may be worried because we only proved the existence of $\lim_i |\nabla D(x_i)|$ for a subsequence, but what will save us is that for almost every choice of $Q \in E$, the left-hand side $L = |\nabla D_\infty(X_\infty)|$ does not depend on $\{x_i\}$ or the choice of subsequences. Then it will follow that all the accumulation points of $|\nabla D(x_i)|$, where $x_i \in \Gamma_{R,\eta}(Q)$ and x_i tends to Q, are equal to the number L (take a sequence $\{x_i\}$, so that $|\nabla D(x_i)|$ tends to a given accumulation point, and then proceed as above). The existence of the non-tangential limit n.t.$\lim_{x \to Q}^\eta |\nabla D(x)| = L$ will follow.

So we look for $Q \in E$ such that $|\nabla D_\infty(X_\infty)|$ above does not depend on $\{x_i\}$, the choice of subsequences, or X_∞ for that matter.

Since E is rectifiable, E has an approximate tangent d-plane P' at almost every point $Q \in E$ (see 15.19 in [Ma]). Since E is Ahlfors regular, and by Exercise 41.21 in [Dv], for instance, P' is a true tangent plane, and any limit E_∞ that we get from extraction is the vector plane P parallel to P'. In addition, 16.5 in [Ma] says that (for almost every $Q \in E$), all the blow-up limits of $\sigma = \mathcal{H}^d_{|E}$ are flat measures, and in fact of the form $\sigma_\infty = \mathcal{H}^d_{|P}$, because the density of σ is 1 almost everywhere. In addition, $\mu = f\sigma$ for some function f such that $C^{-1} \leq f \leq C$, and if Q is a Lebesgue density point for f, all the blow-up limits of μ at Q are of the form $\mu_\infty = f(Q)\lambda_P$.

Thus for almost every point $Q \in E$, we have no choice: in (5.4), $|\nabla D_\infty(X_\infty)|$ must be the constant value of $|\nabla D|$ associated to the plane P and the measure $\mu_\infty = f(Q)\lambda_P$. The existence of n.t.$\lim_{x \to Q}^\eta |\nabla D(x)|$, and Theorem 5.3, follow. $\square$

What follows is the converse to Theorem 5.3. However, we note that in order to prove the rectifiability of E, we need the non-tangential limit to exist inside cones of *all* apertures, as opposed to checking the existence inside cones of any given aperture.

Theorem 5.4. *Let E be a set supporting the d-Ahlfors regular measure, μ, with $d < n$ (not necessarily an integer) and let $\beta > 0$. Assume that for μ-almost every $Q \in E$, the non-tangential limit* n.t.$\lim_{x \downarrow Q}^\eta |\nabla D_{\beta,\mu}(x)|$ *exists for every aperture $\eta \in (0,1)$. Then d is an integer and E is d-rectifiable.*

Proof. We will show that at μ-almost every $Q \in E$, every tangent measure to μ is flat (i.e., is a multiple of the restriction of $\mathcal{H}^d$ to a d-plane). This, combined with a powerful theorem of Preiss, will show that μ is d-rectifiable and thus (since μ is Ahlfors regular) that E is d-rectifiable.

Let $Q \in E$ be a point such that the non-tangential limit of $|\nabla D_{\beta,\mu}|$ exists for every aperture and let $\{r_i\}$ be any sequence of positive numbers that tends to 0. Then define E_i, μ_i and $D_i \equiv D_{\mu_i,\beta}$ as in (5.1). Lemma 5.1 shows that, passing to subsequence if needed, we may

assume that E_i tends to a limit E_∞, μ_i has a weak limit μ_∞, and D_i converges, uniformly on compact subsets of $\mathbb{R}^n \setminus E_\infty$, to $D_\infty \equiv D_{\beta,\mu_\infty}$.

We now want to show that $|\nabla D_\infty|$ is constant on $\Omega_\infty \equiv \mathbb{R}^n \setminus E_\infty$ and is equal to $L = \text{n.t.}\lim_{x \to Q}|\nabla D_{\beta,\mu}(x)|$. Fix $Y \in \Omega_\infty \cap B(0,1)$ and set $\kappa = \text{dist}(Y, E_\infty) \in (0,1)$. Then let $Z \in \Omega_\infty$ be any other point and let $\eta_Z = \text{dist}(Z, E_\infty)/(2\|Z\|) \in (0,1)$. By the convergence of E_i to E_∞, we have $Z \in \Omega_i$ for i large enough and $\text{dist}(Z, E_i) \geq \text{dist}(Z, E_\infty)/2$. Therefore, $r_i Z + Q \in \Omega$ and $\text{dist}(r_i Z + Q, E) = r_i \text{dist}(Z, E_i) \geq r_i \text{dist}(Z, E_\infty)/2 = r_i \eta_Z \|Z\|$. Thus $r_i Z + Q \in \Gamma_{\eta_Z,1}(Q)$ for i large enough. Similarly $r_i Y + Q \in \Gamma_{\kappa/2,1}(Q)$ for i large enough. We can assume that $\eta_Z < \kappa/2$ so that $\Gamma_{\kappa/2,1}(Q) \subset \Gamma_{\eta_Z,1}(Q)$. Observe that $L = \text{n.t.}\lim_{x \to Q}^{\eta_Z}|\nabla D_{\beta,\mu}(x)|$ because the nontangential convergence was in every sector. We can then write

$$|\nabla D_{\beta,\mu_\infty}(Z)| = \lim_{i \to +\infty}|\nabla D_{\beta,\mu}(r_i Z + Q)| = L = \lim_i |\nabla D_{\beta,\mu}(r_i Y + Q)| = |\nabla D_{\beta,\mu_\infty}(Y)|.$$

We conclude that $|\nabla D_{\beta,\mu_\infty}|$ is constant on Ω_∞. If that constant is zero then by the fact that D_{β,μ_∞} vanishes on E_∞ we get that $D_{\beta,\mu_\infty} \equiv 0$. This contradicts (1.4)-(1.5). Thus, $|\nabla D_{\beta,\mu_\infty}|$ is a non-zero constant on Ω_∞ and by Corollary 3.2, E_∞ is a d-dimensional affine space and μ_∞ is a constant multiple of $\mathcal{H}^d$ restricted to E_∞. Thus μ_∞ is flat.

In the language of Preiss's theorem, all the tangents to μ at Q are flat measures. Furthermore by Ahlfors regularity the upper density of μ is bounded away from infinity and the lower density of μ is bounded away from zero. Thus we can invoke Preiss's theorem ([Pr], see also Theorem 17.6 in [Ma]) and conclude that the support of μ is a d-rectifiable set. Since E is the support of μ we are done. $\square$

Finally, we can compute the non-tangential limit of $|\nabla D_{\mu,\beta}|$ at a point at which the d-density of μ exists (call it $\Theta^d(\mu, Q)$) and E has a unique tangent plane. At that point

$$D_\infty = D_{\Theta^d(\mu,Q)\mathcal{H}^d|_{T_Q E}, \beta} = c_{\beta,n,d}\Theta^d(\mu, Q)^{-1/\beta}\delta_{T_Q E},$$

which implies that

(5.5) $$\text{n.t.}\lim_{x \to Q}|\nabla D_{\mu,\beta}| = c_{\beta,n,d}\Theta^d(\mu, Q)^{-1/\beta}.$$

6. D_α FOR "MAGIC-α"

Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set and let μ be a d-Ahlfors regular measure supported on E. If the numbers n, $d < n$ (not necessarily integer), and $\alpha > 0$ are such that

(6.1) $$n = d + 2 + \alpha$$

then it turns out that the function $D_{\alpha,\mu}$ defined in (1.5) is a solution of

$$L_{\alpha,\mu}u \equiv -\text{div}\left(\frac{1}{D_{\alpha,\mu}^{n-d-1}}\nabla u\right) = 0$$

in $\Omega = \mathbb{R}^n \setminus E$.

We can check this (in the classical sense) in Ω simply by differentiating the smooth function $D_{\alpha,\mu}$ (recall (2.8)),

$$
\begin{aligned}
L_{\alpha,\mu}D_{\alpha,\mu} &= \operatorname{div}D_{\alpha,\mu}^{-n+d+1}\nabla D_{\alpha,\mu} = -\frac{1}{\alpha}\operatorname{div}D_{\alpha,\mu}^{-n+d+1}R_{\alpha,\mu}^{-\frac{1}{\alpha}-1}\nabla R_{\alpha,\mu} \\
&= -\frac{1}{\alpha}\operatorname{div}D_{\alpha,\mu}^{-n+d+1}D_{\alpha,\mu}^{1+\alpha}\nabla R_{\alpha,\mu} = -\frac{1}{\alpha}\Delta R_{\alpha,\mu}
\end{aligned}
$$

(6.2)

by (1.5) and (6.1). Then by (1.3), (and (6.1))

$$
R_{\alpha,\mu}(x) \equiv \int_{y\in E}|x-y|^{-d-\alpha}d\mu(y) = \int_{y\in E}|x-y|^{2-n}d\mu(y); \tag{6.3}
$$

we recognize the Green kernel (notice that $n > 2$ by (6.1)); hence $L_{\alpha,\mu}D_{\alpha,\mu} = 0$ on Ω.

We want to say that $D_{\alpha,\mu}$ is the "Green function with pole at infinity" associated to the operator $L \equiv L_{\alpha,\mu}$ (indeed, it is a solution which behaves like distance to the boundary). To do so properly however, we need to define the Green function with a pole at infinity (and the corresponding harmonic measure). We will then show that in the complement of any d-Ahlfors regular set, E, these objects exist and are unique up to multiplication by a positive scalar. Throughout, we will use some of the elliptic regularity and potential theory studied in [DFM2], in particular we will assume the reader is comfortable the existence and properties of a Green function and associated harmonic measure with finite pole.

Before we begin we must recall the weighted Sobolev spaces introduced in [DFM2]. Throughout this section E will be a closed d-Ahlfors regular set and $\delta(x)$ will denote the distance from x to the closest point in E.

Definition 6.1. *[see [DFM2]] Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set for some $d < n-1$ (not necessarily an integer). Set $w(x) \equiv \delta(x)^{-(n-d-1)}$ and define the weighted Sobolev space*

$$
W \equiv W_w^{1,2} \equiv \{u \in L_{\mathrm{loc}}^1(\mathbb{R}^n\backslash E): \ \nabla u \in L^2(\mathbb{R}^n\backslash E, wdx)\}.
$$

We can then localize these Sobolev spaces: for any open $\mathcal{O} \subset \mathbb{R}^n$ we define

$$
W_r(\mathcal{O}) = \{u \in L_{\mathrm{loc}}^1(\mathcal{O}), \ \varphi f \in W \text{ for all } \varphi \in C_0^\infty(\mathcal{O})\}.
$$

These Sobolev spaces are the setting in which the elliptic estimates and potential theory established in [DFM2] hold. We can now define the Green function and harmonic measure with a pole at infinity.

Definition 6.2. *Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set for some $d < n-1$ (not necessarily an integer), $\beta \in (0,1)$ and μ be a d-Ahlfors regular measure supported on E. Let $\Omega = \mathbb{R}^n\backslash E$, $D \equiv D_{\mu,\beta}$ be as in (1.5) and $L \equiv L_{\mu,\beta}$ be the associated degenerate elliptic operator. We say that u_∞, ω_∞ are the Green function and harmonic measure with pole at infinity, respectively (associated to β,μ), if $u_\infty \in W_r(B(Q,R)) \cap C(\mathbb{R}^n)$ for every $Q \in E$ and $R > 0$ and the*

following holds:

$$u_\infty > 0, \ in \, \Omega,$$
$$u_\infty = 0, \ on \, E,$$
(6.4) $$\qquad Lu_\infty = 0, \ in \, \Omega,$$
$$\int_\Omega D^{-(n-d-1)} \nabla u_\infty \cdot \nabla \varphi \, dX = \int_E \varphi \, d\omega_\infty, \ \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

Before we can show the existence and uniqueness of these objects, we must recall the comparison principle for solutions, stated and proven in our setting in [DFM2] (see also, e.g., [JK] for the co-dimension one statement). Recall from [DFM2] that there exists an $M > 1$ such that for $Q \in E$ and $r > 0$ there exists a point $A_r(Q)$ with

(6.5) $$\qquad |A_r(Q) - Q| \le r \le M\delta(A_r(Q)).$$

We call $A_r(Q)$ a **corkscrew point** for Q at scale $r > 0$.

Theorem 6.3. *[Theorem 11.146 in [DFM2]] Let $Q \in E$ and $r > 0$ and let $X_0 = A_r(Q) \in \Omega$ be the corkscrew point for Q at scale r. Let $u, v \in W_r(B(Q, 2r))$ be non-negative, not identically zero, solutions of $L_{\beta,\mu} u = L_{\beta,\mu} v = 0$ in $B(Q, 2r)$, $\beta > 0$, such that $Tu = Tv = 0$ on $E \cap B(Q, 2r)$. Then there exists a constant $C > 0$ depending on n, d, and Ahlfors regularity constants, such that*

(6.6) $$\qquad C^{-1} \frac{u(X_0)}{v(X_0)} \le \frac{u(X)}{v(X)} \le C \frac{u(X_0)}{v(X_0)}, \forall X \in \Omega \cap B(Q, r).$$

The comparison theorem leads naturally to Hölder regularity of quotients at the boundary. Our proof below is inspired by [FSY] (Theorem 4.5 there), who show this regularity for solutions of a parabolic problem. The "usual" elliptic proof (cf. [JK]) relies on interior approximating domains, which are difficult in the co-dimension greater than one setting because of the presence of boundaries with mixed dimension (see the discussion at the beginning of Section 11 of [DFM2]).

Corollary 6.4. *Let u, v, Q, r_0 be as in Theorem 6.3. There exists $c > 0, \gamma \in (0, 1)$ (depending only on the Ahlfors regularity of E, n, d and β) such that*

(6.7) $$\qquad \left| \frac{u(X)v(Y)}{u(Y)v(X)} - 1 \right| \le c \left(\frac{\rho}{r_0} \right)^\gamma,$$

for all $X, Y \in B(Q, \rho) \cap \Omega$, as long as $\rho < r_0/4$.

Proof. We claim there exists some $\theta \in (0, 1)$ (independent of Q and r) such that

(6.8) $$\qquad \mathrm{osc}_{B(Q,r/2)} \frac{u}{v} \le \theta \, \mathrm{osc}_{B(Q,r)} \frac{u}{v},$$

for all $r < r_0/2$. That the claim implies (6.7) follows from iterating (6.8) and appealing to Theorem 6.3.

Let $\inf_{B(Q,r)} \frac{u}{v} = c_1$ and $\sup_{B(Q,r)} \frac{u}{v} = c_2$ and replace u by $U = \frac{u - c_1 v}{c_2 - c_1}$ (if $c_2 = c_1$ then $u = v$ and the result is trivial). It follows that $L_{\beta,\mu} U = 0$ and $U \ge 0$ in $B(Q, r)$ and $U = 0$ on

$E \cap B(Q, r)$. So we can apply Theorem 6.3 with $2r$ replaced by r. Also note that

$$0 \leq \frac{U}{v}(Z) \leq 1 = \mathrm{osc}_{B(Q,r)} \frac{U}{v}, \ \forall Z \in B(Q, r) \cap \Omega.$$

A small computation shows that if estimate (6.8) holds for U, v it also holds for u, v.

Let $A_{r/2}(Q)$ be the corkscrew point for Q at scale $r/2$. If $\frac{U}{v}(A_{r/2}(Q)) < C^{-2}$ (where C is the constant from Theorem 6.3), then by Theorem 6.3 we would have

$$0 \leq \frac{U}{v}(Z) \leq C\frac{U}{v}(A_{r/2}(Q)) < \frac{1}{C}, \ \forall Z \in B(Q, r/2) \cap \Omega,$$

which would imply that $\mathrm{osc}_{B(Q,r/2)} \frac{U}{v} < \frac{1}{C}$, and hence, the desired result (with $\theta = \frac{1}{C}$).

If, on the other hand, $\frac{U}{v}(A_{r/2}(Q)) > C^{-2}$ we apply Theorem 6.3 to obtain

$$\inf_{B(Q,r/2)} \frac{U}{v} \geq C^{-1}\frac{U}{v}(A_{r/2}(Q)) > C^{-3}.$$

This implies that $\mathrm{osc}_{B(Q,r/2)} \frac{U}{v} \leq 1 - C^{-3}$ and so (6.8) holds with $\theta = 1 - C^{-3}$.   □

Finally, using an argument inspired by [KT1], Lemma 3.7 and Corollary 3.2, we can show the existence and uniqueness of the Green function and harmonic measure with pole at infinity.

Lemma 6.5. *For any E, β, μ as in Definition 6.2, there exist an associated harmonic measure and Green function with pole at infinity. Furthermore, they are both unique up to multiplication by a positive scalar.*

Proof. First we show existence of the Green function with pole at infinity. Fix $Q \in E$ and let $X_i = A_{2^i}(Q) \in \Omega$ denote a corkscrew point for Q at the scale 2^i (i.e., $M\delta(X_i) > |X_i - Q| \geq 2^i$, see (6.5)). Define (for $i > 1$) $g_i(X) \equiv \frac{g(X,X_i)}{g(X_1,X_i)}$, where $g(X, Y)$ is the Green function for $L_{\beta,\mu}$ with pole at Y (cf. Section 10 in [DFM2]). Similarly define $\omega_i(S) \equiv \frac{\omega^{X_i}(S)}{g(X_1,X_i)}$. These are somewhat arbitrary normalizations (recall that the Green function that we want to construct will only be unique modulo a multiplicative function).

We claim that for any $K \subset\subset \mathbb{R}^n$ there exists a $C > 0$ (depending on K) such that $g_i(X) < C$ for all $X \in K$ and $i > i_0(K) \geq 0$ large enough (so that X_i lies away from K). Indeed, this follows from the fact that $g_i(X_1) \equiv 1$, the Harnack inequality (Lemma 8.42 in [DFM2]), and the existence of Harnack chains in Ω (Lemma 2.1 in [DFM2]). It then follows (after the extraction of a diagonal subsequence) that the $g_i \to g_\infty$ where the convergence is uniform on compacta in the continuous topology.

Furthermore, for any $Q \in E$ and $R > 0$, if $i > i_0(Q, r) \geq 0$ is large enough we know that $X_i \notin B(Q, 4R)$. Thus we can estimate

$$\int_{B(Q,R)} |\nabla_x g_i(x)|^2 w(x)dx \leq CR^{-2} \int_{B(Q,2R)} g_i(x)^2 w(x)dx \leq C_R,$$

where the first inequality follows from Lemma 8.47 in [DFM2] (a Cacciopoli type estimate) and the second inequality follows from the fact that $|g_i| < C_R$ on $B(Q, 2R)$ by the argument in the above paragraph (and the fact that $w(x)$ is locally integrable). Thus the g_i are in $W_r(B(Q, R))$ with uniformly controlled norms for all i large enough and, perhaps passing to

another subsequence, we may assume that $g_i \to g_\infty$ in $W_r(B(Q, R))$. In particular, g_∞ is in $W_r(B(Q, R))$ for all $Q \in E$ and $R > 0$.

As the g_i are in $W_r(B(Q, R))$ with uniformly controlled norms, it follows from taking limits that $L_{\beta,\mu} g_i = 0$ and hence $L_{\beta,\mu} g_\infty = 0$ in Ω. From the uniform convergence two paragraphs above we know that $g_\infty \geq 0$ in Ω and the continuity of the trace operator gives $g_\infty = 0$ on E (see [DFM2], Section 3). Furthermore, by the Harnack inequality and $g_\infty(X_1) = 1$ it must be that $g_\infty > 0$ in Ω.

We will now show that g_∞ is the unique positive solution to $L_{\beta,\mu}$ which vanishes on E and is in $W_r(B(Q, R))$ for all $Q \in E$ and $R > 0$ (up to scalar multiplication). Indeed, assume there existed some other f which was positive in Ω, zero on E, in $W_r(B(Q, R))$ for all $R > 0, Q \in E$ and satisfied $L_{\beta,\mu} f = 0$. We can multiply f by a positive scalar such that $f(X_1) = 1$. Then by Corollary 6.4 applied at larger and larger scales it is clear that $f(X) = g_\infty(X)$ for all $X \in \Omega$: starting from

$$\left| \frac{g_\infty(X)}{f(X)} - 1 \right| \leq C \left(\frac{|X - X_1|}{R} \right)^\gamma,$$

take $R \to \infty$.

It is time to establish the existence of ω_∞; let $Q \in E$ and $R > 0$. If i is big enough then by Lemma 11.78 in [DFM2] we have $\omega^{X_i}(B(Q, R)) \leq CR^{d-1} g(X_i, A_R(Q))$ where $A_R(Q)$ is the corkscrew point for Q at scale R. Thus $\omega_i(B(Q, R)) \leq CR^{d-1} g_i(A_R(Q)) \leq C_{R,Q} < \infty$ by the Harnack argument above. This implies that the sequence of measure $\{\omega_i\}$ is uniformly bounded on any compact set and so, perhaps passing to a subsequence, there is an ω_∞ such that $\omega_i \rightharpoonup \omega_\infty$.

We note that by the definition of g_i and ω_i we have that

$$\int_\Omega D_{\beta,\mu}^{-(n-d-1)} \nabla g_i \nabla \varphi \, dx = \int_E \varphi \, d\omega_i,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n \backslash X_i)$ (cf. Section 9 in [DFM2]). Fix φ and let $i \to \infty$ on both sides; using the fact that $\omega_i \rightharpoonup \omega_\infty$ and $g_i \to g_\infty$ in $W_r(K)$ for any compact K we get that

$$\int_\Omega D_{\beta,\mu}^{-(n-d-1)} \nabla g_\infty \nabla \varphi \, dx = \int_E \varphi \, d\omega_\infty,$$

as desired.

The uniqueness of ω_∞ then follows from its integral relationship with g_∞ (cf. (6.4)) and the uniqueness of g_∞. $\qquad \square$

We shall now show that for the magic value of $\alpha = n - d - 2$, D_α is the Green function with pole at infinity.

Corollary 6.6. *Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set for $d < n - 2$ (not necessarily an integer) and let μ be a d-Ahlfors regular measure supported on E. If $\alpha = n - d - 2$ then $D_{\alpha,\mu}$ is the Green function with pole at infinity for E (cf. Definition 6.2).*

We remark that the Green function with pole at infinity is unique modulo a multiplicative constant, hence, strictly speaking, the Corollary above assures that any such Green function is either $D_{\alpha,\mu}$ or its multiple.

Proof. We have seen earlier that $D_{\alpha,\mu}$ is a positive solution to the degenerate elliptic operator, which vanishes on the boundary; because of Lemma 6.5, it suffices to show that $D_{\alpha,\mu} \in W_r(B(Q,R))$ for all $Q \in E$ and $R > 0$ However, we know from (4.2) that $|\nabla D_{\alpha,\mu}| \leq C$. Since $w(X)$ is locally integrable in $\mathbb{R}^n$, the desired result follows.   □

The fact that we are able to explicitly write down the Green function with pole at infinity for magic α allows us to easily compute and bound the associated harmonic measure. The next theorem shows that, for any Ahlfors regular set E and magic $\alpha = n-d-2$, the harmonic measure $\omega_{\alpha,\mu}$ is comparable to surface measure. As a corollary we have the analogous result for harmonic measure with finite pole. Thus, as mentioned in the introduction, there is absolutely no converse to the theorem that $\omega^X \in A_\infty(\sigma)$ when E is uniformly rectifiable. In fact, our result holds even when $d < n - 2$ is not an integer.

Theorem 6.7. *Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set (for $d < n - 2$, not necessarily an integer) and let μ be a d-Ahlfors regular measure supported on E. If $\alpha = n - d - 2$ then the harmonic measure with pole at infinity $\omega_{\alpha,\mu}$ is comparable to $\sigma \equiv \mathcal{H}^d|_E$; that is, there is a constant $C > 0$ (depending only on n, d and the Ahlfors regularity constants for μ and σ) such that if we normalize $\omega_{\alpha,\mu}$ as we did in the construction*

$$(6.9) \qquad C^{-1} \leq \frac{d\omega_{\alpha,\mu}}{d\sigma}(Q) \leq C, \ \forall Q \in E.$$

For the rest of the section we will use the notation $a \simeq b$ if there is a constant C, depending only on n, d and the Ahlfors regularity of μ, such that $C^{-1} \leq \frac{a}{b} \leq C$.

Proof. For the sake of brevity we will write $\omega = \omega_{\alpha,\mu}, D = D_{\alpha,\mu}$. Recall Lemma 11.78 from [DFM2]: there exists a $C > 0$ (depending on n, d, and the Ahlfors regularity constants of E, μ) such that if $Q \in E, r > 0$ and $X \in \Omega \setminus B(Q, 2r)$ then

$$(6.10) \qquad C^{-1}r^{d-1}g(X, X_0) \leq \omega^X(B(Q, r)) \leq Cr^{d-1}g(X, X_0),$$

where $X_0 = A_r(Q)$ is a corkscrew point for Q at scale r, $g(-, X_0)$ is the Green function with pole at X_0 associated to $L_{\alpha,\mu}$ and ω^X is the harmonic measure with pole at X associated to $L_{\alpha,\mu}$. Divide (6.10) by $g(X_i, X_1)$ for $i > 1$, where $X_i = A_{4^i r}(Q)$, and take $X = X_i$. This yields

$$C^{-1}r^{d-1}\frac{g(X_i, X_0)}{g(X_i, X_1)} \leq \frac{\omega^{X_i}(B(Q,r))}{g(X_i, X_1)} \leq Cr^{d-1}\frac{g(X_i, X_0)}{g(X_i, X_1)}.$$

Then we let i tend to $+\infty$; we claim that

$$(6.11) \qquad C^{-1}r^{d-1}D(X_0) \leq \omega(B(Q, r)) \leq Cr^{d-1}D(X_0).$$

Indeed, arguing as in Lemma 6.5 we can pass to limits and show that the $\frac{g(X_i, -)}{g(X_i, X_1)}$ and $\frac{\omega^{X_i}}{g(X_i, X_1)}$ converge (uniformly on compacta and weakly respectively) to a Green function with pole at infinity and a harmonic measure with pole at infinity. Using the uniqueness of the Green function and harmonic measure with pole at infinity we can say that $\frac{g(X_i, -)}{g(X_i, X_1)}$ converges to a multiple of D (we can even compute the constant, which is $D(X_1)$), and $\frac{\omega^{X_i}}{g(X_i, X_1)}$ to a multiple of ω. This proves (6.11).

From (6.11) the conclusion of the Lemma is easy: notice that $D(X_0) \simeq \delta(X_0) \simeq r$. It follows that $\omega(B(Q, r)) \simeq r^d$ for any $Q \in E$ and any $r > 0$.   □

There is an analogue to Theorem 6.7 for a harmonic measure with finite pole associated to $L_{\alpha,\mu}$ (where α is magic). The proof essentially follows from the boundedness of quotients and the comparability of the Green function with the harmonic measure (Theorem 11.146 and Lemma 11.78 in [DFM2] respectively).

Corollary 6.8. *Let $E \subset \mathbb{R}^n$ be an Ahlfors regular set of dimension d (not necessarily integer), and μ be an Ahlfors regular measure with support E. Assume that $\alpha = n - d - 2 > 0$, and define $D = D_{\alpha,\mu}$ and $L = L_{\alpha,\mu}$ as above. Finally let $X \in \Omega = \mathbb{R}^n \setminus E$ be given, and denote by ω^X the associated harmonic measure with pole at X. Set $R = \text{dist}(X, E)$ Then there is a constant C, that depends only on n, d, and the Ahlfors regularity constant for μ, such that*

$$(6.12) \quad C^{-1}\mu(A) \leq R^d \omega^X(A) \leq C\mu(A) \quad \text{for every measurable set } A \subset E \cap B(X, 100R).$$

We remark that (6.12) is a correct, homogeneous, finite pole version of the statement that the harmonic measure is proportional to the Hausdorff measure. A reader might be more accustomed to see it as a strengthened version of the A^∞ condition: for all $Q \in E$, $X \in \Omega$, $R = \text{dist}(X, E)$,

$$(6.13) \qquad C^{-1}\frac{\mu(A)}{\mu(B(X, 100R))} \leq \frac{\omega^X(A)}{\omega^X(B(X, 100R))} \leq C\frac{\mu(A)}{\mu(B(X, 100R))},$$

for every measurable set $A \subset E \cap B(X, 100R)$.

Proof. Of course there is nothing special about the radius $100R$, but the result for larger R could be obtained by a change of pole.

Rather than proving (6.12), we will find it more convenient to prove that for all $Q \in E, r > 0$ such that $B(Q, r) \subset B(X, 100R)$ and $r < R/4$,

$$(6.14) \qquad\qquad C^{-1}r^d \leq R^d \omega^X(B(Q, r)) \leq Cr^d.$$

By Lemma 11.78 in [DFM2] we know that $\omega^X(B(Q, r)) \simeq r^{d-1}g(X, A_r(Q))$ where g is the Green function associated to the operator $L_{\alpha,\mu}$. Note that $g(X, -)$ and $D(-)$ are both positive solutions to L in $B(Q, R/2)$ which vanish on $B(Q, R/2) \cap E$, thus we can apply Theorem 6.3 and get that

$$(6.15) \quad \frac{\omega^X(B(Q, r))}{r^d} \simeq \frac{g(X, A_r(Q))}{r} \simeq \frac{g(X, A_r(Q))}{D(A_r(Q))} \simeq \frac{g(X, A_{R/2}(Q))}{D(A_{R/2}(Q))} \simeq \frac{g(X, A_{R/2}(Q))}{R},$$

so (6.14) will follow once we check that $g(X, A_{R/2}(Q)) \simeq R^{1-d}$.

Choose $Y \in \Omega$ so that $R/20 < |Y - X| < R/10$ and consider $g(X, Y)$. It is clear that $\text{dist}(Y, E) \simeq R \simeq \text{dist}(A_{R/2}(Q), E) \simeq |A_{R/2}(Q) - Y|$. Thus by the existence of Harnack chains (Lemma 2.1 in [DFM2]), because we can find a chain from Y to $A_{R/2}(Q)$ that does not get close to X, and by the Harnack inequality, we have $g(X, A_{R/2}(Q)) \simeq g(X, Y)$.

Finally, by equations (10.89) and (10.96) in [DFM2] we have that $g(X, Y) \simeq R^{1-d}$. Plugging this into (6.15) gives (6.14); then (6.12) follows by a covering argument. $\qquad\square$

When E is rectifiable the non-tangential limit of $|\nabla D_{\alpha,\mu}|$ exists (Theorem 5.3), and, much as in the co-dimension one setting, gives us the Poisson kernel (for magic α).

Lemma 6.9. *Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set with $d < n - 2$ an integer. Assume that E is d-rectifiable, let μ be a d-Ahlfors regular measure whose support is E and let $\alpha = n - d - 2$. Then for σ-a.e. $Q \in E$, the density of $\omega_{\alpha,\mu}$ is given by $\Theta^d(\mu, Q)$ modulo a multiplicative constant. To be precise, if we fix the constants so that the Green function with pole at infinity is the function $D_{\alpha,\mu}$ that was constructed above, then there exist $c_{n,d} > 0$ and $\widetilde{c}_{n,d} > 0$ such that*

$$\frac{d\omega_{\alpha,\mu}}{d\sigma}(Q) = \widetilde{c}_{n,d} \, \text{n.t.} \lim_{x \to Q} |\nabla D_{\alpha,\mu}(x)|^{-(n-d-2)} \overset{(5.5)}{=} c_{n,d}\Theta^d(\mu, Q).$$

Proof. For simplicity write $D \equiv D_{\alpha,\mu}$ and ω the associated harmonic measure with pole at infinity. For any $\varphi \in C_c^\infty$ we know that

$$(6.16) \qquad \int_\Omega D^{-(n-d-1)}\nabla D \cdot \nabla\varphi \, dx = \int_E \varphi d\omega.$$

Let $Q \in E$ be a point where the non-tangential limit of $|\nabla D|$ exists and where there is a unique tangent to E and tangent measure for μ (call it μ_∞). Such a $Q \in E$ can be found σ-a.e. (by the theory of rectifiable sets and Theorem 5.3 above). Let φ be a smooth approximation of $\chi_{B(0,1)}$ and for $r_i \downarrow 0$ define $\varphi_i(x) \equiv \frac{\varphi(\frac{x-Q}{r_i})}{r_i^d}$. Adapting notation as in (5.1) we get

$$\int_\Omega D^{-(n-d-1)}\nabla D \cdot \nabla\varphi_i \, dx = \frac{1}{r_i^{d+1}} \int_\Omega D^{-(n-d-1)}(x)\nabla D(x) \cdot (\nabla\varphi)\left(\frac{x-Q}{r_i}\right) dx$$

$$= \int_{\Omega_i} D_i^{-(n-d-1)}(y)\nabla D_i(y) \cdot \nabla\varphi(y)dy,$$

by a change of variables $y = \frac{x-Q}{r_i}$.

We now take the limit in i. Recall that E has a unique tangent d-plane T at Q, and that there is a non-tangential limit $L = \text{n.t.} \lim_{x \to Q} |\nabla D|$. In addition, by the discussion in Section 5, D_i tends to $D_\infty(x) = L\delta_T(x)$, and this convergence happens uniformly up to T. The convergence of ∇D_i to ∇D_∞ is only uniform on compact sets of $\mathbb{R}^n \setminus T$, but close to T the integrals are controlled uniformly because the gradients are bounded, so we get that

$$(6.17) \qquad \lim_{i \to \infty} \int_\Omega D^{-(n-d-1)}\nabla D \cdot \nabla\varphi_i \, dx = L^{-(n-d-2)} \int_{\mathbb{R}^n \setminus T} (\delta_T)^{-(n-d-1)}\nabla\delta_T \cdot \nabla\varphi dx.$$

Split the integral on the right hand side of (6.17) into two pieces; one on a neighborhood T_ε of radius $\varepsilon > 0$ around T and the other outside of T_ε. The integral on T_ε goes to zero as $\varepsilon > 0$ goes to zero, by the Lipschitz character of δ_T and φ and the local integrability of $\delta^{-(n-d-1)}$. For the integral on $\mathbb{R}^n \setminus T_\varepsilon$ we can integrate by parts. Notice that δ_T is a distance to the d-tangent plane, hence, it is a radial function in a space with $n - d$ dimensions, and

hence, δ_T^{-n+d+2} is harmonic. Then

$$
\begin{aligned}
\int_{\mathbb{R}^n \setminus T_\varepsilon} (\delta_T)^{-(n-d-1)} \nabla \delta_T \cdot \nabla \varphi \, dx &= \int_{\{x \mid \operatorname{dist}(x,T)=\varepsilon\}} \varepsilon^{-(n-d-1)} \varphi \, d\mathcal{H}^{n-1} \\
&= c_1 \int_T \int_{\varepsilon \mathbb{S}^{n-d-1}} \varepsilon^{-(n-d-1)} \varphi \, d\mathcal{H}^{n-d-1} d\mathcal{H}^d \\
&= c_2 \int_T \varphi \, d\mathcal{H}^d + O(\varepsilon),
\end{aligned}
$$
(6.18)

where c_1 and c_2 are dimensional constants that we shall never need to compute. We let ε tend to 0, return to (6.17), and get that

$$
\lim_{i \to \infty} \int_\Omega D^{-(n-d-1)} \nabla D \cdot \nabla \varphi_i \, dx = c_2 L^{-(n-d-2)} \int_T \varphi \, d\mathcal{H}^d. \tag{6.19}
$$

Now we let φ tend to $\chi_{B(0,1)}$ (as BV functions); the right-hand side tends to $c_2 L^{-(n-d-2)} \omega_d$, where ω_d is the volume of the d-dimensional unit ball. For the left-hand side, notice that by (6.16),

$$
\int_\Omega D^{-(n-d-1)} \nabla D \cdot \nabla \varphi_i \, dx = \int_E \varphi_i \, d\omega = \int_E \varphi_i \frac{d\omega}{d\sigma} \, d\sigma.
$$

When i tends to $+\infty$ and Q is a point of density for $\frac{d\omega}{d\sigma}$ (which is true σ-a.e.), the quantity $\int_E \varphi_i(Z) \left| \frac{d\omega}{d\sigma}(Z) - \frac{d\omega}{d\sigma}(Q) \right| d\sigma(Z)$ tends to 0; we are left with $\frac{d\omega}{d\sigma}(Q) \int_E \varphi_i d\sigma$. If Q is also a point of density 1 for σ (which is again true σ-a.e.), $\int_E \varphi_i d\sigma$ tends to $\int_T \varphi d\mathcal{H}^d$. Now we let φ tend to $\chi_{B(0,1)}$ and get that the left-hand side of (6.19) tends to $c_3 \frac{d\omega}{d\sigma}(Q)$, where c_3 may depend on how we normalize $\mathcal{H}^d$ with respect to the Lebesgue measure. Thus (6.19) implies that $\frac{d\omega}{d\sigma}(Q) = c_4 L^{-(n-d-2)}$. This is the desired result. $\qquad\square$

In the specific case where $\mu = \sigma \equiv \mathcal{H}^d|_E$ and E is d-rectifiable, (5.5) tells us that n.t.$\lim_{x \to Q} |\nabla D_{\alpha,\sigma}| = c_{n,d} \Theta^d(\sigma, Q)^{-1/\alpha} = c_{n,d}$ by the fact that $\Theta^d(\sigma, Q) = 1$ for σ-a.e. Q in any d-rectifiable set. Thus we can conclude that $\omega_{\alpha,\sigma}$ is proportional to σ.

Corollary 6.10. *Let E, d, n, α be as in Lemma 6.9. Then there exists a constant $c > 0$ such that $\omega_{\alpha,\sigma} = c\sigma$, where, as above, $\sigma = \mathcal{H}^d|_E$.*

REFERENCES

[AMT] J. Azzam, M. Mourgoglou, X. Tolsa. *Harmonic measure and quantitative connectivity: geometric characterization of the L^p solvability of the Dirichlet problem. Part II.* Preprint arXiv:1803.07975, March 2018.

[Bo] J. M. Borwein. *Proximality and Chebyshev sets.* Optim. Lett. **Vol 1** (2007), 21–32.

[Da] B. E. J. Dahlberg, *Estimates of harmonic measure.* Arch. Rational Mech. Anal. **65** (1977), no. 3, 275–288.

[Dv] G. David, *Singular sets of minimizers for the Mumford-Shah functional.* Progress in Mathematics 233 (2005), Birkhäuser Verlag, Basel, 1–581.

[DFM1] G. David, J. Feneuil, S. Mayboroda. *Harmonic measure on sets of codimension larger than one.* C. R. Acad. Sci. Paris, Ser I (2017).

[DFM2] G. David, J. Feneuil, S. Mayboroda. *Elliptic theory for sets with higher co-dimensional boundaries.* Preprint, arXiv:1702.05503.

[DFM3] G. David, J. Feneuil, S. Mayboroda. *Dahlberg's theorem in higher co-dimension.* Preprint, arXiv:1704.00667

[DM] G. David, S. Mayboroda. Uniform rectifiability and absolute continuity of harmonic measure in higher co-dimensions, project.

[DS1] G. David, S. Semmes. *Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs.* Astérisque **193** (1991).

[DS2] G. David, S. Semmes. *Analysis of and on uniformly rectifiable sets.* Mathematical Surveys and Monographs, **38**. American Mathematical Society, Providence, RI, 1993.

[FSY] E. Fabes, M. V. Safonov, Y. Yuan. *Behavior near the Boundary of Positive Solutions of Second Order Parabolic Equations: II.* Trans. Am. Math. Soc. **351** (1999), 4947-4961.

[FKP] R. Fefferman, C. Kenig and J. Pipher, *The theory of weights and the Dirichlet problem for elliptic equations*, Ann. of Math. (2) **134** (1991), no. 1, 65–124.

[HM] S.. Hofmann, J. M. Martell. *Harmonic measure and quantitative connectivity: geometric characterization of the L^p-solvability of the Dirichlet problem. Part I.* Preprint arXiv:1712.03696, Dec 2017.

[JK] D. Jerison, C.E. Kenig. *Boundary Behavior of Harmonic Functions in Non-Tangentially Accessible Domains.* Adv. Math. **46** (1982) pp 80-147.

[Jo] P. W. Jones. *Rectifiable sets and the traveling salesman problem.* Invent. Math. **102** (1990) 1-15.

[KT1] C.E. Kenig, T. Toro. *Free Boundary Regularity for Harmonic Measures and Poisson Kernels.* Ann. of Math. **150** (1999), pp 369-454.

[KT2] C.E. Kenig, T. Toro. *On the free boundary regularity theorem of Alt and Caffarelli.* Disc. Cont. Dyn. Sys. **10** (2004) 397-422.

[Ma] P. Mattila, Geometry of sets and measures in Euclidean space. Cambridge Studies in Advanced Mathematics 44, Cambridge University Press l995.

[MZ] S. Mayboroda, Z. Zhao. *Square function estimates, BMO Dirichlet problem, and absolute continuity of harmonic measure on lower-dimensional sets.* Preprint, arXiv: 1802.09648.

[NTV] F. Nazarov, X. Tolsa and A. Volberg, *On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1.* Acta Math. **213** (2014), no. 2, 237–321.

[Pr] D. Preiss. *Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities.* Ann. of Math. **125** (1987), 537-643.

[RR] F. & M. Riesz. *Über die randwerte einer analtischen funktion.* Compte Rendues du Quatrième Congrès des Mathématiciens Scandinaves, Stockholm 1916, Almqvists and Wilksels, Upsala, 1920.

[To] X. Tolsa. *Uniform rectifiability, Calderón-Zygmund operators with odd kernel, and quasiorthogonality.* Proc. Lond. Math. Soc. (3), **98** (2009), no. 2, 393–426.

[To1] X. Tolsa. *Principal values for Riesz transforms and rectifiability.* J. Func. Anal. **254** (2008), 1811-1863.

Univ Paris-Sud, Laboratoire de Mathématiques, UMR 8658 Orsay, F-91405
E-mail address: Guy.David@math.u-psud.fr

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, 02139, USA.
E-mail address: maxe@mit.edu

School of Mathematics, University of Minnesota,University of Minnesota, Minneapolis, MN, 55455, USA.
E-mail address: svitlana@math.umn.edu