Some Aspects of the Schwarzian Derivative

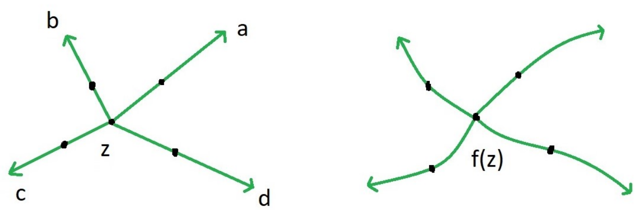
- Definition and properties
- Criteria for univalence and quasiconformal extension
- Conformal Schwarzian
- Ahlfors' Schwarzian

1. Definition and Properties

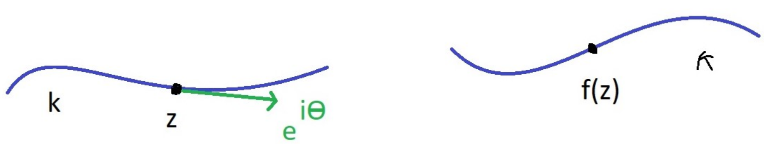
- $Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \ , \ f' \neq 0$
- $Sf = 0 \iff f$ Möbius
- $S(g \circ f) = [(Sg) \circ f](f')^2 + Sf$
- $S(T \circ f) = Sf \ , \ T$ Möbius

Note: Named after H. A. Schwarz, was used by Kummer as early as 1836 in connection with the hypergeometric equation.

Local distortion



- $(f(z+ta), f(z+tb), f(z+tc), f(z+td)) = (a,b,c,d)\left[1 + (a-b)(c-d)\frac{1}{6}Sf(z)t^2 + \cdots\right]$



- $|f'(z)|\kappa = k - \text{Re}\left\{Pf(z)e^{i\theta}\right\} \ , \ Pf = \frac{f''}{f'}$
- $|f'(z)|^2\frac{d\kappa}{ds} = \frac{dk}{dt} - \text{Im}\left\{Sf(z)e^{2i\theta}\right\}$

2. Conditions for Univalence and Qc-extension

f locally injective in $\Omega \subset \mathbb{C}$, $Sf = 2p$

- then $f = \dfrac{u}{v}$ for u, v linearly independent solutions of $u'' + pu = 0$ $(*)$
- f is injective in $\Omega \iff$ every solution u of $(*)$ vanishes in Ω at most once

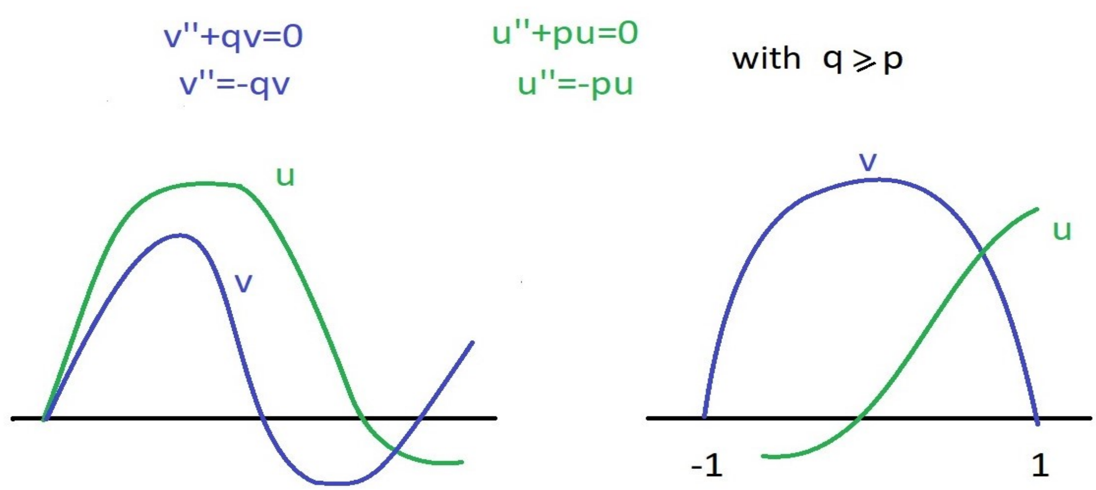
Several criteria in $\mathbb{D}$, the following three are due to Nehari [17, 1949], [18, 1954], and (3) announced by Pokornyi [21, 1951]

(1) $|Sf(z)| \leq \dfrac{2}{(1-|z|^2)^2}$

(2) $|Sf(z)| \leq \dfrac{\pi^2}{2}$

(3) $|Sf(z)| \leq \dfrac{4}{1-|z|^2}$

The absence of multiple zeros of solutions of $(*)$ can be understood from the Sturm theory:



Other important criteria: Ahlfors [2, 1974], Epstein [13, 1984], Anderson-Hinkkanen [4, 1991], Becker [5, 1972] (for Pf)

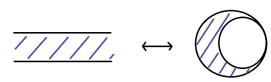
Ahlfors-Weill [3, 1962]

If $(1-|z|^2)^2|Sf(z)| \leq 2t$, $0 \leq t < 1$, then f admits a quasiconformal extension.

Gehring-Pommerenke [14, 1984]

If $(1-|z|^2)^2|Sf(z)| \leq 2$ then f admits a continuous extension to $\overline{\mathbb{D}}$ and $f(\mathbb{D})$ is a Jordan domain except for a mapping onto a parallel strip.

$$L(z) = \log(1+z)/(1-z) \quad , \quad SL(z) = 2(1-z^2)^{-2}$$



They also showed:

- if $|Sf(z)| \leq \rho(z)$ implies univalence then $|Sf(z)| \leq t\rho(z)$, $0 \leq t < 1$ implies qc-extension
- if $f(\mathbb{D})$ is a Jordan domain and $\limsup_{|z|\to 1}(1-|z|^2)^2|Sf(z)| < 2$ then $f(\mathbb{D})$ is a quasidisk

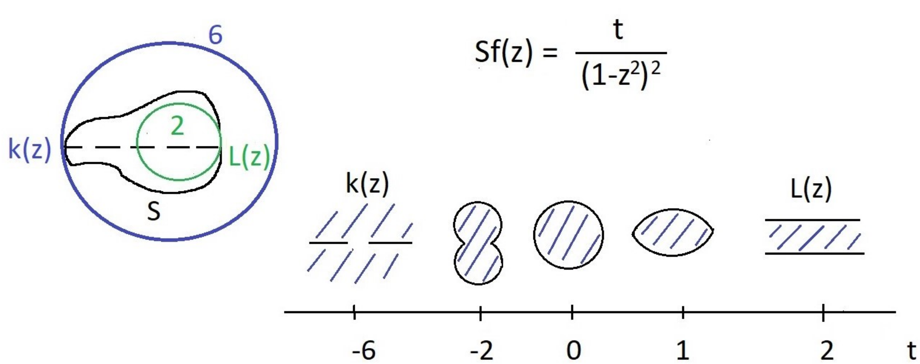
The Gehring-Pommerenke phenomenon holds also for (2) and (3). It follows that if f satisfies (2) or (3) and $f(\mathbb{D})$ is a Jordan domain then $f(\partial\mathbb{D})$ is WP because

$$\iint\limits_{|z|<1} (1-|z|^2)^2|Sf(z)|dxdy < \infty \,.$$

There is one necessary condition for univalence in $\mathbb{D}$ due to Kraus [15, 1932]:

$$|Sf(z)| \leq \frac{6}{(1-|z|^2)^2} \,.$$

Equality at one point implies f maps onto the complement of a slit (Koebe mapping).



3. Conformal Schwarzian

Osgood-Stowe [20, 1992] and Carne [6, 1990]: Let $f : (M, g) \to (N, h)$ be a conformal local diffeomorphism between Riemannian n-manifolds, $\sigma = \log |Df|$

$$S_g(f) = \mathrm{Hess}(\sigma) - d\sigma \otimes d\sigma - \frac{1}{n}\left\{\Delta\sigma - |\nabla\sigma|^2\right\} g$$

- there is a chain rule
- when f is holomorphic we recover the classical definition in the form of a matrix

$$\begin{pmatrix} \mathrm{Re}\{Sf\} & -\mathrm{Im}\{Sf\} \\ -\mathrm{Im}\{Sf\} & -\mathrm{Re}\{Sf\} \end{pmatrix}$$

Theorem A [19, 1990]: *Let (M, g) be a Riemannian manifold with scalar curvature $s(g)$ such that very two points can be joined by a geodesic of length at most δ for some $0 < \delta \le \infty$. If $f : (M, g) \to \mathbb{S}^n$ is a conformal local diffeomorphism and*

$$||S(f)|| \le \frac{2\pi^2}{\delta^2} - \frac{s(g)}{n(n-1)}$$

then f is injective.

4. Ahlfors' Schwarzian

Let $\phi : I \to \mathbb{R}^n$ be a parametrized curve with $\phi' \ne 0$. Then

$$S_1\phi = \frac{\langle \phi', \phi''' \rangle}{|\phi'|^2} - 3\frac{\langle \phi', \phi'' \rangle^2}{|\phi'|^4} + \frac{3}{2}\frac{|\phi''|^2}{|\phi'|^2}\,,$$

$$S_2\phi = \frac{\phi' \wedge \phi'''}{|\phi'|^2} - 3\frac{\langle \phi', \phi'' \rangle}{|\phi'|^4}\phi' \wedge \phi''\,,$$

$\langle \vec{a}, \vec{b} \rangle$: standard inner product

$\vec{a} \wedge \vec{b}$: antisymmetric bivector with components $(\vec{a} \wedge \vec{b})_{ij} = a_i b_j - a_j b_i$

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \quad \Longrightarrow$$

$$\mathrm{Re}\{Sf\} = \frac{1}{|f'|^2}\mathrm{Re}\{f'''\,\overline{f'}\} + \cdots \quad , \quad \mathrm{Im}\{Sf\} = \frac{1}{|f'|^2}\mathrm{Im}\{f'''\,\overline{f'}\} + \cdots$$

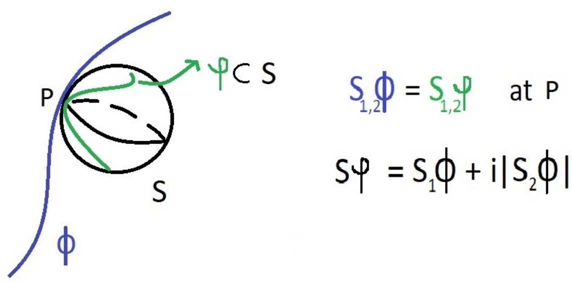
In [7, 2004] it was shown:

$$S_1\phi = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 + \frac{1}{2}v^2k^2 = Ss + \frac{1}{2}v^2k^2$$

$$S_2\phi = vk'(\hat{t}\wedge\hat{n}) + v^2k\tau(\hat{t}\wedge\hat{b})$$

where $v = |\phi'|$ = speed, $s' = v$, k = curvature, τ = torsion

- $S_1\phi + i|S_2\phi|$ has a natural interpretation on the osculating sphere and is Möbius invariant



Theorem B: *Let $\phi : I \to \mathbb{R}^n$ be a C^3 curve with $\phi' \neq 0$ and $S_1\phi \leq 2p$ for some function $p = p(x)$ for which*

$$u'' + pu = 0$$

does not admit solutions with more than one zero on I. Then $\phi(I)$ is simple.

Combining the definitions of conformal Scharzian, Ahlfors' real Schwarzian and Theorem B yields:

Theorem C [9, 2007]: *Let $f : \mathbb{D} \to \mathbb{R}^3$ be a conformal minimal immersion with conformal factor $|Df| = e^\sigma$, and let K stand for the Gaussian curvature of the minimal surface. If*

$$|Sf| + e^{2\sigma}|K| \leq \frac{2}{(1-|z|^2)^2} \qquad (**)$$

then f is injective.

- Sf may be represented by the complex number $2(\sigma_{zz} - \sigma_z^2)$
- since $(**)$ is invariant under compositions $f \circ T$ with an automorphism T of $\mathbb{D}$ it suffices to analize univalence on $(-1, 1)$
- if $\phi(x) = f(x)$ then $(**)$ implies $S_1\phi \leq \dfrac{2}{(1-x^2)^2}$, hence ϕ is 1-1 by Theorem B

References

[1] L.V. Ahlfors, *Cross-ratios and Schwarzian derivatives in* $\mathbb{R}^n$, Complex Analysis: Articles dedicated to Albert Pfluger on the occasion of his 80th birthday, Birkhäuser Verlag, Basel, 1989, 1-15.

[2] L. V. Ahlfors, *Sufficient conditions for quasiconformal extensions*, Discontinuous Groups and Riemann Surfaces, Ann. Math. Studies 79, 1974, Princeton University Press.

[3] L. V. Ahlfors and G. Weill, *A uniqueness theorem for Beltrami equations*, Proc. Amer. Math. Soc. 12 (1962), 975-978.

[4] J. M. Anderson and A. Hinkkanen, *Univalence criteria and quasiconformal extensions*, Trans. Amer. Math. Soc. 324 (1991), 823-842.

[5] Becker, J., *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. 255 (1972), 23-43.

[6] K. Carne, *The Schwarzian derivative for conformal maps*, J. Reine Angew. Math. 408 (1990), 10-33.

[7] M. Chuaqui and J. Gevirtz, *Simple curves in* $\mathbb{R}^n$ *and Ahlfors' Schwarzian derivative*, Proc. Amer. Math. Soc. 132 (2004), 223-230.

[8] M. Chuaqui, *On Ahlfors' Schwarzian derivative and knots*, Pacific J. Math. 231 (2007), 51-62.

[9] M. Chuaqui, P. Duren and B. Osgood, *Univalence criteria for lifts of harmonic mappings to minimal surfaces*, J. Geom. Anal. 17 (2007), 49-74.

[10] M. Chuaqui, P. Duren and B. Osgood, *Injectivity criteria for holomorphic curves in* $\mathbb{C}^n$, Pure Appl. Math. Quarterly 7 (2011), 223-251.

[11] M. Chuaqui, P. Duren and B. Osgood, *Quasiconformal Extensions to Space of Weierstrass-Enneper Lifts*, J. Analyse Math. 135 (2018), 487-526.

[12] M. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976.

[13] Epstein, Ch., *The hyperbolic Gauss map and quasiconformal reflections*, J. Reine Angew. Math. 380 (1987), 196-214.

[14] F. W. Gehring and Ch. Pommerenke, *On the Nehari univalence criterion and quasicircles*, Comm. Math. Helv. 59 (1984), 226-242.

[15] W. Kraus, *Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängenden Bereiches mit der Kreisabildung*, Mitt. Math. Sem. Giesen 21 (1932), 1-28.

[16] O. Martio and J. Sarvas, *Injectivity theorems in the plane*, Ann. Acad. Scie. Fenn., Ser. A.I (1978/1979), 383.401.

[17] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. 55 (1949), 545-551

[18] Z. Nehari, *Some criteria of univalence*, Proc. Amer. Math. Soc. 5 (1949), 700-704.

[19] B. Osgood and D. Stowe, *A generalization of Nehari's univalence criterion*, Comm. Math. Helv. 65 (1990), 234-242.

[20] B. Osgood and D. Stowe, *The Schwarzian derivative and conformal mapping of Riemannian manifolds*, Duke Math. J. 67 (1992), 57-97.

[21] V. Pokornyi, *On some sufficient conditions for univalence*, Dokl. Akad. Nauk. SSSR 79 (1951), 743-746.

[22] D. Stowe, *An Ahlfors derivative for conformal immersions*, J. Geom. Anal. 25 (2015), 592-615.

[23] D.J. Struik, *Lectures on Classical Differential Geometry*, Addison-Wesley, Reading, 1950.