Ahlfors-Weill Extensions of Conformal Mappings and Critical Points of the Poincaré Metric

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1 Introduction

Nehari showed in [9] that if f is analytic in the unit disk D and if its Schwarzian derivative $Sf = (f''/f')' - (1/2)(f''/f')^2$ satisfies

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}\,, \tag{1.1}$$

then f is univalent in the disk. Ahlfors and Weill showed in [1] that if the Schwarzian satisfies the stronger inequality

$$|Sf(z)| \leq \frac{2t}{(1-|z|^2)^2} \tag{1.2}$$

for some $0 \leq t < 1$ then, in addition, f has a quasiconformal extension to the sphere. They gave an explicit formula for the extension.

The class of analytic functions satisfying either of these conditions is quite large. It was shown by Paatero in [12] that any convex univalent function satisfies (1.1). This was later established in a different way by Nehari in [10], and he went on to prove that a bounded convex function satisfies the Ahlfors-Weill condition.

In [5], Gehring and Pommerenke made a careful study of Nehari's original univalence criterion and showed, among other things, that the condition (1.1) implies that $f(D)$ is a Jordan domain except when f is a Möbius conjugation of the logarithm,

$$F_0(z) = \frac{1}{2}\log\frac{1+z}{1-z}. \tag{1.3}$$

The function F_0 has $SF_0(z) = 2/(1-z^2)^2$, and $F_0(D)$ is an infinite parallel strip.

For topological reasons, it then follows from the Gehring-Pommerenke theorem that other than in the exceptional case f has a homeomorphic extension to the sphere. See also [4]. The main result in this paper is that the same Ahlfors-Weill formula defines a homeomorphic extension of f, though it will not in general be a quasiconformal extension. We discuss this phenomenon via a relationship between the Ahlfors-Weill extension and the Poincaré metric of the image of f. This may be of independent interest.

For economy of notation, though at the risk of sinking a crowded ship, we introduce explicitly several subclasses of univalent functions associated with Nehari type bounds. Thus we let N denote the set of analytic functions in the disk satisfying (1.1), N^* the elements of N other than Möbius conjugations of the function F_0, and N^t those functions satisfying (1.2). We use the notation N_0, N_0^*, N_0^t to indicate that a function f in any of the classes has the normalization $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$. If $f(z) = z + a_2 z^2 + \dots$ is in any of the classes, then $f/(1 + a_2 f)$ is in the corresponding class of normalized functions, the point being that the normalized function is still analytic, [2]. The function F_0 is normalized in this way. Furthermore, according to [3], Lemma 4, functions in N_0^* are bounded. The family of normalized extremals for the Ahlfors-Weill condition (1.2) is

$$A_t(z) = \frac{1}{\alpha} \frac{(1+z)^\alpha - (1-z)^\alpha}{(1+z)^\alpha + (1-z)^\alpha}, \quad \alpha = \sqrt{1-t}. \tag{1.4}$$

2 Preliminary Estimates

Several distortion theorems for the classes N_0 and N_0^t were proved in [2] using comparison theorems for the second order, ordinary differential equation associated with the Schwarzian. We continue somewhat in the same vein here for a few basic estimates. We refer to our earlier paper for further background.

Lemma 1 *If $f \in N_0$ then*

$$\left| \frac{f''}{f'}(z) \right| \leq \frac{2|z|}{1 - |z|^2}. \tag{2.1}$$

Equality holds at a single $z \neq 0$ if and only if f is a rotation of $F_0(z)$. If $f \in N_0^t$ then

$$\left| \frac{f''}{f'}(z) \right| \leq \frac{2t|z|}{1 - |z|^2}. \tag{2.2}$$

The inequality (2.2) is not sharp. The proof will show how one may obtain a sharp estimate, but it is not as convenient and explicit as the one given here.

Proof: Let $y = f''/f'$. Then

$$y' = \frac{1}{2} y^2 + 2p, \quad y(0) = 0,$$

with $2p(z) = Sf(z)$. We consider the real equation

$$w' = \frac{1}{2} w^2 + \frac{2}{(1 - x^2)^2}, \quad w(0) = 0$$

on $(-1, 1)$, whose solution is $w(x) = 2x/(1 - x^2)$. We want to show that $|y(z)| \leq w(|z|)$.

Fix z_0, $|z_0| = 1$ and let

$$\varphi(\tau) = |y(\tau z_0)|, \quad 0 \leq \tau < 1.$$

Unless $f(z) = z$ identically the zeros of φ are isolated. Away from these zeros φ is differentiable and $\varphi'(\tau) \leq |y'(\tau z_0)|$. Since $|p(\tau z_0)| \leq 1/(1-\tau^2)^2$ we obtain

$$\begin{aligned} \frac{d}{d\tau}(\varphi(\tau) - w(\tau)) \leq |y'(\tau z_0)| - w'(\tau) &\leq \frac{1}{2}(|y(\tau z_0)|^2 - w^2(\tau)) \\ &= \frac{1}{2}(\varphi(\tau) - w(\tau))(\varphi(\tau) + w(\tau)). \end{aligned}$$

This, together with $\varphi(0) - w(0) = 0$, implies that $\varphi(\tau) - w(\tau)$ can never become positive.

Now suppose that equality holds in (2.1) at $z_1 \neq 0$. Let $z_0 = z_1/|z_1|$ and let $\varphi(\tau)$ be defined as above. Then $\varphi(|z_1|) = w(|z_1|)$ which, by the previous analysis, can happen only if $\varphi(\tau) = w(\tau)$, first on $[0, |z_1|]$, and then for all $\tau \in [0, 1)$ since both functions are analytic. Hence $y(\tau z_0)$ is of the form $e^{i\theta(\tau)}w(\tau)$. Since all inequalities above must be equalities, it follows easily that $\theta(\tau)$ must be constant. From this, it follows in turn that $y(z) = cw(\bar{z}_0 z)$ for all $|z| < 1$, with $|c| = 1$. Integrating this equation and appealing to the normalizations on f shows that $f(z) = e^{-i\theta}F_0(e^{i\theta}z)$. This proves the first part of the Lemma.

Next, suppose that $f \in N_0^t$. The proof that $|f''/f'|$ has the bound in (2.2) proceeds exactly as above with the single difference that the comparison equation is

$$w' = \frac{1}{2}w^2 + \frac{2t}{(1-x^2)^2}, \quad w(0) = 0.$$

The solution is given by

$$w(x) = \frac{2x}{1-x^2} - \frac{2\alpha^2}{1-x^2}A_t(x),$$

where $A_t(x)$ is defined in (1.4). It can be checked that $A_t(x)$ is convex on $[0, 1]$, and hence

$$\min_{0\leq x\leq 1} \frac{A_t(x)}{x} = {A_t}'(0) = 1.$$

Hence

$$\frac{1-x^2}{2x}w(x) \leq 1 - \alpha^2 = t,$$

which proves (2.2).

3 Bounds for the Poincaré metric

The Poincaré metric $\lambda_\Omega|dw|$ of a simply connected domain Ω is defined by

$$\lambda_\Omega(f(z))|f'(z)| = \lambda_D(z) = \frac{1}{1-|z|^2}, \tag{3.1}$$

where $f\colon D \to \Omega$ is a conformal mapping of the unit disk onto Ω. From Schwarz's lemma and the Koebe $1/4-$theorem one has the the sharp inequalities

$$\frac{1}{4}\frac{1}{d(z,\partial\Omega)} \leq \lambda_\Omega(z) \leq \frac{1}{d(z,\partial\Omega)},$$

where $d(z, \partial\Omega)$ denotes the Euclidean distance from z to the boundary.

Writing $w = f(z)$ and taking the $\partial_z = \partial/\partial z$ derivative of the logarithm of (3.1) gives

$$\frac{\partial_w \lambda_\Omega}{\lambda_\Omega}(f(z))f'(z) = \frac{\bar{z}}{1-|z|^2} - \frac{1}{2}\frac{f''}{f'}(z). \tag{3.2}$$

Observe that for a normalized function $f \in N_0$ the Poincaré metric λ_Ω of the image Ω has a critical point at $w = 0$, and that this must be the unique critical point if f is not a rotation of the logarithm F_0. In the latter case Ω is a parallel strip and the critical points of λ_Ω are all the points of the axis of symmetry of Ω.

Lemma 2 *If $f \in N$ is bounded, then λ_Ω has a unique critical point.*

Proof: Since $\Omega = f(D)$ is bounded and $\lambda_\Omega(w) \to \infty$ as $w \to \partial\Omega$, λ_Ω must have at least one critical point. By replacing f by $f \circ T_1$ where T_1 is a Möbius transformation of the disk to itself, and then by $T_2 \circ f \circ T_1$, where T_2 is an affine transformation, we may assume that one such critical point is $w = 0 = f(0)$, and that $f'(0) = 1$. The identity (3.2) then forces $f''(0) = 0$, i.e., that $f \in N_0^*$. Hence, as above, $w = 0$ is the unique critical point for λ_Ω since f cannot be a rotation of the log.

From the fact that N contains the convex conformal mappings we obtain a result of Kim and Minda [6].

Corollary 1 *If Ω is a bounded, convex domain, then λ_Ω has a unique critical point.*

In [11] it was shown that

$$|\nabla \log \lambda_\Omega| \leq 4\lambda_\Omega \tag{3.3}$$

as a consequence of (3.2) and the classical bound for $|f''/f'|$ that holds for any univalent function in the disk. The inequality (3.3) is equivalent to the coefficient inequality $|a_2| \leq 2$.

We now give some lower bounds for $|\nabla \log \lambda_\Omega|$.

Lemma 3 *If $f \in N_0^*$, then there exists a constant $c > 0$ such that*

$$|\nabla \log \lambda_\Omega(w)| \geq c|w|\lambda_\Omega(w)^{1/2}. \tag{3.4}$$

If $f \in N_0^t$, then

$$|\nabla \log \lambda_\Omega(w)| \geq 2(1-t)^{3/2}|w|\lambda_\Omega(w). \tag{3.5}$$

Recall that a function $f \in N_0^*$ is bounded. The constant in (3.4) depends on the bound for f. In an appendix we will give an example to show that the exponent $1/2$ is essentially best possible in (3.4).

Proof: The estimate (3.4) is implicit in [5]. We show how it can be deduced, adopting the notation used there. Let h be the inverse of F_0 and let $g = f \circ h$. For $\tau \in \mathbf{R}$ we have

$2|g'(\tau)| = (1-|h(\tau)|^2)|f'(h(\tau))| = \lambda_\Omega(g(\tau))^{-1}$. It was shown in [5] that $v = |g'|^{-1/2}$ is convex, with $v(0)=1$, $v'(0)=0$. It is not constant when f is not equal to F_0. Now,

$$2\frac{v'}{v}(\tau) = \frac{d}{d\tau}\log\lambda_\Omega(g(\tau)) \le |\nabla\log\lambda_\Omega(g(\tau))||g'(\tau)| = |\nabla\log\lambda_\Omega(g(\tau))|v(\tau)^{-2},$$

hence

$$|\nabla\log\lambda_\Omega(g(\tau))| \ge 2v(\tau)v'(\tau) = 2^{3/2}v'(\tau)\lambda_\Omega(g(\tau))^{1/2}.$$

Since v is not constant and f is bounded, it follows that there exists a constant $a>0$ such that $v'(\tau) \ge a|g(\tau)|$ for $\tau \ge 0$. The estimates can be made uniformly on different rays from the origin by considering $f(e^{i\theta}h)$. This proves (3.4).

Now suppose that $f \in N_0^t$ and write $w = f(z)$. Using (3.2),

$$\frac{1}{2}|\nabla\log\lambda_\Omega(w)| = \frac{|(\partial_w\lambda_\Omega)(f(z))|}{\lambda_\Omega(f(z))} = \frac{\left|\bar{z} - \frac{1}{2}(1-|z|^2)\frac{f''}{f'}(z)\right|}{(1-|z|^2)|f'(z)|}.$$

From Lemma 1, (2.2) we then obtain

$$|\nabla\log\lambda_\Omega(w)| \ge \frac{2(1-t)|z|}{(1-|z|^2)|f'(z)|} = 2(1-t)|z|\lambda_\Omega(w),$$

with $w = f(z)$. But from [2], a function in N_0^t is subject to the sharp bound $|f(z)| \le A_t(|z|)$, where A_t was defined in (1.4). This can be rearranged to

$$|z| \ge \frac{(1+\alpha|w|)^{1/\alpha} - (1-\alpha|w|)^{1/\alpha}}{(1+\alpha|w|)^{1/\alpha} + (1-\alpha|w|)^{1/\alpha}} = \psi(\alpha|w|), \quad \alpha = \sqrt{1-t}.$$

The function $\psi(s)$ is concave on $[0,1]$ with $\psi(0)=0$ and $\psi(1)=1$. Hence $\psi(s) \ge s$ and (3.5) follows.

4 Homeomorphic Extensions

Let $f \in N$ with $f(z) = z + a_2z^2 + \ldots$. It was shown in [2] that $-1/a_2 \notin f(D)$, and it follows from Lemma 4 in [3] that unless f is conjugate to F_0 the point $-1/a_2$ will actually lie outside $\overline{f(D)}$. For a fixed $\zeta \in D$ renormalize in the usual way to

$$g(z) = \frac{f\left(\dfrac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)},$$

which is again in N, and which has $g(0)=0$ and $g'(0)=1$. To say that $-2/g''(0) \notin g(D)$ is equivalent to saying that

$$E_f(\zeta) = f(\zeta) + \frac{(1-|\zeta|^2)f'(\zeta)}{\bar{\zeta} - \frac{1}{2}(1-|\zeta|^2)\frac{f''}{f'}(\zeta)} \notin f(D), \tag{4.1}$$

and, again, if f is not conjugate to F_0 then

$$E_f(\zeta) \notin \overline{f(D)}. \tag{4.2}$$

E_f is precisely the Ahlfors-Weill extension. For f satisfying $|Sf(z)| \le 2t(1-|z|^2)^{-2}$ the function

$$F(z) = \begin{cases} f(z) & |z| \le 1 \\ E_f(1/\bar{z}) & |z| > 1 \end{cases} \tag{4.3}$$

is a $\frac{1+t}{1-t}$−quasiconformal mapping which extends f. (A function in N_0^t is already $\sqrt{1-t}$−Hölder continuous in D. In particular, it can be extended to $\overline{D}$, see [2].)

The extension E_f has the other interesting property that it commutes with Möbius transformations of f. If T is a Möbius transformation, then

$$E_{T\circ f} = T(E_f). \tag{4.4}$$

This can be checked directly from the definition, first for affine transformations and then, less obviously, for an inversion.

In terms of the Poincaré metric, E_f has the expression

$$E_f(z) = f(z) + \frac{1}{\partial_w(\log\lambda_\Omega)(f(z))}, \tag{4.5}$$

by (3.2). We point out one quick consequence of this. If $f \in N$ then, using (4.1), since $E_f(z) \notin f(D) = \Omega$ for $z \in D$, it must be that $|E_f(z) - f(z)| \ge d(f(z), \partial\Omega)$. Thus (4.5) implies $|\nabla \log\lambda_\Omega(w)| \le 1/d(w, \partial\Omega)$. This becomes an equality for the upper half-plane. This estimate was proved for convex domains by Minda in [8].

Theorem 1 *If $f \in N^*$ then the function F defined in (4.3) is a homeomorphism of the sphere extending f.*

Proof: First, recall that by the Gehring-Pommerenke theorem f has a homeomorphic extension to $\overline{D}$. Next, using (4.4) we may normalize further and first assume that $f \in N_0^*$. It is clear from

$$E_f(z) = f(z) + \frac{(1-|z|^2)f'(z)}{\bar{z} - \frac{1}{2}(1-|z|^2)\frac{f''}{f'}(z)}$$

that F is continuous; from Lemma 1 the denominator vanishes only at $z = 0$, which corresponds to ∞ under the reflection in $|z| = 1$.

We next show that E_f is injective. Suppose that $E_f(z_1) = E_f(z_2)$. Appealing again to (4.4) we may change f to $T \circ f$ by an appropriate Möbius transformation T and assume that this common value is ∞. But (4.2) now implies that f must be bounded, while on the other hand (4.5) shows that an infinite value of E_f must be a critical point of $\log\lambda_\Omega$. By Lemma 2 such a critical point must be unique, hence $z_1 = z_2$ because f is univalent.

The mapping E_f is continuous and injective and hence a homeomorphism onto its range. Since $f(D)$ is a Jordan domain, to complete the proof we must see that E_f matches with f

along $|z|=1$. We use (4.4) one more time to go back to the normalization $f \in N_0^*$. Then $f(D)=\Omega$ is bounded and it suffices to show that $E_f(z)-f(z) \to 0$ as $|z| \to 1$. This is equivalent to $|\nabla \log \lambda_\Omega(w)| \to \infty$ as $w \to \partial\Omega$ which follows from the first part of Lemma 3. This completes the proof of the Theorem.

The complex dilatation $\mu_F = \partial_{\bar z}F/\partial_z F$ of the Ahlfors-Weill extension at a point ζ in the exterior of the disk is

$$\mu_F(\zeta) = -\frac{1}{2}(1-|z|^2)^2 Sf(z),$$

where $z = 1/\bar\zeta$. It will therefore not define a quasiconformal mapping at points where $|Sf(z)|$ is at least $2/(1-|z|^2)^2$. In the case when $|Sf(z)| \le 2t(1-|z|^2)^2$, $t<1$, i.e., $f \in N^t$, the Gehring-Pommerenke result and the fact that we obtain a homeomorphic extension via E_f gives a somewhat more direct approach to the Ahlfors-Weill theorem than the original proof. There it was necessary to dilate so the map becomes regular on the boundary, and to use a topological argument based on the Monodromy Theorem to get global injectivity of the extended mapping; see also [7].

It is also intresting to note that all the functions f with $Sf(z) = -2t/(1-z^2)^2$ for $1 \le t < 3$ map the disk onto quasidisks, and hence do have quasiconformal extensions. Normalized examples are the functions $A_{-t}(z)$, $1 \le t < 3$, where A_t is defined in (1.4). They map the disk onto the interior of the union of the circles through the points $1/\alpha$, $-1/\alpha$ and $\pm i(1/\alpha)\tan(\pi\alpha/4)$, where $\alpha = \sqrt{1+t}$.

Appendix: An Example

We return to the first part of Lemma 3. We want to construct a function $f \in N_0^*$ showing that the exponent $1/2$ in the bound $|\nabla \log \lambda_\Omega(w)| \ge c|w|\lambda_\Omega(w)^{1/2}$, $\Omega = f(D)$, is, in general, best possible. As the proof of Lemma 3 shows, this will be the case provided the convex function v, introduced in the proof, has bounded derivative.

The extremal F_0 maps the disk onto the strip $-\pi/4 < \operatorname{Im} w < \pi/4$. We want to construct g, analytic in this strip, so that $f = g \circ F_0$ will be in N_0^*, and $v(\tau) = |g'(\tau)|^{-1/2}$ will be convex with bounded derivative for τ on the real axis.

Let $a>0$, to be chosen, and let

$$g'(\zeta) = \frac{a}{a+\zeta^2}.$$

If $a > \sqrt{\pi}/2$ then g' will be regular in the strip and $v(\tau)$, $\tau \in \mathbf{R}$, will be a convex function with bounded derivative. We compute the Schwarzian of g to be

$$Sg(\zeta) = \frac{-2a}{(a+\zeta^2)^2}.$$

Then $f = g \circ F_0$ is normalized and

$$Sf(z) = Sg(F_0(z))({F_0}'(z))^2 + SF_0(z) = \frac{2}{(1-z^2)^2}\left\{1 - \frac{4a}{(a+\zeta^2)^2}\right\}, \quad \zeta = F_0(z).$$

It is not hard to show that if a sufficiently large then

$$\left|1 - \frac{4a}{(a+\zeta^2)^2}\right| \leq 1,$$

so that $f \in N_0^*$.

References

[1] L. Ahlfors and G. Weill. A uniqueness theorem for Beltrami equations. *Proc. Amer. Math. Soc.*, 13:975–978, 1962.

[2] M. Chuaqui and B. Osgood. Sharp distortion theorems associated with the Schwarzian derivative. *Jour. London Math. Soc.(2)*, 48:289–298, 1993.

[3] M. Chuaqui and B. Osgood. An extension of a theorem of Gehring and Pommerenke. *Israel Jour. Math.*, pages 33–48, 1995.

[4] P.L. Duren and O. Lehto. Schwarzian derivatives and homeomorphic extensions. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 477:3–11, 1970.

[5] F.W. Gehring and Ch. Pommerenke. On the Nehari univalence criterion and quasicircles. *Comment. Math. Helv.*, 59:226–242, 1984.

[6] S.-A. Kim and D. Minda. The hyperbolic and quasihyperbolic metrics in convex regions. *J. Analysis*, 1:109–118, 1993.

[7] O. Lehto. *Univalent functions and Teichmüller spaces.* Springer-Verlag, New York, 1987.

[8] D. Minda. Applications of hyperbolic convexity to euclidean and spherical convexity. *Jour. d'Analyse Math.*, 49:90–105, 1987.

[9] Z. Nehari. The Schwarzian derivative and schlicht functions. *Bull. Amer. Math. Soc.*, 55:545–551, 1949.

[10] Z. Nehari. A property of convex conformal maps. *Jour. d'Analyse Math.*, 30:390–393, 1976.

[11] B. Osgood. Some properties of f''/f' and the Poincaré metric. *Indiana Univ. Math. Jour.*, 31:449–461, 1982.

[12] V. Paatero. Über die konforme Abbildung von Gebieten, deren Ränder von beschränkter Drehung sind. *Ann. Acad. Sci. Fenn. Ser. A XXXIII*, 9:1–79, 1931.