ON A SHARP INEQUALITY CONCERNING THE DIRICHLET INTEGRAL

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1. Introduction. If f is an analytic function defined in the unit disk, Δ, let

$$\mathfrak{D}(f) = \left(\iint_{\Delta} |f'(z)|^2 \frac{dx\,dy}{\pi}\right)^{1/2} \qquad (z = x + iy)$$

be the Dirichlet integral of f. In this note, we will answer the following question mentioned in a paper of J. Moser [3]: Does there exist a constant $C < \infty$ such that for all functions f analytic in Δ with $\mathfrak{D}(f) \leq 1$ and $f(0) = 0$

$$\int_0^{2\pi} e^{|f(e^{i\theta})|^2} \frac{d\theta}{2\pi} \leq C?$$

This integral, of course, can be written in terms of the distribution function of f. For each $M > 0$, let $E_M = \{\theta \in [0, 2\pi] : |f(e^{i\theta})| > M\}$ and let $|E_M|$ denote the normalized Lebesgue measure of E_M. Then

$$(1.1) \qquad \int_0^{2\pi} e^{|f(e^{i\theta})|^2} \frac{d\theta}{2\pi} = 1 + 2\int_0^{\infty} |E_M|\, e^{M^2} M dM.$$

In his influential study of majorization and the length-area principle, A. Beurling [1] proved the following estimate for such functions f:

$$(1.2) \qquad |E_M| \leq e^{-M^2+1}.$$

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Moreover, he proved that for the function

$$B_a(e^{i\theta}) = \left(\log \frac{1}{1 - ae^{i\theta}}\right) \Big/ \sqrt{\log \frac{1}{1 - a^2}}, \qquad 0 < a < 1,$$

the estimate (1.2) is sharp in the sense that there is a constant c (independent of a) such that

$$|E_M| \geq ce^{-M^2} \quad \text{if} \quad M = \sqrt{\log \frac{1}{1 - a^2}}.$$

If we let $\mathfrak{D}$ denote the set of analytic functions f on Δ with $f(0) = 0$ and $\mathfrak{D}(f) \leq 1$, then from these facts, one easily concludes that

$$\sup_{\mathfrak{D}} \int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} \frac{d\theta}{2\pi} \tag{1.3}$$

is finite if $\alpha < 1$ and is infinite if $\alpha > 1$. A further observation (pointed out to us by J. Garnett) from Beurling's estimate is that for each $\alpha > 0$ and each $f \in \mathfrak{D}$ it is true that

$$\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} \frac{d\theta}{2\pi} < \infty.$$

Indeed if $f = \Sigma_{n=1}^{\infty} a_n z^n$ then $\mathfrak{D}(f) = (\Sigma_{n=1}^{\infty} n|a_n|^2)^{1/2}$, so we may find a polynomial p and an analytic function g with $f = p + g$ and $\mathfrak{D}(g) < 1/\sqrt{3\alpha}$. Thus if we let $A = \|p\|_\infty$,

$$\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2} d\theta \leq e^{2\alpha A^2} \int_0^{2\pi} e^{2\alpha|g(e^{i\theta})|^2} d\theta < \infty.$$

At this point we will formulate our result in a conformally invariant form and compare it to a famous theorem in real analysis. The quantity $\mathfrak{D}(f)$ is the area of the image of Δ under the map f, counting multiplicity, and so $\mathfrak{D}(f \circ \tau) = \mathfrak{D}(f)$ for any conformal map τ of Δ onto itself.

THEOREM 1. *There is a constant $C < \infty$ such that if f is analytic on Δ and $\mathfrak{D}(f) \leq 1$ then*

$$\sup_{z\in\Delta}\int_0^{2\pi} e^{\alpha|f(e^{i\theta})-f(z)|^2} P_z(\theta)\,\frac{d\theta}{2\pi} \le C$$

where P_z is the Poisson kernel for the point $z \in \Delta$ and $0 \le \alpha \le 1$. If $\alpha > 1$, the integral can be made arbitrarily large with the functions

$$B_a(z) = \log\frac{1}{1-az}\bigg/\sqrt{\log\frac{1}{1-a^2}}, \qquad 0 < a < 1.$$

COROLLARY 1. *There is a constant $C < \infty$ such that if f is analytic on Δ, $f(0) = 0$ and $\mathfrak{D}(f) \le 1$ and if $E_M = \{\theta : |f(e^{i\theta})| > M\}$ then*

$$\sum_{M=1}^{\infty} |E_M| e^{M^2} \le C.$$

The following corollary is a real-variable version of Theorem 1. For f a real valued function in $L^1(d\theta)$, we will use the same notation $f(z)$, $z \in \Delta$, to denote the harmonic extension of f at z, suppose $f(0) = 0$ and let

$$\mathfrak{D}(f) = \left(\iint_D (f_x^2 + f_y^2)\,\frac{dx\,dy}{\pi}\right)^{1/2}.$$

COROLLARY 2. *There is a constant $C < \infty$ such that if $f \in L^1$ is real-valued with $\int_0^{2\pi} f(e^{i\theta})d\theta = 0$ and $\mathfrak{D}(f) \le 1$, then*

$$\sup_{z\in\Delta}\int_0^{2\pi} e^{\alpha|f(e^{i\theta})-f(z)|^2} P_z(\theta)\,\frac{d\theta}{2\pi} \le C$$

where $0 \le \alpha \le 1$. If $\alpha > 1$, the integral can be made arbitrarily large by the functions

$$\operatorname{Re} B_a(z) = \log\left|\frac{1}{1-az}\right|\bigg/\sqrt{\log\frac{1}{1-a^2}}, \qquad 0 < a < 1.$$

This corollary of Theorem 1 should be compared to the following well-known theorem of Helson-Szegö and Hunt, Muckenhoupt, Wheedon (cf., e.g., Garnett [2, Chapter VI]), which also motivates out initial interest in the problem.

THEOREM 2. *The following are equivalent for a real-valued function* $f \in L^1$:

(a) $\sup_{z\in\Delta} \int_0^{2\pi} e^{|f(e^{i\theta})-f(z)|} P_z(\theta) \frac{d\theta}{2\pi} < \infty$

(b) $f = u + \tilde{v}$ *for some* $u, v \in L^\infty$ *with* $\|v\|_\infty < \pi/2$,

where $\tilde{v}$ *is the boundary value function of the harmonic conjugate of* v.

One way of proving that (b) implies (a) in Theorem 2 is to prove that if $f = u + \tilde{v}$ with $\|v\|_\infty \le \pi/2$ then

$$m_z(\{\theta : |f(e^{i\theta}) - f(z)| > \lambda\}) \le Ce^{-\lambda}, \tag{1.4}$$

for all $z \in \Delta$ and for some constant c, independent of z, where m_z is the measure $P_z(\theta)\, d\theta/2\pi$. Notice that (1.4) implies

$$\sup_{z\in\Delta} \int_0^{2\pi} e^{\alpha|f(e^{i\theta})-f(z)|} P_z(\theta) d\theta < \infty \tag{1.5}$$

for all $\alpha < 1$. So starting with $f = u + \tilde{v}$, $\|v\|_\infty < \pi/2$, we may apply (1.5) to $(1 + \epsilon)f$ for some $\epsilon > 0$ and obtain (a). The analogous statement to (1.4) for functions f with $\int_0^{2\pi} f(e^{i\theta})d\theta = 0$ and $\mathfrak{D}(f) \le 1$ which comes from Beurling's estimate (1.2) is

$$m_z(\{\theta : |f(e^{i\theta}) - f(z)| > \lambda\}) \le e \cdot e^{-\lambda^2}. \tag{1.6}$$

Comparing the inequalities (1.4) and (1.6), the statements (1.3) and (1.5), and observing that the strict inequality $\|v\|_\infty < \pi/2$ is essential in Theorem 2, we were led to believe that Theorem 1 would be incorrect at the critical index $\alpha = 1$. Also, J. Garnett has shown that if

$$f(x) = \left(\log \frac{\pi}{x}\right)^{1/2}, \qquad \text{for} \quad 0 \le x \le \pi,$$

and

$$f(x) = -\left(\log \frac{\pi}{2\pi - x}\right)^{1/2}, \qquad \text{for} \quad \pi \le x \le 2\pi,$$

then f satisfies (1.6) yet $\int_0^{2\pi} e^{|f(x)|^2} dx = \infty$. This says that (1.6) alone is not sufficient to bound the integral (1.1).

On the other hand the result in Moser [3], where our problem was mentioned, led us to believe that the problem had a positive answer. Moser proved a sharp form of an inequality originally studied by N. Trudinger in connection with embedding Sobolev spaces into Orlicz spaces. Suppose D is an open domain in $\mathbf{R}^n$, let $\mathring{W}_n^1(D)$ denote the Banach space obtained from the C^1-functions u with compact support in D by completion with the norm

$$\mathfrak{D}_n(u) = \left(\int_D |u_x|^n dx\right)^{1/n}.$$

Moser proved that there exist constants α_n and c_n which depend only on n such that

$$\int_D e^{\alpha u^{p}(x)} dx \leq c_n m(D), \tag{1.7}$$

where $p = n/n - 1$, $m(D) = \int_D dx$, for each $\alpha \leq \alpha_n$ and for all functions u in the unit ball $\{u : \mathfrak{D}_n(u) \leq 1\}$ of $\mathring{W}_n^1(D)$. Furthermore if $\alpha > \alpha_n$, the integral can be made arbitrarily large by an appropriate choice of u. As Moser states [3]: "The remarkable phenomenon is that the inequality still holds for the critical value (α_n) itself." The difference between Moser's result, for $n = 2$, and ours, is that his integral (1.7) takes place on the disk and is concerned with real-valued functions. Thus he is able to symmetrize his functions and reduce his problem to a problem about C^1 functions on the interval $[0, \infty)$. Our integral (1.1) takes place on the boundary of the disk and is concerned with analytic functions.

We wish to emphasize here the influence of the works of Beurling and Moser on our proof of Theorem 1. As the discerning reader will undoubtedly notice, it is Beurling's proof of (1.2) that we modify and it is Moser's method of splitting and estimating the integral (1.7) that motivates our splitting and estimating the analogous integral (1.1). We are grateful to John Garnett for several discussions.

2. The Beurling Functions. Since the functions

$$B_a(z) = \log \frac{1}{1 - az} \Bigg/ \sqrt{\log \frac{1}{1 - a^2}},$$

which Beurling considered, yield the extreme cases for the estimate (1.2) of the distribution function $|E_M|$, we will first show that Theorem 1 holds for these functions. This is accomplished by improving the estimate (1.2) when

$$M \leq \sqrt{\log \frac{1}{1 - a^2}}.$$

LEMMA 1. *There exist constants c_0 and a_0 so that if $a_0 \leq a \leq 1$ and if $M \geq 1$*

$$|\{\theta : |B_a(e^{i\theta})| > M\}| \leq c_0 e^{-M\sqrt{N_a}}$$

where $N_a = \log(1/(1 - a^2))$.

Proof. Fix a, $0 < a < 1$, and let $N = N_a$. Then

$$\{\theta : |B_a(e^{i\theta})| > M\} \subseteq \left\{\theta : \left|\log \frac{1}{|1 - ae^{i\theta}|}\right| > M\sqrt{N}\right\}.$$

Thus if $|B_a(e^{i\theta})| > M$, then either $|1 - ae^{i\theta}| < e^{-M\sqrt{N}}$ or $|1 - ae^{i\theta}| > e^{M\sqrt{N}}$. In the second case, we have $2 \geq |1 - ae^{i\theta}| > e^{M\sqrt{N}}$ which does not occur when $M \geq 1$ and $N \geq 1$ (i.e. when $a \geq a_0 \geq (1 - 1/e)^{1/2}$). In the first case, we have

$$4a \frac{|\theta|^2}{\pi^2} \leq 4a \sin^2 \frac{\theta}{2} \leq |1 - ae^{i\theta}|^2 \leq e^{-2M\sqrt{N}}.$$

Thus $|\{\theta : |B_a(e^{\theta})| > M\}| \leq c_0 e^{-M\sqrt{N}}$ with $c_0 = \pi/2a_0^{1/2}$.

We can now see that the Beurling functions B_a satisfy the conclusion of Theorem 1. While this fact follows from the proof of Theorem 1 below, we mention it here because the division of the integral (1.1) for general $f \in \mathfrak{D}$ will result in similar integrals. By Lemma 1 and Beurling's estimate (1.2), for the function B_a, we have

$$\int_1^\infty |E_M| e^{M^2} M dM \leq \int_1^{\sqrt{N_a}} c_0 e^{M^2 - M\sqrt{N}} M dM + \int_{\sqrt{N_a}}^{\|B_a\|_\infty} eM dM. \tag{2.1}$$

Looking at the Taylor series expansion of B_a, we know that

$$\|B_a\|_\infty^2 - N_a = \left(\frac{\log \dfrac{1}{1-a}}{\sqrt{N_a}}\right)^2 - N_a = \left(\frac{\log(1+a)}{\sqrt{N_a}} + \sqrt{N_a}\right)^2 - N_a$$

$$\leq 1 + 2\log 2$$

if $N_a \geq N_{a_0} \geq 1$, and therefore the last integral in (2.1) is uniformly bounded. That the first integral on the right-hand side of (2.1) is uniformly bounded follows from the lemma below, which we will use several times in the proof of Theorem 1.

LEMMA 2. *If $B > A \geq 0$, then*

$$\int_A^B e^{(M-A)(M-B)} M dM \leq \frac{2(A+B)}{B-A}.$$

Proof. On the interval $A \leq M \leq (A+B)/2$, estimate the parabola $y = (M-A)(M-B)$ by the line $y = -(M-A)(B-A)/2$, and on the interval $(A+B)/2 \leq M \leq B$, estimate the parabols by the line $y = (M-B)(B-A)/2$. Integrate each term by parts.

For the Beurling function B_a, we use this lemma with $A = 0$ and $B = \sqrt{N} \geq \sqrt{N_{a_0}} \geq 1$.

3. A technical lemma. Let $f = \Sigma_{n=1}^\infty a_n z^n$ be analytic in Δ with $\mathfrak{D}(f) = (\Sigma_{n=1}^\infty n|a_n|^2)^{1/2} \leq 1$, and as before let $E_M = \{\theta : |f(e^{i\theta})| > M\}$. To prove the theorem, we will show $\int_1^\infty |E_M| e^{M^2} M dM \leq C$ for some constant C, independent of f. We first notice that it suffices to assume $\Sigma|a_n| < \infty$. Indeed, if $f_r(e^{i\theta}) = f(re^{i\theta})$ then

$$\int e^{|f_r|^2} d\theta = \sum_{n=0}^\infty \frac{1}{n!} \int |f_r|^{2n} d\theta.$$

Since the L^{2n} norms of f_r increase with r to the L^{2n} norm of f, we see that

$$\lim_{r\to 1} \int e^{|f_r|^2} d\theta = \int e^{|f|^2} d\theta.$$

Secondly, we remark that J. Clunie has observed that we may suppose that $a_n \geq 0$ for all n. Indeed

$$\int |f|^{2n} d\theta = \sum_{k=1}^{\infty} \left| \sum_{j_1+\cdots+j_n=k} a_{j_1} \cdots a_{j_n} \right|^2$$

which is not decreased if each a_n is replaced by $|a_n|$. Clearly this replacement doesn't affect

$$\mathfrak{D}(f) = \left(\sum_{n=1}^{\infty} n |a_n|^2 \right)^{1/2}.$$

We do not need this latter observation in what follows, but it simplifies the presentation.

We now give the main technical lemma which is a slight variation of the original argument of Beurling. We also include a proof since [1] is not readily available.

LEMMA 3. *If* $1 - r^2 = |E_M|/e$ *then*

$$M \leq 2\left(\sum_{n=1}^{\infty} n |a_n|^2 (1 - r^{2n}) \right)^{1/2} + \sum_{n=1}^{\infty} |a_n| r^n. \tag{3.1}$$

Proof. For each subset $E \subset \partial\Delta$ and $r \leq 1$ let

$$I_E(r) = \frac{1}{|E|} \int_E \left(\int_0^r |f'(te^{i\theta})| \, dt \right)^2 \frac{d\theta}{2\pi}.$$

Then

$$\begin{aligned} I_E'(r) &= \frac{1}{|E|} \int_E 2\left(\int_0^r |f'(te^{i\theta})| \, dt \right) |f'(re^{i\theta})| \frac{d\theta}{2\pi} \\ &\leq \frac{2}{|E|} \left(\int_E \left(\int_0^r |f'(te^{i\theta})| \, dt \right)^2 \frac{d\theta}{2\pi} \right)^{1/2} \left(\int_E |f'(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2} \\ &\leq 2\left(\frac{I_E(1)}{|E|} \right)^{1/2} \left(\int_0^{2\pi} |f'(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2}. \end{aligned}$$

Hence

$$
\begin{aligned}
I_E(1) - I_E(r) &= \int_r^1 I_E'(t)\,dt \\
&\le 2\left(\frac{I_E(1)}{|E|}\right)^{1/2} \int_r^1 \left(\int_0^{2\pi} |f'(te^{i\theta})|^2 \frac{d\theta}{2\pi}\right)^{1/2} dt \\
&\le 2\left(\frac{I_E(1)}{|E|}\right)^{1/2} \left(\int_r^1 \int_0^{2\pi} |f'(te^{i\theta})|^2 \frac{d\theta}{2\pi}\,dt\right)^{1/2} \left(\int_r^1 dt\right)^{1/2} \qquad (3.2)\\
&\le 2\left(\frac{I_E(1)}{|E|} \sum_{n=1}^{\infty} n|a_n|^2(1 - r^{2n})(1 - r)\right)^{1/2}.
\end{aligned}
$$

We also have that

$$
(3.3) \qquad I_E(r) \le \frac{1}{|E|} \int_E \left(\int_0^r \sum_{n=1}^{\infty} n|a_n|t^{n-1}dt\right)^2 \frac{d\theta}{2\pi} \le \left(\sum_{n=1}^{\infty} |a_n|r^n\right)^2.
$$

We deviate here from Beurling by using the estimate $A^2 \le AB + C^2$ implies $A \le B + C$ to obtain from (3.2) and (3.3)

$$
(3.4) \qquad (I_E(1))^{1/2} \le 2\left(\frac{1-r}{|E|} \sum_{n=1}^{\infty} n|a_n|^2(1 - r^{2n})\right)^{1/2} + \sum_{n=1}^{\infty} |a_n|r^n.
$$

If $E = E_M$, choose r such that $1 - r^2 = |E_M|/e$ and then observe

$$
M \le \frac{1}{|E|} \int_{E_M} |f(e^{i\theta})| \frac{d\theta}{2\pi} \le \frac{1}{|E_M|} \int_{E_M} \left(\int_0^1 |f'(te^{i\theta})|\,dt\right) \frac{d\theta}{2\pi} \le (I_E(1))^{1/2}.
$$

Together with (3.4), this gives the desired estimate.

4. Proof of Theorem 1. Suppose $f = \Sigma_{n=1}^{\infty} a_n z^n$ is an analytic function on Δ with $f(0) = 0$ and

$$
\mathfrak{D}^2(f) = \sum_{n=1}^{\infty} n|a_n|^2 \le 1
$$

and, without loss of generality, $a_n \geq 0$ for each n and $\Sigma |a_n| < \infty$. We will begin by comparing our f with the Beurling functions

$$B_a(z) = \log \frac{1}{1 - az} \Big/ \sqrt{\log \frac{1}{1 - a^2}} = \sum_{n=1}^{\infty} \frac{a^n z^n}{n\sqrt{N_a}}$$

where

$$N_a = \log \frac{1}{1 - a^2}.$$

For this purpose, let

$$\sup_{0 \leq r \leq 1} \frac{\left(\sum_1^\infty a_n r^n\right)^2}{\log \dfrac{1}{1 - r^2}} = 1 - \delta. \tag{4.1}$$

Since

$$(\Sigma\, a_n r^n)^2 \leq (\Sigma\, n a_n^2)\left(\Sigma\, \frac{1}{n} r^{2n}\right) \leq \log \frac{1}{1 - r^2},$$

we always have $\delta \geq 0$. Since $\Sigma |a_n| < \infty$, there is a number a, $0 \leq a < 1$, such that

$$\frac{\left(\sum_1^\infty |a_n| a^n\right)^2}{\log \dfrac{1}{1 - a^2}} = 1 - \delta. \tag{4.2}$$

If we write $B_a(z) = \Sigma_{n=1}^\infty b_n z^n$, (4.2) says that

$$\mathfrak{D}^2(f - B_a) = \sum_{n=1}^{\infty} n(a_n - b_n)^2 = 2 - 2 \sum_{n=1}^{\infty} n a_n b_n \tag{4.3}$$

$$= 2(1 - (1 - \delta)^{1/2}) \leq 2\delta.$$

In other words, the Dirichlet norm of $f - B_a$ is at most $\sqrt{2\delta}$. Henceforth, let

$$N = \log \frac{1}{1-a^2}.$$

We have already noted that in Section 1 the integral (1.1) is uniformly bounded for $f = B_a$, i.e. in case $\delta = 0$. Let δ_0 be a fixed small number to be chosen later. We begin by considering the case $\delta \geq \delta_0$, then by Lemma 3, if we choose r so that $1 - r^2 = |E_M|/e$, then

$$M \leq 2 + \sum_{n=1}^{\infty} a_n r^n \leq 2 + (1-\delta)^{1/2}\left(\log \frac{e}{|E_M|}\right)^{1/2}$$
$$\leq 2 + (1-\delta_0)^{1/2}\left(\log \frac{e}{|E_M|}\right)^{1/2}.$$

Thus we have

$$\frac{1}{e}|E_M|e^{M^2} \leq e^{M^2-(M-2)^2/1-\delta_0} \leq e^{M^2(1-1/(1-\delta_0)^{1/2})} \tag{4.4}$$

when $M \geq 2/(1-(1-\delta_0)^{1/4})$. From Beurling's estimate (1.2), we also have that

$$|E_M|e^{M^2} \leq e \tag{4.5}$$

when $M \leq 2/(1-(1-\delta_0)^{1/4})$. From (4.4) and (4.5) it is easy to obtain that

$$\int_0^{\infty} |E_M|e^{M^2}MdM \leq c_1 < \infty$$

where c_1 is a constant depending only on δ_0.

To further orient the reader we consider another simple, special case. We suppose that $N \leq N_{a_0} \equiv N_0$, with $N_0 \geq 1$. In other words, we suppose $a \leq a_0$, where a_0 is to be chosen later. We assume also in this case that $\delta \leq \delta_0 < 1/8$. Then

$$|B_a(e^{i\theta})| \leq \log \frac{1}{1-a} / \sqrt{N} = \sqrt{N} + \log(1+a)/\sqrt{N} \leq \sqrt{N} + 1 \leq \sqrt{N_0} + 1.$$

Thus by (4.3) and estimate (1.2) we have

$$\begin{aligned} |E_M| &\leq |\{\theta : |f(e^{i\theta}) - B_a(e^{i\theta})| > M - \sqrt{N_0} - 1\}| \\ &\leq e^{-(M-\sqrt{N_0}-1)^2/2\delta} \end{aligned}$$

Thus if $M > 2(\sqrt{N_0} + 1)$ then

$$|E_M| e^{M^2} \leq e^{M^2(1-1/8\delta)}. \tag{4.6}$$

Using Beurling's estimate (1.2) again when $M \leq 2(\sqrt{N_0} + 1)$ we obtain

$$\int_0^\infty |E_M| e^{M^2} M dM \leq c_2 < \infty$$

where c_2 is a constant depending only on a_0 and δ_0.

Henceforth we will suppose $\delta \leq \delta_0$ and $N \geq N_0 \geq 4$. Following Moser [3], we will choose some constant $c = c_3$ and split our integral into five pieces:

$$\begin{aligned} \int_1^\infty |E_M| e^{M^2} M dM = \int_1^{\sqrt{N}/2} + \int_{\sqrt{N}/2}^{(1-c\delta)\sqrt{N}} + \int_{(1-c\delta)\sqrt{N}}^{(1+c\delta)\sqrt{N}} \\ + \int_{(1+c\delta)\sqrt{N}}^{3\sqrt{N}} + \int_{3\sqrt{N}}^{\infty}. \end{aligned} \tag{4.7}$$

We remark here that the estimate which we will use below to estimate the first and fifth terms in (4.7) really works for the integrals $\int_1^{(1-c\sqrt{\delta})\sqrt{N}}$ and $\int_{(1+c\sqrt{\delta})\sqrt{N}}^{\infty}$ respectively. But it is the delicate method of estimation in Moser [3] that we adapt to our integrals which allows the passage from $\sqrt{\delta}$ to δ in the central terms of (4.7).

On the first term $1 \leq M \leq \sqrt{N}/2$. Here we use the fact that f is close to B_a in the Dirichlet norm and that the distribution functions of B_a can be easily controlled via Lemma 1. We have

$$E_M \subset \left\{\theta : |f - B_a| > \frac{M}{2}\right\} \cup \left\{\theta : |B_a| > \frac{M}{2}\right\}.$$

Applying Beurling's estimate (1.2) and Lemma 1, we observe that

$$|E_M| \leq e \cdot e^{-M^2/8\delta} + c_0 e^{-M\sqrt{N}/2}.$$

By Lemma 2, for $\delta_0 < 1/8$ we obtain

$$\int_1^{\sqrt{N}/2} |E_M| e^{M^2} M dM \leq \frac{e}{2\left(\dfrac{1}{8\delta_0} - 1\right)} + 2c_0. \tag{4.8}$$

In each of the next three integrals we will use the estimate provided by Lemma 3. Henceforth we suppose $1 - r^2 = |E_M|/e$. In each case we estimate the first term in (3.1) by comparing the coefficients a_n with the coefficients of the Beurling function $b_n = a^n/n\sqrt{N}$ and using (4.3) as follows:

$$\begin{aligned} 2\left(\sum_1^\infty n a_n^2 (1 - r^{2n})\right)^{1/2} &\leq 2\left(\sum_1^\infty n(a_n - b_n)^2 (1 - r^{2n})\right)^{1/2} \\ &\quad + 2\left(\sum_1^\infty n b_n^2 (1 - r^{2n})\right)^{1/2} \\ &\leq 2(2\delta)^{1/2} + 2\left(\log \frac{1 - a^2 r^2}{1 - a^2}\Big/ N\right)^{1/2}. \end{aligned} \tag{4.9}$$

The second term in (3.1) can be estimated similarly by

$$\begin{aligned} \Sigma\, a_n r^n &= \Sigma\, a_n a^n + \Sigma\, b_n (r^n - a^n) + \Sigma\, (a_n - b_n)(r^n - a^n) \\ &\leq (1 - \delta)^{1/2} N^{1/2} + \log\left(\frac{1 - a^2}{1 - ra}\right)\Big/ N^{1/2} \\ &\quad + \left(\sum_1^\infty n(a_n - b_n)^2\right)^{1/2} \left(\Sigma \frac{(r^n - a^n)^2}{n}\right)^{1/2} \\ &\leq N^{1/2} + \log \frac{1 - a^2}{1 - ra}\Big/ N^{1/2} \\ &\quad + (2\delta)^{1/2} \left(\log \frac{(1 - ra)^2}{(1 - r^2)(1 - a^2)}\right)^{1/2} \end{aligned} \tag{4.10}$$

$$= \left(\log \frac{1}{1 - ra}\right) \Big/ N^{1/2}$$

$$+ (2\delta)^{1/2} \left(\log \frac{(1 - ra)^2}{(1 - r^2)(1 - a^2)}\right)^{1/2}.$$

On the second term $(1/2)\sqrt{N} \leq M \leq (1 - c\delta)\sqrt{N}$. Here $c = c_3$ is a constant to be chosen later. Choose $M_0 > 0$ with $|E_{M_0}|/e = (1 - a^2)^{(1-c\delta)^2}$. Such an M_0 exists because $(1 - a^2)^{(1-c\delta)^2} \leq (1 - a_0^2)^{(1-c\delta_0)^2} < 1/e$, provided we choose a_0 sufficiently large, dependent on $c = c_3$ andδ_0. Notice that $(1 - a^2)^{(1-c\delta)^2} = |E_M|/e \leq e^{-M_0^2}$ by (1.2), and hence $M_0 \leq (1 - c\delta)\sqrt{N}$.

We will first consider the interval where $(1/2)\sqrt{N} \leq M \leq M_0$. Using estimate (1.2) again

$$M^2 \leq \log \frac{e}{|E_M|} = \log \frac{1}{1 - r^2} \leq \log \frac{e}{|E_{M_0}|} = (1 - c\delta)^2 N \leq \log \frac{1}{1 - a^2}.$$

We may rewrite this as two inequalities:

$$r \leq a \quad \text{and} \quad \log \frac{e}{|E_M|} \leq (1 - c\delta)^2 N. \tag{4.11}$$

Since $r \leq a$, we have $1 - a^2 r^2 = 1 - a^2 + a^2(1 - r^2) \leq 2(1 - r^2)$ and $1/(1 - ar) \leq 1/(1 - r) \leq 2/(1 - r^2)$ and $(1 - ra)^2 \leq (1 - r^2)^2$. By Lemma 3, (4.9) and (4.10) we obtain

$$M \leq 2(2\delta)^{1/2} + 2\left(\frac{\log 2 + N - \log \frac{e}{|E_M|}}{N}\right)^{1/2} + \frac{\log 2}{N^{1/2}} + \frac{\log \frac{e}{|E_M|}}{N^{1/2}}$$

$$+ (2\delta)^{1/2}\left(N - \log \frac{e}{|E_M|}\right)^{1/2}.$$

Multiplying by $N^{1/2}$ and using the estimate $A^{1/2} + B^{1/2} \leq \sqrt{2}(A + B)^{1/2}$, we obtain

$$(4.12) \qquad M\sqrt{N} \le 2((\delta N)^{1/2} + 1)\left(N - \log \frac{e}{|E_M|} + 4\right)^{1/2} + \log 2 + \log \frac{e}{|E_M|}.$$

By (4.11), if $\delta \le \delta_0 \le 1/c$, then $c\delta N \le N - \log e/|E_M|$ and hence

$$2((\delta N)^{1/2} + 1) \le \frac{2\sqrt{2}}{c^{1/2}}(c\delta N + c)^{1/2} \le \frac{2\sqrt{2}}{c^{1/2}}\left(N - \log \frac{e}{|E_M|} + c\right)^{1/2}.$$

So by (4.12), if $c \ge 128$,

$$M\sqrt{N} \le \frac{1}{4}\left(N - \log \frac{e}{|E_M|} + c\right) + \log 2 + \log \frac{e}{|E_M|}.$$

Thus for $\sqrt{N}/3 \le M \le M_0$

$$(4.13) \qquad |E_M| e^{M^2} \le c_4 e^{M^2 - (4/3)M\sqrt{N} + (1/3)N} = c_4 e^{(M - \sqrt{N}/3)(M - \sqrt{N})}$$

where $c_4 = e^{1 + c/3 + (4/3)\log 2}$.

Hence

$$(4.14) \qquad \int_{\sqrt{N}/2}^{M_0} |E_M| e^{M^2} dM \le c_4 \int_{\sqrt{N}/2}^{\sqrt{N}} e^{(M - \sqrt{N}/3)(M - \sqrt{N})} M dM$$

$$\le 4c_4 \qquad \text{by Lemma 2.}$$

Now if $M_0 \le M \le (1 - c\delta)\sqrt{N}$ then

$$\log \frac{|E_M|}{e} \le \log \frac{|E_{M_0}|}{e} = -(1 - c\delta)^2 N.$$

Hence

$$(4.14)' \qquad \int_{M_0}^{(1-c\delta)\sqrt{N}} |E_M| e^{M^2} M dM \le e \cdot e^{-(1-c\delta)^2 N} \int_{M_0}^{(1-c\delta)\sqrt{N}} e^{M^2} M dM$$

$$\le \frac{e}{2}.$$

We have finished the estimate of the second term by combining (4.14) and (4.14)′.

On the third term $(1 - c\delta)\sqrt{N} \le M \le (1 + c\delta)\sqrt{N}$. We first consider the simple special case, when $N\delta \le c_5$ where c_5 is a constant to be chosen later. By Beurling's estimate (1.2) we have

$$\int_{(1-c\delta)\sqrt{N}}^{(1+c\delta)\sqrt{N}} |E_M| e^{M^2} M dM \le e(1 + c\delta)\sqrt{N} \cdot 2c\delta\sqrt{N} \le 4ecc_5.$$

In the other case, when $N\delta > c_5$, we observe

$$1 - r^2 = \frac{|E_M|}{e} \le \frac{|E_{(1-c\delta)\sqrt{N}}|}{e} \le e^{-(1-c\delta)^2 N} \le (1 - a^2)^{1-2c\delta}.$$

Hence

$$\left(\log \frac{1 - a^2 r^2}{1 - a^2}\right)\Big/ N \le \left(\log \frac{2(1 - a^2)^{1-2c\delta}}{1 - a^2}\right)\Big/ N = \frac{\log 2}{N} + 2c\delta.$$

For this integral, instead of (4.10) we use the simpler estimate that comes from (4.1) together with (4.9) to obtain from Lemma 3,

(4.15)

$$M \le 2(2\delta)^{1/2} + 2\left(\frac{\log 2}{N}\right)^{1/2} + 2(2c\delta)^{1/2} + (1 - \delta)^{1/2}\left(\log \frac{e}{|E_M|}\right)^{1/2}$$

$$\le c_6 \delta^{1/2} + (1 - \delta)^{1/2}\left(\log \frac{e}{|E_M|}\right)^{1/2}$$

where $c_6 = 2\sqrt{2} + 2(\log 2)^{1/2}/c_5^{1/2} + 2\sqrt{2}\, c^{1/2} \ge 1$. Then (4.15) implies that

$$\frac{|E_M|}{e} \le e^{-(M - c_6 \delta^{1/2})^2 / 1 - \delta}.$$

We thus obtain

$$\int_{(1-c\delta)\sqrt{N}}^{(1+c\delta)\sqrt{N}} |E_M| e^{M^2} M dM \le \int_{(1-c\delta)\sqrt{N}}^{(1+c\delta)\sqrt{N}} e^{(-\delta/(1-\delta))(M-c_6/(\delta^{1/2}))^2+c_6^2} M dM$$

$$\le e^{(-\delta/(1-\delta))((1-c\delta)\sqrt{N}-c_6/\delta^{1/2})^2} e^{c_6^2}(1+c\delta)\sqrt{N}\cdot 2c\delta\sqrt{N}.$$

Since $ye^{-y} \le 1/e$ for all $y \ge 0$, we obtain

$$\le \frac{e^{c_6^2}}{e\left(\frac{\delta}{1-\delta}\right)\left((1-c\delta)\sqrt{N}-\frac{c_6}{\delta^{1/2}}\right)^2}(1+c\delta)\cdot 2c\delta N$$

$$\le \frac{e^{c_6^2} 4c\delta N}{((1-c\delta)\sqrt{N}\sqrt{\delta}-c_6)^2} \le 64ce^{c_6^2}. \tag{4.16}$$

When we choose $c_3 = c < 1/2\delta_0$ and c_5 with $N\delta \ge c_5 \ge 16c_6^2$.

On the fourth term $(1 + c\delta)\sqrt{N} \le M \le 3\sqrt{N}$. We first notice that by Beurling's estimate (1.2),

$$\log\frac{1}{1-r^2} = \log\frac{e}{|E_M|} \ge \log\frac{e}{|E_{(1+c\delta)\sqrt{N}}|} \ge (1+c\delta)^2 N \ge \log\frac{1}{1-a^2}.$$

We may rewrite this as two inequalities:

$$r \ge a \quad \text{and} \quad \log\frac{e}{|E_M|} \ge (1+c\delta)^2 N. \tag{4.17}$$

Following the same pattern of proof as in the estimate of the second term, we note that since $r \ge a$,

$$1-a^2r^2 \le 1-r^2+r^2(1-a^2) \le 2(1-a^2) \quad \text{and} \quad \frac{1}{1-ra} \le \frac{1}{1-a} \le \frac{2}{1-a^2}$$

and $(1-ra)^2 \le (1-a^2)^2$. By Lemma 3, (4.9) and (4.10), we obtain

$$M \le 2(2\delta)^{1/2} + 2\left(\frac{\log 2}{N}\right)^{1/2} + \frac{\log 2}{N^{1/2}} + N^{1/2} + (2\delta)^{1/2}\left(\log\frac{e}{|E_M|} - N\right)^{1/2}.$$

Multiplying by $N^{1/2}$ we obtain

$$M\sqrt{N} \le 3 + N + (2\delta N)^{1/2}\left(\log \frac{e}{|E_M|} - N + 4\right)^{1/2}.$$

By (4.17) we have $2c\delta N \le \log e/|E_M| - N$. Thus for $c \ge 16$

$$M\sqrt{N} \le 3 + N + \frac{1}{4}\left(\log \frac{e}{|E_M|} - N + 4\right) = \frac{1}{4}\log \frac{e}{|E_M|} + \frac{3}{4}N + 4.$$

Thus

$$|E_M| e^{M^2} \le e^{17} e^{M^2 - 4M\sqrt{N} + 3N}$$

and by Lemma 2,

$$\text{(4.18)} \quad \int_{(1+c\delta)\sqrt{N}}^{3\sqrt{N}} |E_M| e^{M^2} M dM \le e^{17} \int_{\sqrt{N}}^{3\sqrt{N}} e^{M^2 - 4M\sqrt{N} + 3N} M dM \le 4e^{17}.$$

On the fifth term $3\sqrt{N} \le M < \infty$. We first observe that

$$|B_a(e^{i\theta})| \le \frac{\log \dfrac{1}{1-a}}{\sqrt{N}} = \sqrt{N} + \frac{\log 1 + a}{\sqrt{N}} \le \sqrt{N} + 1 \le 2\sqrt{N}$$

when $N \ge N_0 > 1$ so that $E_M \subset \{\theta : |f - B_a| > M - 2\sqrt{N}\}$. By Beurling's estimate (1.2) and by the estimates $M \le 3(M - 2\sqrt{N})$ and $\delta \le \delta_0 < 1/24$ we have

(4.19)

$$\int_{3\sqrt{N}}^{\infty} |E_M| e^{M^2} M dM \le \int_{3\sqrt{N}}^{\infty} e \cdot e^{M^2 - (M - 2\sqrt{N})^2/2\delta} M dM$$

$$\le \int_{3\sqrt{N}}^{\infty} 3e \cdot e^{-3(M - 2\sqrt{N})^2} (M - 2\sqrt{N}) dM \le \frac{e^{-11}}{2}.$$

Combining (4.7), (4.8), (4.14), (4.16), (4.18), and (4.19) we see that if we choose $c = 2^7$, $\delta_0 = 2^{-8}$ and $a_0 = (1 - e^{-4})^{1/2}$ (i.e. $N_0 = 4$) then all the estimates are satisfied and we get the desired estimate of the integral (1.1) and we have finished the proof of Theorem 1.

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