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DOES FINITE KNOT ENERGY LEAD TO DIFFERENTIABILITY?

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ABSTRACT

In this article, we raise the question if curves of finite (j,p)-*knot energy* introduced by O'Hara are at least pointwise differentiable. If we exclude the highly singular range $(j-2)p \geq 1$, the answer is *no* for $jp \leq 2$ and *yes* for $jp > 2$. In the first case, which also contains the most prominent example of the *Möbius energy* ($j = 2$, $p = 1$) investigated by Freedman, He and Wang, we construct counterexamples. For $jp > 2$, we prove that finite-energy curves have in fact a Hölder continuous tangent with Hölder exponent $\frac{1}{2}(jp-2)/(p+2)$. Thus, we obtain a complete picture as to what extent the (j,p)-energy has self-avoidance and regularizing effects for $(j,p) \in (0,\infty)\times(0,\infty)$. We provide results for both closed and open curves.

Keywords: Knot energy; Möbius energy; differentiability; regularity.

Mathematics Subject Classification 2000: Primary 53A04; Secondary 26A27, 57M25

1. Introduction

A *knot energy* is a functional that is bounded from below and *self-repulsive*, i.e. blows up on sequences of embedded curves converging to a curve with a self-intersection [11, Definition 1.1]. One motivation to study such functionals is to find a "nicer", that is, less entangled shape for a given knot in order to determine its knot type, e.g. by following the negative gradient flow of the knot energy up to a local minimum [5]. By claiming self-repulsion, one hopes[a] not to run into the danger of leaving the ambient isotopy class during this process. Global minimizers within a prescribed knot class may be regarded as optimal representatives of this knot class exhibiting an "ideal" shape; see the nice illustrations of minimizing curves in various knot classes in [6].

The idea of considering energy functionals on knots goes back to Fukuhara. He thought of the motion of a non-elastic string with electrons on it lying in a viscous

[a]In general, self-repulsion does not penalize "pulling tight" of small knots, cf. [9, Theorem 3.1(2)].

liquid absorbing kinetic energy. In a first paper [3], he treated the case of knotted polygons.

In 1991, O'Hara [7] introduced the knot energy[b]

$$E(\gamma) := \iint_{X\times X} \left(\frac{1}{|\gamma(s)-\gamma(t)|^2} - \frac{1}{D_\gamma(s,t)^2}\right) |\dot\gamma(s)||\dot\gamma(t)| \mathrm{d}s\, \mathrm{d}t, \tag{1.1}$$

which may be viewed as a functional on the set $AC_{\mathrm{reg}}(X,\mathbb{R}^3)$ of regular absolutely continuous curves defined on an interval $X \subset \mathbb{R}$ or a circle $X = \mathbb{R}/(\ell\mathbb{Z})$. Here, $D_\gamma(s,t)$ denotes the distance of $\gamma(s)$ and $\gamma(t)$ on γ. The factor $|\dot\gamma(s)||\dot\gamma(t)|$ guarantees the invariance under reparametrization, which allows us to restrict our attention to curves parametrized by arc-length, that are naturally Lipschitz continuous.

In 1994, Freedman, He and Wang proved in their seminal paper [2] the existence and $C^{1,1}$-regularity[c] of E-minimizers in prescribed prime knot classes using the invariance of this particular knot energy under Möbius transformations in $\mathbb{R}^3$. Due to that they coined the name *Möbius energy*. Among many other things, they proved that E takes finite values on sufficiently smooth embeddings of $\mathbb{S}^1$ [2, Proposition 1.5], [4, Theorem 1.5.1], and that, on the other hand, any curve with finite Möbius energy has locally a bi-Lipschitz constant arbitrarily close to 1 [2, Corollary 1.3]. This means that the Möbius energy exhibits a "regularizing" effect: Finite energy excludes corner points and even more so cusps on the curve. This fact led to the question if finite energy implies differentiability.

In his (unfortunately unpublished) lecture notes on the Möbius energy [4, pp. 14–19], He constructed an open finite-energy curve (of "spiral" shape) that is not differentiable at a boundary point and asked about the differentiability at interior points [4, Problem 1.6.3].[d]

The answer to this question is contained as a special case in our Theorem 1.1 below, which in fact deals with an entire family of energies, the so-called (j,p)-*energies*

$$E^{j,p}(\gamma) := \mathcal{L}(\gamma)^{jp-2} \iint_{X\times X} \left(\frac{1}{|\gamma(s)-\gamma(t)|^j} - \frac{1}{D_\gamma(s,t)^j}\right)^p |\dot\gamma(s)||\dot\gamma(t)| \mathrm{d}s\mathrm{d}t, \tag{1.2}$$

where $\mathcal{L}(\gamma)$ denotes the length of γ. These energies were introduced and investigated by O'Hara [8–11]. The Möbius energy (1.1) corresponds to the case $j = 2$, $p = 1$. According to the definition at the beginning, the general (j,p)-energy is a true knot energy if and only if $jp \geq 2$, see [9, Theorems 1.9, 2.3].

[b]In fact, O'Hara's first version of a knot energy equals $\frac{1}{2}E - 2$.

[c]Later, using the machinery of pseudo-differential operators, He sketched a proof of C^∞-smoothness of the E-minimizers [5, Corollary 5.3], thus resolving completely the regularity theory for minimizers of this particular knot energy.

[d]In [2, Remark to Corollary 1.3] the authors conjectured the existence of such a curve.

In 2003, Abrams, Cantarella, Fu, Ghomi and Howard showed that circles are the unique minimizers of all (j,p)-energies among closed curves if $p \geq 1$ and $(j-2)p < 1$ [1, Corollary 3]. Their proof of minimality also works for $(j-2)p \geq 1$, but in this case circles are only "*weak* minimizers", since their energy (and hence the energy of all closed curves) is infinite in this case, which is a consequence of Lemma 2.1(ii). In the $jp < 2$ section, we will see that circles are no longer minimizers if $p < 1$ and $(j-2)p \geq 1$.

Although knot energies are usually defined only in the context of closed curves, the corresponding functionals are obviously also well-defined for open curves, so we always present statements in terms of open and closed curves.

In the present paper, we prove the following

Theorem 1.1 (Differentiability of finite-energy curves). *Let* $j, p \in (0, \infty)$.

(i) *If* $jp \leq 2$ *and* $(j-2)p < 1$, *there are finite-energy curves that are not differentiable. Furthermore, these curves can be chosen of "infimal" energy, i.e. with energy arbitrarily close to the energy of a circle or a line respectively.*
(ii) *If* $jp > 2$, *all finite-energy curves are of class* $C^{1,\alpha/2}$, *where* $\alpha = (jp-2)/(p+2)$.

With this result we obtain a complete picture of the regularizing effects of O'Hara's (j,p)-energies for $(j,p) \in (0,\infty) \times (0,\infty)$; see Fig. 1. The graphs of the three functions $jp = 2$, $(j-2)p = 1$ and $p = 1$ partition the parameter space

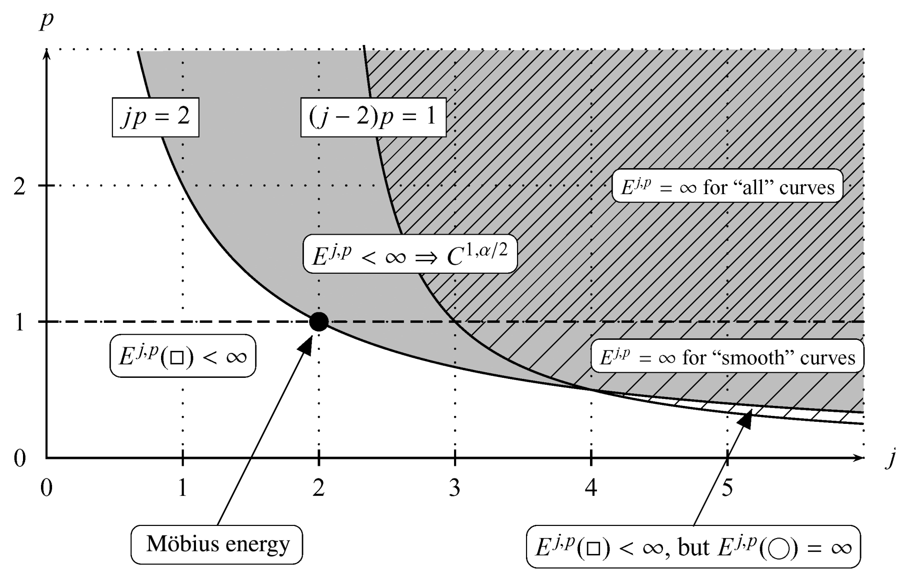


Fig. 1. Range of (j,p).
$\alpha = (jp-2)/(p+2)$.

$(j,p) \in (0,\infty) \times (0,\infty)$ into several regions of completely different behavior: In the white region, the (j,p)-energy has no regularizing effect, see Part (i) of Theorem 1.1. In the gray region, finite energy does lead to Hölder continuous first derivatives (Theorem 1.1, Part (ii)), although in the hatched region above the line $p = 1$, one cannot hope to find *any* closed curve with finite energy.[e] (In fact, we expect the same behavior also in the gray hatched region below $p = 1$; at least sufficiently smooth curves except lines can be shown to have infinite energy there too, cf. Lemma 2.1(ii).[f]) In addition, we have indicated in Fig. 1 the bizarre effect that squares have always finite energy in the white region, whereas the seemingly ideal shape of the circle leads to infinite energy in the white region between the graphs $(j-2)p = 1$ and $jp = 2$.

We do not know whether the Hölder exponent $\frac{1}{2}\alpha$ is sharp, but we cannot expect C^2-regularity, for there is a finite energy curve[g] in the range $\{(j-2)p < 1\}$ which belongs to $C^{1,1}\backslash C^2$.

The paper is structured as follows. In Sec. 2, we briefly discuss the range of high singularity $(j-2)p \geq 1$ using techniques that will also appear later on. By means of [1, Corollary 3], any closed curve has infinite (j,p)-energy if in addition $p \geq 1$. Furthermore, if an open or closed curve belongs to $C^{3,1}$ on some open subdomain with non-vanishing curvature, its energy is infinite. Note also that by Part (ii) of Theorem 1.1, the energy of polygons (except lines[h]) is infinite in the case $jp > 2$.

In Sec. 3, we observe for $jp < 2$ curves of finite energy that are not differentiable, e.g. squares. We obtain the same fact for the boundary case $j = 0$, $p = \infty$ which corresponds to Gromov's distortion [9, Example 1.3]. Our result from Sec. 2 is used to understand the bizarre effect in $\{jp < 2, (j-2)p \geq 1\}$ mentioned above.

Section 4 is devoted to the situation $jp = 2$, where the involved construction of a non-differentiable curve of finite energy is carried out. This curve possesses a local bi-Lipschitz constant arbitrarily close to 1, cf. Corollary 4.3. We briefly explain the idea of our proof before going into detail. After constructing an open curve that is not differentiable at one inner point and additionally has an arbitrarily small energy, we "glue" it on a cylinder obtaining a closed curve that is not differentiable at one inner point and whose energy is arbitrarily close to that of a circle, which is known [1, Corollary 3] to be the unique minimizer for $j \leq 2$. This technique of deriving an energy bound for the "closure" of an open curve applies to a wide range of curves.

[e] This observation also shows that the assumption $(j-2)p < 1$ in Part (i) of Theorem 1.1 is in the case of closed curves not a restriction at all.

[f] To be more precise, on $\{(j-2)p \geq 1, jp > 2\}$ we know that all curves which are at least $C^{3,1}$ (except lines, since their curvature vanishes completely) and, on the other hand, all curves which fail to be $C^{1,\alpha/2}$ have infinite energy, see Lemma 2.1(ii) and Part (ii) of Theorem 1.1.

[g] By modifying the computation in Lemma 4.5, one can see that the finite "hockey stick" $\bar{\zeta}_\alpha|_{[-r,\alpha]}$, $0 < \alpha \ll 1$, $r > 0$, is the desired curve.

[h] The energy of a line amounts to 0 for all $j, p > 0$.

In Sec. 5, we deal with $jp > 2$. In this case, any finite energy curve belongs[i] to $C^{1,\alpha/2}$, where $\alpha = (jp-2)/(p+2)$. There are two key ingredients for this proof. The first one is a kind of "quantified" bi-Lipschitz constant that was derived in [10, Proposition 1.6]. The second, Lemma 5.1, was originally stated by Semmes and will allow us to develop a technical tool in Lemma 5.2, which proves the regularity of curves fulfilling the quantified bi-Lipschitz estimate.

2. $(j-2)p \geq 1$

Lemma 2.1 (Range of high singularity). *Let $\gamma : X \to \mathbb{R}^3$ be an open or closed curve in arc-length parametrization, where X denotes an interval of $\mathbb{R}$ or a circle $\mathbb{R}/(\ell\mathbb{Z})$, and $j, p > 0$ with $(j-2)p \geq 1$. Then the conditions*

(i) *$p \geq 1$ and $X = \mathbb{R}/(\ell\mathbb{Z})$ or*
(ii) *$\gamma \in C^{3,1}$ on an arbitrary open subdomain of X, where additionally $\ddot{\gamma} \neq 0$,*

both imply $E^{j,p}(\gamma) = \infty$.

Proof. (i) According to [1, Corollary 3], circles are *strict* minimizers for $p \geq 1$, $(j-2)p < 1$ among all closed curves of length 2π in arc-length parametrization. From the proof of this result, one can derive that this statement also holds for *all* $p \geq 1$, while the circle is only a "*weak* minimizer" for $(j-2)p \geq 1$, since its energy (and hence the energy of all closed curves) is infinite in this case, which is a consequence from Part (ii). So this gives a rigorous proof of the fact that all closed curves in arc-length parametrization have infinite energy for $(j-2)p \geq 1$.

(ii) We start with the Taylor expansion of γ; for $s, t \in \mathbb{R}$, we obtain

$$\gamma(s) - \gamma(t) = \dot{\gamma}(t)(s-t) + \frac{1}{2}\ddot{\gamma}(t)(s-t)^2 + \frac{1}{6}\dddot{\gamma}(t)(s-t)^3 + R_4(s-t),$$

where $R_4(s-t) \leq \frac{1}{4!}\|\gamma^{(4)}\|_{L^\infty(R/(\ell\mathbb{Z}),\mathbb{R}^3)}|s-t|^4$. The arc-length parametrization implies $|\dot{\gamma}| \equiv 1$, $\langle\dot{\gamma},\ddot{\gamma}\rangle \equiv 0$, $\langle\dot{\gamma},\dddot{\gamma}\rangle \equiv -|\ddot{\gamma}|^2$, so

$$\frac{|\gamma(s)-\gamma(t)|^2}{|s-t|^2} \leq 1 - \frac{1}{12}|\ddot{\gamma}(t)|^2(s-t)^2 + C_1|s-t|^3 \tag{2.1}$$

for some $C_1 < \infty$ depending on $\|\gamma^{(i)}\|_{L^\infty}$, $i = 1,2,3,4$ and ℓ. Since $x \mapsto x^{-j/2}$ is convex on $(0,\infty)$, we obtain $y^{-j/2} \geq 1 - \frac{j}{2}(y-1)$ for all $y \in (0,\infty)$. This leads to

$$\begin{aligned}\frac{1}{|\gamma(s)-\gamma(t)|^j} - \frac{1}{|s-t|^j} &= \frac{1}{|s-t|^j}\left[\left(\frac{|\gamma(s)-\gamma(t)|^2}{|s-t|^2}\right)^{-j/2} - 1\right] \\ &\geq \frac{j}{2}\left(\frac{1}{12}|\ddot{\gamma}(t)|^2 - C_1|s-t|\right)|s-t|^{2-j}.\end{aligned}$$

[i] Hence, O'Hara's results for $jp > 2$ [10] automatically extend to the larger class of regular absolutely continuous curves.

Without loss of generality we may assume that there are $c_1, \varepsilon_0 > 0$ such that $|\ddot\gamma(t)| \geq c_1$ for all $t \in [0, \varepsilon_0]$. Now, $\varepsilon := \min\left(\varepsilon_0, \frac{c_1^2}{24C_1}, \frac{1}{2}\ell\right)$ yields

$$E^{j,p}(\gamma) \geq \int_0^{\varepsilon}\int_t^{\varepsilon+t}\left(\frac{1}{|\gamma(s)-\gamma(t)|^j} - \frac{1}{|s-t|^j}\right)^p \mathrm{d}s\,\mathrm{d}t \geq \left(\frac{j}{48}c_1^2\right)^p \varepsilon\int_0^{\varepsilon} u^{(2-j)p}\mathrm{d}u,$$

which is infinite if $(j-2)p \geq 1$. □

3. $jp < 2$

The (j,p)-energy of the unit square $Q : [0,4] \to \mathbb{R}^2$, $t \mapsto (t,0)$ for $t \in [0,1]$, $t \mapsto (1, t-1)$ for $t \in [1,2]$, $t \mapsto (3-t, 1)$ for $t \in [2,3]$, $t \mapsto (0, 4-t)$ for $t \in [3,4]$, is finite. The only interesting point is the interaction of neighboring segments, which leads to

$$\iint_{[0,1]\times[1,2]}\left(\frac{1}{|Q(s)-Q(t)|^j} - \frac{1}{|s-t|^j}\right)^p \mathrm{d}s\,\mathrm{d}t$$

$$\leq \iint_{[0,1]\times[1,2]}\frac{\mathrm{d}s\,\mathrm{d}t}{|Q(s)-Q(t)|^{jp}} = \iint_{[0,1]^2}\frac{\mathrm{d}s\,\mathrm{d}t}{(s^2+t^2)^{jp/2}}$$

$$\leq \int_0^{\pi/2}\int_0^{\sqrt{2}}\frac{r\,\mathrm{d}r\,\mathrm{d}\varphi}{(r^2)^{jp/2}} = \frac{\frac{\pi}{2}}{2-jp}\cdot 2^{(2-jp)/2}.$$

So there is a curve of finite (j,p)-energy which is not differentiable.

The same calculation yields that $E^{j,p}$ is not self-repulsive, so this case does not model a knot energy, cf. [9, Theorem 1.9].

To obtain an open curve with arbitrary small energy, take two lines and join them by a sufficiently large angle, i. e. $\pi - \varepsilon$. For closed curves, one may replace a small piece of a circle by an angle and adapt the arguments given in the proof of Proposition 4.12. If $j \in (0,4)$, $p \in \left[1, \frac{2}{j}\right)$ the (j,p)-energy of the curves constructed in the next section are bounded by means of Hölder's inequality in terms of their $\left(j, \frac{2}{j}\right)$-energy which can be chosen arbitrarily small.

For the boundary case $j = 0$, $p = \infty$ which corresponds to Gromov's distortion [9, Example 1.3], we furthermore obtain

$$\operatorname{distort}(Q) = \sup_{0\leq s<t\leq 4}\frac{D_Q(s,t)}{|Q(s)-Q(t)|} = 2,$$

so that Q is also an example for a non-differentiable curve whose distortion is finite.

In the set $\{(j,p) \in \mathbb{R}^2 | j > 4, \frac{1}{j-2} < p < \frac{2}{j}\}$, we obtain the bizarre situation that the energy of a square is finite while the energy of a circle is infinite according to Lemma 2.1(ii), so we cannot expect that [1, Eq. (3)] also holds for all $p < 1$, i.e. that circles are still minimizers of the (j,p)-functional.

4. $jp = 2$

Let $E^{[j]} := E^{j,2/j}$. Note that this notation is (for $j \neq 2$) different from $E^{(j)} := E^{j,1}$ that is used by O'Hara in [11].

Theorem 4.1 (Finite $(j, \frac{2}{j})$-energy does not imply differentiability). *For any $\varepsilon > 0$ and $j \in (0,4)$, there is an open curve $\mathbb{R} \to \mathbb{R}^2$ of $(j, \frac{2}{j})$-energy $\leq \varepsilon$ parametrized by arc-length that is not differentiable at 0 and coincides outside a neighborhood of $0_{\mathbb{R}^2}$ with the x_1-axis.*

Corollary 4.2 (Version for closed curves). *Let $\zeta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ denote the circle of length 1. For any $\varepsilon > 0$ and $j \in (0,4)$, there is a closed curve $\mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ in arc-length parametrization whose $(j, \frac{2}{j})$-energy lies in the ε-neighborhood of $E^{j,2/j}(\zeta)$, but which is not differentiable at 0.*

Using the fact that finite Möbius energy curves possess a local bi-Lipschitz property [2, Corollary 1.3], we immediately deduce

Corollary 4.3 (Arbitrarily "small" *local* bi-Lipschitz constant does not imply differentiability). *There is an open curve $\mathbb{R} \to \mathbb{R}^2$ in arc-length parametrization that is not differentiable at 0, but satisfies the following condition:*

> *For any $\varepsilon_0 > 0$, there is a $\delta > 0$ such that for any $y \in \mathbb{R}$ the restriction to $[y - \delta, y + \delta]$ is bi-Lipschitz continuous with constant $1 + \varepsilon_0$.*

There is also a closed curve $\mathbb{S}^1 \to \mathbb{R}^2$ with a similar property.

Before presenting the rigorous proof in several steps, we give a brief

Outline of the proof. The main idea is to construct a basic component κ_α, that lies on the x-axis outside a neighborhood of the origin.

In a smaller neighborhood of the origin κ_α lies on a line segment that meets the x-axis at the origin in an angle of α as drawn in Fig. 2. The energy of κ_α amounts to $O(\alpha^{4/j})$ by Proposition 4.6.

Now let $\alpha^* < \alpha$ and replace (a part of) the line segment by a copy of κ_{α^*} scaled down and restricted to a suitable neighborhood of the origin, so it fits into the gap. See Fig. 3.

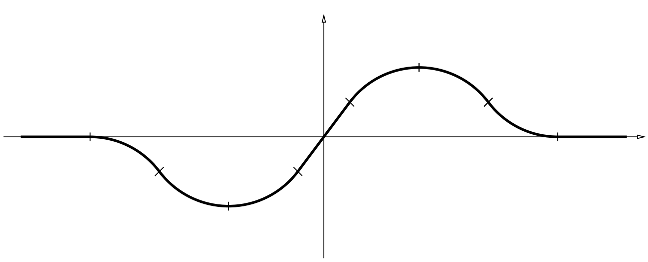


Fig. 2.

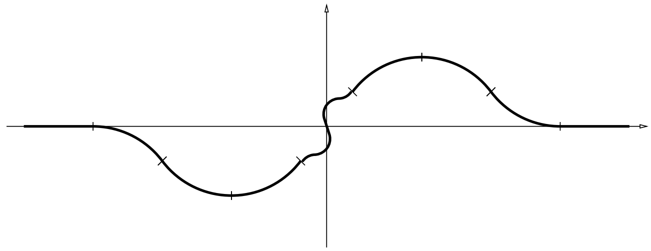


Fig. 3.

Because of the scaling invariance, we can choose this copy arbitrarily small, i.e. we change the original curve κ_α only on a very small subdomain at the origin.

The composite curve meets the origin in an angle of $\alpha + \alpha^*$. It turns out that its energy can be estimated in terms of $E^{[j]}(\kappa_\alpha)$, $E^{[j]}(\kappa_{\alpha^*})$ and some other terms that depend on quantities which can be controlled by the scaling parameter of κ_{α^*}, cf. Proposition 4.7. So we obtain essentially $O(\alpha^{4/j}) + O((\alpha^*)^{4/j})$ as energy of the composite curve.

By repeating this process inductively for $\alpha_k := \frac{1}{k}$, we obtain a limit curve that performs a rotation of $\sum_{k\in\mathbb{N}} k^{-1} = \infty$ near 0 but has energy $\sum_{k\in\mathbb{N}} k^{-4/j} < \infty$. $\square$

For the rigorous proof of Theorem 4.1, we will collect a few tools at first which will be used later on, starting with the following fact.

Lemma 4.4. *Let $j > 0$. There is a constant C_2 depending only on $j > 0$ such that for $\lambda, \mu > 0$*

$$\int_{-\infty}^{-\lambda}\int_{\lambda}^{\infty}\left(\frac{1}{(s-t)^j} - \frac{1}{(s-t+\mu)^j}\right)^{2/j} \mathrm{d}s\,\mathrm{d}t \le C_2\left(\frac{\mu}{2\lambda}\right)^{2/j} \tag{4.1}$$

holds.

Proof. Let $q(j,x) := (1-jx)(1+x)^j$. Since we have

$$\frac{d}{dj}q(j,x) = (1+x)^j\left[-x + (1-jx)\log(1+x)\right] < 0 \quad \text{for all } j, x > 0$$

and $j \mapsto q(j,x)$ is continuous on $[0,\infty)$, we arrive at $q(j,x) \le q(0,x) = 1$, i.e.

$$(1-jx)(1+x)^j \le 1 \quad \text{for all } j, x > 0. \tag{4.2}$$

For $s, t > 0$ and $x = \frac{\mu}{s+t}$, we obtain

$$1 - j\frac{\mu}{s+t} \le \frac{1}{\left(1+\frac{\mu}{s+t}\right)^j} = \left(\frac{s+t}{s+t+\mu}\right)^j$$

$$\Longrightarrow\ 1 - \left(\frac{s+t}{s+t+\mu}\right)^j \le j\frac{\mu}{s+t}. \tag{4.3}$$

This yields

$$\begin{aligned}
&\iint_{[\lambda,\infty)^2} \left(\frac{1}{(s+t)^j} - \frac{1}{(s+t+\mu)^j}\right)^{2/j} \mathrm{d}s\,\mathrm{d}t \\
&\quad = \iint_{[\lambda,\infty)^2} \frac{1}{(s+t)^2} \left(1 - \left(\frac{s+t}{s+t+\mu}\right)^j\right)^{2/j} \mathrm{d}s\,\mathrm{d}t \\
&\quad \overset{(4.3)}{\leq} (j\mu)^{2/j} \iint_{[\lambda,\infty)^2} (s+t)^{-2-2/j} \mathrm{d}s\,\mathrm{d}t \\
&\quad = \frac{1}{\frac{2}{j}} \cdot \frac{1}{1+\frac{2}{j}} \cdot j^{2/j} \left(\frac{\mu}{2\lambda}\right)^{2/j}.
\end{aligned}$$

□

By Taylor approximation, there is a $c_3 > 0$, such that, for any $\alpha \in (0, c_3]$,

$$\begin{aligned}
&0 < \alpha - \sin\alpha \leq \frac{1}{3}\alpha^3, && \frac{1}{2}\alpha \leq \sin\alpha \leq \alpha, \\
&0 < 1 - \cos\alpha \leq \alpha^2, && \alpha \leq \tan\alpha \leq 2\alpha. \\
&0 < \tan\alpha - \sin\alpha \leq \alpha^3, &&
\end{aligned} \tag{4.4}$$

Let $\alpha \in (0, c_3]$ and define

$$\begin{aligned}
x(\alpha) &:= \frac{1-\cos\alpha}{\tan\alpha} < y(\alpha) := \frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2} \leq \alpha, \\
\xi_\alpha &:= 3\sin\alpha + x(\alpha) < \eta_\alpha := 3\alpha + y(\alpha) \leq 4\alpha.
\end{aligned} \tag{4.5}$$

This yields by (4.4)

$$\begin{aligned}
0 &\leq y(\alpha) - x(\alpha) = (1-\cos\alpha)\frac{\tan\alpha - \sin\alpha}{\sin\alpha\tan\alpha} \leq 2\alpha^3, \\
0 &\leq \eta_\alpha - \xi_\alpha \leq 3\alpha^3.
\end{aligned} \tag{4.6}$$

Now we are going to construct the components we will later insert into one another inductively.

The basic component $\kappa_a : \mathbb{R} \to \mathbb{R}^2$ is defined as drawn in Fig. 4 and characterized by the following explicit formula in arc-length parametrization.

$$t \mapsto \begin{cases}
\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix} t & \text{for } t \in [0, y(\alpha)], \\
\begin{pmatrix}\sin(t-\alpha-y(\alpha)) \\ \cos(t-\alpha-y(\alpha))\end{pmatrix} + \begin{pmatrix}x(\alpha)+\sin\alpha \\ 1-2\cos\alpha\end{pmatrix} & \text{for } t \in [y(\alpha), y(\alpha)+2\alpha], \\
\begin{pmatrix}\sin(t-\eta_\alpha) \\ -\cos(t-\eta_\alpha)\end{pmatrix} + \begin{pmatrix}\xi_\alpha \\ 1\end{pmatrix} & \text{for } t \in [y(\alpha)+2\alpha, \eta_\alpha], \\
\begin{pmatrix}t \\ 0\end{pmatrix} - \begin{pmatrix}\eta_\alpha - \xi_\alpha \\ 0\end{pmatrix} & \text{for } t \in [\eta_\alpha, \infty), \\
-\kappa_\alpha(-t) & \text{for } t \in (-\infty, 0).
\end{cases}$$

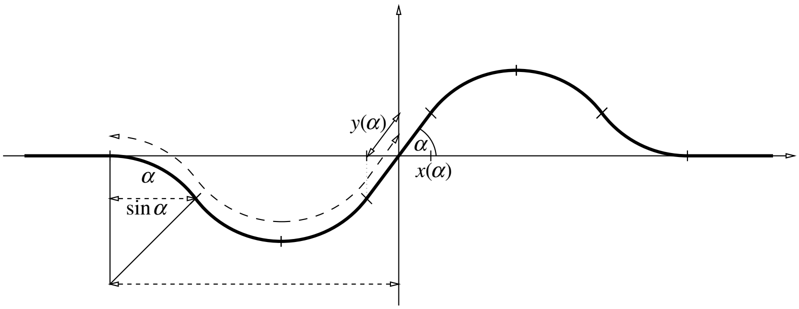


Fig. 4.

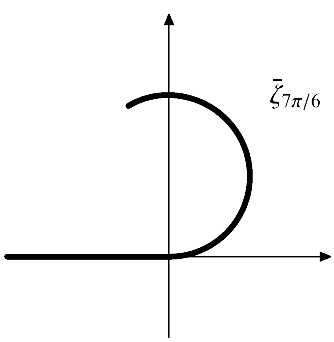
To compute $E^{[j]}(\kappa_\alpha)$, we need an estimate for the energy of an arc joined with a half line. This curve $\bar{\zeta}_\alpha$ was first introduced by He in [4, Example 1.1.2 and p. 15]; his calculations lead to $E^{[2]}(\bar{\zeta}_\alpha) = 2 - \alpha \cot \frac{\alpha}{2} = O(\alpha^2)$.

Lemma 4.5 (Energy of a "hockey stick"). *There are constants* $c_4 \in (0, 2\pi)$, $C_4 < \infty$, *depending only on* $j > 0$, *such that for* $\alpha \in (0, c_4]$ *the curve* $\bar{\zeta}_\alpha : (-\infty, \alpha] \to \mathbb{R}^2$ *given by*

$$t \mapsto \begin{cases} \begin{pmatrix} t \\ 0 \end{pmatrix} & \text{for } t \leq 0, \\ \begin{pmatrix} \sin t \\ 1 - \cos t \end{pmatrix} & \text{for } t \in [0, \alpha] \end{cases}$$

satisfies

$$E^{[j]}(\bar{\zeta}_\alpha) \leq C_4 \alpha^{4/j}.$$



Proof. By modifying the arguments from Lemma 2.1(ii) (estimate towards the other direction) and applying them to the circle segment, we will show that there are constants $c_5 > 0, C_5 < \infty$ such that $E^{[j]}(\bar{\zeta}_\alpha|_{[0,\alpha]}) \leq C_5 \alpha^{4/j}$ for any $\alpha \in (0, c_5]$. The arc $\zeta := \bar{\zeta}_\alpha|_{[0,\alpha]}$ obviously fulfills the assumptions of Lemma 2.1(ii), so we arrive at

$$\frac{|\zeta(s)-\zeta(t)|^2}{|s-t|^2} \geq 1 - \frac{1}{12}|\ddot{\zeta}(t)|^2(s-t)^2 - C_1|s-t|^3 \tag{2.1$'$}$$

for $C_1 < \infty$ depending only on α, since $\|\zeta^{(i)}\|_{L^\infty} = 1$ for all $i \in \mathbb{N}$.

Since $(1+x)^{-j/2} = 1 - \frac{j}{2}\int_0^x (1+\xi)^{-j/2-1}\mathrm{d}\xi$, we obtain

$$y^{-j/2} \leq 1 - \frac{j}{2}y^{-j/2-1}(y-1) \leq 1 + \frac{j}{4}(1-y) \quad \text{for all } y \in [2^{-1/(j/2+1)}, 1].$$

This leads to

$$\begin{aligned}\frac{1}{|\zeta(s)-\zeta(t)|^j} - \frac{1}{|s-t|^j} &= \frac{1}{|s-t|^j}\left[\left(\frac{|\zeta(s)-\zeta(t)|^2}{|s-t|^2}\right)^{-j/2} - 1\right] \\ &\leq \frac{j}{4}\left(\frac{1}{12}|\ddot{\zeta}(t)|^2 + C_1|s-t|\right)|s-t|^{2-j} \\ &\leq \frac{j}{4}\left(\frac{1}{12} + C_1\alpha\right)|s-t|^{2-j}\end{aligned}$$

provided $\frac{|\zeta(s)-\zeta(t)|^2}{|s-t|^2} \geq 2^{-1/(j/2+1)}$. But since ζ has a (uniform) local bi-Lipschitz constant arbitrarily close to 1, there is a $c_5 > 0$ such that this requirement holds for all $|s-t| \leq c_5$. Now

$$\begin{aligned}E^{j,p}(\zeta) &\leq \int_0^\alpha\int_0^\alpha \left(\frac{1}{|\zeta(s)-\zeta(t)|^j} - \frac{1}{|s-t|^j}\right)^p \mathrm{d}s\,\mathrm{d}t \\ &\leq \left[\frac{j}{4}\left(\frac{1}{12} + C_1\alpha\right)\right]^p \cdot 2\int_0^\alpha\int_0^{\alpha-t} u^{(2-j)p}\mathrm{d}u \\ &\leq \frac{\left[\frac{j}{4}\left(\frac{1}{12} + C_1\alpha\right)\right]^p}{[(2-j)p+1]\,[(2-j)p+2]} \cdot \alpha^{(2-j)p+2} \overset{jp=2}{=:} C_5\alpha^{4/j}.\end{aligned}$$

Since

$$|\bar{\zeta}_\alpha(s) - \bar{\zeta}_\alpha(t)| \leq |\bar{\zeta}_{2\alpha}(s+\alpha) - \bar{\zeta}_{2\alpha}(t+\alpha)| \quad \text{for } s,\, t \in [-\alpha, \alpha]\,,$$

we even obtain $E^{[j]}(\bar{\zeta}_\alpha|_{[-\alpha,\alpha]}) \leq C_5(2\alpha)^{4/j}$ for any $\alpha \leq \frac{1}{2}c_5$. Certainly, $E^{[j]}(\bar{\zeta}_\alpha|_{(-\infty,0]}) = 0$. It remains to study the interaction of the intervals $(-\infty, -\alpha]$ and $[0,\alpha]$. Let $s \in [0,\alpha]$, $t \in [\alpha,\infty)$, $\alpha \leq c_4 := \min\left(c_3, \frac{1}{2}c_5, 1, (2j)^{-1/2}\right)$, which implies

$$(1+\tau)^j \overset{(5)}{\leq} \frac{1}{1-j\tau} \leq 1 + 2j\tau \quad \text{for any } \tau \in [0, c_4^2]. \tag{4.7}$$

Since

$$\left(\frac{(s+t)^2}{(\sin s + t)^2 + (1-\cos s)^2}\right)^{1/2} \overset{(4.4)}{\leq} \frac{s+t}{s-s^3+t} \leq 1 + \frac{s^3}{t}$$

and $0 \le \frac{s^3}{t} \le \alpha^2 \le c_4^2$, we arrive at

$$\begin{aligned}
&\int_{-\infty}^{-\alpha}\int_0^{\alpha}\left(\frac{1}{|\bar\zeta_\alpha(s)-\bar\zeta_\alpha(t)|^j}-\frac{1}{(s-t)^j}\right)^{2/j}\mathrm{d}s\,\mathrm{d}t\\
&\quad\le \int_\alpha^\infty\int_0^\alpha \frac{1}{(s+t)^2}\left[\left(\frac{(s+t)^2}{(\sin s+t)^2+(1-\cos s)^2}\right)^{j/2}-1\right]^{2/j}\mathrm{d}s\,\mathrm{d}t\\
&\quad\overset{(4.6)}{\le} (2j)^{2/j}\int_\alpha^\infty\int_0^\alpha \frac{1}{(s+t)^2}\left(\frac{s^3}{t}\right)^{2/j}\mathrm{d}s\,\mathrm{d}t\\
&\quad\le (2j)^{2/j}\int_\alpha^\infty\int_0^\alpha \frac{s^{6/j}}{t^{2/j+2}}\mathrm{d}s\,\mathrm{d}t = \frac{(2j)^{2/j}}{\left(\frac{6}{j}+1\right)\left(\frac{2}{j}+1\right)}\cdot\alpha^{4/j}.
\end{aligned}$$

Now $E^{[j]}(\bar\zeta_\alpha) \le 2^{4/j}C_5\alpha^{4/j} + 0 + 2\cdot\frac{(2j)^{2/j}}{(6/j+1)(2/j+1)}\cdot\alpha^{4/j} =: C_4\alpha^{4/j}$. □

Proposition 4.6 (Energy of a segment). *There are constants $c_0 \in (0, c_4]$, $C_0 < \infty$ depending only on $j > 0$ such that*

$$E^{[j]}(\kappa_\alpha) \le C_0\alpha^{4/j} \quad \textit{for any } \alpha \in (0, c_0].$$

Proof. For $\alpha \in (0, c_4]$ consider the curve $\bar\zeta_\alpha$ defined in Lemma 4.5 in arc-length parametrization. Because of

$$|\kappa_\alpha(s) - \kappa_\alpha(t)| \ge |\bar\zeta_{2\eta_\alpha}(s+\eta_\alpha) - \bar\zeta_{2\eta_\alpha}(t+\eta_\alpha)|$$

for $s, t \in (-\infty, \eta_\alpha]$, we obtain

$$E^{[j]}(\kappa_\alpha|_{(-\infty,\eta_\alpha]}) \le E^{[j]}(\bar\zeta_{2\eta_\alpha}) \le C_4(2\eta_\alpha)^{4/j} \overset{(4.5)}{\le} 8^{4/j}C_4\alpha^{4/j} \quad \text{for } \alpha \le c_0 := \frac{1}{8}c_4.$$

By symmetry we obtain the same estimate for $E^{[j]}(\kappa_\alpha|_{[-\eta_\alpha,\infty)})$.

It remains to treat the case

$$\int_{-\infty}^{-\eta_\alpha}\int_{\eta_\alpha}^{\infty}\left(\frac{1}{|\kappa_\alpha(s)-\kappa_\alpha(t)|^j}-\frac{1}{(s-t)^j}\right)^{2/j}\mathrm{d}s\,\mathrm{d}t \quad \left(\text{respectively } \int_{\eta_\alpha}^{\infty}\int_{-\infty}^{-\eta_\alpha}\cdots\right).$$

Because of $|\kappa_\alpha(s) - \kappa_\alpha(t)| = (s-t) - 2(\eta_\alpha - \xi_\alpha)$ for $s, (-t) \in [\eta_\alpha, \infty)$, the first integral equals

$$\begin{aligned}
&\int_{-\infty}^{-\xi_\alpha}\int_{\xi_\alpha}^{\infty}\left(\frac{1}{(s-t)^j}-\frac{1}{(s-t+2(\eta_\alpha-\xi_\alpha))^j}\right)^{2/j}\mathrm{d}s\,\mathrm{d}t\\
&\quad\overset{(4.1)}{\le} C_2\left(\frac{\eta_\alpha-\xi_\alpha}{\xi_\alpha}\right)^{2/j} \overset{(4.4)-(4.6)}{\le} 2^{2/j}C_2\alpha^{4/j}.
\end{aligned}$$

By symmetry we arrive at the same estimate for the second integral. Summing up we conclude $E(\kappa_\alpha) \le 2\cdot 8^{4/j}C_4\alpha^{4/j} + 2\cdot 2^{2/j}C_2\alpha^{4/j} =: C_0\alpha^{4/j}$. □

Proposition 4.7 (Joining curves). *Let $j \ge 2, L \ge \Lambda \ge \frac{1}{2}\Lambda \ge \ell \ge \lambda > 0$, $\beta \in [0, 2\pi)$.*

Let $\gamma : [-L-\Lambda, L+\Lambda] \to \mathbb{R}^2$ *be an open curve in arc-length parametrization satisfying*

$$\gamma(t) = \begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix} t \quad \textit{for all } |t| \le \Lambda \quad \textit{and}$$
$$|\gamma(t)| \ge \Lambda \qquad \textit{for all } |t| \ge \Lambda.$$

Let $\delta : [-(\Lambda-\lambda)-\ell, (\Lambda-\lambda)+\ell] \to \mathbb{R}^2$ *be an open curve in arc-length parametrization satisfying*

$$\delta(t) = \begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}(t+(\ell-\lambda)) \quad \textit{for all } t \le -\ell,$$
$$|\delta(t)| \le \lambda \qquad \textit{for all } |t| \le \ell, \quad \textit{and}$$
$$\delta(t) = \begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix}(t-(\ell-\lambda)) \quad \textit{for all } t \ge \ell.$$

Then the energy of the open arc-length parametrized curve

$$\gamma \curvearrowleft \delta : [-L-(\Lambda-\lambda)-\ell, L+(\Lambda-\lambda)+\ell] \to \mathbb{R}^2$$

$$t \mapsto \begin{cases} \gamma(t+\ell-\lambda) & \textit{for } t \in [-L-(\Lambda-\lambda)-\ell, -\ell],\\ \delta(t) & \textit{for } t \in [-\ell, \ell],\\ \gamma(t-\ell+\lambda) & \textit{for } t \in [\ell, L+(\Lambda-\lambda)+\ell], \end{cases}$$

fulfills the estimate $E^{[j]}(\gamma \curvearrowleft \delta) \le E^{[j]}(\gamma) + E^{[j]}(\delta) + 2C_2\left(\frac{\ell-\lambda}{\lambda}\right)^{2/j} + 32\frac{L\ell}{\Lambda^2}$.

In the proof we will need the fact that, for $a \ge b \ge 0$, $p \in [0,1]$,

$$a^p - b^p \le (a-b)^p \tag{4.8}$$

holds, which can easily be verified by showing that $\sigma^p + (1-\sigma)^p$ is monotone increasing in σ on $[0, \frac{1}{2}]$. The fact that there is no similar estimate for $p > 1$ is the reason for the restriction to $j \ge 2$.

Proof. Let

$$\begin{aligned} \mathcal{A} &:= [-L-(\Lambda-\lambda)-\ell, -(\Lambda-\lambda)-\ell],\\ \mathcal{B} &:= [-(\Lambda-\lambda)-\ell, -\ell],\\ \mathcal{C} &:= [-\ell, \ell],\\ \mathcal{D} &:= [\ell, (\Lambda-\lambda)+\ell],\\ \mathcal{E} &:= [(\Lambda-\lambda)+\ell, L+(\Lambda-\lambda)+\ell]. \end{aligned}$$

We divide the integration domain into 25 sub-domains. Because of symmetry we may restrict ourselves to 15 of them (see Table 1), but we have to keep in mind to multiply by 2 off the diagonal. The term $E^{[j]}(\gamma)$ absorbs cases 1, 2, 6, 13, 14, 15, $E^{[j]}(\delta)$ treats cases 7, 8, 10, 11; in both cases the symmetric domains are already considered. See Fig. 5.

Table 1.

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$
$\mathcal{A}$	1	2	3	4	5
$\mathcal{B}$		6	7	8	9
$\mathcal{C}$			10	11	12
$\mathcal{D}$				13	14
$\mathcal{E}$					15

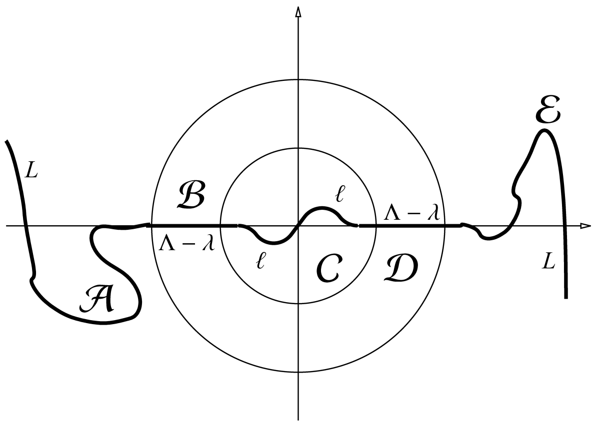


Fig. 5.

Now we consider the cases 4, 5, 8, 9.

$$\iint_{(\mathcal{A}\cup\mathcal{B})\times(\mathcal{D}\cup\mathcal{E})} \left(\frac{1}{|(\gamma \curvearrowleft \delta)(s) - (\gamma \curvearrowleft \delta)(t)|^j} - \frac{1}{(s-t)^j} \right)^{2/j} \mathrm{d}s\,\mathrm{d}t$$

$$= \int_{-L-\Lambda+\lambda-\ell}^{-\ell} \int_{\ell}^{L+\Lambda-\lambda+\ell} \left(\frac{1}{|\gamma(s-\ell+\lambda) - \gamma(t+\ell-\lambda)|^j} - \frac{1}{(s-t)^j} \right)^{2/j} \mathrm{d}s\,\mathrm{d}t$$

$$= \int_{-L-\Lambda}^{-\lambda} \int_{\lambda}^{L+\Lambda} \left(\frac{1}{|\gamma(s) - \gamma(t)|^j} - \frac{1}{(s-t+2(\ell-\lambda))^j} \right)^{2/j} \mathrm{d}s\,\mathrm{d}t\,.$$

The $\int_{-L-\Lambda}^{-\lambda} \int_{\lambda}^{L+\Lambda}$ part of $E^{[j]}(\gamma)$ was not already used in the absorbing process, so referring to (4.8), it suffices to examine

$$\int_{-L-\Lambda}^{-\lambda} \int_{\lambda}^{L+\Lambda} \left(\frac{1}{|\gamma(s) - \gamma(t)|^j} - \frac{1}{(s-t+2(\ell-\lambda))^j} \right)^{2/j} \mathrm{d}s\,\mathrm{d}t$$

$$\qquad - \int_{-L-\Lambda}^{-\lambda} \int_{\lambda}^{L+\Lambda} \left(\frac{1}{|\gamma(s) - \gamma(t)|^j} - \frac{1}{(s-t)^j} \right)^{2/j} \mathrm{d}s\,\mathrm{d}t$$

$$\overset{(4.8)}{\leq} \int_{-L-\Lambda}^{-\lambda} \int_{\lambda}^{L+\Lambda} \left(\frac{1}{(s-t)^j} - \frac{1}{(s-t+2(\ell-\lambda))^j} \right)^{2/j} \mathrm{d}s\,\mathrm{d}t \overset{(4.1)}{\leq} C_2 \left(\frac{\ell-\lambda}{\lambda} \right)^{2/j}.$$

For case 3, let $s \in \mathcal{A}$, $t \in \mathcal{C}$. Now $|(\gamma \frown \delta)(s) - (\gamma \frown \delta)(t)| \geq \Lambda - \lambda$ leads to

$$\int_{\mathcal{A}} \int_{\mathcal{C}} \left(\frac{1}{|(\gamma \frown \delta)(s) - (\gamma \frown \delta)(t)|^j} - \frac{1}{(s-t)^j} \right)^{2/j} \mathrm{d}s\mathrm{d}t \leq \frac{|\mathcal{A}||\mathcal{C}|}{(\Lambda - \lambda)^2} \leq 8\frac{L\ell}{\Lambda^2}.$$

Since we arrive at the same situation in the remaining case 12, we obtain $2 \cdot 8L\ell/\Lambda^2$ together for both cases. Summing up and remembering that we have to multiply by 2 off the diagonal, we are finally led to the formula stated above.

□

Proposition 4.8 (Extending a curve). *Let $j, L^*, \Lambda^*, \ell^*, \lambda^*, \delta^*$ as in Proposition 4.7, and $\beta^* := 0$ and $\gamma^*(t) := \binom{1}{0}t$ for all $t \in [-L^* - \Lambda^*, L^* + \Lambda^*]$. Then there is a constant $C_6 > 0$ depending only on j such that*

$$E^{[j]}(\gamma^* \frown \delta^*) \leq E^{[j]}(\delta^*) + 2C_2 \left(\frac{\ell^* - \lambda^*}{\lambda^*} \right)^{2/j} + 4C_6 \left(\frac{2\ell^*}{\Lambda^* - \lambda^*} \right)^{1+2/j}.$$

Of course, Propositions 4.7 and 4.8 are also true for an analogous situation in $\mathbb{R}^n$.

Proof. We treat cases 1, 2, 4, 5, ..., 11, 13, 14, 15 as in the proof of Proposition 4.7. In case 3 (and analogously in case 12), we obtain for $s \in \mathcal{A}$, $t \in \mathcal{C}$ the estimates

$$|(\gamma^* \frown \delta^*)(s) - (\gamma^* \frown \delta^*)(t)| \geq |s| - \ell^* \quad \text{and} \quad |s - t| \leq |s| + \ell^*.$$

This leads to

$$\int_{\mathcal{A}} \int_{\mathcal{C}} \left(\frac{1}{|(\gamma^* \frown \delta^*)(s) - (\gamma^* \frown \delta^*)(t)|^j} - \frac{1}{(s-t)^j} \right)^{2/j} \mathrm{d}t\,\mathrm{d}s$$

$$\leq |\mathcal{C}| \int_{-L^* - (\Lambda^* - \lambda^*) - \ell^*}^{-(\Lambda^* - \lambda^*) - \ell^*} \left(\frac{1}{(-s - \ell^*)^j} - \frac{1}{(-s + \ell^*)^j} \right)^{2/j} \mathrm{d}s$$

$$\leq 2\ell \int_{\Lambda - \lambda}^{\infty} \left(\frac{1}{s^j} - \frac{1}{(s + 2\ell)^j} \right)^{2/j} \mathrm{d}s = 2\ell \int_{\Lambda - \lambda}^{\infty} \frac{1}{s^2} \left[1 - \left(\frac{s}{s + 2\ell} \right)^j \right]^{2/j} \mathrm{d}s$$

$$\overset{(4.3)}{\leq} (2\ell)^{1+2/j} j^{2/j} \int_{\Lambda - \lambda}^{\infty} s^{-2-2/j} \mathrm{d}s = \frac{j^{2/j}}{1 + \frac{2}{j}} \cdot \left(\frac{2\ell}{\Lambda - \lambda} \right)^{1+2/j}$$

$$=: C_6 \left(\frac{2\ell}{\Lambda - \lambda} \right)^{1+2/j}.$$

Notice that $E^{[j]}(\gamma^*) = 0$. □

Proof of Theorem 4.1. For $j, x, y > 0$, the function $j \mapsto \left(x^{-j} - (x+y)^{-j}\right)^{2/j}$ is monotone increasing since, for $0 < j_1 < j_2$,

$$\left(\frac{1}{x^{j_2}} - \frac{1}{(x+y)^{j_2}}\right)^{\frac{2}{j_2}} = \left(\frac{1}{x^{j_2}} - \frac{1}{(x+y)^{j_2}}\right)^{\frac{2}{j_1}\cdot\frac{j_1}{j_2}} \overset{(4.8)}{\geq} \left(\frac{1}{x^{j_2\cdot\frac{j_1}{j_2}}} - \frac{1}{(x+y)^{j_2\cdot\frac{j_1}{j_2}}}\right)^{\frac{2}{j_1}}.$$

This yields that $E^{[j]}$ is monotone increasing in j, so we may restrict ourselves to $j \in [2, 4)$. Fix $k_0 \in \mathbb{N}$, such that

$$k_0 > \max\left(\frac{1}{c_0}, 1 + \left(\frac{C_7}{\varepsilon}\right)^{c_7}\right) \tag{4.9a}$$

for constants $c_7 > 0$, $C_7 < \infty$ that depend only on j and will be defined later in this proof. Let

$$L_{k_0} := 2\left(1 + \frac{4}{3}\frac{1}{k_0 - 1}\right)\sqrt{k_0 - 1} > \Lambda_{k_0} := 1, \tag{4.9b}$$

and define for $k \in \mathbb{N}$, $k \geq k_0$, the following positive quantities, whose estimates will be proven by induction.

$$\alpha_k := \frac{1}{k}, \tag{4.9c}$$

$$\beta_k := \sum_{\kappa=k_0}^{k-1} \frac{1}{\kappa}, \quad \beta_{k_0} := 0, \tag{4.9d}$$

$$r_k := \frac{\Lambda_k^2 \alpha_k^2}{2L_k \eta_{\alpha_k}} \leq \frac{\alpha_k}{6} \leq \frac{1}{6}, \tag{4.9e}$$

$$\ell_k := r_k \eta_{\alpha_k} \leq \frac{1}{2}\Lambda_k, \tag{4.9f}$$

$$\lambda_k := r_k \xi_{\alpha_k} \leq \ell_k, \tag{4.9g}$$

$$L_{k+1} := L_k + (\Lambda_k - \lambda_k) + \ell_k - \Lambda_{k+1} \geq L_k \geq 1, \tag{4.9h}$$

$$\Lambda_{k+1} := r_k y(\alpha_k) \leq \ell_k \leq 1. \tag{4.9i}$$

The estimate in (4.9g), the first one in (4.9i) and the last one in (4.9e) are immediate.

For the initial step $k = k_0$ the inequality $r_{k_0} < \alpha_{k_0}^2 / (2 \cdot 3\alpha_{k_0})$ verifies (4.9e),

$$\ell_{k_0} = \frac{\Lambda_{k_0}^2 \alpha_{k_0}^2}{2L_{k_0}} \leq \frac{\Lambda_{k_0}}{2} = \frac{1}{2} < 1$$

implies (4.9f) and the last inequality in (4.9i). $L_{k_0+1} - L_{k_0} = (\Lambda_{k_0} - \Lambda_{k_0+1}) + (\ell_{k_0} - \lambda_{k_0}) \geq 0 + 0$ yields (4.9h).

For the step $k \to k+1$, we obtain as above

$$r_{k+1} = \frac{\Lambda_{k+1}^2 \alpha_{k+1}^2}{2L_{k+1}\eta_{\alpha_{k+1}}} \overset{(4.5)}{\leq} \frac{\alpha_{k+1}^2}{6\alpha_{k+1}} \leq \frac{\alpha_{k+1}}{6},$$

$$\ell_{k+1} = \frac{\Lambda_{k+1}^2 \alpha_{k+1}^2}{2L_{k+1}} \leq \frac{\Lambda_{k+1}}{2},$$

$$L_{k+2} - L_{k+1} = \underbrace{\Lambda_{k+1} - \Lambda_{k+2}}_{\geq \Lambda_{k+1} - \ell_{k+1}} + \underbrace{\ell_{k+1} - \lambda_{k+1}}_{\geq 0} \geq \Lambda_{k+1} - \frac{1}{2}\Lambda_{k+1} \geq 0.$$

Notice that $(L_k)_{k\in\mathbb{N}}$ is a monotone increasing sequence; we will see in (4.11) that it converges, whereas $(\beta_k)_{k\in\mathbb{N}}$ is the diverging harmonic sequence. All other sequences are monotone decreasing sequences that converge to 0.

We consider the sequence of functions

$$\delta_k : [-\Lambda_k + \lambda_k - \ell_k, \Lambda_k - \lambda_k + \ell_k] \to \mathbb{R}^2,$$

$$t \mapsto \begin{pmatrix} \cos\beta_k & -\sin\beta_k \\ \sin\beta_k & \cos\beta_k \end{pmatrix} r_k \kappa_{\alpha_k}\left(\frac{1}{r_k}t\right)$$

and start with $\gamma_{k_0} : [-2, 2] \to \mathbb{R}^2$, $t \mapsto \binom{t}{0}$.

If we define $\gamma_{k+1} := \gamma_k \curvearrowleft \delta_k$ for all $k \in \mathbb{N}$, $k \geq k_0$ (see Fig. 6), the conditions of Proposition 4.7 are true in any step, since $\Lambda_{k+1} \leq y(\alpha_k)$ by (4.9e). Now, for $K \geq k_0$,

$$E^{[j]}(\gamma_K) \leq E^{[j]}(\gamma_{k_0}) + \sum_{k=k_0}^{K} E^{[j]}(\delta_k) + 2C_2 \sum_{k=k_0}^{K} \left(\frac{\ell_k - \lambda_k}{\lambda_k}\right)^{2/j} + 32 \sum_{k=k_0}^{K} \frac{L_k \ell_k}{\Lambda_k^2},$$

where $E^{[j]}(\gamma_{k_0}) = 0$, $E^{[j]}(\delta_k) \leq C_0/k^{4/j}$ by Proposition 4.6,

$$\frac{\ell_k - \lambda_k}{\lambda_k} \overset{(4.4)}{\leq} \frac{\eta_{\alpha_k} - \xi_{\alpha_k}}{3 \cdot \frac{1}{2}\alpha_k} \overset{(4.6)}{<} \frac{2}{k^2}, \tag{4.10}$$

$L_k \ell_k / \Lambda_k^2 = 1/(2k^2)$ by (4.9e), and therefore

$$E^{[j]}(\gamma_K) < (C_0 + 2^{2/j+1}C_2 + 16) \sum_{k=k_0}^{K} \frac{1}{k^{4/j}} \leq \underbrace{\frac{C_0 + 2^{2/j+1}C_2 + 16}{\frac{4}{j} - 1}}_{=:C_8} \cdot \frac{1}{(k_0 - 1)^{4/j-1}}.$$

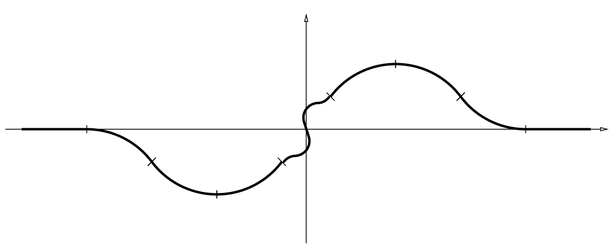


Fig. 6.

Because of

$$\begin{aligned} L_K - L_{k_0} &\leq \sum_{k=k_0}^{K} \ell_k + \sum_{k=k_0}^{K} \Lambda_k \overset{(4.9i)}{\leq} 1 + 2\sum_{k=k_0}^{K+1} \ell_k \\ &\overset{(4.5),(4.9e),(4.9f)}{\leq} 1 + 2\sum_{k=k_0}^{K+1} \frac{\alpha_k}{6} \cdot 4\alpha_k \leq 1 + \frac{4}{3}\sum_{k=k_0}^{K+1} \frac{1}{k^2} \leq 1 + \frac{4}{3}\frac{1}{k_0 - 1} \end{aligned} \tag{4.11}$$

the limit $L_\infty := \lim_{K\to\infty} L_K < \infty$ exists.

The reparametrizations $\tilde{\gamma}_k : [-1,1] \to \mathbb{R}^2$, $\tilde{\gamma}_k(t) := \gamma_k((L_k + \Lambda_k)t)$, of γ_k to constant velocity (depending on k) form a C^0-Cauchy sequence by (4.11) and $\Lambda_k \searrow 0$ that converges to some limit curve $\tilde{\gamma}_\infty \in C^0([-1,1],\mathbb{R}^2)$. Now, by [2, Lemma 4.2], we obtain $E^{[j]}(\tilde{\gamma}_\infty) \leq \liminf_{k\to\infty} E^{[j]}(\gamma_k) < \infty$. Let $\gamma_\infty : [-L_\infty, L_\infty] \to \mathbb{R}^2$, $t \mapsto \tilde{\gamma}(t/L_\infty)$.

The proof that γ_∞ is not differentiable at 0 is deferred to Proposition 4.10.

Now we are going to extend $\gamma_\infty : [-L_\infty, L_\infty] \to \mathbb{R}^2$ to $\mathbb{R} \to \mathbb{R}^2$ via Proposition 4.8. Let $\Lambda^* := L_{k_0} + \Lambda_{k_0} = L_{k_0} + 1$, $\ell^* := L_\infty - L_{k_0}$, $\lambda^* := \Lambda_{k_0} = 1$, $\delta^* := \gamma_\infty$, and $L^* \geq \Lambda^*$ arbitrary. The fact that our estimates will not depend on L^* will allow us to finally take $L^* \to \infty$.

We compute $\ell^* = L_\infty - L_{k_0} \overset{(4.11)}{\leq} 1 + \frac{4}{3}\frac{1}{k_0-1}$ and

$$\frac{2\ell^*}{\Lambda^* - \lambda^*} \overset{(4.9b)}{=} \frac{2\left(1 + \frac{4}{3}\frac{1}{k_0-1}\right)}{2\left(1 + \frac{4}{3}\frac{1}{k_0-1}\right)\sqrt{k_0 - 1}} = \frac{1}{\sqrt{k_0 - 1}}.$$

Summing up and choosing $c_7 := 1/\min(\frac{4}{j} - 1, \frac{2}{j}, \frac{1}{2} + \frac{1}{j}) < \infty$, we obtain

$$\begin{aligned} E^{[j]}(\gamma^* \frown \delta^*) &\leq E^{[j]}(\delta^*) + 2C_2\left(\frac{\ell^* - \lambda^*}{\lambda^*}\right)^{2/j} + 4C_6\left(\frac{2\ell^*}{\Lambda^* - \lambda^*}\right)^{1+2/j} \\ &\leq \frac{C_8}{(k_0 - 1)^{4/j-1}} + 2C_2\left(\frac{\frac{4}{3}}{k_0 - 1}\right)^{2/j} + \frac{4C_6}{(k_0 - 1)^{1/2+1/j}} \\ &\leq \left(C_8 + 2\left(\frac{4}{3}\right)^{2/j} C_2 + 4C_6\right) \cdot \frac{1}{(k_0 - 1)^{1/c_7}} \\ &=: \frac{C_7}{(k_0 - 1)^{1/c_7}} \overset{(4.9a)}{<} \varepsilon. \end{aligned}$$

□

Remark 4.9. In the previous proof, the limit curve γ_∞ was defined as $\gamma_\infty(t) := \lim_{k\to\infty} \gamma_k(\frac{L_k - \Lambda_k}{L_\infty} t)$. The fact that all curves γ_k are parametrized by arc-length yields

$$\gamma_\infty(t) = \lim_{\substack{k\to\infty \\ L_k + \Lambda_k > t}} \gamma_k(t) = \gamma_K\left(t - \sum_{k=K}^{\infty} (\ell_k - \lambda_k)\right) \tag{4.12}$$

for $t > 0$, where K is chosen so large that $\Lambda_K < t$. The second identity is due to the fact that $\gamma_k(t) = \gamma_{k+1}(t + (\ell_k - \lambda_k))$ for all $k \ge k_0$ and $\lambda_k \le t \le L_k + \Lambda_k$. — Because of $r_{k+1} \le \frac{1}{6}\Lambda_{k+1}^2 \alpha_k \le (6k)^{-2} r_k$ the sequence $(r_k)_{k\in\mathbb{N}}$ decreases "*super-exponentially*".

Proposition 4.10. *The curve* γ_∞ *constructed in the proof of Theorem* 4.1 *is not differentiable at* 0.

Proof. Let $\tau_k := \ell_k + \sum_{\kappa=k}^{\infty}(\ell_\kappa - \lambda_\kappa) \in (0, L_\infty)$. Since $\tau_k \ge \ell_k \ge \lambda_k$, we infer from (4.12) that $\gamma_\infty(\tau_k) = \gamma_k(\ell_k)$ and note $\measuredangle(\gamma_\infty(\tau_\kappa), e_1) \equiv \beta_k \mod 2\pi$, where $\measuredangle(\cdot,\cdot) \in [0, 2\pi)$, $e_1 := \binom{1}{0}$ and $x \equiv y \mod 2\pi$ if and only if there is an $m \in \mathbb{Z}$ with $y - x = 2\pi m$. We compute

$$\tau_k \le 2\sum_{\kappa=k}^{\infty} \ell_\kappa \overset{(4.11)}{\le} \frac{4}{3}\frac{1}{k-1} \searrow 0,$$

$$\xi_{\alpha_k} \overset{(4.4)}{\ge} \frac{3}{2}\alpha_k \overset{(4.6)}{\Longrightarrow} \frac{\ell_k}{\lambda_k} = \frac{\eta_{\alpha_k}}{\xi_{\alpha_k}} \overset{(4.5)}{\le} \frac{\xi_{\alpha_k} + 3\alpha_k^3}{\xi_{\alpha_k}} \le 1 + \frac{2}{k^2} \searrow 0$$

and

$$\frac{\tau_k}{\lambda_k} \le \frac{\ell_k}{\lambda_k} + \sum_{\kappa=k}^{\infty}\frac{\ell_\kappa - \lambda_\kappa}{\lambda_\kappa} \overset{(4.10)}{<} \frac{\ell_k}{\lambda_k} + 2\sum_{\kappa=k}^{\infty}\frac{1}{\kappa^2} \le 1 + \frac{2}{k^2} + \frac{2}{k-1} \searrow 1$$

for $k \to \infty$. Since $|\gamma_k(\ell_k)| = \lambda_k$ and $\gamma_\infty(0) = 0$, we arrive at

$$\cos\beta_k = \cos\measuredangle(\gamma_\infty(\tau_k), e_1) = \frac{\langle\gamma_\infty(\tau_k), e_1\rangle}{|\gamma_\infty(\tau_k)|} = \frac{\tau_k}{\lambda_k}\left\langle\frac{\gamma_\infty(\tau_k) - \gamma_\infty(0)}{\tau_k}, e_1\right\rangle.$$

If γ_∞ were differentiable at 0, the right-hand side would converge. But the left-hand side does not converge since the sequence $(\beta_k \mod 2\pi)_{k\in\mathbb{N}, k\ge k_0}$ is dense in $[0, 2\pi)$, as we will see now.

For any $\delta > 0$ choose $K > \max(k_0, \frac{1}{\sqrt{\delta}})$. Since the harmonic series does not converge and $\frac{1}{k} - \frac{1}{k+1} \in (0, \delta)$ for all $k \ge K$, for any $y \in [\beta_K, \infty)$, there is some $k_y \ge K$ with $\beta_{k_y} - y \in [0, \delta)$. So, for any $x \in [\delta, 2\pi - \delta]$, we may find some $y \in [\beta_K, \infty)$ with $x \equiv y \mod 2\pi$, which implies $|(\beta_{k_y} - 2\pi m) - x| = \beta_{k_y} - y < \delta$ for some $m \in \mathbb{Z}$. □

We will proceed to the situation of closed curves.

For the Möbius energy $(j, p) = (2, 1)$, we obtain our result immediately by applying the Möbius invariance. Let $\varepsilon > 0$. Carrying out an inversion on a circle whose center does not lie on the curve referred to by Theorem 4.1, we obtain a closed curve of energy $4 + \varepsilon$ according to [2, Theorem 2.1]. Remember that the Möbius energy of a circle amounts to 4.

If we just want to obtain a closed curve of finite energy that is not differentiable at one point, i.e. skipping the "infimal" property, we might extend the curve constructed in Theorem 4.1 as indicated in Fig. 7, using estimates as in Proposition 4.7.

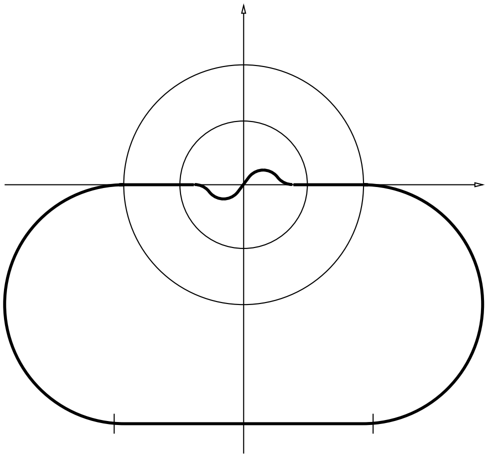


Fig. 7.

The basic idea in proving the "infimal" property is to "glue" the curve constructed in the foregoing proofs on a cylinder; for a sphere we would arrive at the same situation. As we mentioned in Sec. 1, our technique applies to a wide range of open curves, more precisely to planar curves δ that are admittable in sense of Proposition 4.7.

Proposition 4.11 (Projecting a curve onto a cylinder). *For $j \in (0,4)$, $\ell \in (0, \frac{1}{4}]$, let $\gamma : [-\ell, \ell] \to \mathbb{R}^2$ be a curve in arc-length parametrization, $E^{[j]}(\gamma) < \infty$. Then, the energy of the projection $\gamma^* : [-\ell, \ell] \to \mathbb{R}^3$ of γ onto a cylinder of radius $\frac{1}{2\pi}$ satisfies*

$$E^{[j]}(\gamma) \leq E^{[j]}(\gamma^*) \leq E^{[j]}(\gamma) + \omega(\ell, E^{[j]}(\gamma)),$$

where $\omega : [0, \frac{1}{4}] \times [0, \infty) \to [0, \infty)$ is a continuous function with $\omega(0, \cdot) \equiv 0$ depending only on j.

By *projection* we mean that the plane in which γ lies is "glued" onto the cylinder, so lengths are preserved, i.e. $D_{\gamma^*} = D_\gamma$, and since $\sin \pi x \geq \pi x - 2\pi x^3$ on $(0, \infty)$, we obtain

$$|\gamma^*(s) - \gamma^*(t)| \geq \frac{1}{\pi} \sin(\pi |\gamma(s) - \gamma(t)|) \geq |\gamma(s) - \gamma(t)| - 2|\gamma(s) - \gamma(t)|^3. \quad (4.13)$$

This result can be extended to $jp \geq 2$, but due to the lack of the bi-Lipschitz property (4.16) our proof fails for $jp < 2$.

In the proof of Theorem 4.1, we used the fact that the energy of a line amounts to zero for all j together with the monotonicity of $j \mapsto E^{[j]}$ to transfer the "infimal" property from $j \in [2, 4)$ to $(0, 4)$. In the case of closed curves, we face the problem that the $(j, \frac{2}{j})$-energy of the circle depends on j. So, for $j \in (0, 2)$, we will make use of Minkowski's inequality. Applying the mean value theorem to $x \mapsto x^{1/p}$, we arrive at

$$a - b \le a^{1-1/p} p \left(a^{1/p} - b^{1/p}\right) \quad \text{for } a \ge b \ge 1,\ \ p \in [1,\infty). \tag{4.14}$$

Proof. The first inequality is an immediate consequence of (4.13). For the second, we start computing for $x \in (0, \frac{1}{2}]$

$$\begin{aligned}\frac{1}{(x-2x^3)^j} - \frac{1}{x^j} &= \frac{1}{(x-2x^3)^j}\left(1-(1-2x^2)^j\right) \overset{(4.2)}{\le} \frac{1}{x^j(1-2x^2)^j}\cdot\frac{2jx^2}{1-2x^2}\\ &= 2j\cdot\frac{x^{2-j}}{(1-2x^2)^{1+j}} \le 2^{2+j} j x^{2-j}.\end{aligned} \tag{4.15}$$

If $j \le 2$, we obtain using the Minkowski inequality

$$\begin{aligned}&E^{[j]}(\gamma^*)^{j/2} - E^{[j]}(\gamma)^{j/2}\\ &\quad\le \left[\iint_{[-\ell,\ell]^2}\left(\frac{1}{|\gamma^*(s)-\gamma^*(t)|^j} - \frac{1}{|\gamma(s)-\gamma(t)|^j}\right)^{2/j} \mathrm{d}s\,\mathrm{d}t\right]^{j/2}\\ &\quad\overset{(4.13),(4.15)}{\le} \left[(2\ell)^2\left(2^{2+j} j\,(2\ell)^{2-j}\right)^{2/j}\right]^{j/2} \le 2^{4+j} j\ell^2,\end{aligned}$$

which implies $E^{[j]}(\gamma^*) - E^{[j]}(\gamma) \le \left(E^{[j]}(\gamma)^{j/2} + 2^j j\right)^{2/j-1} \cdot 2^{5+j}\ell^2$ by (4.14).

In case $j \in (2,4)$, we arrive at

$$\begin{aligned}E^{[j]}(\gamma^*) - E^{[j]}(\gamma) &\overset{(4.8)}{\le} \iint_{[-\ell,\ell]^2}\left(\frac{1}{|\gamma^*(s)-\gamma^*(t)|^j} - \frac{1}{|\gamma(s)-\gamma(t)|^j}\right)^{2/j} \mathrm{d}s\,\mathrm{d}t\\ &\overset{(4.13),(4.15)}{\le} 2^{2+4/j} j^{2/j} \iint_{[-\ell,\ell]^2} |\gamma(s)-\gamma(t)|^{4/j-2}\mathrm{d}s\,\mathrm{d}t\\ &\overset{(4.16)}{\le} 2^{2+4/j} j^{2/j} K(E^{[j]}(\gamma))^{4/j-2} \iint_{[-\ell,\ell]^2} |s-t|^{4/j-2}\mathrm{d}s\,\mathrm{d}t\\ &\le \frac{2^{1+8/j} j^{2+2/j}}{4-j}\cdot K(E^{[j]}(\gamma))^{4/j-2}\cdot \ell^{4/j}.\end{aligned}$$

□

Proposition 4.12 (Modifying a closed curve). *Let $\zeta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ be a circle of length 1 parametrized in arc-length by $t \mapsto \frac{1}{2\pi}(\sin(2\pi t), 1-\cos(2\pi t), 0)$, and $\lambda \in (0, \frac{1}{8}]$, $\ell > 0$, $j \in (0,4)$. If we change ζ on $(-\lambda, \lambda)$ by inserting a curve segment (parametrized by constant velocity) of length 2ℓ that is completely contained in*

$$\left\{\frac{1}{2\pi}(\sin(2\pi t), 1-\cos(2\pi t), u) \mid t \in [-\lambda,\lambda], u \in \mathbb{R}\right\} \cap B_{\sin(\pi\lambda)/\pi}(0),$$

which is a ball in the cylinder induced by ζ, the new curve $\tilde{\zeta} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ satisfies

$$|E^{[j]}(\tilde{\zeta}) - E^{[j]}(\zeta)| \le E^{[j]}(\tilde{\zeta}|_{[-\frac{1}{4},\frac{1}{4}]}) + \tilde{\omega}\left(\ell + \left(\frac{\ell}{\lambda} - 1\right)\right),$$

where $\tilde{\omega} : [0,\infty) \to [0,\infty)$ *is a continuous function with* $\tilde{\omega}(0) = 0$ *depending only on* j.

For $j \le 2$, the estimate $E^{[j]}(\zeta) \le E^{[j]}(\tilde{\zeta})$ immediately follows from [1, Corollary 3] implying $E^{[j]}(\zeta) \le E^{[j]}(\tilde{\zeta}) \le E^{[j]}(\zeta) + E^{[j]}(\tilde{\zeta}|_{[-\frac{1}{4},\frac{1}{4}]}) + \tilde{\omega}\left(\ell + \left(\frac{\ell}{\lambda} - 1\right)\right)$.

Proof. We will again split up the integration domain, but since we intend to treat closed curves, we have to consider a *parallelogram*

$$\mathcal{R} := \left\{ (s,t) \in \mathbb{R}^2 \middle| s \in \left[-\frac{1}{2}, \frac{1}{2}\right],\ t \in \left[s + \frac{1}{2}, s - \frac{1}{2}\right] \right\},$$

which yields $D_\zeta(s,t) = |s-t|$ for $(s,t) \in \mathcal{R}$. Due to symmetry, we may restrict to $s \ge 0$ and cover $\mathcal{R} \cap ([0,\infty) \times \mathbb{R})$ by $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup [-\frac{1}{4}, \frac{1}{4}]^2$, where

$$\mathcal{A} := \left(\left[\frac{1}{4}, \frac{1}{2}\right] \times [-\lambda, \lambda]\right) \cap \mathcal{R},$$

$$\mathcal{B} := \left([0,\lambda] \times \left[-\frac{1}{2}, -\frac{1}{4}\right]\right) \cap \mathcal{R},$$

$$\mathcal{C} := \left([0,\lambda] \times \left[\frac{1}{4}, \frac{1}{2} + \lambda\right]\right) \cap \mathcal{R},$$

$$\mathcal{D} := \left(\left[\frac{1}{2} - \lambda, \frac{1}{2}\right] \times [1-\lambda, 1]\right) \cap \mathcal{R},$$

$$\mathcal{E} := \left(\left[\lambda, \frac{1}{2}\right] \times [\lambda, 1-\lambda]\right) \cap \mathcal{R},$$

$$\mathcal{F} := \left(\left[\lambda, \frac{1}{2} - \lambda\right] \times \left[-\frac{1}{2} + \lambda, -\lambda\right]\right) \cap \mathcal{R}.$$

See Fig. 8. Note that $\tilde{\zeta}|_Y = \zeta|_Y$, where $Y = [-\frac{1}{2}, -\lambda]$ or $[\lambda, \frac{1}{2}]$. Furthermore, $|\dot{\tilde{\zeta}}(s)| = \frac{\ell}{\lambda}$ for $s \in [-\lambda, \lambda]$ and $|\dot{\tilde{\zeta}}(s)| = |\dot{\zeta}(s)| = 1$ elsewhere.

Since the subdomains $\mathcal{Y} = \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are bounded away from the diagonal, we may estimate the corresponding integrals to the measure of their domain obtaining

$$\iint_{\mathcal{Y}} \left(\frac{1}{|\tilde{\zeta}(s) - \tilde{\zeta}(t)|^j} - \frac{1}{D_{\tilde{\zeta}}(s,t)^j} \right)^{2/j} |\dot{\tilde{\zeta}}(s)||\dot{\tilde{\zeta}}(t)| \mathrm{d}t\, \mathrm{d}s \le C_9 \ell$$

for some generic $C_9 < \infty$. The identity $D_{\tilde{\zeta}} = D_\zeta$ on $\mathcal{E}$ yields

$$\iint_{\mathcal{E}} \left(\frac{1}{|\tilde{\zeta}(s) - \tilde{\zeta}(t)|^j} - \frac{1}{D_{\tilde{\zeta}}(s,t)^j} \right)^{2/j} |\dot{\tilde{\zeta}}(s)||\dot{\tilde{\zeta}}(t)| \mathrm{d}t\, \mathrm{d}s$$

$$= \iint_{\mathcal{E}} \left(\frac{1}{|\zeta(s) - \zeta(t)|^j} - \frac{1}{D_\zeta(s,t)^j} \right)^{2/j} \mathrm{d}t\, \mathrm{d}s.$$

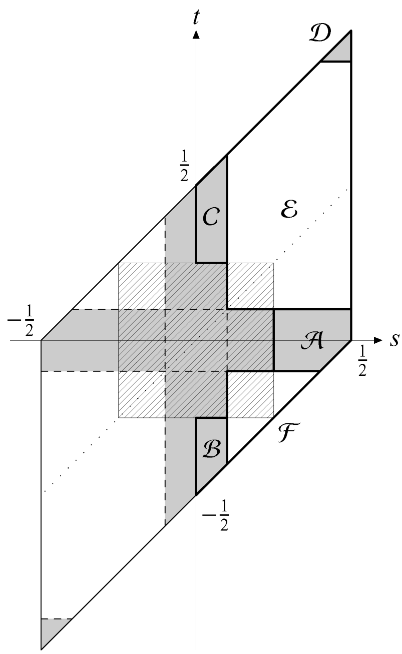


Fig. 8.

In the remaining case $\mathcal{F}$, where $D_{\tilde{\zeta}}(s,t) \leq (s-t) + 2(\ell - \lambda)$ for $(s,t) \in \mathcal{F}$, we first consider $j \leq 2$ and apply Minkowski's inequality

$$\left[\iint_{\mathcal{F}} \left(\frac{1}{|\tilde{\zeta}(s) - \tilde{\zeta}(t)|^j} - \frac{1}{D_{\tilde{\zeta}}(s,t)^j}\right)^{2/j} |\dot{\tilde{\zeta}}(s)||\dot{\tilde{\zeta}}(t)| \mathrm{d}t\, \mathrm{d}s\right]^{j/2}$$

$$- \left[\iint_{\mathcal{F}} \left(\frac{1}{|\zeta(s) - \zeta(t)|^j} - \frac{1}{D_{\zeta}(s,t)^j}\right)^{2/j} \mathrm{d}t\, \mathrm{d}s\right]^{j/2}$$

$$\leq \left[\iint_{\mathcal{F}} \left(\frac{1}{(s-t)^j} - \frac{1}{(s-t+2(\ell-\lambda))^j}\right)^{2/j} \mathrm{d}t\, \mathrm{d}s\right]^{j/2} \overset{(4.1)}{\leq} C_2^{j/2}\left(\frac{\ell}{\lambda} - 1\right),$$

which implies

$$\iint_{\mathcal{F}} \left(\frac{1}{|\tilde{\zeta}(s) - \tilde{\zeta}(t)|^j} - \frac{1}{D_{\tilde{\zeta}}(s,t)^j}\right)^{2/j} |\dot{\tilde{\zeta}}(s)||\dot{\tilde{\zeta}}(t)| \mathrm{d}t\, \mathrm{d}s$$

$$- \iint_{\mathcal{F}} \left(\frac{1}{|\zeta(s) - \zeta(t)|^j} - \frac{1}{D_{\zeta}(s,t)^j}\right)^{2/j} \mathrm{d}t\, \mathrm{d}s$$

$$\leq \left[E^{[j]}(\zeta)^{j/2} + C_2^{j/2}\left(\frac{\ell}{\lambda} - 1\right)\right]^{2/j-1} \cdot \frac{2}{j} \cdot C_2^{j/2}\left(\frac{\ell}{\lambda} - 1\right)$$

by (4.14). For $j \in (2,4)$, we obtain

$$\iint_{\mathcal{F}} \left(\frac{1}{|\tilde{\zeta}(s) - \tilde{\zeta}(t)|^j} - \frac{1}{D_{\tilde{\zeta}}(s,t)^j} \right)^{2/j} |\dot{\tilde{\zeta}}(s)||\dot{\tilde{\zeta}}(t)| \mathrm{d}t\, \mathrm{d}s$$

$$- \iint_{\mathcal{F}} \left(\frac{1}{|\zeta(s) - \zeta(t)|^j} - \frac{1}{D_{\zeta}(s,t)^j} \right)^{2/j} \mathrm{d}t\, \mathrm{d}s$$

$$\overset{(4.8)}{\leq} \iint_{\mathcal{F}} \left(\frac{1}{(s-t)^j} - \frac{1}{(s-t+2(\ell-\lambda))^j} \right)^{2/j} \mathrm{d}t\, \mathrm{d}s \overset{(4.1)}{\leq} C_2 \left(\frac{\ell}{\lambda} - 1 \right)^{2/j},$$

which concludes the proof of $E^{[j]}(\tilde{\zeta}) - E^{[j]}(\zeta) \leq \cdots$.

For $E^{[j]}(\zeta) - E^{[j]}(\tilde{\zeta}) \leq \cdots$ we may use the same estimates as above. □

Proof of Corollary 4.2. Let $\varepsilon > 0$ and take the open curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ that was constructed in the proof of Theorem 4.1 choosing $E^{[j]}(\gamma) < \varepsilon/3$. Recall that $E^{j,p}$ is scaling invariant, so scaling down γ, we may assume that $\gamma|_{\mathbb{R}\setminus[-\mu,\mu]}$ lies on the the x-axis for some $\mu \in (0, \lambda]$ without affecting its energy. We denote the length of the curve $\gamma_{[-\lambda,\lambda]}$ by ℓ which tends to 0 as $\lambda \searrow 0$. By choosing μ sufficiently small, the ratio $\frac{\ell}{\lambda}$ of curve-length 2ℓ and diameter 2λ tends to 1. So the term $\ell + \left(\frac{\ell}{\lambda} - 1\right)$ may be chosen arbitrary small, i.e. the quantities $\omega(\ell, E^{[j]}(\gamma))$ and $\tilde{\omega}\left(\ell + \left(\frac{\ell}{\lambda} - 1\right)\right)$ are both bounded by $\varepsilon/3$. Applying Propositions 4.11 and 4.12, we obtain $|E^{[j]}(\tilde{\zeta}) - E^{[j]}(\zeta)| < \varepsilon$. □

Lemma 4.13 (Bi-Lipschitz continuity of finite energy curves for $jp \geq 2$). *Let $jp \geq 2$. There is a continuous monotone decreasing function $K = K_{j,p}$:$[0,\infty) \to (0,\infty)$ such that, for any finite-energy $\gamma \in C^{0,1}(X, \mathbb{R}^3)$ and $s, t \in X$*

$$|\gamma(s) - \gamma(t)| \geq K(E^{j,p}(\gamma)) \cdot D_\gamma(s,t). \tag{4.16}$$

Moreover, the energy of non-injective curves is infinite.

The proof in [9, Theorem 2.3] or [11, Theorem 2.4.1(2)], which restricts to closed curves of length 1, also holds for open curves. Since $E^{j,p}$ is invariant under scaling and reparametrization by definition, we obtain the claim for arbitrary curves of finite energy.

5. $jp > 2$

For the case $jp > 2$, O'Hara showed that finite-energy C^1-curves are in fact $C^{1,\alpha/2}$ [10, Theorem 1.11]. We will prove that the same is true also for $C^{0,1}$-curves. By our framework used for arriving at pointwise differentiability, we obtain the step "$C^1 \Rightarrow C^{1,\alpha/2}$" almost immediately, without carrying out a geometric argument as the treatment of "solid cylinders" conducted in O'Hara's proof [10, Sublemma 1.10], cf. [11, pp. 67–74] for a more detailed version.

Again we will provide results for both closed and open curves. Let $X_1 := \mathbb{R}/\mathbb{Z}$ or $[0,1]$ respectively. Proving the second part of Theorem 1.1, we start with some technical preliminaries.

Lemma 5.1 ([13, Lemma 8.5 revised]). *For $\ell > 0$, let $a : [0,\ell] \to \mathbb{R}^n$ be a curve parametrized by arc-length, where $P := a(0), Q := a(\ell)$. Then we obtain for all $t \in [0,\ell]$*

$$\left|a(t) - \left(P + \frac{t}{\ell}(Q-P)\right)\right| \le 3\ell\left(\frac{\ell - |P-Q|}{\ell}\right)^{1/2}.$$

The norm on the left-hand side cannot be estimated to a power of $\ell - |P-Q|$ greater than $\frac{1}{2}$, for example, the arc-length parametrization of $a_h : [0,2] \to \mathbb{R}^2$, $t \mapsto h(t, \min(t, 2-t))$, satisfies

$$\begin{aligned}\frac{\left|a_h(1) - \left(P + \frac{1}{2}(Q-P)\right)\right|}{(\ell - |P-Q|)^\beta} &= h^{1-\beta}\left(\frac{h}{2\sqrt{1+h^2}-2}\right)^\beta \\ &= h^{1-\beta}\left(\frac{\sqrt{1+h^2}+1}{2h}\right)^\beta \ge h^{1-2\beta}\end{aligned}$$

which tends to infinity as $h \searrow 0$ if $\beta > \frac{1}{2}$. See Fig. 9.

Proof. Applying a rotation and a translation, we may assume $P = 0$, $Q = |P-Q|e_n$. For $t \in [0,\ell]$, we find the following estimate for the vector $\hat{a}(t) := (a_1(t), \ldots, a_{n-1}(t)) \in \mathbb{R}^{n-1}$.

$$\begin{aligned}|\hat{a}(t)| &\le \int_0^\ell |(\dot{a}_1(t), \ldots, \dot{a}_{n-1}(t))| \mathrm{d}t \le \sqrt{\ell}\left(\int_0^\ell |(\dot{a}_1(t), \ldots, \dot{a}_{n-1}(t))|^2 \mathrm{d}t\right)^{1/2} \\ &\overset{|\dot{a}|=1}{=} \sqrt{\ell}\left(\int_0^\ell \left(1 - \dot{a}_n(t)^2\right) \mathrm{d}t\right)^{1/2} \le \sqrt{\ell}\left(2\int_0^\ell (1 - \dot{a}_n(t))\, \mathrm{d}t\right)^{1/2} \\ &= \sqrt{2\ell}\,(\ell - |P-Q|)^{1/2} \le \sqrt{2}\ell\left(\frac{\ell - |P-Q|}{\ell}\right)^{1/2}.\end{aligned}$$

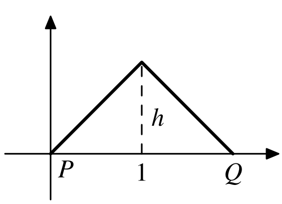


Fig. 9.

Now, $|a_n(\ell)| - a_n(t) \le |a_n(\ell) - a_n(t)| \le \ell - t$ yields $a_n(t) \ge |P-Q| - (\ell - t)$, which leads to

$$a_n(t) - \frac{t}{\ell}|P-Q| \ge (\ell - |P-Q|)\left(\frac{t}{\ell} - 1\right) \ge -(\ell - |P-Q|)\,.$$

On the other hand, $a_n(t) \le |a(t)| \le t$ implies

$$a_n(t) - \frac{t}{\ell}|P-Q| \le t - \frac{t}{\ell}|P-Q| = \frac{t}{\ell}(\ell - |P-Q|) \le \ell - |P-Q|,$$

hence, $|a_n(t) - \frac{t}{\ell}|P-Q|| \le \ell(\frac{\ell-|P-Q|}{\ell})$. Using the estimate for $\hat{a}(t)$, we conclude

$$\left|a(t) - \frac{t}{\ell}(P-Q)\right| \le \ell\left(\frac{\ell - |P-Q|}{\ell}\right) + \sqrt{2}\ell\left(\frac{\ell-|P-Q|}{\ell}\right)^{1/2}.$$

By $x \le \sqrt{x}$ for $x \in [0,1]$, we obtain the result. □

Lemma 5.2. *Let $\gamma \in C^{0,1}(X_1, \mathbb{R}^3)$ be parametrized by arc-length. Suppose that there are numbers $\alpha > 0$, $\varrho_0 \in (0, \frac{1}{2}]$, $C < \infty$ such that for any $\varrho \in (0, \varrho_0]$ the quantity*

$$\kappa(\varrho) \quad := \quad \sup\left\{ \frac{|s-t|}{|\gamma(s)-\gamma(t)|} - 1 \,\middle|\, s,t \in X_1, |s-t| \le \varrho \right\}$$

fulfills the estimate

$$\kappa(\varrho) \le C\varrho^\alpha. \tag{5.1}$$

Then, there is an $\varepsilon_0 = \varepsilon_0(\alpha, \varrho_0, C) > 0$, such that all $x, z, \xi, \zeta \in X_1$ with $|x-z| \le \varepsilon_0$, $x \le \xi < \zeta \le z$, and $|\xi - \zeta| \ge \frac{1}{2}|x-z|$ satisfy

$$\left|\frac{\gamma(z)-\gamma(x)}{|\gamma(z)-\gamma(x)|} - \frac{\gamma(\zeta)-\gamma(\xi)}{|\gamma(\zeta)-\gamma(\xi)|}\right| \le 48\sqrt{C}\cdot|x-z|^{\alpha/2}.$$

Note that the restriction $|s-t| \le \varrho_0 \le \frac{1}{2}$ implies $|s-t| = D_\gamma(s,t)$ and

$$|s-t| \le (1 + \kappa(|s-t|))\cdot|\gamma(s)-\gamma(t)| \quad \text{for all } |s-t| \le \varrho_0. \tag{5.2}$$

Proof. We choose $\varepsilon_0 \in (0, \varrho_0]$ so small that

$$\kappa(\varepsilon_0) \le 1. \tag{5.3}$$

For $x, z, \xi, \zeta \in X_1$ as in the assumptions, we set

$$a := \gamma(\zeta) - \gamma(\xi),$$
$$b := \frac{\zeta - \xi}{z - x}[\gamma(z) - \gamma(x)]\,.$$

Decomposing

$$a-b=\left[\gamma(\zeta)-\left(\frac{\zeta-x}{z-x}\gamma(z)+\frac{z-\zeta}{z-x}\gamma(x)\right)\right]-\left[\gamma(\xi)-\left(\frac{\xi-x}{z-x}\gamma(z)+\frac{z-\xi}{z-x}\gamma(x)\right)\right],$$

we apply Lemma 5.1 with $\ell = z-x$, $a=\gamma(t-x)$, and $t=\zeta$ or ξ respectively. This yields

$$\begin{aligned}|a-b| &\le 6|z-x|\left(\frac{|z-x|-|\gamma(z)-\gamma(x)|}{|z-x|}\right)^{1/2}\\ &\overset{(5.2)}{\le} 6|z-x|\left(\frac{|z-x|-\frac{|z-x|}{1+\kappa(|z-x|)}}{|z-x|}\right)^{1/2}\\ &= 6|z-x|\left(1-\frac{1}{1+\kappa(|z-x|)}\right)^{1/2} \le 6|z-x|\cdot\kappa(|z-x|)^{1/2}\\ &\overset{(5.1)}{\le} 6\sqrt{C}|z-x|^{\alpha/2+1}.\end{aligned}$$

Since

$$|\gamma(\xi)-\gamma(\zeta)| \overset{(5.2)}{\ge} \frac{|\xi-\zeta|}{1+\kappa(|\xi-\zeta|)} \overset{(5.3)}{\ge} \frac{1}{2}|\xi-\zeta| \ge \frac{1}{4}|z-x|,$$

we obtain recalling $|\frac{a}{|a|}-\frac{b}{|b|}| \le 2\frac{|a-b|}{|a|}$

$$\left|\frac{\gamma(z)-\gamma(x)}{|\gamma(z)-\gamma(x)|}-\frac{\gamma(\zeta)-\gamma(\xi)}{|\gamma(\zeta)-\gamma(\xi)|}\right| \le 2\,\frac{|a-b|}{|a|} \le 48\sqrt{C}|z-x|^{\alpha/2}. \qquad \square$$

Lemma 5.3. *The hypotheses of Lemma* 5.2 *imply* $\gamma \in C^{1,\alpha/2}(X_1,\mathbb{R}^3)$.

Proof. For $x,z\in X_1$ with $x<z\le x+\varepsilon_0$ and $x\le s<t\le z$, let $k\in\mathbb{N}$ be such that

$$2^{-k+1}|z-x| \ge |t-s| > 2^{-k}|z-x|.$$

Then, there are $\xi_l,\zeta_l\in[x,z]$, $\xi_l<\zeta_l$, $l=0,\ldots,k$ satisfying $[\xi_0,\zeta_0]=[x,z]$, $[\xi_k,\zeta_k]=[s,t]$, $[\xi_l,\zeta_l]\subset[\xi_{l-1},\zeta_{l-1}]$, and $|\zeta_l-\xi_l|=\frac{1}{2}|\zeta_{l-1}-\xi_{l-1}|$ for $l=0,\ldots,k-1$ and $|\zeta_k-\xi_k|\ge\frac{1}{2}|\zeta_{k-1}-\xi_{k-1}|$. Applying Lemma 5.2 to

$$\nu_l := \frac{\gamma(\zeta_l)-\gamma(\xi_l)}{|\gamma(\zeta_l)-\gamma(\xi_l)|},$$

we arrive at

$$|\nu_{l-1}-\nu_l| \le 48\sqrt{C}\cdot 2^{(-l+1)\alpha/2}\cdot|z-x|^{\alpha/2}, \tag{5.4}$$

for $l = 1, \ldots, k$. We now compute

$$\begin{aligned}
&\left|\frac{\gamma(z)-\gamma(x)}{z-x} - \frac{\gamma(t)-\gamma(s)}{t-s}\right| \\
&\quad \le \left|\frac{\gamma(z)-\gamma(x)}{z-x} - \frac{\gamma(z)-\gamma(x)}{|\gamma(z)-\gamma(x)|}\right| \\
&\qquad + \left|\frac{\gamma(z)-\gamma(x)}{|\gamma(z)-\gamma(x)|} - \frac{\gamma(t)-\gamma(s)}{|\gamma(t)-\gamma(s)|}\right| + \left|\frac{\gamma(t)-\gamma(s)}{|\gamma(t)-\gamma(s)|} - \frac{\gamma(t)-\gamma(s)}{t-s}\right| \\
&\quad \overset{(5.4)}{\le} \frac{\frac{|z-x|}{|\gamma(z)-\gamma(x)|}-1}{\frac{|z-x|}{|\gamma(z)-\gamma(x)|}} + \sum_{l=1}^{k} |\nu_{l-1}-\nu_l| + \frac{\frac{|t-s|}{|\gamma(t)-\gamma(s)|}-1}{\frac{|t-s|}{|\gamma(t)-\gamma(s)|}} \\
&\quad \overset{(5.2)}{\le} \kappa(|z-x|) + 48\sqrt{C}\cdot|z-x|^{\alpha/2}\sum_{l=0}^{\infty}(2^{-\alpha/2})^l + \kappa(|t-s|) \\
&\quad \overset{(5.1)}{\le} 2C\cdot|z-x|^{\alpha} + 48\sqrt{C}\cdot|z-x|^{\alpha/2}\cdot\frac{1}{1-2^{-\alpha/2}} \\
&\quad \le C_\alpha|z-x|^{\alpha/2}, \qquad (5.5)
\end{aligned}$$

where C_α is a constant depending only on α and C.

Now let $y \in \mathbb{R}$ or $y \in (0,1)$ respectively and choose $\delta \in (0, \varepsilon_0]$ so small that $y \pm \delta \in X_1$. Using the last inequality with $x = y - \delta$ and $z = y + \delta$, we obtain

$$\left|\frac{\gamma(t)-\gamma(s)}{t-s} - \frac{\gamma(y+\delta)-\gamma(y-\delta)}{2\delta}\right| \le C_\alpha(2\delta)^{\alpha/2}$$

for all $y - \delta \le s < t \le y + \delta$. Thus, γ is differentiable in y.

Finally, we obtain $\gamma \in C^{1,\alpha/2}$ by

$$\begin{aligned}
|\dot\gamma(z) - \dot\gamma(x)| &\le \left|\frac{\gamma(z)-\gamma(x)}{z-x} - \dot\gamma(z)\right| + \left|\frac{\gamma(z)-\gamma(x)}{z-x} - \dot\gamma(x)\right| \\
&\le \lim_{h\searrow 0}\left|\frac{\gamma(z)-\gamma(x)}{z-x} - \frac{\gamma(z)-\gamma(z-h)}{h}\right| \\
&\qquad + \lim_{h\searrow 0}\left|\frac{\gamma(z)-\gamma(x)}{z-x} - \frac{\gamma(x+h)-\gamma(x)}{h}\right| \\
&\overset{(5.5)}{\le} 2C_\alpha|z-x|^{\alpha/2}
\end{aligned}$$

for all $x, z \in X_1$ with $|z - x| \le \varepsilon_0$ and

$$|\dot\gamma(z) - \dot\gamma(x)| \le 2\varepsilon_0^{-\alpha/2} D_\gamma(z,x)^{\alpha/2}$$

for all $x, z \in X_1$ with $D_\gamma(z,x) \ge \varepsilon_0$. □

Proposition 5.4 (Quantified bi-Lipschitz estimate for finite energy curves of unit-length, [10, Proposition 1.6]). *For* $jp > 2$, *let* $\gamma \in C^{0,1}(X_1, \mathbb{R}^3)$ *be parametrized by arc-length with* $E^{(j,p)}(\gamma) \le B$ *for some* $B > 0$. *Then there is an* $A = A(j,p,B) > 0$, *such that, provided* $|s-t| \le \varrho_0 := \min\left((2A)^{-(p+2)/(jp-2)}, \frac{1}{2}\right)$,

$$|s-t| \le \frac{|\gamma(s)-\gamma(t)|}{1 - A|\gamma(s)-\gamma(t)|^{(jp-2)/(p+2)}}.$$

The proof which may be found in [10, pp. 49–51] or [11, Corollary 4.2.3(1)] also holds for open curves. The condition $|s-t| \le \min\left((2A)^{-(p+2)/(jp-2)}, \frac{1}{2}\right)$ guarantees that the denominator of the right-hand side is bounded below by $\frac{1}{2}$ and that $|s-t| = D_\gamma(s,t)$ holds.

Proof of Theorem 1.1(ii). As mentioned in Sec. 1, we may restrict ourselves to arc-length parametrized curves. Due to scaling invariance, we may furthermore assume that the length of our curve is 1. So let $\gamma \in C^{0,1}(X_1, \mathbb{R}^3)$ be parametrized by arc-length with $E^{j,p}(\gamma) < \infty$. Proposition 5.4 guarantees condition (5.1) with $C = 2A$ and $\alpha = (jp-2)/(p+2)$, for

$$\frac{|s-t|}{|\gamma(s)-\gamma(t)|} - 1 \le \frac{A|\gamma(s)-\gamma(t)|^{(jp-2)/(p+2)}}{1 - A|\gamma(s)-\gamma(t)|^{(jp-2)/(p+2)}} \le 2A|\gamma(s)-\gamma(t)|^{(jp-2)/(p+2)}.$$

Now the claim follows by Lemma 5.3. □

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