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Curves Between Lipschitz and C^1 and Their Relation to Geometric Knot Theory

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Abstract
In this article, we investigate regular curves whose derivatives have vanishing mean oscillations. We show that smoothing these curves using a standard mollifier one gets regular curves again. We apply this result to solve a couple of open problems. We show that curves with finite Möbius energy can be approximated by smooth curves in the energy space $W^{\frac{3}{2},2}$ such that the energy converges which answers a question of He. Furthermore, we prove conjectures by Ishizeki and Nagasawa on certain parts of a decomposition of the Möbius energy and extend a theorem of Wu on inscribed polygons to curves with derivatives with vanishing mean oscillation. Finally, we show that the result by Scholtes on the Γ-convergence of the discrete Möbius energies towards the Möbius energy also holds for curves of merely bounded energy.

Keywords Vanishing mean oscillation · Möbius energy · Gamma convergence

Mathematics Subject Classification 53A04 · 41A30 · 57M25

1 Introduction

Approximating functions by functions with better regularity properties was, is, and will certainly remain to be one of the most important techniques in analysis. In this short note, we want to contribute to this topic. We consider regularly closed curves with regularity somewhere between C^1 and merely Lipschitz continuity. One ends up looking at such curves, if one assumes that the curve is parameterized by arc length

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and lies in some critical fractional Sobolev space $W^{1+s,\frac{1}{s}}$, $s \in (0,1)$—which is known not to embed into C^1. But still the fact that the curve is of class $W^{1+s,\frac{1}{s}}$ gives us some subtle new information on the derivative that we will use in this article. For example, the derivative of the curve $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ then belongs to the space $VMO(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ of all functions with vanishing mean oscillation, i.e.,[1]

$$\lim_{r\to 0}\left(\sup_{x\in\mathbb{R}/\mathbb{Z}}\left(⨍_{B_r(x)} |\gamma'(y) - \overline{\gamma'}_{B_r(x)}|\mathrm{d}y\right)\right) = 0.$$

Here $\overline{\gamma'}_{B_r(x)} := ⨍_{B_r(x)} \gamma'(y)\mathrm{d}y := \frac{1}{2r}\int_{B_r(x)} \gamma'(y)\mathrm{d}y$ denotes the integral mean of the function γ' over the ball $B_r(x)$. Let $\eta \in C^\infty(\mathbb{R}, [0,\infty))$ be such that $\eta \equiv 0$ on $\mathbb{R} \setminus (-1,1)$ and $\int_{\mathbb{R}} \eta(x)\mathrm{d}x = 1$. For $\varepsilon > 0$ we consider the smoothing kernels $\eta_\varepsilon(x) = \frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon})$ and set

$$\gamma_\varepsilon(x) = (\gamma * \eta_\varepsilon)(x) = \int_{\mathbb{R}} \gamma(x-y)\eta_\varepsilon(y)\mathrm{d}y. \tag{1.1}$$

Though for merely regular curves $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ we cannot expect that the smoothed functions γ_ε are regular curves, the situation changes drastically, if we assume that γ' has vanishing mean oscillation. We will start with proving the following theorem:

Theorem 1.1 *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be a curve parameterized by arc length with $\gamma' \in VMO(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Then the speed $|\gamma_\varepsilon'|$ of the convolutions γ_ε defined by (1.1) converges uniformly to $|\gamma'| = 1$ as $\varepsilon \to 0$. So especially, the curves γ_ε are regular if ε is small enough.*

Let us also state a useful uniform bi-Lipschitz estimate for the smoothed functions γ_ε, that we will need in the applications later on.

Lemma 1.2 *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be an injective curve parameterized by arc length with $\gamma' \in VMO(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Then there is an $\varepsilon_0 > 0$ such that*

$$\inf_{x\neq y, 0<\varepsilon<\varepsilon_0} \frac{|\gamma_\varepsilon(x) - \gamma_\varepsilon(y)|}{|x-y|} > 0.$$

Sometimes one might need that also the approximating curves are parameterized by arc length and have the same length as the original curve. In the case that the curve belongs to the fractional Sobolev space $W^{1+s,\frac{1}{s}}$ for some $s \in (0,1)$ the following theorem can help. We denote the length of a curve γ by $L(\gamma)$.

Theorem 1.3 *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \cap W^{1+s,\frac{1}{s}}$ be a curve parameterized by arc length and let γ_ε again denote the convolutions given by (1.1). Furthermore, let $\tilde{\gamma}_\varepsilon : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ be the re-parameterization by arc length of the unit length curve $\frac{1}{L(\gamma_\varepsilon)}\gamma_\varepsilon$ that satisfies $\tilde{\gamma}_\varepsilon(0) = \frac{1}{L(\gamma_\varepsilon)}\gamma_\varepsilon(0)$. Then $\tilde{\gamma}_\varepsilon$ still converges to the curve γ in $W^{1+s,\frac{1}{s}}$.*

[1] We give an elementary argument for this fact at the end of Sect. 2.1

In the last section, we will show how to apply the techniques of this article in order to answer some open questions in the literature and settle some conjectures in the context of knot energies. All the statements of the theorems are known for curves that possess more regularity than we can naturally assume. The approximation techniques above allow to extend these statements to curves of bounded Möbius energy—which is the most natural assumption for these theorems. Let us just present one particular open question due to He here.

O'Hara introduced the Möbius energy [10]

$$E_{\text{möb}}(\gamma) := \iint_{\mathbb{R}/\mathbb{Z}} \left(\frac{1}{|\gamma(x)-\gamma(y)|^2} - \frac{1}{d_\gamma(x,y)^2} \right) |\gamma'(x)||\gamma'(y)| \mathrm{d}x\mathrm{d}y$$

for regular curves $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, which was the first geometric implementation of the concept of knot energy. In the influential paper [6], Freedman et al. discussed many interesting properties of this energy including its invariance under Möbius transformations.

In his article [7], He asked whether any regular curve of bounded Möbius energy can be approximated by smooth curves such that the energy converges. We will use the above approximation result together with the characterization of curves of finite Möbius energy in [3] to give the following answer:

Theorem 1.4 *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be a curve parameterized by arc length such that the Möbius energy $E_{möb}(\gamma)$ is bounded. Then there is a constant $\varepsilon_0 > 0$ such that the γ_ε are smooth regular curves for all $0 < \varepsilon < \varepsilon_0$ converging to γ in $W^{\frac{3}{2},2}$ and in energy, i.e., $E_{möb}(\gamma_\varepsilon) \to E_{möb}(\gamma)$ for $\varepsilon \to 0$.*

We hope that the list of applications, although far from being complete, convinces the reader that the results and techniques developed in this article are of considerable importance for the analysis of critical knot energies for curves.

2 Preliminaries

2.1 Fractional Sobolev Spaces

In the applications, we will use the classification of curves of finite energy E^α in [3] using fractional Sobolev spaces. For $s \in (0, 1)$, $p \in [1, \infty)$, and $k \in \mathbb{N}_0$ the space $W^{k+s,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ consists of all functions $f \in W^{k,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for which

$$\lfloor f^{(k)} \rfloor_{W^{s,p}} := \left(\int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|f^{(k)}(x) - f^{(k)}(y)|^p}{|x-y|^{1+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}$$

is finite. This space is equipped with the norm $\|f\|_{W^{k+s,p}} := \|f\|_{W^{k,p}} + \lfloor f^{(k)} \rfloor_{W^{s,p}}$. For a thorough discussion of the subject of fractional Sobolev spaces, we point the reader to the monograph by Triebel [12], Chapter 7 of [1], and the very nicely written and easily accessible introduction to the subject [5].

The following result is a special case of Theorem 1.1 in [3]:

Theorem 2.1 (Classification of curves with finite Möbius energy) *Let* $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ *be a curve parameterized by arc length. Then the Möbius energy* $E_{möb}(\gamma)$ *is finite if and only if* γ *is bi-Lipschitz and belongs to* $W^{\frac{3}{2},2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

We will also use the well-known fact, that $f \in W^{s,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $s \in (0,1)$, $p = \frac{1}{s}$, implies $f \in VMO(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. This follows, for example, from

$$\begin{aligned}
\frac{1}{2r}\int_{B_r(x)} |f - \overline{f}_{B_r(x)}|\mathrm{d}y &\le \frac{1}{(2r)^2}\int_{B_r(x)}\int_{B_r(x)} |f(y) - f(z)|\mathrm{d}y\mathrm{d}z \\
&\le \frac{1}{(2r)^{2/p}}\left(\int_{B_r(x)}\int_{B_r(x)} |f(y) - f(z)|^p\mathrm{d}y\mathrm{d}z\right)^{\frac{1}{p}} \\
&= \left(\int_{B_r(x)}\int_{B_r(x)} \frac{|f(y) - f(z)|^p}{2r^2}\mathrm{d}y\mathrm{d}z\right)^{\frac{1}{p}} \\
&\le \left(\int_{B_r(x)}\int_{B_r(x)} \frac{|f(y) - f(z)|}{|y - z|^{1+sp}}\mathrm{d}y\mathrm{d}z\right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}/\mathbb{Z}}\int_{B_{2r}(0)} \frac{|f(z+w) - f(z)|^p}{|w|^{1+sp}}\mathrm{d}w\mathrm{d}z\right)^{\frac{1}{p}} \to 0
\end{aligned}$$

for $r \to 0$.

Applying this to $f = \gamma'$, in view of Theorem 2.1, the velocity of a curve parameterized by arc length of finite Möbius energy belongs to VMO. Hence, we can apply Theorem 1.1 in this situation.

2.2 Vitali's Theorem

Apart from the approximation results from the last section, our applications heavily rely on Vitali's characterization of L^1 convergence.

Theorem 2.2 (Vitali's theorem, cf. [2]). *A sequence* f_n, $n \in \mathbb{N}$, *of* L^1 *functions in a measure space* (X, σ, μ) *converges to* f *in* L^1 *if and only if the following three conditions hold:*

(1) f_n *converges in measure to* f, *i.e., for all* $\varepsilon > 0$

$$\lim_{n\to\infty} \mu(|f_n - f| > \varepsilon) = 0.$$

(2) f_n *is uniformly integrable, i.e., for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $\mu(E) < \delta$ *for a measurable* $E \subset X$ *implies*

$$\int_E |f_n| d\mu < \varepsilon$$

for all $n \in \mathbb{N}$.

(3) *f_n is tight, i.e., for every $\varepsilon > 0$ there is a measurable $E \subset X$ of finite measure such that*

$$\int_{X \setminus E} |f_n| d\mu < \varepsilon$$

for all $n \in \mathbb{N}$.

Of course, every sequence of measurable functions on a finite measure space is tight.

3 Approximation by Smooth Curves: Proof of Theorem 1.1, Lemma 1.2, and Theorem 1.3

Proof of Theorem 1.1 Note that $\int_{\mathbb{R}} \eta_\varepsilon(x)\mathrm{d}x = 1$ and $\|\gamma'\|_{L^\infty} \leq 1$ imply

$$|\gamma_\varepsilon'(x)| = \left| \int_{\mathbb{R}} \gamma'(x-y)\eta_\varepsilon(y)\mathrm{d}y \right| \leq \int_{\mathbb{R}} |\eta_\varepsilon(y)|\mathrm{d}y = 1$$

for all $x \in \mathbb{R}/\mathbb{Z}$.

For $r \leq \frac{1}{2}$, let us set

$$VMO(r) = VMO_{\gamma'}(r) = \sup_{x \in \mathbb{R}/\mathbb{Z}} ⨍_{B_r(x)} |\gamma'(y) - \overline{\gamma'}_{B_r(x)}|\mathrm{d}y,$$

where

$$\overline{\gamma'}_{B_r(x)} := ⨍_{B_r(x)} \gamma'(y)\mathrm{d}y$$

denotes the integral mean.

We calculate using the triangle inequality and the estimate above

$$\begin{aligned} |\overline{\gamma'}_{B_r(x)}| &= ⨍_{B_r(x)} |\overline{\gamma'}_{B_r(x)}|\mathrm{d}y \geq ⨍ \Big(|\gamma'| - ||\overline{\gamma'}_{B_r(x)}| - |\gamma'|| \Big) \mathrm{d}y \\ &\geq 1 - ⨍_{B_r(x)} |\overline{\gamma'}_{B_r(x)} - \gamma'|\mathrm{d}y \geq 1 - VMO(r). \end{aligned} \tag{3.1}$$

So we derive

$$\begin{aligned} |\gamma_\varepsilon'(x)| &= \left| \int_{B_\varepsilon(0)} \gamma'(x-y)\eta_\varepsilon(y)\mathrm{d}y \right| \\ &\geq |\overline{\gamma'}_{B_\varepsilon(x)}| - \int_{B_{\varepsilon(0)}} |\gamma'(x-y) - \overline{\gamma'}_{B_\varepsilon(x)}||\eta_\varepsilon(y)|\mathrm{d}y \\ &\geq 1 - (1 + \|\eta\|_{L^\infty}) \cdot VMO(\varepsilon)). \end{aligned} \tag{3.2}$$

Since

$$VMO(\varepsilon) \to 0$$

as $\varepsilon \to 0$ since γ' has vanishing mean oscillation, we deduce that $|\gamma'_\varepsilon| \to |\gamma'| = 1$ uniformly as $\varepsilon \to 0$. So especially γ_ε is a regular curve for $\varepsilon > 0$ small enough. This completes the proof of Theorem 1.1.

Proof of Lemma 1.2 We first note that the bound in VMO of the first derivative for γ is inherited by the curves γ_ε; more precisely, we have (using Fubini twice and substituting variables appropriately)

$$\begin{aligned}
&\fint_{B_r(x)} |\gamma'_\varepsilon(y) - \overline{(\gamma'_\varepsilon)}_{B_r(x)}|\mathrm{d}y \\
&= \fint_{B_r(x)} \left| \int_{\mathbb{R}} \left(\gamma'(y-\zeta) - \fint_{B_r(x)} \gamma'(z-\zeta)\mathrm{d}z \right) \eta_\varepsilon(\zeta) d\zeta \right| \mathrm{d}y \\
&\leq \int_{\mathbb{R}} \left(\fint_{B_r(x)} \left| \gamma'(y-\zeta) - \fint_{B_r(x)} \gamma'(z-\zeta)\mathrm{d}z \right| \mathrm{d}y \, \eta_\varepsilon(\zeta) \right) d\zeta \\
&\leq \int_{\mathbb{R}} \left(\fint_{B_r(x-\zeta)} |\gamma'(y) - \overline{\gamma'}_{B_r(x-\zeta)}|\mathrm{d}y \, \eta_\varepsilon(\zeta) \right) d\zeta \\
&\leq VMO_{\gamma'}(r).
\end{aligned}$$

So we have

$$VMO_{\gamma'_\varepsilon} \leq VMO_{\gamma'}. \tag{3.3}$$

For $x \neq y$, $r = \frac{|x-y|}{2}$, and $z = \frac{x+y}{2}$, we can now estimate

$$\begin{aligned}
|\gamma_\varepsilon(x) - \gamma_\varepsilon(y)| &= | \int_{B_r(z)} \gamma'_\varepsilon(\tau) d\tau | \\
&\geq \int_{B_r(z)} |\gamma'_\varepsilon(\tau)| d\tau - \int_{B_r(z)} |\gamma'_\varepsilon - \overline{(\gamma'_\varepsilon)}_{B_r(z)}| d\tau | \\
&\geq 2r(1 - C(VMO_{\gamma'}(\varepsilon) + VMO_{\gamma'}(r))) \\
&= |x-y| \left(1 - C \left(VMO_{\gamma'}(\varepsilon) + VMO_{\gamma'} \left(\tfrac{|x-y|}{2} \right) \right) \right),
\end{aligned} \tag{3.4}$$

where we have used (3.1) with γ replaced by γ_ε, (3.2), and (3.3). If we now choose $r_0 > 0$ and $\varepsilon_0 > 0$ small enough, we get

$$|\gamma_\varepsilon(x) - \gamma_\varepsilon(y)| \geq \frac{1}{2}|x-y|$$

for all $0 < \varepsilon < \varepsilon_0$ and $0 < r < r_0$.

Since γ is injective, continuous, and $K := \{(x, y) \in (\mathbb{R}/\mathbb{Z})^2 : |x - y| \geq r_0\}$ is compact, we furthermore have

$$\inf \left\{ \frac{|\gamma(x) - \gamma(y)|}{|x - y|} : (x, y) \in K \right\} =: 4d > 0.$$

Making $\varepsilon_0 > 0$ smaller if necessary, we can guarantee that

$$\|\gamma_\varepsilon - \gamma\|_{L^\infty} < r_0 d$$

for all $\varepsilon < \varepsilon_0$ which together with the last estimate yields

$$\inf \left\{ \frac{|\gamma_\varepsilon(x) - \gamma_\varepsilon(y)|}{|x - y|} : (x, y) \in K, \varepsilon < \varepsilon_0 \right\} \geq 4d - \frac{2r_0 d}{r_0} \geq 2d > 0.$$

Hence

$$\inf_{x \neq y, 0 < \varepsilon < \varepsilon_0} \frac{|\gamma_\varepsilon(x) - \gamma_\varepsilon(y)|}{|x - y|} \geq \min \left\{ 2d, \tfrac{1}{2} \right\} > 0.$$

Remark 3.1 If we assume that our curve $\gamma \in C^{0,1}$ is not parameterized by arc length but uniformly regular in the sense that

$$m := \operatorname*{essinf}_{x \in \mathbb{R}/\mathbb{Z}} |\gamma'(x)| > 0$$

then the argument of the proof above still shows that γ_ε is a regular curve for all $\varepsilon > 0$ small enough.

Proof of Theorem 1.3 Let us now consider the curves $\tilde{\gamma}_\varepsilon$ which obviously converge to γ uniformly and hence especially in L^p for $p = \frac{1}{s}$. We now show that the derivatives of these curves satisfy

$$\lim_{\varepsilon \to 0} \lfloor \tilde{\gamma}'_\varepsilon - \gamma' \rfloor_{W^{s,p}} = 0 \tag{3.5}$$

using Vitali's theorem where

$$\lfloor f \rfloor^2_{W^{s,p}} = \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|f(x) - f(y)|^p}{|x - y|^{1+sp}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{|f(x) - f(y)|^p}{|x - y|^2} \mathrm{d}x\mathrm{d}y$$

denotes the Gagliardo semi-norm that we introduced in Sect. 2.1.

We therefore consider the integrand

$$I_\varepsilon(x, y) := \frac{|(\tilde{\gamma}'_\varepsilon(x) - \gamma'(x)) - (\tilde{\gamma}'_\varepsilon(y) - \gamma'(y))|^p}{|x - y|^2}.$$

To show that for a sequence $\varepsilon_i \downarrow 0$ the integrands I_{ε_i} are uniformly integrable, we use the inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ to estimate these integrands from above by

$$2^{p-1}\left(\frac{|\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}_\varepsilon'(y)|^p}{|x-y|^2} + \frac{|\gamma'(x) - \gamma'(y)|^p}{|x-y|^2}\right).$$

Let us now consider the transformation

$$\begin{aligned} \psi_\varepsilon : (\mathbb{R}/\mathbb{Z})^2 &\to (\mathbb{R}/\mathbb{Z})^2 \\ (x, y) &\mapsto (s(x), s(y)) \end{aligned}$$

where $s = s_\varepsilon$ denotes the re-parameterization of $\frac{1}{L(\gamma_\varepsilon)}\gamma_\varepsilon$ by arc length such that $s(0) = 0$. Note that since γ_ε' is uniformly bounded away from 0 by Theorem 1.1, these transformations are uniformly bi-Lipschitz for $\varepsilon > 0$ small enough. For $E \subset (\mathbb{R}/\mathbb{Z})^2$, we therefore have

$$\begin{aligned} \iint_E \frac{|\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}_\varepsilon'(y)|^p}{|x-y|^2} \mathrm{d}x\mathrm{d}y &\leq C \iint_{\psi_\varepsilon^{-1}(E)} \frac{|\gamma_\varepsilon'(x) - \gamma_\varepsilon'(y)|^p}{|x-y|^2} |\gamma_\varepsilon'(x)||\gamma_\varepsilon'(y)| \mathrm{d}x\mathrm{d}y \\ &\leq C \iint_{\psi_\varepsilon^{-1}(E)} \frac{|\gamma_\varepsilon'(x) - \gamma_\varepsilon'(y)|^p}{|x-y|^2} \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Let us show that the integrands

$$\frac{|\gamma_{\varepsilon_i}'(x) - \gamma_{\varepsilon_i}'(y)|^p}{|x-y|^2}$$

are uniformly integrable. For this, for $\varepsilon_0 > 0$, we first chose an $i_0 \in \mathbb{N}$ such that

$$\lfloor \gamma_{\varepsilon_i}' - \gamma' \rfloor_{W^{s,p}} \leq \frac{(\varepsilon_0)^{\frac{1}{p}}}{2},$$

for all $i > i_0$. We then chose $\delta > 0$ such that for every set $F \subset (\mathbb{R}/\mathbb{Z})^2$ with $|F| \leq \delta$ we have

$$\iint_F \frac{|\gamma'(x) - \gamma'(y)|^p}{|x-y|^2} \mathrm{d}x\mathrm{d}y \leq \varepsilon_0$$

and

$$\iint_F \frac{|\gamma_{\varepsilon_i}'(x) - \gamma_{\varepsilon_i}'(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y \leq \frac{\varepsilon_0}{2^p}$$

for the finite set of indices $i \in \{1, 2, \ldots, i_0\}$. We hence get for $i \in \mathbb{N}$ with $i > i_0$ using the triangle inequality in L^2

$$\left(\iint_F \frac{|\gamma'_{\varepsilon_i}(x)-\gamma'_{\varepsilon_i}(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} \le \left(\iint_F \frac{|\gamma'(x)-\gamma'(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}} + \lfloor \gamma'_{\varepsilon_i}-\gamma'\rfloor_{W^{s,p}} \le \sqrt{\varepsilon_0}$$

for all $F \subset (\mathbb{R}/\mathbb{Z})^2$ with $|F| \le \delta$. This proves that the integrands

$$\frac{|\gamma'_{\varepsilon_i}(x)-\gamma'_{\varepsilon_i}(y)|^p}{|x-y|^2}$$

are indeed uniformly integrable.

Hence, for every $\varepsilon_0 > 0$ there is a $\delta > 0$ such that $|\psi_{\varepsilon_i}^{-1}(E)| \le \delta$ implies

$$\iint_{\psi_\varepsilon^{-1}(E)} \frac{|\gamma'_{\varepsilon_i}(x)-\gamma'_{\varepsilon_i}(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y \le \varepsilon_0$$

for all $i \in \mathbb{N}$. But, as the ψ_ε^{-1} are uniformly Lipschitz for $\varepsilon > 0$ small enough, we get that there is a $\tilde{\delta} > 0$ such that $|E| \le \tilde{\delta}$ implies $|\psi_\varepsilon^{-1}(E)| \le \delta$ and hence

$$\iint_E \frac{|\tilde{\gamma}'_{\varepsilon_i}(x)-\tilde{\gamma}'_{\varepsilon_i}(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y \le C \iint_{\psi_\varepsilon^{-1}(E)} \frac{|\gamma'_{\varepsilon_i}(x)-\gamma'_{\varepsilon_i}(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y \le C\varepsilon_0.$$

Thus, the I_{ε_i} are uniformly integrable.

To show that I_ε converges in measure to 0 we show that $\tilde{\gamma}'_\varepsilon$ converges to γ' in L^p. This can be seen from the estimate

$$\|\tilde{\gamma}'_\varepsilon - \gamma'\|_{L^p} \le \|\tilde{\gamma}'_\varepsilon - \gamma' \circ s_\varepsilon\|_{L^p} + \|\gamma' \circ s_\varepsilon - \gamma'\|_{L^p}.$$

Since the functions s_ε are uniformly bi-Lipschitz for $\varepsilon > 0$ small, we get

$$\|\tilde{\gamma}'_\varepsilon - \gamma' \circ s_\varepsilon\|_{L^p} \le C\|\gamma'_\varepsilon - \gamma'\|_{L^p} \xrightarrow{\varepsilon\to 0} 0.$$

Furthermore, for a smooth function f we have

$$\|f \circ s_\varepsilon - f\|_{L^p} \xrightarrow{\varepsilon\to 0} 0.$$

Using again that the s_ε are uniformly bi-Lipschitz for small ε together with the triangle inequality, we get

$$\begin{aligned}\|\gamma' \circ s_\varepsilon - \gamma'\|_{L^p} &\le \|\gamma' \circ s_\varepsilon - f' \circ s_\varepsilon\|_{L^p} + \|f' \circ s_\varepsilon - f'\|_{L^p} + \|f' - \gamma'\|_{L^p} \\ &\le C\|f' - \gamma'\|_{L^p} + \|f' \circ s_\varepsilon - f'\|_{L^p} \\ &\xrightarrow{\varepsilon\to 0} C\|f' - \gamma'\|_{L^p}\end{aligned}$$

and choosing smooth functions f converging to γ in $W^{1,p}$ we get that

$$\|\gamma' \circ s_\varepsilon - \gamma'\|_{L^p} \xrightarrow{\varepsilon \to 0} 0.$$

Hence, the integrands I_ε converge locally in L^1 to 0 on $\{(x, y) \in (\mathbb{R}/\mathbb{Z})^2 : x \neq y\}$ and hence in measure. As I_{ε_i} converges to 0 in measure and is uniformly integrable, we can apply Vitali's theorem (Theorem 2.2) to prove the claim.

4 Applications

We want to present several applications of Theorem 1.1. We will start with analyzing the convergence of the Möbius energy and the parts of its decomposition found by Ishizeki and Nagasawa if the original curve has bounded Möbius energy. Unfortunately, the smoothed curves γ_ε in general do not converge in $W^{1,\infty}$—so we cannot apply the fact that the Möbius energy is C^1 in $W^{\frac{3}{2},2} \cap W^{1,\infty}$ [4, Theorem II]. We will show how to use the convergence of $|\gamma_\varepsilon'|$ from Theorem 1.1 together with bi-Lipschitz estimates in order to prove convergence in energy.

4.1 Convergence of Some Critical Knot Energies

4.1.1 The Möbius Energy

As a first application, we want to answer a question due to He [7, Question 8 in Sect. 7]. He asked, whether a curve of bounded Möbius energy can be approximated by smooth curves such that the energies of these curves converge to the energy of the initial curve. The following lemma shows that this is indeed the case and that one can just use the mollified curves γ_ε. This lemma together with Theorem 1.1 obviously proves Theorem 1.4.

Lemma 4.1 (Convergence of the Möbius energy) *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be parameterized by arc length of finite Möbius energy. Then we have $E_{möb}(\gamma_\varepsilon) \to E_{möb}(\gamma)$.*

Proof We use Vitali's convergence theorem to prove this lemma. Setting $I_\gamma(x, w) := \left(\frac{1}{|\gamma(x+w) - \gamma(x)|^2} - \frac{1}{d_\gamma(x, x+w)^2} \right) |\gamma'(x)|\, |\gamma'(x+w)|$, we get

$$E_{\text{möb}}(\gamma) = \int_{\mathbb{R}/\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} I_\gamma(x, w)\, \mathrm{d}x \mathrm{d}w.$$

As $|\gamma_\varepsilon'|$ converges pointwise to $|\gamma'|$ by Theorem 1.1 and γ_ε converges to γ pointwise, the integrand $I_{\gamma_\varepsilon}(x, w)$ also converges to $I_\gamma(x, w)$ pointwise. Let us show that the integrands are uniformly integrable. For this purpose, we only have to consider points close to the diagonal, i.e., we will only integrate over $x, y \in \mathbb{R}/\mathbb{Z}$ with $|x - y| \leq \frac{1}{4}$, since on the rest of the domain the bi-Lipschitz estimate gives us a uniform bound on the integrand.

We have for $\varepsilon > 0$ small enough and $|w| \le \frac{1}{4}$ that $d_{\gamma_\varepsilon}(x+w, x) = \int_0^1 |\gamma_\varepsilon'(x+sw)| ds$. Together with the identity $\gamma_\varepsilon(x+w) - \gamma_\varepsilon(x) = w \int_0^1 \gamma_\varepsilon'(x+sw) ds$, we get

$$\begin{aligned}
&I_{\gamma_\varepsilon}(x, w) \\
&= \left(\frac{1}{|\gamma_\varepsilon(x+w) - \gamma_\varepsilon(x)|^2} - \frac{1}{d_{\gamma_\varepsilon}(x, x+w)^2} \right) \\
&= \frac{|w|^4}{|\gamma_\varepsilon(x+w) - \gamma_\varepsilon(x)|^2 d_{\gamma_\varepsilon}(x, x+w)^2} \left(\frac{d_{\gamma_\varepsilon}(x, y)^2 - |\gamma_\varepsilon(x+w) - \gamma_\varepsilon(x)|^2}{|w|^4} \right) \\
&= \frac{|w|^4}{|\gamma_\varepsilon(x+w) - \gamma_\varepsilon(x)|^2 d_{\gamma_\varepsilon}(x, x+w)^2} \\
&\quad \times \left(\frac{\int_0^1 \int_0^1 \left(|\gamma_\varepsilon'(x+s_1 w)||\gamma_\varepsilon'(x+s_2 w)| - \langle \gamma_\varepsilon'(x+s_1 w), \gamma_\varepsilon'(x+s_2 w) \rangle \right) \mathrm{d}s_1 \mathrm{d}s_2}{|w|^2} \right).
\end{aligned}$$

Using the uniform bi-Lipschitz estimate Lemma 1.2, we get

$$\begin{aligned}
&I_{\gamma_\varepsilon}(x, w) \\
&\le C \left(\frac{\int_0^1 \int_0^1 \left(|\gamma_\varepsilon'(x+s_1 w)||\gamma_\varepsilon'(x+s_2 w)| - \langle \gamma_\varepsilon'(x+s_1 w) \gamma_\varepsilon'(x+s_2 w) \rangle \right) \mathrm{d}s_1 \mathrm{d}s_2}{|w|^2} \right)
\end{aligned}$$

for all $\varepsilon > 0$ small enough.

As all vectors $a, b \in \mathbb{R}^n \setminus \{0\}$ satisfy

$$|a||b| - \langle a, b \rangle = \frac{|a||b|}{2} \left| \frac{a}{|a|} - \frac{b}{|b|} \right|^2$$

and

$$\left| \frac{a}{|a|} - \frac{b}{|b|} \right| \le \frac{|a-b|}{|a|} + |b| \left| \frac{1}{|a|} - \frac{1}{|b|} \right| \le \frac{2|a-b|}{|a|},$$

we get

$$|a||b| - \langle a, b \rangle \le 2 \frac{|b|}{|a|} |a-b|^2.$$

Applying this inequality to $a = \gamma'(x+s_1 w)$ and $b = \gamma'(x+s_2 w)$, we arrive at

$$\begin{aligned}
|I_{\gamma_\varepsilon}(x, w)| &\le C \left(\frac{\int_0^1 \int_0^1 |\gamma_\varepsilon'(x+s_1 w) - \gamma_\varepsilon'(x+s_2 w)|^2 \mathrm{d}s_1 \mathrm{d}s_2}{|w|^2} \right) \\
&=: C \tilde{I}_{\gamma_\varepsilon}(x, w)
\end{aligned} \tag{4.1}$$

for all $|w| \le \frac{1}{4}$ and $\varepsilon > 0$ small enough. Let us now show that $\tilde{I}_{\gamma_\varepsilon}(x, w)$ converges to $\tilde{I}_\gamma(x, w)$ in $L^1(\mathbb{R}/\mathbb{Z} \times [-\frac{1}{2}, \frac{1}{2}])$ which implies that $I_{\gamma_\varepsilon}(x, w) \le C \tilde{I}_{\gamma_\varepsilon}(x, w)$ is uniformly integrable.

Jensen's inequality followed by Fubini's theorem and the substitutions $x = x+s_2 w$, $w = (s_1 - s_2)w$ give

$$\begin{aligned}
&\int_{\mathbb{R}/\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\tilde{I}_{\gamma_\varepsilon}(x, y) - \tilde{I}_\gamma(x, w)| \mathrm{d}w \mathrm{d}x \\
&= \int_0^1 \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{R}/\mathbb{Z}} \\
&\quad \times \left(\frac{|\gamma_\varepsilon'(x+s_1 w) - \gamma_\varepsilon'(x+s_2 w)|^2 - |\gamma'(x+s_1 w) - \gamma'(x+s_2 w)|^2}{|w|^2} \right) \\
&\quad \times \mathrm{d}w \mathrm{d}x \mathrm{d}s_2 \mathrm{d}s_1 \\
&\leq \int_0^1 \int_0^1 |s_1 - s_2| \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{R}/\mathbb{Z}} \\
&\quad \times \left(\frac{|\gamma_\varepsilon'(x+w) - \gamma_\varepsilon'(x)|^2 - |\gamma'(x+w) - \gamma'(x)|^2}{|w|^2} \right) \mathrm{d}w \mathrm{d}x \mathrm{d}s_2 \mathrm{d}s_1 \\
&\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{R}/\mathbb{Z}} \left(\frac{|\gamma_\varepsilon'(x+w) - \gamma_\varepsilon'(x)|^2 + |\gamma'(x+w) - \gamma'(x)|^2}{|w|^2} \right) \mathrm{d}w \mathrm{d}x.
\end{aligned}$$

We set

$$f(x, w) := \frac{\gamma_\varepsilon'(x+w) - \gamma_\varepsilon'(x)}{w} \quad \text{and} \quad g(x, w) := \frac{\gamma'(x+w) - \gamma'(x)}{w}.$$

Combining $||f|^2 - |g|^2| = (|f| + |g|)||f| - |g|| \leq (|f| + |g|)|f - g|$ with Cauchy's inequality, we get

$$\int_{\mathbb{R}/\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} ||f(x, w)|^2 - |g(x, w)|^2| \mathrm{d}x \mathrm{d}w \leq \left(\|f\|_{L^2} + \|g\|_{L^2} \right) \|f - g\|_{L^2}.$$

Spelling out the above inequality gives

$$\begin{aligned}
&\int_{\mathbb{R}/\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{|\gamma_\varepsilon'(x+w) - \gamma_\varepsilon'(x)|^2}{w^2} - \frac{|\gamma'(x+w) - \gamma'(x)|^2}{w^2} \right| \mathrm{d}w \mathrm{d}x \\
&\leq (\lfloor \gamma_\varepsilon' \rfloor_{W^{\frac{1}{2},2}} + \lfloor \gamma' \rfloor_{W^{\frac{1}{2},2}}) \lfloor \gamma_\varepsilon' - \gamma' \rfloor_{W^{\frac{1}{2},2}} \xrightarrow{\varepsilon \to 0} 0.
\end{aligned}$$

This shows that the family of functions $\tilde{I}_{\gamma_\varepsilon}$ converge to $\tilde{I}_\gamma$ in L^1. Hence, for every sequence $\varepsilon_i \downarrow 0$, the integrands $I_{\gamma_{\varepsilon_i}}$ are uniformly integrable and Vitali's theorem (Theorem 2.2) implies $E(\gamma_{\varepsilon_i}) \xrightarrow{i \to \infty} E(\gamma)$. □

4.1.2 Ishizeki's and Nagasawa's Decomposition of the Möbius Energy

In [9], Ishizeki and Nagasawa found the decomposition

$$E_{\text{möb}}(\gamma) = E^1(\gamma) + E^2(\gamma) + 4$$

of the Möbius energy where

$$E^1(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^2} \frac{|\tau(x) - \tau(y)|^2}{2|\gamma(x) - \gamma(y)|^2} |\gamma'(x)||\gamma'(y)| \mathrm{d}x \mathrm{d}y$$

$\tau = \frac{\gamma'}{|\gamma'|}$ and

$$E^2(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^2} \frac{2}{|\gamma(x) - \gamma(y)|^2} \det \begin{pmatrix} \langle \tau(x), \tau(y) \rangle & \langle (\gamma(x) - \gamma(y)), \tau(x) \rangle \\ \langle (\gamma(x) - \gamma(y)), \tau(y) \rangle & |\gamma(x) - \gamma(y)|^2 \end{pmatrix} \times |\gamma'(x)||\gamma'(y)| \mathrm{d}x \mathrm{d}y.$$

As in the proof of Lemma 4.1, we can show the following.

Lemma 4.2 *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be a curve of bounded Möbius energy. Then*

$$\lim_{\varepsilon \to 0} E^1(\gamma_\varepsilon) = E^1(\gamma) \quad \text{and} \quad \lim_{\varepsilon \to 0} E^2(\gamma_\varepsilon) = E^2(\gamma).$$

Proof It is enough to show the convergence for E^1, as the statement for E^2 follows from the decomposition

$$E_{\text{möb}} = E^1 + E^2 + 4$$

by Ishizeki and Nagasawa [9]. As γ has bounded Möbius energy, we know that $\gamma' \in VMO$. As in the proof of Theorem 1.3 one shows that the integrand in the definition of E^1 converges in measure. From the uniform bi-Lipschitz estimate in Lemma 1.2 and the estimate

$$\left| \frac{a}{|a|} - \frac{b}{|b|} \right| = \frac{||b|(a-b) + (|b| - |a|)b|}{|a||b|} \leq 2 \frac{|a-b|}{a}$$

and the fact that $|\gamma_\varepsilon'|$ is uniformly bounded away from 0 for all ε sufficiently small, we get

$$\frac{|\tau_\varepsilon(x) - \tau_\varepsilon(y)|^2}{2|\gamma_\varepsilon(x) - \gamma_\varepsilon(y)|^2} |\gamma_\varepsilon'(x_1)||\gamma_\varepsilon'(x_2)| \leq C \frac{|\gamma_\varepsilon'(x) - \gamma_\varepsilon'(y)|^2}{|x-y|^2}. \tag{4.2}$$

We have shown in the proof of Lemma 4.1 that the right-hand side of this inequality is uniformly integrable for every sequence $\varepsilon_i \downarrow 0$—and thus the integrands in the definition of E^1 are uniformly integrable and Vitali's theorem (Theorem 2.2) implies the assertion. □

4.2 Proof of a Conjecture by Ishizeki and Nagasawa

In [9], Ishizeki and Nagasawa proved that for all curves γ in $C^{1,1}$ we have $E^1(\gamma) \geq 2\pi^2$ and conjectured that the same is also true under the weaker but more natural condition $\gamma \in W^{\frac{3}{2},2}$. Using the techniques we developed so far, we can now prove this conjecture quite easily.

Theorem 4.3 (A conjecture by Nagasawa and Ishizeki). *We have $E^1(\gamma) \geq 2\pi^2$ for all regular curves $\gamma \in W^{\frac{3}{2},2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$.*

Proof Let $\gamma_\varepsilon = \gamma * \eta_\varepsilon$. Since

$$E^1(\gamma_\varepsilon) \to E^1(\gamma)$$

and $E(\gamma_\varepsilon) \geq 2\pi^2$ as the inequality holds for smooth curves, we get $E^1(\gamma) \geq 2\pi^2$. □

In the same paper, Ishizeki and Nagasawa also showed the Möbius invariance of the energies E^1 and E^2 for curves of bounded Möbius energy except for one important case: the case of an inversion in a sphere centered on the curve. For applications this seems to be one of the most important cases. We can now show that in this last case the energy E^1 decreases by $2\pi^2$, whereas E^2 increases by the same amount

Theorem 4.4 *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ be a regular curve with bounded Möbius energy and I be an inversion in a sphere centered on γ. Then*

$$E^1(I \circ \gamma) = E^1(\gamma) - 2\pi^2 \quad \textit{and} \quad E^2(I \circ \gamma) = E^2(\gamma) + 2\pi^2.$$

Proof We only have to show the statement for E^1 as due to a theorem of Ishizeki and Nagasawa the sum

$$E^1 + E^2$$

is known to be invariant under all Möbius transformations [8].

Let us assume that γ is parameterized by arc length. We set $\gamma_\varepsilon := \gamma * \eta_\varepsilon$ and assume without loss of generality, that 0 is the center of the inversion I. Then we can find $x_\varepsilon \to 0$ such that $0 \in \gamma_\varepsilon(\mathbb{R}/\mathbb{Z}) + x_\varepsilon$. Let us denote by $\tilde{\gamma}_\varepsilon : \mathbb{R} \to \mathbb{R}^n$ a re-parameterization of $I \circ (\gamma_\varepsilon - x_\varepsilon)$ by arc length such that $\tilde{\gamma}_\varepsilon(0) = (I \circ \gamma_\varepsilon)(0)$ and let $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^n$ a re-parameterization of $I \circ \gamma$ by arc length such that $\tilde{\gamma}(0) = (I \circ \gamma)(0)$. Then $\tilde{\gamma}_\varepsilon$ converges pointwise to $\tilde{\gamma}$.

The proof now relies on the following

Claim 4.5 *We have*

$$\lim_{\varepsilon\to 0}\lfloor \tilde{\gamma}_\varepsilon' - \tilde{\gamma}'\rfloor_{W^{\frac{1}{2},2}(\mathbb{R})} = 0,$$

where

$$\lfloor f\rfloor_{W^{\frac{1}{2},2}(\mathbb{R})} := \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}$$

denotes the Gagliardo semi-norm on $\mathbb{R}$.

Let us use this claim to prove the statement for E^1 in Theorem 4.4. On the one hand, Lemma 4.2 and the Möbius invariance for smooth curves imply

$$E^1(\tilde{\gamma}_\varepsilon) + 2\pi^2 = E^1(I\circ(\gamma_\varepsilon + x_\varepsilon)) + 2\pi^2 = E^1(\gamma_\varepsilon + x_\varepsilon) \xrightarrow{\varepsilon\downarrow 0} E^1(\gamma). \quad (4.3)$$

Note that $\tilde{\gamma}_\varepsilon$ have uniformly bounded Möbius energy and are thus uniformly bi-Lipschitz. So on the other hand, we can use the estimate

$$\frac{|\tilde{\tau}_\varepsilon(x)-\tilde{\tau}_\varepsilon(y)|^2}{2|\tilde{\gamma}_\varepsilon(x)-\tilde{\gamma}_\varepsilon(y)|^2}|\tilde{\gamma}_\varepsilon'(x)||\tilde{\gamma}_\varepsilon'(y)| \le C\frac{|\tilde{\gamma}_\varepsilon'(x)-\tilde{\gamma}_\varepsilon'(y)|^2}{|x-y|^2}$$

and follow the argument in the proof of Lemma 4.1 to see that the integrands in the definition of the energies $E^1(\tilde{\gamma}_\varepsilon)$ satisfy the assumptions of Vitali's theorem. Hence,

$$\lim_{\varepsilon\to 0} E^1(\tilde{\gamma}_\varepsilon) = E^1(\tilde{\gamma}). \quad (4.4)$$

But (4.3) and (4.4) imply

$$E^1(\tilde{\gamma}) + 2\pi^2 = E(\gamma).$$

□

Proof of Claim 4.5 We will show that the integrands appearing in the definition of

$$\lfloor \tilde{\gamma}_\varepsilon' - \tilde{\gamma}'\rfloor_{W^{\frac{1}{2},2}(\mathbb{R})}$$

are tight and uniformly integrable on compact subsets and converge in measure on compact subsets to 0. Then the claim essentially follows from Vitali's theorem. These integrands are

$$\frac{|(\tilde{\gamma}_\varepsilon'(x)-\tilde{\gamma}'(x)) - (\tilde{\gamma}_\varepsilon'(y)-\tilde{\gamma}'(y))|^2}{|x-y|^2}.$$

As in the proof of Theorem 1.3, one sees that γ_ε' converge in measure to γ' and hence the integrands converge in measure to 0 on compact subsets.

Let us first deal with the point ∞ and show that for every $\delta > 0$ there is an $R > 0$ such that

$$\iint_{\mathbb{R}^2 \setminus (B_R(0))^2} \frac{|\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}_\varepsilon'(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y \leq \delta \tag{4.5}$$

for all $\varepsilon > 0$ small enough. For this, we use the Möbius invariance of the Möbius energy [6, Theorem 2.1]. Together with Fatou's lemma the latter implies

$$E_{\text{möb}}(\tilde{\gamma}) \leq \lim_{\varepsilon \to 0} E_{\text{möb}}(\tilde{\gamma}_\varepsilon) = \lim_{\varepsilon \to 0} E_{\text{möb}}(\gamma_\varepsilon) - 4 = E_{\text{möb}}(\gamma) - 4 = E_{\text{möb}}(\tilde{\gamma}).$$

Hence,

$$\lim_{\varepsilon \to 0} E_{\text{möb}}(\tilde{\gamma}_\varepsilon) = E_{\text{möb}}(\tilde{\gamma}). \tag{4.6}$$

For $\delta > 0$, we now choose $R > 0$ such that

$$E_{B_R(0)}(\tilde{\gamma}) := \int_{B_R(0)} \int_{B_R(0)} \left(\frac{1}{|\tilde{\gamma}(x) - \tilde{\gamma}(y)|^2} - \frac{1}{|x-y|^2} \right) \mathrm{d}x\mathrm{d}y \geq E_{\text{möb}}(\tilde{\gamma}) - \delta.$$

Then

$$E_{B_R(0)}(\tilde{\gamma}_\varepsilon) \geq E_{\text{möb}}(\tilde{\gamma}) - 2\delta$$

for $\varepsilon > 0$ small enough, since else the lower semi-continuity of the Möbius energy would imply

$$E_{B_R(0)}(\tilde{\gamma}) \leq \liminf_{\varepsilon \to 0} E_{B_R(0)}(\tilde{\gamma}_\varepsilon) \leq E_{\text{möb}}(\tilde{\gamma}) - 2\delta \leq E_{B_R(0)}(\tilde{\gamma}) - \delta.$$

In view of (4.6), we even obtain

$$E_{\text{möb}}(\tilde{\gamma}) + 2\delta \geq E_{\text{möb}}(\tilde{\gamma}_\varepsilon) \geq E_{B_R(0)}(\tilde{\gamma}_\varepsilon) \geq E_{\text{möb}}(\tilde{\gamma}) - 2\delta$$

for all $\varepsilon > 0$ sufficiently small and hence

$$\begin{aligned} &\iint_{\mathbb{R}^2 \setminus (B_R(0))^2} \left(\frac{1}{|\tilde{\gamma}_\varepsilon(x) - \tilde{\gamma}_\varepsilon(y)|^2} - \frac{1}{|x-y|^2} \right) \mathrm{d}x\mathrm{d}y \\ &\quad = E_{\text{möb}}(\tilde{\gamma}_\varepsilon) - E_{B_R(0)}(\tilde{\gamma}_\varepsilon) \leq 4\delta. \end{aligned} \tag{4.7}$$

So the energy does not concentrate at the point infinity. Let us translate this into a statement for the Gagliardo semi-norm.

One estimates

$$
\begin{aligned}
&\frac{1}{|\tilde{\gamma}_\varepsilon(x)-\tilde{\gamma}_\varepsilon(y)|^2}-\frac{1}{|x-y|^2}\\
&\quad=\frac{|x-y|^2}{|\tilde{\gamma}_\varepsilon(x)-\tilde{\gamma}_\varepsilon(y)|^2}\left(\frac{1-\frac{|\tilde{\gamma}_\varepsilon(x)-\tilde{\gamma}_\varepsilon(y)|^2}{|x-y|^2}}{|x-y|^2}\right)\\
&\quad\geq\frac{1-\frac{|\tilde{\gamma}_\varepsilon(x)-\tilde{\gamma}_\varepsilon(y)|^2}{|x-y|^2}}{|x-y|^2}\\
&\quad=\frac{1-\int_0^1\int_0^1\langle\tilde{\gamma}_\varepsilon'(x+t_1(y-x)),\tilde{\gamma}_\varepsilon'(x+t_2(y-x))\rangle dt_1dt_2}{|x-y|^2}.
\end{aligned}
$$

With $f(t){:=}\tilde{\gamma}_\varepsilon'(x+tw)$, we find using that $\tilde{\gamma}_\varepsilon$ is parameterized by arc length that

$$
1-\langle f(t_1),f(t_2)\rangle=\frac{1}{2}|f(t_1)-f(t_2|^2
$$

and hence

$$
\begin{aligned}
&\iint_{\mathbb{R}^2\setminus(B_R(0))^2}\left(\frac{1}{|\tilde{\gamma}_\varepsilon(x)-\tilde{\gamma}_\varepsilon(y)|^2}-\frac{1}{|x-y|^2}\right)\mathrm{d}x\mathrm{d}y\\
&\quad\geq\frac{1}{2}\iint_{\mathbb{R}^2\setminus(B_R(0))^2}\\
&\quad\frac{\int_0^1\int_0^1|\tilde{\gamma}_\varepsilon'(x+t_1(y-x))-\tilde{\gamma}_\varepsilon'(x+t_2(y-x))|^2dt_1dt_2}{|y-x|^2}\mathrm{d}y\mathrm{d}x \qquad (4.8)\\
&\quad\geq\frac{1}{2}\int_{\mathbb{R}^2\setminus(B_R(0))^2}\int_{\mathbb{R}}\\
&\quad\frac{\int_0^\sigma\int_{1-\sigma}^1|\tilde{\gamma}_\varepsilon'(x+t_1(y-x))-\tilde{\gamma}_\varepsilon'(x+t_2(y-x))|^2dt_1dt_2}{|y-x|^2}\mathrm{d}wd.
\end{aligned}
$$

Applying Lemma 2.2 in [3] with $q=2$ and $\varepsilon=\frac{1}{2}$, we get

$$
|f(t_1)-f(t_2)|^2\geq\frac{1}{2}|f(0)-f(1)|^2-C\left(|f(0)-f(t_1)|^2+|f(1)-f(t_2|^2\right).
$$

Thus the right-hand side of (4.8) can further be estimated from below by

$$
\begin{aligned}
&\frac{\sigma^2}{4}\iint_{\mathbb{R}^2\setminus(B_R(0))^2}\frac{|\tilde{\gamma}_\varepsilon'(x)-\tilde{\gamma}_\varepsilon'(y)|^2}{|y-x|^2}\mathrm{d}x\mathrm{d}y\\
&\quad-C\iint_{\mathbb{R}^2}\frac{\int_0^\sigma\int_{1-\sigma}^1|\tilde{\gamma}_\varepsilon'(x)-\tilde{\gamma}_\varepsilon'(x+t_1(y-x))|^2+|\tilde{\gamma}_\varepsilon'(x+t_2(y-x))-\tilde{\gamma}_\varepsilon'(y)|^2dt_1dt_2}{|y-x|^2}\mathrm{d}w\mathrm{d}x.
\end{aligned}
$$

Using Fubini and substituting $w = t_1(y-x)$ and $w = (1-t_2)(y-x)$, respectively, we get the estimate

$$\iint_{\mathbb{R}^2} \frac{\int_0^\sigma \int_{1-\sigma}^1 |\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}_\varepsilon'(x+t_1(y-x))|^2 + |\tilde{\gamma}_\varepsilon'(x+t_2(y-x)) - \tilde{\gamma}_\varepsilon'(y)|^2 dt_1 dt_2}{|y-x|^2} \mathrm{d}y\mathrm{d}x$$
$$\leq 2\sigma^3 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}_\varepsilon'(x+w)|^2}{|w|^2} \mathrm{d}w\mathrm{d}x.$$

Plugging these estimates into (4.8), we get for $\sigma \in (0, 1)$

$$\iint_{\mathbb{R}^2 \setminus (B_R(0))^2} \frac{|\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}_\varepsilon'(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y$$
$$\leq \frac{2}{\sigma^2} \iint_{\mathbb{R}^2 \setminus (B_R(0))^2} \left(\frac{1}{|\tilde{\gamma}_\varepsilon(x) - \tilde{\gamma}_\varepsilon(y)|^2} - \frac{1}{|x-y|^2} \right) \mathrm{d}x\mathrm{d}y + C\sigma \lfloor \tilde{\gamma}_\varepsilon \rfloor^2_{W^{\frac{3}{2},2}}.$$

The Gagliardo semi-norm on the right-hand side can be bounded by the Möbius energy which is uniformly bounded for our curves. Choosing first $\sigma \in (0, 1)$ small enough and then $R > 0$ big enough we get (4.5).

We will now deduce that

$$\lim_{\varepsilon \to 0} \int_{B_R(0)} \int_{B_R(0)} \frac{|(\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}'(x)) - (\tilde{\gamma}_\varepsilon'(y) - \tilde{\gamma}'(y))|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y = 0 \tag{4.9}$$

again using Vitali's theorem. As noted before, we know that the integrand converges to 0 in measure.

To show uniform integrability of the integrands, we use $|a+b| \leq 2(|a|^2 + |b|^2)$ valid for $a, b \in \mathbb{R}^n$ to get the estimate

$$\frac{|(\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}'(x)) - (\tilde{\gamma}_\varepsilon'(y) - \tilde{\gamma}'(y))|^2}{|x-y|^2} \leq 2 \left(\frac{|\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}_\varepsilon'(y)|^2}{|x-y|^2} + \frac{|\tilde{\gamma}'(x) - \tilde{\gamma}'(y)|^2}{|x-y|^2} \right).$$

Of course, one only has to show that the first summand is uniformly integrable for a sequence $\varepsilon_i \downarrow 0$. This can be done using the same arguments as in the proof of Theorem 1.3.

Hence, Vitali's theorem (Theorem 2.2) implies that

$$\lim_{\varepsilon \to 0} \int_{B_R(0)} \int_{B_R(0)} \frac{|(\gamma_\varepsilon'(x) - \tilde{\gamma}'(x)) - (\gamma_\varepsilon'(y) - \tilde{\gamma}'(y))|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y = 0.$$

Let us now conclude the proof of the claim. For $\delta > 0$ we first use (4.5) to get an $R > 0$ such that

$$\iint_{\mathbb{R}^2 \setminus (B_R(0))^2} \frac{|\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}_\varepsilon'(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y \leq \delta$$

for all $\varepsilon > 0$ small enough and

$$\iint_{\mathbb{R}^2 \setminus (B_R(0))^2} \frac{|\tilde{\gamma}'(x) - \tilde{\gamma}'(y)|^2}{|x-y|^2} \mathrm{d}x \mathrm{d}y \leq \delta.$$

Then (4.9) implies

$$\limsup_{\varepsilon \to 0} \lfloor \tilde{\gamma}_\varepsilon' - \tilde{\gamma}' \rfloor^2_{W^{\frac{1}{2},2}} = \limsup_{\varepsilon \to 0} \iint_{\mathbb{R}^2 \setminus (B_R(0))^2} \frac{|\tilde{\gamma}_\varepsilon'(x) - \tilde{\gamma}_\varepsilon'(y)|^2}{|x-y|^2} \mathrm{d}x \mathrm{d}y + \iint_{\mathbb{R}^2 \setminus (B_R(0))^2} \frac{|\tilde{\gamma}'(x) - \tilde{\gamma}'(y)|^2}{|x-y|^2} \mathrm{d}x \mathrm{d}y \leq \delta.$$

and thus

$$\lim_{\varepsilon \to 0} \lfloor \tilde{\gamma}_\varepsilon' - \tilde{\gamma}' \rfloor_{W^{\frac{1}{2},2}} = 0.$$

With the help of Theorem 4.4, we can now also discuss the case of equality in Theorem 4.3 to get the following extension of Corollary 4.1 in [9].

Theorem 4.6 *We have $E^1(\gamma) \geq 2\pi^2$ for all regular curves $\gamma \in W^{\frac{3}{2},2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ with equality if and only if γ is a circle.*

We omit the proof of Theorem 4.6 as it is literally the same as the proof of Corollary 4.1 in [9] where one only uses Theorem 4.4 instead of Theorem 1.2 in [9].

4.3 Inscribing Equilateral polygons

With the tools we have at hand, we can also extend a result of Wu [13] on inscribed equilateral polygons in the following way

Theorem 4.7 *Let $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^d)$ be a uniformly regular curve, i.e.,*

$$\operatorname*{essinf}_{x \in \mathbb{R}/\mathbb{Z}} |\gamma'(x)| > 0$$

with $\gamma' \in VMO$ that is injective. Then for every $n \in \mathbb{N}$, $n \geq 2$, and any $x_0 \in \mathbb{R}/\mathbb{Z}$, there is an inscribed equilateral n-gon with the starting point $\gamma(x_0)$.

The proof is based on the fact, that there is a lower bound $c_n > 0$ of the Gromov distortion of equilateral n-gons as the infimum of the Gromov distortion is attained for an equilateral n-gon and thus cannot be 0. We approximate γ by the smooth curves γ_ε. Then Wu's theorem guarantees the existence of an inscribed equilateral n-gon with the starting point $\gamma_\varepsilon(x_0)$ and the fact above will show that these polygons subconverge to an non-vanishing equilateral n-gon that has all the desired properties.

Proof Let $\gamma_\varepsilon = \gamma * \eta_\varepsilon$ be the standard mollified curves and $\varepsilon > 0$ be so small that γ_ε is a regular curve (cf. Remark 3.1). Furthermore, let p_ε be the in γ_ε inscribed equilateral n-gon through point $\gamma_\varepsilon(x_0)$. Its existence is guaranteed by Wu's theorem. We first note that $\inf_{\varepsilon>0} \operatorname{diam} p_\varepsilon = 0$ would imply due to the uniform bi-Lipschitz estimate from Lemma 1.2

$$\sup_{x\neq y\in\mathbb{R}/\mathbb{Z}, |x-y|\leq \operatorname{diam} p_\varepsilon} \frac{d_{\gamma_\varepsilon}(x,y)}{|\gamma_\varepsilon(x)-\gamma_\varepsilon(y)|} \geq \sup_{x\neq y\in\mathbb{R}/\mathbb{Z}, |\gamma_\varepsilon(x)-\gamma_\varepsilon(y)|\leq \frac{1}{C} \operatorname{diam} p_\varepsilon} \frac{d_{\gamma_\varepsilon}(x,y)}{|\gamma_\varepsilon(x)-\gamma_\varepsilon(y)|} \geq c_n > 0, \tag{4.10}$$

where c_n is a lower bound on the Gromov distortion of equilateral n-gons.

On the other hand, inequality (3.4) implies

$$|\gamma_\varepsilon(x)-\gamma_\varepsilon(y)| \geq |x-y| \left(1 - C\cdot VMO_{\gamma'}\left(\tfrac{|x-y|}{2}\right) - C\cdot VMO(\varepsilon)\right)$$

which implies

$$\sup_{x\neq y\in\mathbb{R}/\mathbb{Z}, |x-y|\leq r} \frac{d_{\gamma_\varepsilon}(x,y)}{|\gamma_\varepsilon(x)-\gamma_\varepsilon(y)|} \leq \frac{c_n+1}{2}$$

for all $r > 0$ and $\varepsilon > 0$ sufficiently small. But this contradicts inequality (4.10).

So we have shown that $\inf_{\varepsilon>0} \operatorname{diam} p_\varepsilon > 0$. As the vertices of the polygons p_ε belong to a bounded set, we can chose a subsequence $\varepsilon_i \to 0$ such that the vertices of the polygons p_{ε_i} converge in $\mathbb{R}^d$ to the vertices of an equilateral n-gon. As furthermore γ_ε converges uniformly to γ as $\varepsilon \downarrow 0$, the equilateral n-gon is inscribed in γ with starting point $\gamma(x_0)$. □

4.4 Γ-Convergence of the Discrete Möbius Energies by Scholtes

Let us extend the Γ-convergence result by Scholtes in [11]. Scholtes introduced the discretized Möbius energy

$$E_n(p) = \sum_{i,j=1}^{m} \left(\frac{1}{|p(a_i)-p(a_j)|^2} - \frac{1}{d_p(a_i,a_j)^2} \right) d_p(a_{i+1},a_i) d_p(a_{j+1},a_j)$$

of a polygon $p : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ with vertices $p(a_i)$, $a_i \in [0,1)$, $i = 1,\ldots,m$.

Theorem 4.8 (Γ-convergence of discrete Möbius energies) *Let $q \in [1,\infty)$. We have*

$$E_n \xrightarrow{\Gamma} E_{\text{möb}}$$

on the space of curves $C^{0,1}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ of unit velocity equipped with the L^q and $W^{1,q}$-norm.

Scholtes proved this theorem for curves that are in C^1, a property that is not implied by bounded Möbius energy. The respective lim inf-inequality was already shown by Scholtes to hold in our more general setting. We will give two proofs of the lim sup-inequality. The first one combines [11, Corollary 1.4] with our extension of Wu's result in the last section. The second proof reduces the problem to the known lim sup-inequality for C^∞ functions approximating the curve by smooth curves using Theorem 1.3.

So the second proof does neither use the full strength of the results by Scholtes nor does it rely on our extension of Wu's theorem. It only relies on the fact that we can approximate our curves and a lim sup-inequality for smooth functions—and hence the method of proof in contrast to the first one should be applicable in other situations as well.

Proof 1 of Theorem 4.8 By Theorem 4.7 we find an inscribed equilateral n-gon p_n in γ. [11, Corollary 1.4] then tells us that

$$\lim_{n\to\infty} E_n(p_n) = E_{\text{möb}}(\gamma).$$

So the only thing left to show is that the p_n converge to γ in $W^{1,q}$ for all $q \in [1,\infty)$. But this follows from the observation by Scholtes, that p_n converges to γ in $W^{1,2}$. Since both p_n and γ are uniformly bounded in $W^{1,\infty}$ we get the convergence in $W^{1,q}$, $q > 2$ using the estimate

$$\|f\|_{L^q}^q = \int |f|^{q-2}|f|^2 \mathrm{d}x \le \|f\|_{L^\infty}^{q-2}\|f\|_{L^2}^2.$$

Proof 2 of Theorem 4.8 Since the lim inf-inequality was already shown by Scholtes, we again only have to prove the lim sup inequality. Scholtes has already shown that the lim sup inequality holds for C^1 curves parameterized by arc length. If now γ is a regular curves with bounded Möbius energy, we can consider the smoothed curves $\gamma_\varepsilon = \gamma * \eta_\varepsilon$ and let $\tilde{\gamma}_m$ the re-parameterizations of the curves $\frac{1}{L(\gamma_{\frac{1}{m}})}\gamma_{\frac{1}{m}}$ by arc length. By Lemma 4.1 we have $\lim_{m\to\infty} E_{\text{möb}}(\tilde{\gamma}_m) = E_{\text{möb}}(\gamma)$.

By the lim sup-inequality of Scholtes, we can find in $\tilde{\gamma}_m$ inscribed equilateral k-gons $p_{m,k}$ with $\limsup_{k\to\infty} E_k(p_{m,k}) \le E_{\text{möb}}(\tilde{\gamma}_m)$. We observe that for all k, $\tilde{m}$, and $m' \ge \tilde{m}$, we have

$$\inf_{m\ge\tilde{m}} E_k(p_{m,k}) \le E_k(p_{m',k}).$$

Taking the limes superior with respect to k of this inequality, we get

$$\limsup_{k\to\infty}\left(\inf_{m\ge\tilde{m}} E_k(p_{m,k})\right) \le \limsup_{k\to\infty} E_k(p_{m',k}) \le E_{\text{möb}}(\tilde{\gamma}_{m'})$$

for all $m' \ge \tilde{m}$. Hence,

$$\limsup_{k\to\infty}\left(\inf_{m\ge\tilde{m}} E_k(p_{m,k})\right) \le \inf_{m\ge\tilde{m}} E_{\text{möb}}(\tilde{\gamma}_m) = E_{\text{möb}}(\gamma)$$

for all $\tilde{m} \in \mathbb{N}$.

If now for every $\tilde{m}, k \in \mathbb{N}$ we pick $m_{\tilde{m},k} \in \mathbb{N}$ such that $E_k(p_{m_{\tilde{m},k}k}) \leq \inf_{m \geq \tilde{m}} E_k(p_{m,k}) + \frac{1}{k}$, we get

$$\begin{aligned}\limsup_{k\to\infty} E_k(p_{m_{\tilde{m},k},k}) &\leq \limsup_{k\to\infty} \left(\inf_{m\geq\tilde{m}} E_k(p_{m,k}) + \frac{1}{k} \right) \\ &= \limsup_{k\to\infty} \left(\inf_{m\geq\tilde{m}} E_k(p_{m,k}) \right) \leq E_{\text{möb}}(\gamma).\end{aligned}$$

Now we are ready to inductively define our sequence of polygons. We let p_k be equal to $p_{m_{1,k},k}$ until $E_k(p_{m_{2,k}}) \leq E_{\text{möb}}(\gamma) + 1$ for all bigger k. Then, we let p_k be $p_{m_{2,k},k}$ until $E_k(p_{m_{3,k}}) \leq E_{\text{möb}}(\gamma) + \frac{1}{2}$ for all bigger k and so on.

This leads to a sequence p_k of k-gons inscribed in the curves $\tilde{\gamma}_{m_k}$ such that both

$$\limsup_{m\to\infty} E_k(p_k) \leq E_{\text{möb}}(\gamma)$$

and $m_k \to \infty$.

We finally have to prove that the polygons $p_k = p_{m_k,k}$ converge to γ in $W^{1,q}$ for $k \to \infty$ for all $q \in [1, \infty)$. By construction, we know that the $p_{m_k,k}$ are uniformly bounded in $W^{1,\infty}$. Let $p_{m_k,k}(x_1^k) = \gamma_m(x_1^k), \ldots, p_{m_k,k}(x_k^k) = \gamma(x_k^k)$ be the vertices of the k-gon $p_{m_k,k}$. Due to the uniform bi-Lipschitz estimate, we have $\frac{1}{Ck} \leq |x_{i+1}^k - x_i^k| \leq \frac{C}{k}$ for a constant $C < \infty$ and hence we get for $x \in [x_i^k, x_{i+1}^k]$ using Taylor's approximation of first order in x_i^k

$$|p_k(x) - \tilde{\gamma}_m(x)| \leq C|x - x_i^k| \leq \frac{C}{k}.$$

So, the p_k converge uniformly to γ.

To get the convergence of the derivatives, we use $p_k' = \overline{(\tilde{\gamma}_{m_k}')}_{[x_i^k, x_{i+1}^k]}$ for $x \in [x_i^k, x_{i+1}^k]$ to estimate

$$\begin{aligned}\int_{\mathbb{R}/\mathbb{Z}} |p_k' - \tilde{\gamma}_{m_k}'| \mathrm{d}x &= \sum_{i=1}^{k-1} \int_{x_i^k}^{x_{i+1}^k} |\overline{(\tilde{\gamma}_{m_k}')}_{[x_i^k, x_{i+1}^k]} - \tilde{\gamma}_{m_k}'| \mathrm{d}y \\ &\leq \sum_{i=1}^{k-1} |x_{i+1}^k - x_i^k| \cdot VMO_{\tilde{\gamma}_{m_k}'}\left(\tfrac{C}{k}\right) \\ &\leq C \cdot VMO_{\tilde{\gamma}_{m_k}'}\left(\tfrac{C}{k}\right).\end{aligned}$$

From (3.4) and (3.3), we get $VMO_{\tilde{\gamma}_{m_k}'} \leq C \cdot VMO_{\gamma_{m_k}'} \leq C \cdot VMO_{\gamma'}$. So $VMO_{\tilde{\gamma}_{m_k}'}(r)$ goes uniformly to zero as $r \to 0$ and hence $p_k' - \tilde{\gamma}_{mk}'$ converges to 0 as k goes to ∞. Since γ_{m_k}' converges to γ' in L^1, we deduce that p_k' converges to γ' in L^1. Since the polygons are furthermore uniformly bounded in $W^{1,\infty}$, we get convergence in $W^{1,q}$ for all $q \in [1, \infty)$ as at the end of proof 1.

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