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## CURVATURE ESTIMATES FOR MINIMAL SURFACES IN 3-MANIFOLDS

BY MICHAEL T. ANDERSON

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This paper is concerned with compactness results for spaces of minimal immersions of surfaces of a fixed topological type in Riemannian 3-manifolds. Given an appropriate area bound, there is a well-known compactness theorem for integral currents or varifolds; this provides a general setting in which to obtain stronger compactness results. However, a sequence of smoothly immersed minimal surfaces of a given topological type will not converge, in general, to a smooth minimal surface of the same type.

The compactness of a class  $C$  of minimal immersions of bounded area is equivalent to the existence of an *a priori* curvature estimate for  $C$ . The aim of the paper is to find natural classes of minimal surfaces admitting such a curvature estimate. Without being too precise at this time, one may see, using appropriate scaling arguments, that if the class  $C$  has such an estimate, then the only complete minimal surfaces in  $\mathbb{R}^3$  in  $C$  are planes. Thus, one seeks a characterization of the planes in  $\mathbb{R}^3$  in terms of a larger class of minimal surfaces.

The prototype of such a characterization is the Bernstein Theorem [B]: a complete minimal graph in  $\mathbb{R}^3$  is a plane. The corresponding curvature estimate is due to Heinz [H] and Osserman [Os]. More recently, Schoen-Simon-Yau [SSY] and Schoen-Simon [SS<sub>1</sub>] have obtained curvature estimates for stable minimal hypersurfaces in  $\mathbb{R}^n$ ,  $n \leq 7$ . In dimension 3, Schoen [S] has recently obtained optimal bounds for stable minimal surfaces.

The main result of the paper is an interior curvature estimate for minimal embedded discs in 3-manifolds (Theorem 2.1); we emphasize that no assumptions on the stability of the surface are made. This is obtained via the following characterization of the plane in  $\mathbb{R}^3$ : the only complete embedded minimal surface of finite topological type with one end and of quadratic area growth is a plane (Corollary 1.5). Note that the approach is to prove the global theorem first and deduce the local curvature estimate as a consequence. We also obtain a curvature estimate at the boundary for embedded minimal discs (Theorem 2.2); the result can be strengthened in case the boundary is extreme (Theorem 2.3). Also we apply the method above to obtain estimates for

immersed minimal discs with boundary of total curvature  $< 6\pi$  (Proposition 2.5) and relate this to a result of Nitsche [N].

The interior curvature estimate (Theorem 2.1) was proved recently by Schoen and Simon [SS<sub>2</sub>]; in fact their result is more general, expressed in terms of a quasiconformal Gauss map. The proofs are different and proceed in 'opposite' directions: they deduce a special case of our global result (Corollary 1.5) as a consequence of their local theorem. On the other hand Corollary 1.5 is also related to recent work of Jorge-Meeks [JM]; building on Osserman's work, they show that a complete embedded minimal surface of finite total curvature and one end is a plane. Again, our proof is different, relying on geometric measure theory rather than complex analysis.

The above curvature estimates are not valid for surfaces with more than one end or genus greater than zero; for example, the space of minimal embeddings of an annulus  $A$  in  $B^3(1)$ , with  $\partial A \subset S^2(1)$  is not compact in the weak topology. Let  $\mathcal{M}_n$  be the space of minimal embeddings of a surface  $\Sigma$  of Euler characteristic  $\geq n$  in  $B^3(1)$  with  $\partial \Sigma \subset S^2(1)$ . Although  $\mathcal{M}_n$  may be non-compact, we show that its boundary  $\bar{\mathcal{M}}_n - \mathcal{M}_n$  in the weak topology is contained in  $\mathcal{M}_{n/2}$ , counted with multiplicity  $\geq 2$  (Theorems 3.1 and 4.2); we use this to obtain some results on the moduli spaces of minimal embeddings of surfaces in compact Riemannian 3-manifolds; see paragraph 4 for further details.

In paragraph 1, we prove certain global results for complete minimal surfaces in  $\mathbb{R}^3$  of quadratic area growth. Theorems 1.1 and 1.3 are of independent interest and admit generalization to higher dimensions. Lemma 1.2 is used repeatedly throughout the paper.

I wish to thank R. Gulliver for some helpful remarks on boundary branch points, Richard Schoen and Frank Morgan for criticism of an earlier draft of the paper and Bill Dunbar for advice on 3-manifolds.

Our results and proofs make use of geometric measure theory; for background information in this field, we suggest [A<sub>1</sub>] and [L<sub>2</sub>].

### 1. Minimal surfaces in $\mathbb{R}^3$ of quadratic area growth

We consider complete minimal surfaces  $\Sigma$  immersed in  $\mathbb{R}^3$  such that

$$(1.1) \quad \text{area}(\Sigma \cap B(r)) \leq C \cdot r^2,$$

for all  $r$ , where  $C$  is a fixed constant depending on  $\Sigma$ , and  $B(r)$  is the ball of radius  $r$  around 0. Surfaces satisfying (1.1) will be said to have quadratic area growth. Using results of Osserman, it is not difficult to show that any minimal surface of finite total curvature satisfies (1.1). The converse, however, is not true; one of Scherk's surfaces gives a counterexample, *cf.* Remark 4 below. The well-known 'monotonicity theorem' for minimal surfaces  $\Sigma$  in  $\mathbb{R}^3$ , *see e. g.* [L<sub>2</sub>], states that

$$v(r) = \frac{\text{area}(\Sigma \cap B(r))}{r^2},$$

is monotone non-decreasing in  $r$ , with  $v(0)=\pi$  provided  $0 \in \Sigma$ . It follows easily from this that minimal surfaces of quadratic area growth are properly immersed.

Consider the family of surfaces  $\Sigma_r = 1/r(\Sigma \cap B(r))$  contained in the unit ball  $B^3(1)$ . Then  $\text{area}(\Sigma_r) \equiv v(r)$  is non-decreasing in  $r$  and bounded above by  $C$ . It follows from the compactness theorem for stationary integral varifolds [Al<sub>1</sub>] that any sequence  $\{\Sigma_{r_i}\}$  subconverges to a stationary integral 2-varifold  $\Sigma_\infty$  supported in  $B^3(1)$ . The limiting varifold may possibly depend on the choice of the sequence  $\{r_i\}$ ; the varifolds  $\Sigma_\infty$  obtained in this fashion will be called *asymptotic varifolds* of  $\Sigma$ .

In this paper, we study mainly minimal surfaces of finite topological type, i. e. of finite genus and with a finite number of ends.

**THEOREM 1.1.** — *Let  $\Sigma$  be a complete minimal immersion of quadratic area growth in  $\mathbb{R}^3$ . Then there is an asymptotic varifold  $\Sigma_\infty$  which is the cone on a stationary integral 1-varifold  $V$  on  $S^2(1)$ . If further  $\Sigma$  is of finite topological type, then  $V$  is a sum of closed geodesics.*

**Remark 1.** — Theorem 1.1 has been proved by Jorge-Meeks [JM] in the context of surfaces of finite total curvature, using the structure theory developed by Osserman. The proof below generalizes naturally to higher dimensions; in fact the proof of the first part of the Theorem carries over immediately to complete minimal immersions  $M^k \rightarrow \mathbb{R}^n$  with  $\text{vol}(M^k \cap B(r)) \leq C r^k$ . Conditions guaranteeing the regularity at infinity of such submanifolds remain to be found however.

*Proof.* — Let  $l(\partial\Sigma_r)$  denote the total length of the boundary  $\partial\Sigma_r$  in  $S^2(1)$ ; choose a sequence  $\{r_j\} \rightarrow \infty$  so that

$$\lim_{j \rightarrow \infty} l(\partial\Sigma_{r_j}) = \lim_{r \rightarrow \infty} l(\partial\Sigma_r)$$

and such that  $\partial\Sigma_{r_j}$  converges to an integral 1-varifold  $B_\infty$  on  $S^2(1)$  in the weak topology on varifolds. We may assume also that  $\{\Sigma_{r_j}\}$  converges to a stationary 2-varifold  $\Sigma_\infty$  in  $B^3(1)$  in the weak topology on 2-varifolds. Now we make the following two observations. First, for any  $r > 0$ ,

$$(1.2) \quad \text{area}(\Sigma_r) = \frac{1}{2} \int_{\partial\Sigma_r} \langle \text{grad } r, \nu \rangle \leq 1/2 l(\partial\Sigma_r),$$

where  $\nu$  is the unit normal to  $\partial\Sigma_r$  and  $\text{grad } r$  is the gradient of the distance function to 0. On the other hand, by the co-area formula

$$(1.3) \quad \int_0^t s \cdot l(\partial\Sigma_{(sr_j)}) ds \leq \text{area}(\Sigma_{r_j} \cap B_t),$$

for  $t \in (0,1)$ . Letting  $j \rightarrow \infty$  and taking  $\lim$  of (1.3) gives

$$(1.4) \quad \frac{t^2}{2} \lim_{j \rightarrow \infty} l(\partial\Sigma_{r_j}) \leq \lim_{j \rightarrow \infty} \underline{M}(\Sigma_{r_j} \cap B(t)).$$

Set  $t=1$  in (1.4) and use (1.2); one obtains

$$\frac{1}{2} \underline{M}(\Sigma_\infty) = \underline{M}(B_\infty) = \underline{M}(C(B_\infty)),$$

since the mass  $\underline{M}$  is continuous in the weak topology on varifolds;  $C(B_\infty)$  denotes the cone on  $B_\infty$  from 0. Using (1.4) again, one has

$$\frac{\underline{M}(\Sigma_\infty \cap B_t)}{t^2} \geq \frac{1}{2} \underline{M}(B_\infty) = \underline{M}(\Sigma_\infty).$$

By the monotonicity theorem,  $\underline{M}(\Sigma_\infty \cap B(t))/t^2 \equiv \underline{M}(\Sigma_\infty)$ , for all  $t \in [0, 1]$ .

The proof that  $\Sigma_\infty$  is a cone over  $B_\infty$  now follows by well-known methods in the theory of area-minimizing currents, *see e. g.* [L<sub>2</sub>], p. 74ff.; it also follows that  $B_\infty$  is a stationary integral 1-varifold on  $S^2(1)$ . We claim further that  $\text{supp}(B_\infty)$  is a union of closed geodesics on  $S^2$  in the case that  $\Sigma$  is of finite topological type. For this, it is sufficient to show that, for any  $x \in \text{supp}(B_\infty)$ ,  $\exists r$  such that

$$B_\infty \llcorner B_x(r) = \sum \alpha_i,$$

where  $\alpha_i$  is an arc of a geodesic, of multiplicity 1, and  $x$  is in the interior of each  $\alpha_i$ .

Since  $B_\infty$  is a stationary integral 1-varifold,  $B_\infty$  is a finite union of geodesic segments, with a finite number of vertices. Let  $x_0$  be a vertex and choose  $r$  so that  $B_\infty \llcorner B_{x_0}(r)$  contains no other vertices. Now

$$B_\infty \llcorner B_{x_0}(r) = \lim_{j \rightarrow \infty} (\gamma_{r_j} \llcorner B_{x_0}(r))$$

where  $\gamma_{r_j} = \partial \Sigma_{r_j}$  is a collection of smoothly immersed curves on  $S^2(1)$ . Let  $\alpha_j^k$  denote the components of  $\gamma_{r_j} \llcorner B_x(r)$  (the connected components of  $\gamma_{r_j}$  lifted to the tangent bundle  $TS^2$ ). Note that  $\partial \alpha_j^k$  is either empty or lies in  $\partial B_{x_0}(r)$ . We claim there is a bound on the number of components  $c_j^k \subset \{\alpha_j^k\}$  such that  $l(c_j^k) \rightarrow 0$  as  $j \rightarrow \infty$ . For if not, by choosing  $r'$  slightly smaller than  $r$ , it follows that  $\gamma_{r_j} \llcorner B_{x_0}(r')$  contains an unbounded number of circles, the length of each one converging to zero as  $j \rightarrow \infty$ . However, since  $\Sigma$  is a properly immersed surface of finite topological type,  $\Sigma_{r_j} \setminus B_0(1/2)$  for  $j$  large consists of a finite number of immersed annuli, each annulus corresponding to one end of  $\Sigma$ . In particular there is a bound  $N$  on the number of boundary components of  $\Sigma_{r_j}$  on  $S^2(1)$ .

We may assume by relabelling that  $\{\alpha_j^k\}$  converges weakly to an integral 1-varifold with  $\partial \alpha_\infty^k \subset \partial B_{x_0}(r)$ . Ignoring those components converging to zero, we have

$$(1.5) \quad B_\infty \llcorner B_{x_0}(r) = \sum \alpha_\infty^k.$$

It is thus sufficient to prove that  $\alpha_\infty^k$  is stationary w. r. t. compactly supported deformations in  $S^2(1) \cap B_{x_0}(r)$ . Fix  $k$  and set  $\bar{\alpha}_j = \alpha_j^k$ . Let  $\bar{A}_j$  be the annulus in  $\Sigma_{r_j} \setminus B_0(1/2)$

such that  $\partial \bar{A}_j \cap (S^2(1) \cap B_{x_0}(r)) = \bar{\alpha}_j$ . Then  $\underline{M}(\bar{A}_j)$  is bounded away from zero and  $\bar{A}_j$  is stationary w. r. t. its boundary; thus the same is true for  $\lim_{j \rightarrow \infty} \bar{A}_j \subset C(B_\infty)$ . Since a domain in a cone is stationary exactly when its boundary on  $S^2(1)$  is stationary, it follows that  $\bar{\alpha}_\infty$  is stationary. ■

*Remark 2.* — We remark that it is natural to consider the converse to Theorem 1.1: if  $\Sigma$  is a complete, properly immersed minimal surface in  $\mathbb{R}^3$  with  $\lim_{j \rightarrow \infty} \{\text{supp } \Sigma_{r_j}\} \subset C(V)$  for some sequence  $r_j \rightarrow \infty$  and  $V$  an integral 1-varifold on  $S^2(1)$ , then  $\Sigma$  has quadratic area growth. This question is open at this time however.

The following Lemma will be used repeatedly in the work to follow in order to relate the local and global topology of a minimal surface; the result is well known in  $\mathbb{R}^3$  and is a consequence of the convex hull property for minimal surfaces. However, we need a formulation for a general Riemannian manifold.

Let  $\Omega$  be a smooth bounded domain in a Riemannian manifold  $N^3$  such that  $\Omega = \{x \in N^3 : f(x) < 0\}$ , where  $f$  is a convex function defined in a neighborhood of  $\Omega$  in  $N^3$  with  $df \neq 0$  on  $\partial\Omega$ ; in other words,  $f$  is a convex defining function for  $\Omega$ . Let  $\Omega_s = f^{-1}(-\infty, s)$ .

**LEMMA 1.2.** — *Let  $\Sigma \subset \Omega$  be a properly immersed minimal surface of finite topological type, where  $\Omega \subset N^3$  has a convex defining function. Then for generic  $s < 0$ ,  $\Sigma \cap \Omega_s$  is a union of properly immersed minimal surfaces of topological type bounded by that of  $\Sigma$ . In particular, if  $\Sigma$  is simply connected, then  $\Sigma \cap \Omega_s$  is a union of simply connected surfaces.*

*Proof.* — Let  $\psi: \Sigma \rightarrow \Omega$  denote the immersion of  $\Sigma$  and let  $s$  be a regular value of  $f \circ \psi$ . Then  $\psi^{-1}(\partial\Omega_s)$  is a disjoint union of Jordan curves  $\{\gamma_\alpha\}$  on  $\Sigma$ . Consider the compact set  $K = \psi^{-1}(\Omega \setminus \Omega_s)$  with  $\partial K = \{\gamma_\alpha\} \cap \partial\Sigma$ . We claim that every component  $P$  of  $K$  satisfies  $\partial P \cap \partial\Sigma \neq \emptyset$ . For suppose  $P_0$  does not satisfy this condition: then  $\psi(\partial P_0) \subset \partial\Omega_s$ . Now the restriction of a convex function to a minimal surface is a subharmonic function on the surface; in particular the maximum value occurs on the boundary. Applying this to  $P_0$  gives  $P_0 \subset \Omega_s$ , a contradiction. Thus every component of  $K$  contains a component of  $\partial\Sigma$ .  $\Sigma \setminus K$  is then a disjoint union of domains on  $\Sigma$ ; the genus of each component is bounded by the genus of  $\Sigma$ . Further, since the curves  $\{\gamma_\alpha\}$  are not nested, the number of ends of each component is bounded by the genus and number of ends of  $\Sigma$ . ■

Let  $V = \sum_{i=1}^n k_i c_i$  be a stationary 1-varifold as in Theorem 1.1; each  $c_i$  is a closed geodesic with multiplicity 1 on  $S^2(1)$  and  $c_i \neq c_j$  for  $i \neq j$ .

**THEOREM 1.3.** — *There is a neighborhood  $\mathcal{U}$  of  $V = \sum_{i=1}^n k_i c_i$  in the weak topology on 1-varifolds in  $S^2(1)$  and a neighborhood  $\mathcal{W}$  of  $C(V)$  in the weak topology on 2-varifolds in  $B^3(1)$  such that if  $S$  is any properly embedded minimal surface in  $B^3(1)$  of finite topological type with  $\partial S = \bar{S} - S \in \mathcal{U}$ , then  $S \notin \mathcal{W}$  unless  $n=1$ .  $\mathcal{W}$  depends only on the topological type of  $S$ .*

*Proof.* — Since the geodesics of  $V$  intersect in a finite number of points on  $S^2(1)$ , it follows that there is a ball  $B \subset B^3(1)$  such that  $C(V) \cap B$  is a sum of flat discs with common diameter  $L$  forming a stationary varifold in  $B$ . Clearly it is sufficient to prove the theorem for the surface  $S \cap B$ , since the topological type of  $S \cap B$  is bounded by that of  $S$ , by Lemma 1.2, and since  $S \in \mathcal{W}$  implies  $S \cap B \in \mathcal{W} \cap B$ . Thus we suppose  $V$  is a sum  $\sum_1^n k_i c_i$  of geodesics intersecting exactly in a pair of antipodal points  $z^\pm \in S^2(1)$ . Let  $L_z$  be the line through  $z^+$  and  $z^-$  and let  $z$  be the coordinate function on  $\mathbb{R}^3$  determined by  $L_z$ .

For a given direction  $L$  in  $\mathbb{R}^3$ , let  $H_L$  denote the coordinate or height function on  $\mathbb{R}^3$  determined by  $L$ . Then the restrictions  $H_L|_S = h_L$  are harmonic functions on  $S$ ; thus the critical points of  $h_L$  are either of index 1 or degenerate. Note that the critical points of  $h_L$  are precisely the points where  $T_x S$  is normal to  $L$ ; a critical point  $x \in S$  is degenerate if in addition the Gauss curvature  $K_S(x) = 0$ . It follows that for a given  $S$ , there is an open and dense set of directions  $L$  so that  $h_L$  is a Morse function with only critical points of index 1. By elementary Morse theory (capping off the ends of  $S$  by discs), there is a bound on the number of critical points of  $h_L$  in  $S \cap B(1-\varepsilon)$ , depending only on the topological type of  $S$ ; renormalizing, we assume  $\varepsilon = 0$ .

Now suppose the theorem were false, i. e. there exists a sequence of embedded minimal surfaces  $S_i \subset B^3(1)$  of bounded topological type with  $\partial S_i \rightarrow V$  and  $S_i \rightarrow C(V)$  weakly as varifolds. By the above, there is a dense set  $\mathcal{D}$  of directions  $L$  so that  $h_L$  is a Morse function on each  $S_i$  with a bounded number of critical points. For any fixed direction  $L \in \mathcal{D}$ , by passing to subsequences, one may assume that the critical points of  $h_L$  on  $S_i$  converge to a finite collection of points.

We claim there is a ball  $\tilde{B}_x \subset B^3(1)$ , centered at  $x \in L_z$ , so that the height function  $z = H_{L_z}$  on  $S_j$  has no critical points in  $\tilde{B}_x$ , for some subsequence  $\{j\} \subset \{i\} \rightarrow \infty$ . This is clear if  $L_z \in \mathcal{D}$ . If  $z|_{S_i}$  has a critical point in  $\tilde{B}_x$  for some  $i$ , then the lines  $\tilde{L} \in \mathcal{D}$  sufficiently close to  $L$  define functions  $H_{\tilde{L}}|_{S_i}$  having critical points in  $\tilde{B}_x$ ; this follows from the fact that the Gauss map is an open map. Thus, if the claim were false, it would follow that given any ball  $\tilde{B}_x(r)$ , for all  $i$  sufficiently large  $H_{L_z}|_{S_i}$  has a critical point in  $\tilde{B}_x(r)$ ; since each  $H_L|_{S_i}$  with  $L \in \mathcal{D}$  has a uniformly bounded number of critical points, this is impossible.

Now we consider the embedded surfaces  $S_j \cap \tilde{B}_x$ ; as  $j \rightarrow \infty$

$$S_j \cap \tilde{B}_x \rightarrow C(V) \cap \tilde{B}_x$$

and  $C(V)$  is a union of flat discs with common diameter  $L_z$ . Furthermore, the slices

$$S_j \cap \tilde{B}_x \cap H_{L_z}^{-1}(t),$$

for  $t \in [-1, 1]$  form a collection of disjoint arcs  $\gamma_j^t(t)$  converging weakly to the slices  $C(V) \cap \tilde{B}_x \cap H_{L_z}^{-1}(t)$ . The latter are a collection of line segments  $\{T_i\}_1^n$  intersecting at the point  $p_t = z^{-1}(t) \cap L_z$ , forming a stationary 1-cone (with generating directions independent of  $t$ ). Fix  $j$  and consider the collection of curves  $\{\gamma_j^t(t)\}_t$  as  $t$  varies. One sees

that at no value  $t$  do any pair of curves intersect; if they did, there would exist two linearly independent tangent vectors at the intersection  $x_0$ , spanning  $T_{x_0} S_j \cap z^{-1}(t)$ . Thus  $x_0$  would be a critical point of  $z|_{S_j}$  in  $\tilde{B}_x$ , a contradiction. It follows that each component  $\gamma_j^l(t)$  traces out an embedded disc  $D_j^l$  as  $t$  runs from  $-1$  to  $1$ .

For all  $j$  sufficiently large,  $S_j \cap \tilde{B}_x$  is thus a disjoint union of  $n$  embedded discs  $\{D_j^l\}_{l=1}^n$ , converging to  $C(V)$  weakly as  $n \rightarrow \infty$ . However, passing to subsequences if necessary, each  $\{D_j^l\}_{j=1}^\infty$  converges to a stationary 2-varifold  $D_\infty^l$  in  $\tilde{B}_x$  and  $C(V) = \sum_1^n D_\infty^l$ . Clearly this is possible only if  $n=1$ . ■

*Remark 3.* — An examination of the proof shows that Theorem 1.3 is valid for arbitrary stationary integral 1-varifolds  $V$ ; such  $V$  are composed of a definite number of geodesic arcs meeting in a finite number of vertices.

As an application of the above, we easily obtain the following.

**THEOREM 1.4.** — *Let  $\Sigma$  be a complete, connected, embedded minimal surface in  $\mathbb{R}^3$  of finite topological type and quadratic area growth. Then the asymptotic varifold  $\Sigma_\infty$  of Theorem 1.1 is a flat disc with multiplicity  $k$ , where  $k$  is the number of ends of  $\Sigma$ .*

*Proof.* — By Theorem 1.1, there is an asymptotic varifold  $\Sigma_\infty$  which is the cone  $C(V)$  on a sum  $V = \sum_1^n k_i c_i$  of closed geodesics on  $S^2(1)$ . The sequence of boundaries  $\{\partial \Sigma_{r_j}\} \subset S^2(1)$  converges weakly to  $V$ ; since  $\{\Sigma_{r_j}\}$  is of bounded topological type (by Lemma 1.2) and converges weakly to  $C(V)$ , it follows from Theorem 1.3 that  $n=1$ , i. e.  $\Sigma_\infty$  is a flat disc with multiplicity  $k \geq 1$ .

Now recall from the proof of Theorem 1.1 that each end  $E^i$  of  $\Sigma$  corresponds to an annulus  $A_j^i$  in  $\Sigma_{r_j} \setminus B(1/2)$ , for  $j$  large. It follows easily from the monotonicity formula that area  $(A_j^i)$  is bounded below and thus

$$l(\partial A_j^i \cap S^2(1)) > c > 0, \quad \forall j \text{ large}, \quad \forall i.$$

For fixed  $i$ , the Jordan curves  $\gamma_j^i = \partial A_j^i \cap S^2(1)$  have a non-zero limit and thus converge to the closed geodesic  $\partial D$  with multiplicity  $m_i$  as  $j \rightarrow \infty$ .

Suppose  $m = m_1 = 2$ ; then there is an arc  $\alpha_j$  in  $\gamma_j^i$  with endpoints  $x_i, y_i$ , of length  $1/10$ , such that  $\alpha_j$  converges to a geodesic arc  $\gamma_\infty$  on  $D$  of length  $1/20$ , but counted twice. If  $v_1$  and  $v_2$  are the endpoints of  $\gamma_\infty$ , then  $x_i$  and  $y_i$  both converge to either  $v_1$  or  $v_2$ , say  $v_1$ . By the proof of Theorem 1.1 [see in particular (1.5) and below],  $\gamma_\infty$  must be stationary with respect to deformations supported in  $B_{v_1}(1/10)$ , which is clearly not the case.

Similar arguments rule out the cases  $m_j > 2$ ; thus each end gives rise to a disc of multiplicity 1. ■

*Remark 4.* — Theorem 1.4 is false if one drops the hypothesis of finite topological type. In fact, one of the Scherk surfaces, given as  $\sin z = \sin hx \cdot \sin hy$  is of infinite



genus, with one end and of quadratic area growth. It is embedded and yet is asymptotic to a union of two orthogonal discs in  $B^3(1)$ . See [P] for a sketch and further details.

**COROLLARY 1.5.** — *Let  $\Sigma$  be a complete embedded minimal surface in  $\mathbb{R}^3$  of finite topological type and of quadratic area growth. If  $\Sigma$  has one end, then  $\Sigma$  is a plane.*

*Proof.* — By Theorem 1.4, there is an asymptotic varifold  $\Sigma_\infty$  which is a disc with multiplicity 1. Thus

$$\pi = \text{area } \Sigma_\infty = \lim_{j \rightarrow \infty} \text{area } (\Sigma_{r_j}) = \frac{1}{r_j^2} \text{area } (\Sigma \cap B(r_j)) \geq \Theta(\Sigma, 0) = \pi,$$

where the inequality follows from the monotonicity formula in  $\mathbb{R}^3$ . Thus  $\text{area } (\Sigma \cap B(r)) = \pi r^2$ , for all  $r$ ; it follows that  $\Sigma$  is a plane. ■

*Remark 5.* — Jorge-Meeks [JM] have proven a similar theorem in the context of finite total curvature.

## 2. Curvature estimates for minimal discs

In this section, we apply the global results of paragraph 1 to obtain estimates on the Gauss curvature  $K$  of embedded minimal discs in Riemannian 3-manifolds. The following compactness theorem is well-known.

**COMPACTNESS THEOREM.** — *Let  $\Omega$  be a bounded domain in a complete Riemannian 3-manifold  $N^3$  and let  $M_i$  be a sequence of minimally immersed surfaces in  $\Omega$ . Suppose there is a constant  $C$  such that the Gauss curvature  $K_{M_i}(x)$  satisfies  $|K_{M_i}(x)| < C$ , for all  $i$ . Then a subsequence of  $\{M_i\}$  converges smoothly (in the  $C^k$ -topology,  $k \geq 2$ ) to an immersed minimal surface  $M_\infty$  (with multiplicity) in  $\Omega$  and  $|K_{M_\infty}(x)| \leq C$ . If each  $M_i$  is embedded, then  $M_\infty$  is also embedded.*

The same result holds more generally for  $k$ -dimensional minimal surfaces in Riemannian  $n$ -manifolds. The proof is obtained by combining the compactness theorem for integral varifolds [Al<sub>1</sub>] with the local regularity given by the curvature bound. The bound implies all minimal surfaces  $M_i$  may be locally graphed over their tangent planes; one then uses standard results from the theory of elliptic P.D.E. to obtain smooth convergence.

*Remark 1.* — There is a similar compactness theorem, without curvature bound, for branched minimal immersions of discs; namely the space of branched minimal immersions of discs in  $N^3$ , with uniformly bounded area, is compact. However, limits of smoothly immersed discs may have branch points (true or false).

Our aim in this section, and those following, is to obtain conditions under which the curvature hypothesis in the Compactness Theorem is derivable from more natural geometric assumptions. We begin with an interior curvature estimate for embedded minimal discs in 3-manifolds. Throughout this section,  $N^3$  denotes a complete oriented Riemannian 3-manifold  $\Omega \subset N^3$  a bounded domain, with convex defining function  $\rho$ .

THEOREM 2.1. — *Let  $D \subset \Omega$  be an embedded minimal disc with  $\partial D \cap \Omega = \emptyset$ . Then there is a constant  $M$ , depending only on area  $(D)$  and the geometry of  $\Omega$  such that*

$$(2.1) \quad |K_D(x)| \leq \frac{M}{R^2(x)},$$

where  $R(x) = \text{dist}_N(x, \partial\Omega)$ .

*Proof.* — One argues by contradiction; if the theorem were false, there would exist embedded minimal discs  $D_i \subset \Omega$ , with  $\text{area}(D_i) \leq A$  and  $R_i \in [0, \text{diam}(\Omega)]$  such that

$$(2.2) \quad R_i^2 \cdot \sup \{ |K_i(x)| : \text{dist}_N(x, \partial\Omega) \geq R_i \} \xrightarrow{i \rightarrow \infty} \infty.$$

Let  $\bar{K}_i = \sup \{ |K_i(x)| : \text{dist}_N(x, \partial\Omega) \geq R_i \}$  and suppose  $|K_i(x_i)| = \bar{K}_i$ . Let  $\delta_i$  be the geodesic dilation of  $x_i$  by the factor  $\sqrt{\bar{K}_i}$ ; that is, if the metric  $ds^2$  of  $\Omega$  is expressed in geodesic polar coordinates at  $x_i$ , then  $ds_i^2 = \varphi_i(ds^2)$  is the metric obtained by multiplying the radial component by  $\sqrt{\bar{K}_i}$ , leaving the angular component unchanged. Passing perhaps to a subsequence, the Riemannian manifolds  $(\Omega, ds_i^2)$  converge to a domain  $\Omega_\infty$  in  $\mathbb{R}^3$  with a flat metric; by (2.2),  $\Omega_\infty = \mathbb{R}^3$  with a complete flat metric. Let  $B_i(t)$  denote the geodesic  $t$ -ball about  $x_i$  in  $(\Omega, ds_i^2)$ ; then one has

$$(2.3) \quad \frac{\text{area}(\delta_i(D_i) \cap B_i(t))}{t^2} = \text{area}(D_i \cap B_{x_i}(t/\sqrt{\bar{K}_i})) \cdot \frac{1}{(t/\sqrt{\bar{K}_i})^2} \leq C \cdot \text{area}(D_i),$$

where the last inequality follows from the monotonicity formula for minimal surfaces in  $\Omega$  (cf. [L<sub>2</sub>]); the constant  $C$  depends only on the geometry of  $\Omega$ . It now follows by the Remark above, together with Lemma 1.2, that  $\tilde{D}_i = \delta_i(D_i)$  subconverges to a complete embedded minimal disc  $D_\infty$ , possibly with multiplicity, in  $\mathbb{R}^3$ ;  $D_\infty$  is of quadratic area growth by (2.3).

Now within the balls  $\delta_i(B_{R_i})$ , where  $B_{R_i} = \{x \in \Omega : \text{dist}_N(x, \partial\Omega) \geq R_i\}$ , the curvature of the discs  $\tilde{D}_i$  is bounded above in absolute value by 1. By the Compactness Theorem above,  $\tilde{D}_i \cap \delta_i(B_{R_i})$  converges smoothly to  $\tilde{D}_\infty \cap H$ , where  $H = \lim_{i \rightarrow \infty} \delta_i(B_{R_i})$ . It is easily seen that either  $H = \mathbb{R}^3$  or  $H$  is a half-space in  $\mathbb{R}^3$ . In either case, if  $x_\infty = \lim_{i \rightarrow \infty} x_i$ , then

there is a neighborhood of  $x_\infty$  in  $H$  in which  $\tilde{D}_\infty$  has non-zero curvature. This is clear if  $H = \mathbb{R}^3$ ; if  $H$  is a half-space, we may assume that  $|\nabla K_{(\tilde{D}_i)}|$  is bounded away from  $\infty$  near  $x_\infty$  since we may dilate further to achieve this. Thus, by smooth convergence, the same fact holds on  $\tilde{D}_\infty$ . In particular,  $\tilde{D}_\infty$  is not a plane in  $\mathbb{R}^3$ . This stands in contradiction to Corollary 1.5. ■

*Remark 2.* — (1) We note that Theorem 2.1 is a special case of recent work of Schoen and Simon [SS<sub>2</sub>] on surfaces with quasi-conformal Gauss map; the proofs are different however.

(2) There is no general curvature estimate for immersed discs. For instance, if  $E$  denotes Enneper's surface in  $\mathbb{R}^3$ , then the surfaces  $E_r = 1/r(E \cap B(r))$  are a sequence of

immersed discs in  $B^3(1)$  converging to a flat disc with multiplicity three as  $r \rightarrow \infty$ . However, the curvature of  $E_r$  at the origin becomes unbounded as  $r \rightarrow \infty$ .

(3) It is natural to ask if the hypothesis on the area bound in Theorem 2.1 can be removed. However, the 'blow-down' of the helicoid to a sequence of surfaces in  $B^3(1)$  shows this is not possible.

We now turn to a discussion of boundary estimates. First, there is a general estimate on the curvature near the boundary, which depends however on the geometry of the boundary. Let  $\mathcal{J}$  be the space of  $C^{2,\alpha}$  Jordan curves in  $\Omega \subset N^3$ , parametrized according to arc length. We endow  $\mathcal{J}$  with the usual  $C^{2,\alpha}$  topology. We note explicitly that each  $\gamma \in \mathcal{J}$  is an embedding of a circle  $S^1(l)$  of length  $l$  in  $N^3$ ; in particular, a curve of multiplicity  $p > 1$  is not in  $\mathcal{J}$ , but rather in  $\bar{\mathcal{J}}$ .

**THEOREM 2.2.** — *Let  $\Omega$  be a domain in  $N^3$  with a strictly convex defining function and let  $\mathcal{D}$  be a compact subset of the space of Jordan curves in  $\Omega$ . Let  $D$  be an embedded minimal disc in  $\Omega$  with  $\partial D = \gamma \in \mathcal{D}$ . Then there is a constant  $M$ , depending only on  $\mathcal{D}$  and  $\Omega$  such that*

$$(2.4) \quad |K_D(x)| \leq M,$$

for all  $x \in D$ .

*Proof.* — If the theorem were false, it would follow that there exist Jordan curves  $\gamma_i$  converging to  $\gamma$  and embedded minimal discs  $D_i = \text{Im } f_i$ ,  $f_i: \Delta \rightarrow N^3$  with  $\partial D_i = \gamma_i$  and such that

$$(2.5) \quad \sup_{x \in D_i} |K_{D_i}(x)| \xrightarrow{i \rightarrow \infty} \infty.$$

Let  $f$  be the defining function of  $\Omega$ ; then  $D^2 f \geq c \cdot I$ , for some  $c > 0$ . Let  $\Delta$  denote the Laplacian on  $D_i$ ; since  $D_i$  is minimal, one obtains

$$c \cdot \text{area}(D_i) = \int_{D_i} c \cdot 1 \leq \int_{D_i} \Delta f = \int_{\partial D_i} \frac{\partial f}{\partial n},$$

where  $n$  is the outward unit normal. Since  $|df|$  and  $l(\partial D_i) = l(\gamma_i)$  are bounded, it follows that  $\text{area}(D_i)$  is bounded. Thus Theorem 2.1 and the Compactness Theorem imply that  $\{D_i\}$  subconverges smoothly on compact sets  $K \subset \mathring{\Delta}$  to a minimal disc  $D_\infty = \text{Im } f_\infty$ ,  $f_\infty: \Delta \rightarrow N^3$ ;  $f_\infty$  is an embedding on  $\mathring{\Delta}$ . Further, the functions  $|K_{D_i}|$  are uniformly bounded on compact sets  $K \subset \mathring{\Delta}$ .

We claim that (2.5) implies that  $D_\infty$  has a boundary branch point. Supposing this were not so, it would follow that  $D_\infty$  is a smoothly immersed closed disc, of multiplicity 1, with  $\partial D_\infty = \gamma$ ;  $D_\infty$  is embedded in the interior. By the Allard boundary regularity theorem [Al<sub>2</sub>], for any  $p \in \gamma$ , there is a small ball  $B_p(r)$  so that  $B_p(r) \cap D_\infty$  is contained in the graph of a  $C^{2,\alpha}$  function  $f_\infty$ , defined over  $\mathcal{P} \cap B_p(r)$ , where  $\mathcal{P} = \exp_p(T_p D_\infty)$ . (We graph over domains in  $\mathcal{P}$  by the exponential map normal to  $\mathcal{P}$ .) Since

$$\text{area}(D_i \cap B_p(r)) \rightarrow \underline{M}(D_\infty \cap B_p(r)) \quad \text{and} \quad \underline{M}(D_\infty \cap B_p(r)) = \text{area}(D_\infty \cap B_p(r))$$

because  $D_\infty$  has multiplicity 1, we may choose  $r$  so small that, for  $i$  sufficiently large,  $D_i \cap B_p(r)$  is also contained in the graph of a  $C^{2,\alpha}$  function  $f_i$ , defined over domains in  $\mathcal{P} \cap B_p(r)$ ; this follows again by Allard regularity.

It follows that  $f_i \rightarrow f_\infty$  in the  $C^{2,\alpha}$  norm. Since the curvature of  $D_\infty$  is uniformly bounded,  $|K_{D_i}|$  is uniformly bounded in  $B_p(r)$ . Thus, (2.5) implies the existence of a boundary branch point on  $D_\infty$ . On the other hand, it is well known that an embedded disc  $D_\infty = \text{Im } f_\infty : \Delta \rightarrow \mathbb{N}^3$  of multiplicity one has no boundary branch points; see [GL], Lemma 1, for a proof. ■

**Remark 3.** — Examples show that the condition  $\partial D$  be contained in a compact set of Jordan curves is necessary. For instance, one may cut Enneper's surface along a line  $L$  in  $\mathbb{R}^3$  to obtain a 'half' complete minimal surface  $E'$  and use the sequence  $E'_r = 1/r(E' \cap B(r))$  obtain a contradiction if this hypothesis is dropped.

Theorem 2.2 may be strengthened in case the boundary is extreme.

**THEOREM 2.3.** — *Let  $D$  be a minimally embedded disc in a convex domain  $\Omega \subset \mathbb{N}^3$ . Suppose  $\partial D = \gamma$  is a  $C^1$  Jordan curve on  $\partial\Omega$ . Then there is a constant  $C$  such that*

$$\sup_{x \in D} |K_D(x)| \leq C,$$

where  $C$  depends only on the geometry of  $\Omega$ ,  $\text{area}(D)$  and the  $C^1$  norm of  $\gamma$ , when parametrized by arc length.

*Proof.* — We give a brief outline of the proof, since it resembles earlier arguments. If the Theorem were false, the dilations of  $D_i$  by the factor  $\bar{K}_i = \sup_{x \in D_i} |K_{D_i}(x)|$  converge

smoothly to an embedded minimal disc  $\tilde{D}_\infty$  in  $\mathbb{R}^3$  which is either complete or has boundary a complete straight line  $L$  in  $\mathbb{R}^3$ ; this follows from the bound on the  $C^1$  norm of  $\gamma$ . In the former case, the proof follows as in Theorem 2.1. In the latter case, since  $D$  is contained in a convex domain,  $\tilde{D}_\infty$  is contained in a half-space  $H$  with  $L \subset \partial H$ . Reflect  $\tilde{D}_\infty$  through the line  $L$  to obtain  $\tilde{D}_\infty$ . It follows from the reflection principle for minimal surfaces in  $\Omega^3$  that  $\tilde{D}_\infty \cup \tilde{D}_\infty$  is a smooth complete minimal surface, which is embedded since  $\tilde{D}_\infty \subset H$ . By the arguments in Theorem 2.1,  $\tilde{D}_\infty$  is not a plane; however, it has quadratic area growth, and thus contradicts Corollary 1.5. ■

One immediately obtains compactness theorems from Theorems 2.2 and 2.3. For instance:

**COROLLARY 2.4.** — *Let  $C$  be a compact subset of the space of Jordan curves  $\mathcal{J}$  in  $\Omega \subset \mathbb{N}^3$ . Then the space of minimal discs  $D \subset \Omega$ , embedded in the interior, with  $\partial D \in C$  is compact in the weak topology on 2-varifolds.*

**Remark 4.** — Tomi [T] proved that an analytic Jordan curve in  $\mathbb{R}^3$  bounds only finitely many least area discs. Using his method together with Corollary 2.4, one may easily prove that any  $C^{4,\alpha}$  Jordan curve in  $\mathbb{R}^3$  bounds only finitely many stable and embedded minimal discs.

In general, the above estimates have no analogues for immersed minimal discs. In one case however, we are able to obtain an estimate for this larger class of surfaces.

**PROPOSITION 2.5.** — *Let  $\mathcal{D}$  be a compact subset of the space of Jordan curves  $\mathcal{J}$  in  $\mathbb{R}^3$  such that  $\max \kappa_\gamma < 6\pi$ ,  $\gamma \in \mathcal{D}$  where  $\kappa_\gamma$  is the total absolute curvature of  $\gamma$ . Then there is a constant  $M$ , depending only on  $\mathcal{D}$  such that*

$$|K_\Sigma(x)| \leq \frac{M}{R^2(x)},$$

for any minimally immersed disc  $\Sigma$  with  $\partial\Sigma = \gamma$ ;  $R(x) = \text{dist}(x, \gamma)$ .

*Proof.* — As in the proof of Theorem 2.1, if the conclusion were false, there would exist a sequence of curves  $\{\gamma_i\}$  such that  $\gamma_i \rightarrow \gamma \in \mathcal{D}$  and minimally immersed discs  $\Sigma_i$ ,  $\partial\Sigma_i = \gamma_i$ , such that

$$R_i^2 \cdot \sup \{ |K_i(x)| : \text{dist}(x, \gamma_i) \geq R_i \} \xrightarrow{i \rightarrow \infty} \infty,$$

for some sequence  $\{R_i\}$ . Let  $\tilde{\Sigma}_i$  be the dilated discs in  $\mathbb{R}^3$  as in Theorem 2.1; then  $\{\tilde{\Sigma}_i\}$  subconverges to a complete minimally immersed disc  $\tilde{\Sigma}_\infty$ , possibly with isolated branch points, but not a plane.

Suppose that  $\tilde{\Sigma}_\infty$  has no branch points. By Gauss-Bonnet,

$$2\pi - \int_{\gamma_i} \kappa_{\Sigma_i} = \int_{\Sigma_i} K_{\Sigma_i} = \int_{\tilde{\Sigma}_i} K_{\tilde{\Sigma}_i}.$$

By lower semi-continuity of the area integral, applied to the Gauss map,

$$\int_{\tilde{\Sigma}_i \cap B} K_{\tilde{\Sigma}_i} \leq \int_{\tilde{\Sigma}_\infty \cap B} K_{\tilde{\Sigma}_\infty},$$

for any compact ball  $B$ . By a theorem of Osserman [Os], the total curvature of  $\tilde{\Sigma}_\infty$  is  $-4n\pi$ ,  $n \geq 1$  so that, given any  $\varepsilon > 0$ , we may choose  $B$  so that

$$\int_{\tilde{\Sigma}_\infty \cap B} K_{\tilde{\Sigma}_\infty} \leq -(4n\pi - \varepsilon).$$

This gives

$$6\pi - \varepsilon \leq \int_{\gamma_i} \kappa_{\Sigma_i} \leq \int_{\gamma_i} \kappa_{\gamma_i} \xrightarrow{i \rightarrow \infty} \int_{\gamma} \kappa_{\gamma}.$$

Since  $\varepsilon$  is arbitrary, one obtains a contradiction.

If  $\tilde{\Sigma}_\infty$  has a branch point at  $p_\infty$ , then the curvature of  $\tilde{\Sigma}_i$  diverges to  $-\infty$  at some sequence  $\{p_i\}$  converging to  $p_\infty$ . Dilating by this factor of the curvature, we may apply the above argument to obtain a contradiction. ■

*Remark 5.* — In [N] (see also [Ob]), Nitsche proved the following result. Let  $\gamma$  be an analytic Jordan curve in  $\mathbb{R}^3$  with total curvature  $\leq 6\pi$  with the property that no branched minimal immersion of a disc with boundary  $\gamma$  has interior or boundary branch points. Then  $\gamma$  bounds only finitely many minimally immersed discs. Using Proposition 2.5 and his method of proof, if  $\kappa_\gamma < 6\pi$ ,  $\gamma \in C^{4,\alpha}$  and one assumes the absence of boundary branch points, then the same conclusion holds. For example, if  $\gamma$  is extreme, then  $\gamma$  as above bounds only finitely many minimally immersed discs. As another example, if  $\kappa_\gamma < 6\pi$  and  $\gamma \in C^{4,\alpha}$  then  $\gamma$  bounds only finitely many embedded minimal discs.

### 3. Estimates for surfaces of higher genus

The object of this section is to extend the main results of paragraph 2 to embedded minimal surfaces of higher topological type. For this class of surfaces, there is no *a priori* curvature bound; however, if the curvature of a sequence of surfaces does diverge, the limit surfaces retain much of the regularity. As in paragraph 1,  $\Omega \subset \mathbb{N}^3$  is a strictly convex domain with defining function  $f$ , and  $\Omega_{-\varepsilon} = f^{-1}(-\infty, -\varepsilon]$ .

**THEOREM 3.1.** — *Let  $\tilde{\Sigma}_i$  be a collection of embedded minimal surfaces in  $\Omega$ ,  $\partial\tilde{\Sigma}_i \subset \partial\Omega$ ; suppose, for some  $\varepsilon > 0$ , each surface  $\Sigma_i = \tilde{\Sigma}_i \cap \Omega_{-\varepsilon}$  is connected and diffeomorphic to a fixed surface  $\tilde{\Sigma}$  of Euler characteristic  $\chi = n$ . If the area of  $\{\Sigma_i\}$  is uniformly bounded, then either:*

(1)  $\{\Sigma_i\}$  subconverges smoothly to an embedded minimal surface of the same topological type or:

(2)  $\{\Sigma_i\}$  subconverges in the weak topology on varifolds to a smoothly embedded minimal surface  $\Sigma_\infty$  of Euler characteristic  $\chi \geq n/2$ , counted with multiplicity  $\geq 2$ ; the convergence  $\Sigma_i \rightarrow \Sigma_\infty$  is smooth away from a finite number of points  $x_i \in \Sigma_\infty$ .

*Proof.* — Suppose first that for all  $x \in \Omega_{-\varepsilon}$ , there is an  $r = r(x)$ , such that  $\Sigma_i \cap B_x(r)$  is either empty or a disjoint collection of embedded discs. In this case, the curvature estimate Theorem 2.1 and the Compactness Theorem imply that  $\Sigma_i \cap B_x(r)$  subconverges smoothly to a smooth limiting surface  $\Sigma_\infty \cap B_x(r)$ . If the multiplicity of  $\Sigma_\infty$  is 1, it follows that  $\Sigma_\infty$  is diffeomorphic to  $\tilde{\Sigma}$  and 1) holds. Otherwise,  $\text{multi}(\Sigma_\infty) \geq 2$  and 2) holds (with  $\{x_i\} = \emptyset$ ); this may happen if  $\Sigma_\infty$  is not orientable.

If the assumption above does not hold, there exist  $x \in \Omega_{-\varepsilon}$  such that  $\Sigma_i \cap B_x(r)$  has non-simply connected components, for  $r$  arbitrarily small. Let  $\mathcal{N} = \{x : \forall r > 0, \exists I_0 \text{ such that } \Sigma_i \cap B_x(r) \text{ is not simply connected, } \forall i \geq I_0\}$ . We will first show that  $\mathcal{N}$  consists of a finite number of points. To do this, let  $\mathcal{S}_i$  denote the set of essential, simple closed loops  $\gamma_i$  on  $\Sigma_i$ . Define  $\mathcal{S}$  to be the set of equivalence classes  $[\Gamma]$  of sequences  $\{\gamma_i\}_{i=1}^\infty$ ,  $\gamma_i \in \mathcal{S}_i$ , such that  $l(\gamma_i) \rightarrow 0$  as  $i \rightarrow \infty$ ;  $[\Gamma^1]$  is equivalent to  $[\Gamma^2]$  if the loops  $\gamma_i^1$  are freely homotopic to  $\gamma_i^2$ . Next, given  $[\Gamma] = \{\gamma_i\} \in \mathcal{S}$ , let  $\mathcal{D}[\Gamma] = \{[\sigma] = \{\sigma_i\} \in \mathcal{S} : \sigma_i \cap \gamma_i = \emptyset, \text{ for } i \text{ sufficiently large, for some representatives } \{\sigma_i\} = [\sigma] \text{ and } \{\gamma_i\} = [\Gamma]\}$ .

We define inductively a set  $\beta \subset \mathcal{S}$  as follows. First, choose an element  $[\Gamma^1] \in \mathcal{S}$ ; if  $\mathcal{D}[\Gamma^1] = \emptyset$ , let  $\beta = [\Gamma^1]$ . If  $\mathcal{D}[\Gamma^1] \neq \emptyset$ , choose  $[\Gamma^2] \in \mathcal{D}[\Gamma^1]$  and let  $\beta = [\Gamma^1] \cup [\Gamma^2]$  if

$\mathcal{D}[\Gamma^1] \cap \mathcal{D}[\Gamma^2] = \emptyset$ . In case  $\mathcal{D}[\Gamma^1] \cap \mathcal{D}[\Gamma^2] \neq \emptyset$ , choose  $[\Gamma^3] \in \mathcal{D}[\Gamma^2] \cap \mathcal{D}[\Gamma^1]$  and set  $\beta = [\Gamma^1] \cup [\Gamma^2] \cup [\Gamma^3]$  if  $\mathcal{D}[\Gamma^1] \cap \mathcal{D}[\Gamma^2] \cap \mathcal{D}[\Gamma^3] = \emptyset$ ; and so on. Now it is well known that a surface of fixed topological type admits a finite number of disjoint, simple, essential curves in distinct free homotopy classes; ( $3g-3$  in case of a closed surface of genus  $g$ ). It follows that there is a fixed, finite upper bound on the cardinality of  $\beta$ . We choose once and for all fixed representatives  $[\tau^j] = \{\tau_i^j\}$  for the classes of  $\beta$ . We may also pass to subsequences and assume henceforth that the curves  $\{\tau_i^j\}$  converge to a finite set of points  $X = \{x^j\}_{j=1}^n$  in  $\Omega_{-\varepsilon}$ . Since every element  $[\Gamma] \in \mathcal{S} \setminus \beta$  intersects non-trivially any representative of some class  $[\mu] \in \beta$ , it follows that  $X$  is also the set of accumulation points of sequences  $[\Gamma] \in \mathcal{S} \setminus \beta$ .

Now it is straightforward to see that  $\mathcal{N} = X$ . Clearly  $X \subset \mathcal{N}$ ; if  $p \in \mathcal{N}$ , we may choose an essential simple loop  $\sigma_i$  in  $B_p(r) \cap \Sigma_i$ , where  $r \rightarrow 0$  as  $i \rightarrow \infty$ . We may assume  $l(\sigma_i) \rightarrow 0$ . Further, by Lemma 1.2,  $\sigma_i$  does not bound a disc in  $\Sigma_i$ . Thus,  $\{\sigma_i\}$  determines a class  $[\sigma]$  in  $\mathcal{S}$ , with limit point  $p$ . If  $p \notin X$ , we must have  $[\sigma] \in \beta$  and so  $\sigma_i$  is homotopic to  $\tau_i$ , where  $\{\tau_i\}$  is the representative of  $[\tau] \in \beta$  chosen above. Then  $\sigma_i - \tau_i$  is the boundary of an annulus  $A_i \subset \Sigma_i$ , for  $i$  sufficiently large, whose boundary components converge to the zero varifold, but remain a fixed distance apart. Using, for instance, the monotonicity formula in  $\Omega$ , this is easily seen to be impossible. Further, we note that the above argument implies that the curves in  $\partial \Sigma_i \subset \partial \Omega_{-\varepsilon}$  have lengths bounded away from 0. This in turn implies that the points of  $X$  are in  $\bar{\Omega}_{-\varepsilon}$ .

Let  $\Sigma_\infty$  be a weak limit of  $\{\Sigma_i\}$  in the space of varifolds. For any  $x \notin \mathcal{N}$ , there is an  $r$  such that  $\Sigma_i \cap B_x(r)$  is a union of embedded discs; thus  $\Sigma_i \cap B_x(r)$  converges smoothly to  $\Sigma_\infty \cap B_x(r)$ . In particular,  $\Sigma_\infty \cap B_x(r)$  is a union of smoothly embedded discs, with multiplicity. Now suppose  $x \in \mathcal{N}$  and let  $C_x$  be the varifold tangent cone to  $\Sigma_\infty$  at  $x$ :  $C_x$  is the cone on a stationary 1-varifold on  $S^2(1)$ . By definition, there is a sequence  $x_i \in \Sigma_i$  and  $r_i \rightarrow \infty$  such that the dilations of  $\Sigma_i$  at  $x_i$ ,  $\tilde{\Sigma}_i = r_i(\Sigma_i)_{x_i}$ , converge weakly to  $C_x$  (see the proof of Theorem 2.1 for the definition of dilation at  $x_i$ ). The sequence  $\{\tilde{\Sigma}_i\}$  is a sequence of embedded minimal surfaces, of bounded type by Lemma 1.2, converging weakly to  $C(V) = C_x$ . By Remark 3 of paragraph 1,  $V$  is a single geodesic with multiplicity and  $C_x$  is thus a plane with multiplicity; cf. also Proposition 1 of [CS]. It follows now by the Allard regularity Theorem that  $\Sigma_\infty$  is a smoothly embedded minimal surface. Further,  $\Sigma_\infty$  has multiplicity  $\geq 2$ ; in fact, if  $\Sigma_\infty$  had multiplicity 1, the convergence  $\Sigma_i \rightarrow \Sigma_\infty$  would be smooth (by Allard regularity), which implies  $\mathcal{N} = \emptyset$ .

We claim that  $\Sigma_\infty$  results topologically from  $\Sigma_i$  by performing surgery on a subset (perhaps proper) of representatives of  $\mathcal{S}$ , together possibly with a collapsing of certain handles in  $\Sigma_i$ . To see this, let  $x \in X$  and choose  $r$  such that  $\Sigma_\infty \cap B_x(r)$  is a disc in  $D$ , with multiplicity  $m \geq 1$ . Then  $\Sigma_i \cap B_x(r)$  consists of a collection of disjoint embedded surfaces  $Z_i^1, Z_i^2, \dots$  containing loops in  $\mathcal{S}$ , together with a collection of embedded discs.

These latter converge smoothly to  $k \cdot D$ , for some  $k < m$ . The surfaces  $Z_i^j, \dots$  converge weakly to the disc  $D$ , with multiplicity  $q, \dots$  equal to the number of ends of  $Z_i$  in  $B_x(r)$ . This follows from the fact that the convergence is smooth away from  $x$ , so that the multiplicity is at least  $q$ , together with the fact that  $\{Z_i\}$  consists of embedded surfaces, so that the multiplicity is at most  $q$ .

It follows that  $\Sigma_\infty \cap B_x(r)$  is obtained from  $\Sigma_i \cap B_x(r)$  by surgery on a set of loops  $\{\delta_i^1\}, \{\delta_i^2\}, \dots$  on  $Z_i^1, Z_i^2, \dots$ , with  $\{\delta_i^j\} \in \mathcal{S}$ , together with possible collapse of handles in  $Z_i^1, Z_i^2, \dots$  to points. Since the convergence  $\Sigma_i \rightarrow \Sigma_\infty$  is smooth away from  $X$ , the claim above follows. Finally, since surgery on a simple closed curve increases the Euler characteristic by 2, and collapse of handles leads to a further increase, the result follows. ■

The above result leads easily to a compactness theorem for embedded minimal surfaces of fixed topological type, provided the boundary is controlled; let  $\mathcal{J}_k$  be the space whose elements are systems of  $k$  disjoint Jordan curves in  $\partial\Omega \subset \mathbb{N}^3$ .  $\mathcal{J}_k$  is an open set of  $X$  and is given the induced topology, where  $\mathcal{J}$  is topologized as in paragraph 2. We assume as before  $\Omega$  is strictly convex.

Analogous to Corollary 2.4, one has:

**THEOREM 3.2.** — *Let  $\mathcal{D}$  be a compact subset of  $\mathcal{J}_k$ . Let  $\Sigma$  be an embedded minimal surface of genus  $g$ , in  $\Omega$  with  $\partial\Sigma \in \mathcal{D}$ . Then there is a constant  $M$ , depending only on  $\mathcal{D}$ ,  $\Omega$  and  $\chi$  such that*

$$|K_\Sigma(x)| \leq M,$$

for all  $x \in \Sigma$ . In particular, the space of minimal immersions of a surface of genus  $g$ , embedded in the interior, with boundary in  $\mathcal{D}$  is compact in the weak topology on varifolds.

*Proof.* — Let  $\Sigma_i$  be an infinite sequence of embedded minimal surfaces of  $\chi = n$  with  $\partial\Sigma_i \in \mathcal{D}$ ; area  $(\Sigma_i)$  is bounded by the argument used in Theorem 2.2. Applying Theorem 3.1 to  $\{\Sigma_i\}$ , the boundary condition  $\partial\Sigma_i \in \mathcal{D}$  implies that the alternative (2) is impossible. ■

#### 4. Closed minimal surfaces in 3-manifolds

The results of the previous sections give curvature estimates and compactness results for closed embedded minimal surfaces in complete oriented Riemannian 3-manifolds  $\mathbb{N}^3$ .

Let  $\mathcal{M}_n$  denote the space of minimal embeddings  $\Sigma_n$  of a closed surface of Euler characteristic  $\chi \geq n$  in  $\mathbb{N}^3$ ;  $\mathcal{M}_n$  is given the weak topology as a subset of the space of 2-varifolds. Let  $\bar{\mathcal{M}}_n$  denote the closure of  $\mathcal{M}_n$  in this topology; thus  $\mathcal{S} \in \bar{\mathcal{M}}_n$  if and only if  $\mathcal{S} \in \mathcal{M}_n$  or  $\mathcal{S}$  is the varifold limit of a sequence  $\mathcal{S}_i$  in  $\mathcal{M}_n$ .

**THEOREM 4.2.** — *The boundary  $\partial\mathcal{M}_n = \bar{\mathcal{M}}_n - \mathcal{M}_n$  of the space of embedded closed minimal surfaces of Euler characteristic  $\geq n$  in  $\mathbb{N}^3$  is contained in  $\mathcal{M}_{n/2}$ , counted with multiplicity  $\geq 2$ .*

*Proof.* — Let  $\Sigma_i$  be a sequence of minimal surfaces in  $\mathcal{M}_n$ , with no convergent subsequence in  $\mathcal{M}_n$ . Let  $\Sigma_\infty$  be a varifold limit of  $\Sigma_i$ . Choose  $x \in \text{supp } \Sigma_\infty$  and let  $B_x(r)$  be a small convex ball around  $x$ ; then  $\Sigma_i \cap B_x(r)$  is a union of minimal surfaces, of bounded genus. Consider the components  $P_i$  (in the tangent bundle) of  $\Sigma_i \cap B_x(r)$  which intersect  $B_x(r/2)$ . One has  $\partial P_i \cap S_x(r/2)$  is a union of Jordan curves on  $S_x(r/2)$ ,



none of which bound surfaces in the solid annulus  $P_i \cap (B_x(r) - B_x(r/2))$ , by Lemma 1.2. It follows that the number of components of  $\partial P_i \cap S_x(r)$  is at least as large as the number of components of  $\partial P_i \cap S_x(r/2)$ ; by the argument used in the proof of Theorem 1.4, there is a uniform bound on the number of ends of  $P_i$  in  $B_x(r)$  (depending on area  $\Sigma_i$ ).

Applying Theorem 3.1 to sequences of components  $P_i \subset \Sigma_i \cap B_x(r)$ , it follows that  $\Sigma_\infty$  is a smoothly embedded minimal surface.

By assumption,  $\chi(\Sigma_\infty) \geq n$ . By Theorem 3.1 the convergence  $\Sigma_i \rightarrow \Sigma_\infty$  is smooth away from a finite number of points on  $\Sigma_\infty$ ; near these,  $\Sigma_\infty$  results from  $\Sigma_i$  by surgery on essential curves. Thus  $\chi(\Sigma_\infty) \geq n/2$ ; it is clear that multiplicity  $(\Sigma_\infty) \geq 2$ .

*Remark 1.* — We note that it is possible to obtain non-orientable limits (e. g.  $P^2$ ) from sequences of embedded, orientable minimal surfaces (e. g.  $S^2$ ).

*Remark 2.* — The theorem above is not true if one allows  $n$  to become unbounded. Lawson ( $L_1$ ) has constructed a sequence of embedded minimal surfaces  $\xi_{g,1}$  of genus  $g$  in  $S^3$  with  $\xi_{g,1}$  converging weakly to a union of two orthogonal, totally geodesic two-spheres in  $S^3$ . In particular, the limiting surface is not embedded.

It is an interesting question to determine conditions guaranteeing that  $\partial \mathcal{M}_n = \emptyset$ . We have recently learned that Choi and Schoen [CS] have proved that  $\partial \mathcal{M}_n = \emptyset$  for  $N^3 = S^3$ , or more generally  $N^3$  any simply connected 3-manifold with strictly positive Ricci curvature. (Thus  $N^3$  is topologically  $S^3$ , by work of Hamilton.) It is unlikely that  $\partial \mathcal{M}_n = \emptyset$  without imposing ambient conditions on  $N^3$ ; for instance, it is probably the case that  $\partial \mathcal{M}_2 \neq \emptyset$  if one allows metrics of non-negative Ricci curvature on  $S^3$ .

Of course, if  $\mathcal{M}_{n/2} = \emptyset$ , then  $\partial \mathcal{M}_n = \emptyset$ ; in certain situations, global arguments imply that  $\mathcal{M}_{n/2} = \emptyset$ .

**COROLLARY 4.3.** — *Let  $N^3$  be a compact oriented Riemannian 3-manifold, with sectional curvature  $K_N \leq -c < 0$ . Then the space of minimal embeddings  $\mathcal{M}_{-2}$  of an oriented surface of genus 2 is compact in the weak topology.*

*Proof.* — Using the Gauss-Bonnet and Gauss curvature equations, one has

$$-\text{vol}(\Sigma_{-2}) = \int_{\Sigma_{-2}}^{-1} d\text{vol} \geq \frac{1}{c} \int_{\Sigma_{-2}} K_\Sigma d\text{vol} = -\frac{4\pi}{c}.$$

for any minimal surface  $\Sigma_{-2}$  of genus 2 in  $N^3$ . Thus the area function on  $\mathcal{M}_{-2}$  is uniformly bounded. Since there are no closed minimal surfaces in  $\mathcal{M}_{-1}$  in  $N^3$  (by Gauss-Bonnet),  $\mathcal{M}_{-1} = \emptyset$  and the result follows by Theorem 4.2. ■

*Remark 3.* — It is interesting to note that no assumptions on the incompressibility of surfaces in  $\mathcal{M}_{-2}$  are made in Corollary 4.3.

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