Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem

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Introduction

Among all k-dimensional surfaces having a prescribed boundary, is there one having the least k-dimensional area? There have been several formulations of this question in recent years utilizing different definitions of *surface* and *boundary* [A1] [FF] [FL2] [Z]; and there has been success in each case showing the existence of solutions having the geometric properties of compact rectifiable sets, even though the topological type or the set of admissible singularities of these solutions was not prescribed beforehand. A discussion of the phenomena of least area problems appears in [A1, 11.1] and [A2, 1].

There has also been progress in the study of the regularity properties of the various solutions. De Giorgi [DG1] and Reifenberg [R3] have shown that their minimal surfaces are real analytic manifolds except for a set of measure zero on the surfaces. (Reifenberg's surfaces are of all codimensions, while those of De Giorgi are exclusively of codimension one). Reifenberg [R1] [R3] and Fleming [FL1] have shown that solutions to certain formulations of the problem of least area for two dimensional surfaces in R^3 are real analytic manifolds at all non-boundary points. More precisely, regularity is known in case the minimal surfaces in question are two dimensional integral currents [FF] (which include the two dimensional surfaces of De Giorgi), flat 2-chains over the group of integers modulo two [FL2], or the proper minimal surfaces of Reifenberg [R1] with boundary $\supset$ a cyclic subgroup of the one dimensional Cech homology group with coefficients in the group of integers modulo two of the boundary sets.

In this paper we show that three dimensional minimal surfaces in R^4 are three dimensional real analytic submanifolds of R^4, except perhaps at their boundaries, where we mean by minimal surface:

(1) an oriented frontier of least three dimensional measure [DG1, p. 3],

(2) a minimal three dimensional integral current [FF 9.1],

(3) a minimal flat 3-chain over the group of integers modulo two [FL2], or

(4) a proper minimal surface of Reifenberg [R1] with boundary $\supset$ a cyclic

* This work was supported in part by National Science Foundation grants NSF-GP 2439 and NSF-GP 3946.

subgroup of the two dimensional Cech homology group with coefficients in the group of integers modulo two of the boundary set.

Note that there has been no assumption that the minimal surfaces above could be expressed locally in non-parametric form. The fact that we have made no such assumption allows us to use our regularity results for three dimensional minimal surfaces in R^4 to conclude regularity results for non-parametric four dimensional minimal surfaces in R^5. For example, one can use the work of Miranda [M1] to conclude the existence, uniqueness, and interior analyticity of solutions to the boundary value problem for the minimal surface equation over bounded uniformly convex regions in R^4.

A further generalization of Bernstein's theorem is another consequence. Fleming showed [FL2, p. 15] that an n-dimensional generalization of Bernstein's theorem (that a globally defined non-parametric n-dimensional minimal surface in R^{n+1} must be a hyperplane) would be a consequence of an interior regularity theorem for minimal n-dimensional integral currents in R^{n+1}. His two dimensional regularity results thus furnished another proof of the usual two dimensional Bernstein's theorem. De Giorgi then showed [DG2] that an n-dimensional generalization of Bernstein's theorem would follow from an interior regularity theorem for minimal $(n - 1)$-dimensional integral currents in R^n. A three dimensional generalization of Bernstein's theorem then followed from two dimensional regularity, and a four dimensional regularity theorem is a consequence of the three dimensionel regularity theorems of this paper.

The fact that a minimal surface is stationary implies the existence of a non-empty set of tangent cones at each point. The minimality of the surface implies that each of these tangent cones is itself a minimal surface and, in particular, must be stable with respect to its boundary. One can use two dimensional regularity results to conclude that the boundary of each cone is a compact, orientable, two dimensional, manifold analytically imbedded as a minimal surface on the three dimensional sphere. The stability of the cone over this manifold implies that the manifold topologically must be a two dimensional sphere (Lemma 2). We then show that the only way a two dimensional sphere can be imbedded as a minimal surface on the three dimensional sphere is as a great two sphere (Lemma 1). Thus each tangent cone is a three dimensional disk. Regularity of the minimal surface then follows from results of De Giorgi [DG1] and Triscari [T], or from results of Reifenberg [R3].

We include also in this paper a new interior regularity theorem for some two dimensional minimal surfaces in R^n.

The main results of this paper appear as corollaries to Theorem 1. This theorem is stated in the terminology of [FF].

LEMMA 1. *Let* $f\colon S^2 \to S^3 \subset S^4$ *be a real analytic immersion of the two dimensional sphere* S^2 *as a minimal surface in the unit three dimensional sphere* S^3 *in* R^4. *Then* f *imbeds* S^2 *in* S^3, *and* $f(S^2) = S^3 \cap \{x\colon xv = 0\}$ *for some* $v \in S^3$.

PROOF. *Part* 1. Let $f\colon S^2 \to S^3 \subset R^4$ be a real analytic immersion of S^2 as a minimal surface in S^3. Let $g\colon S^2 \to S^3 \subset R^4$ be chosen continuously as a vector field tangent to S^3 (i.e., $fg = 0$, where fg denotes the inner product in R^4 of f with g) and perpendicular to $f(S^2)$ (i.e., if u and v are differentiable coordinates on S^2, then $f_u g = f_v g = 0$, where $f_u = \partial f/\partial u$, $f_v = \partial f/\partial v$). Let $k_1, k_2\colon S^2 \to R$ denote the principal curvatures of $f(S^2)$ with respect to g with the convention that $k_1 \geqq k_2$. Since f satisfies the minimal surface equation with respect to S^3, $k_1 + k_2 = 0$. We will show $k_1 = k_2 = 0$. Our argument depends ultimately on the non-existence of non-zero complex quadratic forms on S^2 considered as a Riemann surface, and is similar to the argument used to prove that a compact two dimensional manifold of genus zero imbedded in R^3 with constant mean curvature is a standard sphere [H IV, 2.1, p. 83].

We define two covariant tensor fields φ, ψ of degree two on S^3. If u, v are local coordinates in a neighborhood U of p in S^3, we set

$$\varphi = f_u f_u du \otimes du + f_{uv}(du \otimes dv + dv \otimes du) + f_v f_v dv \otimes dv$$
$$\psi = f_{uu} g du \otimes du + f_{uv} g(du \otimes dv + dv \otimes du) + f_{vv} g dv \otimes dv\ .$$

φ is, of course, the metric tensor induced on S^3 by its imbedding in R^4, and ψ is a generalization of the sccond fundamental tensor.

We now choose and fix coordinates u, v in a neighborhood U of p in S^2 which are isothermic with respect to the metric φ. We have then

(1) $f_u f_u = f_v f_v$,

(2) $f_u f_v = 0$,

(3) $k_1 k_2 = (f_u f_u)^{-2}[(f_{uu} g)(f_{vv} g) - (f_{uv} g)^2]$, and

(4) $k_1 + k_2 = (f_u f_u)^{-1}[f_{uu} g + f_{vv} g]$.

One verifies (3) and (4) as follows. Choose orthonormal coordinates y^1, y^2, y^3, y^4 for R^4 such that, identifying R^4 with its various tangent spaces, one has $\partial/\partial y^3 = g(p)$, $\partial/\partial y^4 = f(p)$, and for some neighborhood V of p in S^2 and for some $\varepsilon > 0$,

$$\begin{aligned} f(V) = R^4 \cap \{x\colon & |x - f(p)| < \varepsilon\ , \\ & y^3(x) = 2^{-1}k_1(p)y^1(x)^2 + 2^{-1}k_2(p)y^2(x)^2 + O(|x - f(p)|^3)\ , \\ & \text{and}\ \ y^4(x) = -2^{-1}y^1(x)^2 - 2^{-1}y^2(x)^2 + O(|x - f(p)|^3)\}\ . \end{aligned}$$

Let $r = y^1 \circ f$ and $s = y^2 \circ f$. Then r and s are coordinates on V. One computes $f_{rr}(p)g(p) = k_1(p)$, $f_{rs}(p)g(p) = 0$, $f_{ss}(p)g(p) = k_2(p)$, $f_r(p)f_r(p) = f_s(p)f_s(p) = 1$, and $f_r(p)f_s(p) = 0$. The change of coordinates formulas for tensors give

$$\begin{aligned} f_{uu}g &= f_{rr}gr_u^2 + 2f_{rs}gr_us_u + f_{ss}gs_u^2 , \\ f_{uv}g &= f_{rr}gr_ur_v + f_{rs}g(r_us_v + s_ur_v) + f_{ss}gs_us_v , \\ f_{vv}g &= f_{rr}gr_v^2 + 2f_{rs}gr_vs_v + f_{ss}gs_v^2 , \end{aligned}$$

and

$$f_{uu} = f_rf_rr_u^2 + 2f_rf_sr_us_u + f_sf_ss_u^2 .$$

We have thus

$$\begin{aligned} f_{uu}(p)g(p) &= k_1(p)r_u(p)^2 + k_2(p)s_u(p)^2 \\ f_{uv}(p)g(p) &= k_1(p)r_u(p)r_v(p) + k_2(p)s_u(p)s_v(p) \\ f_{vv}(p)g(p) &= k_1(p)r_v(p)^2 + k_2(p)s_v(p)^2 \\ f_u(p)f_v(p) &= r_u(p)^2 + s_u(p)^2 . \end{aligned}$$

The equalities (3) and (4) now follow by direct substitution, utilizing the relations following from the geometrically obvious fact that the matrix

$$\frac{\partial(r, s)}{\partial(u, v)}(p) = \begin{pmatrix} r_u(p) & r_v(p) \\ s_u(p) & s_v(p) \end{pmatrix}$$

is $r_u(p)s_v(p) - s_u(p)r_v(p)$ times an orthogonal matrix.

Part 2. Let f, g, φ be as in part 1, with coordinates u, v defined in a neighborhood U of p in S^2 which are isothermic with respect to the metric φ. Then in U we have

(5) $(f_uf_u)g_u = -(f_{uu}g)f_u - (f_{uv}g)f_v$, and

(6) $(f_uf_u)g_v = -(f_{uv}g)f_u - (f_{vv})f_v$.

We will verify (5). Since $gf = gf_u = 0$, $g_uf = 0$. Since $|g| = 1$, $g_ug = 0$. We can therefore write $g_u = af_u + bf_v$ for some $a, b \colon U \to R$. Since $f_ug = f_vg = 0$,

$$f_{uu}g = -f_ug_u = -af_uf_u - bf_vf_u = -af_uf_u ,$$

and

$$f_{uv}g = -f_vg_u = -af_uf_v - bf_vf_v = -bf_vf_v \qquad \text{(by 2)} .$$

Thus $a = -(f_uf_u)^{-1}(f_{uu}g)$ and $b = -(f_vf_v)^{-1}(f_{uv}g) = -(f_uf_u)^{-1}(f_{uv}g)$, which implies (5), and (6) follows by similar arguments.

Part 3. Let $f, g, \varphi, u, v, U, p$ be as in *Part* 2, and k_1, k_2 as in *Part* 1 with $k_1 + k_2 = 0$. Then the following relations are valid in U.

(7) $f_{uu}g + f_{vv}g = 0$ (follows from (4)),

(8) $f_{uuu}g + f_{uu}g_u + f_{uvv}g + f_{vv}g_u = 0$

(follows by differentiating (7) with respect to u),

(9) $f_{uu}f_u - f_{uv}f_v = 0$ (follows by differentiating (1) with respect to u),

(10) $f_{uv}f_u - f_{vv}f_v = 0$ (follows by differentiating (1) with respect to v),

(11) $f_{uv}f_u + f_{uu}f_v = 0$ (follows by differentiating (2) with respect to u),
(12) $f_{vv}f_u + f_{uv}f_v = 0$ (follows by differentiating (2) with respect to v),
(13) $f_{uu}f_v + f_{vv}f_v = 0$ (subtract (10) from (11))
(14) $f_{uu}f_u + f_{vv}f_u = 0$ (add (9) and (12))
(15) $(f_{uu}g)_u + (f_{uv}g)_v = 0.$

We verify (15):

$$\begin{aligned}
&(f_{uu}g)_u + (f_{uv}g)_v \\
&\quad = f_{uuu}g + f_{uu}g_u + f_{uvv}g + f_{uv}g_v \\
&\quad = f_{uuu}g - (f_uf_u)^{-1}(f_{uu}g)(f_{uu}f_u) - (f_uf_u)^{-1}(f_{uv}g)(f_{uu}f_v) \\
&\qquad - f_{uuu}g - f_{uu}g_u - f_{vv}g_u - (f_uf_u)^{-1}(f_{uv}g)(f_{uv}f_u) \\
&\qquad - (f_uf_u)^{-1}(f_{vv}g)(f_{uv}f_v) && \text{(by (5), (8), (6))} \\
&\quad = -f_{uu}g_u - f_{vv}g_u && \text{(by (7) and (9), and (11))} \\
&\quad = +(f_uf_u)^{-1}[(f_{uu}g)(f_{uu}f_u) + (f_{uv}g)(f_{uu}f_v) \\
&\qquad + (f_{uu}g)(f_{vv}f_u) + (f_{uv}g)(f_{vv}f_v)] && \text{(by (5))} \\
&\quad = 0 && \text{(by (13) and (14))}\,.
\end{aligned}$$

(16) $(f_{uu}g)_v - (f_{uv}g)_u = 0.$

We verify (16):

$$\begin{aligned}
&(f_{uu}g)_v - (f_{uv}g)_u \\
&\quad = f_{uuv}g + f_{uu}g_v - f_{uuv}g - f_{uv}g_u \\
&\quad = (f_uf_u)^{-1}[-(f_{uv}g)(f_{uu}f_u) - (f_{vv}g)(f_{uu}f_v) + (f_{uu}g)(f_{uv}f_u) \\
&\qquad + (f_{uv}g)(f_{uv}f_v)] && \text{(by (5) and (6))} \\
&\quad = 0 && \text{(by (7) and (11), and (9))}\,.
\end{aligned}$$

Part 4. Let $f, g, \varphi, u, v, U, p, k_1, k_2$ be as in *Part* 3. We define complex parameters $w, \bar{w}$ on U by setting

$$w = u + iv\,, \qquad \bar{w} = u - iv\,.$$

We define the complex valued function Φ on U by setting

$$\Phi = f_{uu}g - if_{uv}g$$

which gives

(17) $2|\Phi| = (f_uf_u)\,|k_1 - k_2|.$

We verify (17):

$$\begin{aligned}
|k_1 - k_2|^2 &= k_1^2 - 2k_1k_2 + k_2^2 \\
&= k_1^2 + 2k_1k_2 + k_2^2 - 4k_1k_2 \\
&= (k_1 + k_2)^2 - 4(k_1k_2) \\
&= -4(f_uf_u)^{-2}[(f_{uu}g)(f_{vv}g) - (f_{uv}g)^2] \\
&\qquad \text{(by (3) and our hypothesis that } k_1 + k_2 = 0) \\
&= 4(f_uf_u)^{-2}[(f_{uu}g)^2 + (f_{uv}g)^2] && \text{(by (7))} \\
&= 4(f_uf_u)^{-2}\,|\Phi|^2\,.
\end{aligned}$$

Note that (17) implies that $\Phi(q) = 0$ if and only if $k_1(q) = k_2(q)$ for $q \in U$. We assert that Φ is a complex analytic function. To see this, one observes

$$\begin{aligned}\Phi_{\bar{w}} &= 2^{-1}\big((f_{uu}g)_u + (f_{uv}g)_v\big) + i2^{-1}\big((f_{uu}g)_v - (f_{uv}g)_u\big)\\ &= 0 \qquad \text{(by (15) and (16))}\end{aligned}$$

so that, in particular, the Cauchy-Riemann equations are satisfied. We assert also

(18) $\Phi = -2f_w g_w$.

We verify (18). Since $f_u g = f_v g = 0$,

(19) $f_{uu}g + f_u g_u = 0$,

(20) $f_{uv}g + f_u g_v = 0$,

(21) $f_{uv} + f_v g_u = 0$, and

(22) $f_{vv}g + f_v g_v = 0$.

Thus

$$\begin{aligned}f_w g_w &= [2^{-1}(f_u - if_v)][2^{-1}(g_u - ig_v)]\\ &= 4^{-1}[-2f_{uu}g + 2if_{uv}g] \qquad \text{(by (7), (19), (20), (21), (22))}\\ &= -2^{-1}\Phi .\end{aligned}$$

We assert that (in the terminology which is usual in the theory of Riemann surfaces) $\Phi(dw)^2$ is a complex quadratic differential (where Φ is a function of w and $\bar{w}$). We see this as follows. Suppose x, y are coordinate functions in a neighborhood of p in S^2. Then these new parameters are isothermic with respect to the metric φ if and only if $z = x + iy$ is an analytic function of $w = u + iv$ with non-vanishing derivative; i.e., $z = z(w)$, $z' \neq 0$, and, in particular, the correspondence between the w-plane and the z-plane is conformal. If $\Psi(z, \bar{z})$ is the function analogous to $\Phi(w, \bar{w})$ for the parameters z and $\bar{z}$, then $\Psi = -2f_z g_z$ by (18). But one checks

$$f_w = f_z(dz/dw) , \qquad g_w = g_z(dz/dw) ,$$

which gives $\Phi = \Psi(dz/dw)^2$, which is the change of coordinates formula one must verify to show $\Phi(dw)^2$ is a complex quadratic differential. By [H VI (2.6), p. 85], Φ is identically zero since its domain S^2 is of genus zero. By (17), $k_1 = k_2 = 0$. Using (18), (19), (20), (21), (22), and our observation in the verification of (5), we have

$$g_u f = g_u g = g_u f_u = g_u f_v = 0 ,$$
$$g_v f = g_v g = g_v f_u = g_v f_v = 0 ,$$

which implies that g is constant. Since $fg = 0$,

$$f(S^2) = S^3 \cap \{x\colon xg(p) = 0\} \cong S^2 .$$

Since f is an immersion, f must cover $S^3 \cap \{x\colon xg(p) = 0\}$. Since $S^3 \cap \{x\colon xg(p) =$

$0\} \cong S^2$ is simply connected, f must be one to one.

LEMMA 2. *Let T be a compact orientable real analytic two dimensional manifold which is not topologically S^2, and let $f: T \to S^3 \subset R^4$ be a real analytic imbedding of T into S^3 as a minimal surface in S^3. Then the cone $Of(T)$ over $f(T)$ with vertex at the origin O in S^4 is not stable with respect to S^3.*

PROOF. Let $S = f(T) \subset S^3 \subset R^4$ where f and T are as in the hypotheses. Let $g: S^3 \to R^4$ be chosen of class 4 so that $xg(x) = 0$ for each $x \in S^3$, and $g(y)$ is perpendicular to S with $|g(y)| = 1$ for each $y \in S$. For each $x \in S$, let $k(x)$ denote the absolute value of the principal curvatures of S at x with respect to $g(x)$, which principal curvatures are equal in absolute value since S is minimal with respect to S^3. For each $x \in S$, let $K(x)$ denote the gaussian curvature of S at x. We assert that, for each $x \in S$,

$$K(x) = 1 - k(x)^2 . \tag{1}$$

We see this as follows. Let $p \in S$ and choose orthonormal coordinates y^1, y^2, y^3, y^4 for R^4 so that, identifying R^4 with its various tangent spaces, one has $\partial/\partial y^3 = g(p)$, $\partial/\partial y^4 = p$, and for some neighborhood U of p in R^4,

$$\begin{aligned} S \cap U = U \cap \{x:\ & y^3(x) = 2^{-1}k(p)y^1(x)^2 - 2^{-1}k(p)y^2(x)^2 + O(|x-p|^3) \\ & \text{and } y^4(x) = 2^{-1}y^1(x)^2 - 2^{-1}y^2(x)^2 + O(|x-p|^3)\} . \end{aligned}$$

Then $r = y^1 \circ f$, $s = y^2 \circ f$ are coordinates for T in $V = f^{-1}(U)$. The metric tensor φ on T induced by f is

$$\varphi = f_r f_r dr \otimes dr + f_r f_s (dr \otimes ds + ds \otimes dr) + f_s f_s ds \otimes ds .$$

We compute

$$\begin{aligned} f_r &= (\partial/\partial y^1) + \big(k(p)r + O(r^2+s^2)\big)(\partial/\partial y^3) - \big(r + O(r^2+s^2)\big)(\partial/\partial y^4) \\ f_s &= (\partial/\partial y^2) - \big(k(p)s + O(r^2+s^2)\big)(\partial/\partial y^3) - \big(s + O(r^2+s^2)\big)(\partial/\partial y^4) \end{aligned}$$

which gives

$$\begin{aligned} \varphi = {} & \big(1 + k(p)^2 + r^2 + rO(r^2+s^2)\big) dr \otimes dr \\ & + \big(-k(p)^2 rs + rs + (r+s)O(r^2+s^2)\big)(dr \otimes ds + ds \otimes dr) \\ & + \big(1 + k(p)^2 s^2 + s^2 + sO(r^2+s^2)\big) ds \otimes ds . \end{aligned}$$

In the notation usual for two dimensional manifolds, we set $E = f_r f_r$, $F = f_r f_s$, $G = f_s f_s$ and compute

$$\begin{aligned} H &= (EG - F^2)^{1/2} \\ &= [1 + (r^2+s^2)\big(1 + k(p)^2\big) + 4r^2s^2k(p) + (r+s)O(r^2+s^2)]^{1/2} \end{aligned}$$

$$K = H^{-1}[(2HE)^{-1}(FE_s - EG_r)]_r + H^{-1}[(2HE)^{-1}(2EF_r - FE_r - EE_s)]_s ,$$

from which one computes

$$K(f^{-1}(p)) = K(p) = 0 + F_{rs} = 1 - h(p)^2 .$$

Since T is an orientable two dimensional manifold which is not S^2, the Euler characteristic of T is non-positive. The Gauss-Bonnet formula [AL] implies

$$\int_S [1 - k(x)^2] d\mathbf{H}^2 x \leqq 0 . \tag{2}$$

We choose orthonormal coordinates x^1, x^2, x^3, x^4, for R^4 by setting $x^i = y^i$ for $i = 1, 2, 3$, and $x^4 = y^4 + 1$. Then, in particular, $S^3 = R^4 \cap \{z: \sum_i x^i(z)^2 = 1\}$. We set $g^i = x^i \circ g$ for $i = 1, 2, 3, 4$. Since g is perpendicular to S and of unit length along S, we have in V

$$\begin{aligned} 0 = (g\circ f)f_\sharp(\partial/\partial r) &= (g^1\circ f) + (g^3 \circ f)[k(p)r + O(r^2 + s^2)] \\ &+ (g^4 \circ f)[-r + O(r^2 + s^2)] , \end{aligned} \tag{3}$$

$$\begin{aligned} 0 = (g\circ f)f_\sharp(\partial/\partial s) &= (g^2\circ f) + (g^3 \circ f)[-k(p)r + O(r^2 + s^2)] \\ &+ (g^4 \circ f)[-r + O(r^2 + s^2)] , \end{aligned} \tag{4}$$

and

$$1 = \textstyle\sum_i (g^i \circ f)^2 \tag{5}$$

Thus

$$[\partial(g^1\circ f)/\partial r](f^{-1}(p)) = (\partial g^1/\partial y^1)(p) = -k(p) \tag{6}$$
(follows by differentiating (3) with respect to r) ,

$$[\partial(g^1\circ f)/\partial s](f^{-1}(p)) = (\partial g^1/\partial y^2)(p) = 0 \tag{7}$$
(follows by differentiating (3) with respect to s) ,

$$[\partial(g^2\circ f)/\partial s](f^{-1}(p)) = (\partial g^2/\partial y^2)(p) = k(p) \tag{8}$$
(follows by differentiating (4) with respect to s) ,

$$[\partial(g^3\circ f)/\partial r](f^{-1}(p)) = (\partial g^3/\partial y^1)(p) = 0 \tag{9}$$
(follows by differentiating (5) with respect to r) ,

and

$$[\partial(g^3\circ f)/\partial s](f^{-1}(p)) = (\partial g^3/\partial y^2)(p) = 0 \tag{10}$$
(follows by differentiating (5) with respect to s) .

For each sufficiently small $\varepsilon > 0$, choose $\varphi_\varepsilon: R^+ \to R^+$ of class ∞ such that

$$\begin{aligned} &\varphi_\varepsilon(z) = 1 - z^2 && \text{for } 2\varepsilon \leqq z \leqq 1 - \varepsilon , \\ &\varphi_\varepsilon(z) = 0 && \text{for } 0 \leqq z \leqq \varepsilon \text{ and } 1 \leqq z < \infty , \\ &\operatorname{Lip}(\varphi_\varepsilon \mid \{z: \varepsilon \leqq z \leqq 2\varepsilon\}) \leqq 2\varepsilon^{-1} , \end{aligned}$$

and

$$\operatorname{Lip}(\varphi_\varepsilon \mid \{z: 1 - \varepsilon \leqq z \leqq 1\}) \leqq 3 .$$

We define

$$\begin{aligned} &G_\varepsilon\colon R^4 \longrightarrow R^4 \\ &G_\varepsilon(x) = \varphi_\varepsilon(|x|)g(|x|^{-1}x) \qquad \text{for } x \in R^4 - \{0\} \\ &G_\varepsilon(0) = 0 \end{aligned}$$

and set

$$\begin{aligned} &F_\varepsilon\colon R \times R^4 \longrightarrow R^4 \\ &F_\varepsilon(t, x) = x + tG_\varepsilon(x) \qquad \text{for } x \in R^4 . \end{aligned}$$

Let $x \in R^4$ with $0 < |x| < 1$ and $|x|^{-1}x \in S$. We compute the increment of second variation with respect to F_ε to be

$$\begin{aligned} &[(\partial G_\varepsilon^3/\partial x^1)(x)]^2 + [(\partial G_\varepsilon^3/\partial x^2)(x)]^2 + [(\partial G_\varepsilon^3/\partial x^4)(x)]^2 \\ &\quad + 2[(\partial G_\varepsilon^1/\partial x^1)(x)\cdot(\partial G_\varepsilon^2/\partial x^2)(x) + (\partial G_\varepsilon^1/\partial x^1)(x)\cdot(\partial G_\varepsilon^4/\partial x^4)(x) \\ &\qquad + (\partial G_\varepsilon^2/\partial x^2)(x)\cdot(\partial G_\varepsilon^4/\partial x^4)(x)] \\ &\quad + 2[(\partial G_\varepsilon^1/\partial x^2)(x)\cdot(\partial G_\varepsilon^2/\partial x^1)(x) + (\partial G_\varepsilon^1/\partial x^4)(x)\cdot(\partial G_\varepsilon^4/\partial x^1)(x) \\ &\qquad + (\partial G_\varepsilon^2/\partial x^4)(x)\cdot(\partial G_\varepsilon^4/\partial x^2)(x)] \\ &= [(\partial G_\varepsilon^3/\partial x^4)(x)]^2 + 2[(\partial G_\varepsilon^1/\partial x^1)(x)\cdot(\partial G_\varepsilon^2/\partial x^2)(x)] \\ &\qquad\qquad \text{(by (7), (9), (10) and the fact that } OS \text{ is a cone)} \\ &= (d\varphi_\varepsilon/dz)^2(|x|) - 2[\varphi_\varepsilon(|x|)\cdot|x|^{-1}k(|x|^{-1}x)]^2 \\ &\qquad\qquad \text{(by (6), (8), and the fact that } OS \text{ is a cone)} . \end{aligned}$$

The second variation of the cone OS with respect to the deformation F_ε is thus

$$\begin{aligned} &\int_{x\in \operatorname{spt}(O|S|)-\{0\}} (\varphi_\varepsilon'(|x|)^2 - 2[\varphi_\varepsilon(|x|)|x|^{-1}k(|x|^{-1}x)]^2)d\mathbf{H}^3x \\ &= \int_{0<z\leqq 1}\int_{z^{-1}x\in S} \varphi_\varepsilon'(z)^2 - 2[\varphi_\varepsilon(z)z^{-1}k(z^{-1}x)]^2 d\mathbf{H}^2x\,dz \\ &= \int_{2\varepsilon\leqq z\leqq 1-\varepsilon} z^2\int_{x\in S} 4z^2 - 2z^{-2}(1 - 2z^2 + z^4)k(x)^2 d\mathbf{H}^2x\,dz + O(\varepsilon) \\ &= \int_{2\varepsilon\leqq z\leqq 1-\varepsilon}\int_{x\in S}[4z^4 - 4z^4k(x)^2 + k(x)^2(4z^4 - 2 + 4z^2 - 2z^4)]d\mathbf{H}^2x\,dz \\ &\qquad + O(\varepsilon) \\ &= \int_{2\varepsilon\leqq z\leqq 1-\varepsilon} 4z^4dz\int_{x\in S}(1 - k(x)^2)d\mathbf{H}^2x \\ &\quad + \int_{2\varepsilon\leqq z\leqq 1-\varepsilon}(2z^4 + 4z^2 - 2)dz\int_{x\in S}k(x)^2d\mathbf{H}^2x + O(\varepsilon) \end{aligned}$$

which, in view of (2), is strictly negative for all sufficiently small $\varepsilon > 0$. Thus $OS = Of(T)$ is not stable with respect to S^3.

THEOREM 1. *Let Q, $U \subset R^4$ be open with $Q \subset \mathbf{I}_4(R^4)$, and suppose ∂Q is of* *least mass in* U, i.e., $\mathbf{M}(\partial Q + X) \geqq \mathbf{M}(\partial Q)$ *whenever* $X \in \mathbf{I}_3(R^4)$ *with* $\partial X = 0$ *and* $\operatorname{spt}(X) \subset U$. *Then* $\operatorname{spt}(\partial Q) \cap U$ *is a three dimensional real analytic submanifold of* R^4 *satisfying the minimal surface equation.*

PROOF. Let $p \in \operatorname{spt}(\partial Q) \cap U$, and let $T \in \mathbf{I}_2(S^3)$ be such that the cone OT over T with vertex at the oringin O in R^4 is a tangent cone to ∂Q at p, i.e., for each $\varepsilon > 0$, there exists $0 < r < \varepsilon$ such that

$$\mathbf{F}\big(OT - f(p, r)^{-1}_{\#}(\partial Q) \cap \{x: |x| < 1\}\big) < \varepsilon .$$

Here $f(p, r): R^4 \rightarrow R^4$, $f(p, r)(x) = p + rx$ for each $x \in R^4$. The existence of OT follows from [FF 3.9, 8.13, 9.16]. Note that we are not assuming the existence of a *unique* tangent cone to ∂Q at p. Clearly $\partial T = 0$. Also since ∂Q is of least mass in U, OT is of least mass with respect to T. One verifies the existence of an open set $V \subset R^4$ such that $V \in \mathbf{I}_4(R^4)$ and $\partial V \cap \{x: |x| < 1\} - OT$. The methods of [T] imply that $\operatorname{spt}(OT) \cap \{x: |x| < 1\}$ is a real analytic submanifold of R^4 except possibly at the origin O. In particular then, $\operatorname{spt}(T)$ is a real analytic submanifold of S^3 which is minimal on S^3. Since OT is of least mass with respect to T, OT must be both stationary and stable with respect to $\operatorname{spt}(T)$. Lemmas 1 and 2 imply that $\operatorname{spt}(T)$ is a great two sphere on S^3. Thus $\operatorname{spt}(OT)$ is a unit three disk in R^4. [DG1, p. 56] together with [T, p. 370] imply the theorem since $p \in \operatorname{spt}(\partial Q) \cap U$ is arbitrary.

COROLLARY 1. *Let* $E \subset R^4$ *be measurable and* $U \subset R^4$ *be open. Suppose* E *has a locally oriented frontier of least measure in* U [DG1, p. 3]. *Then* $E \cap U$ *is a three dimensional real analytic submanifold of* R^4 *satisfying the minimal surface equation.*

PROOF. [DG1, p. 56] and [F 2.2] reduce the corollary to the theorem.

COROLLARY 2. *Let* $T \in \mathbf{I}_3(R^4)$ *be minimal* [FF 9.1]. *Then* $\operatorname{spt}(T) - \operatorname{spt}(\partial T)$ *is a three dimensional real analytic submanifold of* R^4 *satisfying the minimal surface equation. If* $M_1, M_2, M_3, \cdots$ *are the components of* $\operatorname{spt}(T) - \operatorname{spt}(\partial T)$, *then each* M_i *is oriented by* T, *and there exist positive integers* $a_1, a_2, a_3 \cdots$ *such that* $T \cap (R^4 - \operatorname{spt}(\partial T)) = \sum_i a_i M_i$. *If* $\mathbf{H}^3(\operatorname{spt}(\partial T)) = 0$, *then* $T = \sum_i a_i M_i$.

PROOF. One writes $T = \sum_i T_i$ in the manner of [FL1, 3.4]. The theorem applies to each T_i. Recall that if

$$(x_1, x_2, x_3) \longrightarrow \big(x_1, x_2, x_3, f_i(x_1, x_2, x_3)\big)$$

define minimal surfaces in R^4 for real analytic functions f_i, $i = 1, 2, \cdots$, in a neighborhood of the origin $(0, 0, 0) \in R^3$ if $f_1 \geqq f_2$, and if $f_1(0, 0, 0) = f_2(0, 0, 0)$, then the maximum principle for elliptic partial differential equations applied to $f_1 - f_2$ guarantees that $f_1 = f_2$ in a neighborhood of $(0, 0, 0)$. Corollary 2 follows by elementary arguments.

COROLLARY 3. *Let T be a flat 3-chain over the integers modulo two in R^4* [FL2] *such that $\mathbf{M}(T) < \infty$, $\mathbf{M}(\partial T) < \infty$, and T is of least mass with respect to ∂T. Then* $\operatorname{spt}(T) - \operatorname{spt}(\partial T)$ *is a real analytic submanifold of R^4 satisfying the minimal surface equation. $T \cap (R^4 - \operatorname{spt}(\partial T))$ is the flat 3-chain over the integers modulo two corresponding to* $\operatorname{spt}(T) - \operatorname{spt}(\partial T)$. *If* $\mathbf{H}^3(\operatorname{spt}(\partial T)) = 0$, *then T corresponds to* $\operatorname{spt}(T) - \operatorname{spt}(\partial T)$.

PROOF. Let U be an open ball in R^4 whose closure does not intersect ∂T such that $\mathbf{M}(\partial(T \cap U)) < \infty$. Set $T_1 = T \cap U$. Then there exists an open set $Q \subset U$ and a flat 3-chain S over the integers modulo two with $\operatorname{spt}(S) \subset \partial U$ and $T_1 = \partial Q + S$. The positive orientation of R^4 orients Q and thus also orients T_1. T_1 so regarded is an oriented frontier of least measure in U in the sense of Corollary 1. Corollary 3 follows.

COROLLARY 4. *Let $U \subset R^4$ be an open bounded uniformly convex region in R^4 and let $\varphi: \partial U \to R$ be continuous. Then among all continuous real valued functions defined on* $\operatorname{clos}(U)$ *which agree with φ on ∂U, there exists a unique function f of least four dimensional area,* i.e., *the Lebesgue area of the function $U \to R^5$, $x \to (x_1, x_2, x_3, x_4, f(x))$, $x \in U$, is least. f will be real analytic on U, and satisfy the minimal surface equation.*

PROOF. Miranda has shown the existence and uniqueness of f in [M1]. The methods of Triscari [T] together with Corollary 2 show that f can have at most isolated singularities. De Giorgi and Stampacchia have shown that singularities of two dimensional measure zero can be eliminated [DS], which gives the interior regularity of f.

COROLLARY 5. *Let $f: R^4 \to R$ be of class* 3 *and satisfy the minimal surface equation. Then f is linear.*

PROOF. The methods of [DG2] reduce the corollary to Corollary 2.

THEOREM 2. *Suppose $A \subset R^4$ is compact, and h is a non-trivial cyclic subgroup of the Cech homology group $H_2(A)$ with coefficients in the group of integers modulo two. Let $S \subset R^4$ be a proper surface of minimum three dimensional area with boundary $\supset h$* [R1, p. 4]. *Then $S - A$ is a three dimensional real analytic submanifold of R^4 satisfying the minimal surface equation.*

PROOF. Let S, A, h be as above. Since $S - A$ is 3-rectifiable, $|S - A| \in \mathbf{IV}_3(R^4)$ [R3] [A1, 5.3, 5.4]. Since S is of minimum area, $(|S - A|, 0)$ is stationary with respect to A [A1, 6.5(1)].

Let $x \in S - A$ and let $X \in \mathbf{IV}_2(S^3)$ be such that the cone OX over X with vertex at the origin O in R^4 is a tangent cone to $|S - A|$ at x, i.e., for each $\varepsilon > 0$ there exists $0 < r < \varepsilon$ such that

$$\mathbf{F}(OX, f(x, r)_{\#}^{-1} \mid S - A \mid \cap \{p : |p| < 1\}) < \varepsilon .$$

Here $f(x, r)$ is as in the proof of Theorem 1. By [A1, 10.2], $\mathbf{P}(S^3)(X, 0) = 0$ [A1, 6.4(10)].

Let $y \in \operatorname{spt}(X) \subset S^3$, and let $Y \in \mathbf{IV}_1(R^4)$ be such that OY is a tangent cone to X at y. Since $\operatorname{spt}(X) \subset S^3$, $\operatorname{spt}(Y)$ is contained in a unit two dimensional sphere $S^2 = S^3 \cap \Pi$ lying in that three dimensional plane Π through the origin O in R^4, which plane is parallel with the three dimensional plane in R^4 which is tangent to S^3 at y. By [A1, 10.2], $\mathbf{P}(S^2)(Y, 0) = 0$.

Let $z \in \operatorname{spt}(Y) \subset S^2$, and let $Z \in \mathbf{IV}_0(R^4)$ be such that OZ is a tangent cone to Y at z. Since $\operatorname{spt}(Y) \subset S^2$, $\operatorname{spt}(Z)$ is contained in a unit circle $S^1 = S^2 \cap \Sigma$ lying in that two dimensional plane Σ through the origin O in R^4, which plane is parallel with the two dimensional plane in R^4 which is tangent to S^2 at z.

Z consits of a finite number of points on S^1, each point having a positive integer multiplicity. Also for each $\varepsilon > 0$, there exist $r_\varepsilon > 0$ and $p_\varepsilon \in R^4$ with $\operatorname{dist}(p_\varepsilon, A) > 2n\, r_\varepsilon$ such that

$$g_\varepsilon^{-1}(S - A) \cap D \subset \{p : \operatorname{dist}(p, C) < \varepsilon\}$$

and

$$\mathbf{F}(g_{\varepsilon\#}^{-1} \mid S - A \mid \cap D, OX \times [0, 1] \times [0, 1]) < \varepsilon .$$

Here

$$C = \operatorname{spt}(OZ) \times [0, 1] \times [0, 1] \subset \Sigma \times R \times R \cong R^4 ,$$
$$D = \Sigma \cap \{q : |q| < 1\} \times [0, 1] \times [0, 1] \subset \Sigma \times R \times R \cong R^4 ,$$

and $g_\varepsilon = f(p_\varepsilon, r_\varepsilon) \circ \theta_\varepsilon$ where $\Sigma \times R \times R$ is identified with R^4, $\theta_\varepsilon : R^4 \to R^4$ is orthogonal, and $f(p_\varepsilon, r_\varepsilon)$ is as above.

For each $\delta > 0$, we can thus find arbitrarily small values of $\varepsilon > 0$ and a Lipschitz mapping $e_\varepsilon : R^4 \to R^4$, leaving A fixed such that $e_\varepsilon(S) \cap g_\varepsilon(D) = g_\varepsilon(C)$ and

$$\mathbf{W}(e_{\varepsilon\#}(\mid S - A \mid)) - \mathbf{H}^3(S - A) < \delta \mathbf{H}^3(g_\varepsilon(C)) .$$

We set $S_\varepsilon = e_\varepsilon(S)$.

We assert that each point in $\operatorname{spt}(Z)$ must have multiplicity one. Suppose not. Let $\delta = (\mathbf{W}(Z) + 1)^{-1}$. Clearly then for sufficiently small $\varepsilon > 0$ chosen with respect to δ as above, $\mathbf{H}^3(S_\varepsilon - A) < \mathbf{H}^3(S - A)$, contradicting the minimality of S.

We assert that Z consists of two antipodal points. We see this as follows. Let σ be a generator of $h \subset H_2(A)$ and

$$\partial C = \operatorname{spt}(Z) \times [0, 1] \times [0, 1] \cup \operatorname{spt}(OZ) \times \{0, 1\} \times [0, 1]$$
$$\cup \operatorname{spt}(OZ) \times [0, 1] \times \{0, 1\} .$$

Since, for arbitrarily small $\varepsilon > 0$, S_ε, being obtained by deforming S, is a surface with boundary $\supset h$, there exists a cycle c modulo two on $g_\varepsilon(\partial C)$ which is homologous to a representative of σ in $(S_\varepsilon - g_\varepsilon(D)) \cup g_\varepsilon(\partial C)$. S_ε is a surface with boundary $\supset h$ if and only if c is homologous to zero in $S_\varepsilon \cap g_\varepsilon(D) = g_\varepsilon(C)$. Since c is a chain modulo two it can be identified with a subset of $g_\varepsilon(\partial C)$. Since c is a cycle modulo two, it is determined by specifying which of the points of $g_\varepsilon(\operatorname{spt}(Z))$ are contained in the subset of $g_\varepsilon(\partial C)$ corresponding to c, and furthermore the number of these points must be even. An argument similar to that of our previous assertion shows that, since S is minimal, those points of $g_\varepsilon(\operatorname{spt}(Z))$ corresponding to c must include all of $g_\varepsilon(\operatorname{spt}(Z))$. Suppose now that there exist points $p, q \in \operatorname{spt}(Z) \subset S^1$ which are not antipodal. Let L denote the straight line segment connecting p to q, and let M denote the two dimensional region of Σ interior to the triangle having vertices $0, p, q$. For each $\rho > 0$ let $h^\rho: \Sigma \to \Sigma$, $h^\rho(s) = \rho s$ for $s \in \Sigma$ and define

$$\begin{aligned} C^\rho = [\operatorname{spt}(OZ) \cap \{s: \rho \leqq s \leqq 1\} \cup h^\rho(\operatorname{spt}(O \,|\, \operatorname{spt}(Z) - \{p, q\}\,|)) \cup L] \\ \times [0,1] \times [0,1] \\ \cup M \times \{0,1\} \times [0,1] \\ \cup M \times [0,1] \times \{0,1\} \\ \subset \Sigma \times R \times R \cong R^4 \,. \end{aligned}$$

For all sufficiently small $\rho > 0$, $\mathbf{H}^3(C^\rho) < \mathbf{H}^3(C)$, and c is homologous to zero in $g_\varepsilon(C^\rho)$. Fix such a value of ρ, and set $\delta = (2\mathbf{H}^3(C))^{-1}(\mathbf{H}^3(C) - \mathbf{H}^3(C^\rho)) > 0$. Then for all sufficiently small $\varepsilon > 0$ chosen with respect to δ as above, $S_\varepsilon^\rho = g_\varepsilon(C^\rho) \cup (S_\varepsilon - g_\varepsilon(D))$ is a surface with boundary $\supset h$ and

$$\begin{aligned} &\mathbf{H}^3(S_\varepsilon^\rho - A) - \mathbf{H}^3(S - A) \\ &\quad = \mathbf{H}^3([S_\varepsilon^\rho \cap g_\varepsilon(D)] - A) - \mathbf{H}^3([S_\varepsilon \cap g_\varepsilon(D)] - A) \\ &\qquad + \mathbf{H}^3([S_\varepsilon \cap g_\varepsilon(D)] - A) - \mathbf{H}^3([S \cap g_\varepsilon(D)] - A) \\ &\qquad + \mathbf{H}^3(S_\varepsilon^\rho - [g_\varepsilon(D) \cup A]) - \mathbf{H}^3(S - [g_\varepsilon(D) \cup A]) \\ &\quad < [\mathbf{H}^3(C)^{-1}\mathbf{H}^3(g_\varepsilon(C))][\mathbf{H}^3(C^\rho) - \mathbf{H}^3(C)] + \delta\mathbf{H}^3(g_\varepsilon(C)) \\ &\quad < 0 \,. \end{aligned}$$

This contradicts the minimality of S and proves the second assertion.

We conclude that OZ consists of a straight line segment with multiplicity one. Hence $\Theta^1(\mathbf{W}y, z) = \Theta^1(\mathbf{W}OZ, 0) = 1$. Since our choice of $s \in \operatorname{spt}(Y)$ was arbitrary, we conclude that Y has density one at each point, and has only intervals as tangent cones. The fact that Y is stationary on S^2 implies that $\operatorname{spt}(Y)$ is a great circle on S^2, and that OY is a unit two dimensional disk with multiplicity one. Hence $\Theta^2(\mathbf{W}X, y) = \Theta^2(\mathbf{W}OY, 0) = 1$. Since our choice of $y \in \operatorname{spt}(X)$ was arbitrary, we conclude that Y has density one at each point and has only disks as tangent cones. It follows then that for each $p \in \operatorname{spt}(OX)$ with $O < |p| < 1$, $\Theta^3(\mathbf{W}OX, p) = 1$.

One uses the methods of [R1, Th., p. 38; Th., p. 64] to conclude that, if $p \in \operatorname{spt}(OX)$ with $O < |p| < 1$, then in some neighborhood U of p, $\operatorname{spt}(OX) \cap U$ is topologically a three dimensional disk. Since S is minimal, one concludes that $\operatorname{spt}(OX) \cap U$ is of least area. If $\operatorname{spt}(OX) \cap U$ could be replaced by a three dimensional disk of smaller area having the same boundary, then arguments similar to those above involving $S \cap g_\varepsilon(D)$ and $g_\varepsilon(C)$ contradict the minimality of S. One can thus use [R2] and [R3] to conclude that $\operatorname{spt}(OX) \cap U$ is a three dimensional real analytic manifold. Since our choice of $p \in \operatorname{spt}(OX)$ with $O < |p| < 1$ was arbitrary, $\operatorname{spt}(OX) \cap \{q\colon 0 < |q| < 1\}$ is a real analytic manifold. Therefore $\operatorname{spt}(X)$ is a real analytic two dimensional submanifold of S^3. Lemmas 1 and 2 and the minimality of OX imply that $\operatorname{spt}(OX)$ is a great two dimensional sphere on S^3. This implies $\Theta^3(\mathbf{W}OX, 0) = 1$ and thus S has three dimensional density equal to one at x. Since our choice of x was arbitrary in $S - A$, S has three dimensional density one at each interior point. [R3] implies the regularity of S.

THEOREM 3. *Let $n \geqq 2$ be an integer.*

(1) *Let T be a flat 2-chain over the integers modulo two in R^n such that $\mathbf{M}(T) < \infty$, $\mathbf{M}(\partial T) < \infty$, and T is minimal with respect to ∂T. Then there exists a real analytic two dimensional manifold M, and a real analytic immersion $f\colon M \to R^n$ satisfying the minimal surface equation such that $f(M) = \operatorname{spt}(T) - \operatorname{spt}(\partial T)$. Furthermore f is an imbedding except at, at most, isolated points on $f(M)$, and $T \cap [R^n - \operatorname{spt}(\partial T)]$ is the flat 2-chain over the integers modulo two corresponding to $f(M)$. If $\mathbf{H}^2(\operatorname{spt}(\partial T)) = 0$, then T is the flat 2-chain over the integers modulo two corresponding to $f(M)$.*

(2) *Suppose $A \subset R^n$ is compact, and h is a non-trivial cyclic subgroup of the Cech homology group $H_1(A)$ with coefficients in the group of integers modulo two. Let $S \subset R^n$ be a proper surface of minimum area with boundary $\supset h$* [R1, p. 4]. *Then there exists a real analytic two dimensional manifold M and a real analytic immersion $f\colon M \to R^n$ such that $f(M) = S - A$. Furthermore f is an imbedding except at, at most, isolated points on $f(M)$.*

PROOF. Let T be as above, and let $p \in \operatorname{spt}(T) - \operatorname{spt}(\partial T)$. Arguments similar to those of the proof of Theorem 2 show that each tangent cone K to T at p is the sum of two dimensional disks $K_1, K_2, \cdots, K_m$ each with multiplicity one.

We assert the existence of an angle $\theta > 0$, depending only on n, such that the angle between K_i and K_j is not less than θ whenever $i \neq j$. We see this as follows. Let K be a unit two dimensional disk in R^n with center at the origin and for each $0 < \alpha < \pi/2$, let K_α denote a unit two dimensional disk in R^n with center at the origin which makes angle α with K. Now regard both K_α and K as flat 2-chains over the integers modulo two. Clearly $\lim_{\alpha\to 0} K_\alpha + K = 0$. This implies that for all sufficiently small values of $\alpha > 0$, $K_\alpha + K$ is not of least mass with respect to $\partial(K_\alpha + K)$. The assertion follows.

We assert that, for each sufficiently small $\varepsilon > 0$, we can write $T = \{x: |x - p| < \varepsilon\} = T_1 + T_2 + \cdots + T_m$ where $\operatorname{spt}(\partial T_i) \in \{x: |x - p| = \varepsilon\}$ for each i, and if $L_1, L_2, \cdots, L_m$ are tangent cones to $T_1, T_2, \cdots, T_3$ respectively at p, then each L_i consists of a single unit disk. We see this as follows. Choose $\varepsilon > 0$ so small that $f(p, r)^{-1}(\operatorname{spt}(T)) \cap \{x: 4^{-1} \leqq x \leqq 1\} \subset \{x: \operatorname{dist}(x, Q_r) < \theta/5\}$ for each $0 < r \leqq \varepsilon$. Here Q_r is chosen corresponding to r to be a union of m unit disks each centered at the origin with no two disks making an angle less than θ with each other, and $f(p, r)$ is as in the proof of Theorem 1. Clearly then $\operatorname{spt}(T) \cap \{x: 4^{-1}\varepsilon \leqq |x - p| \leqq \varepsilon\}$ can be separated into m pairwise disjoint sets, and our separation of $T \cap \{x: |x - p| \leqq \varepsilon\}$ has been completed down to radius $\varepsilon/4$. Also $\operatorname{spt}(T) \cap \{x: 8^{-1}\varepsilon \leqq |x - p| \leqq 2^{-1}\varepsilon\}$ can be separated into pairwise disjoint sets. It is immediate then that $\operatorname{spt}(T) \cap \{x: 8^{-1}\varepsilon \leqq |x - p| \leqq \varepsilon\}$ consists of m pairwise disjoint sets. This completes our separation of $T \cap \{x: |x - p| \leqq \varepsilon\}$ down to radius $8^{-}\varepsilon$. One continues in this manner to separate

$$\operatorname{spt}(T) \cap \{x: 0 < |x - p| < \varepsilon\}$$

into m pairwise disjoint sets $M_1, M_2, \cdots, M_m$. One sets $T_i = T \cap M_i$ for each i. The methods of Reifenberg [R1] [R2] [R3] generalize to show that, for each sufficiently small $r > 0$, $\operatorname{spt}(T_i) \cap \{x: |x - p| < r\}$ is a real analytic manifold. Part 1 of the theorem is then immediate.

A similar method proves Part 2 of the theorem. Here one uses varifold tangent cones as in the proof of Theorem 2.

Remark. In Theorem 2 and 3(2) we require as an hypothesis that the homology subgroup h be cyclic. This is a necessary hypothesis for regularity. The following example of a one dimensional minimal surface in R^2 illustrates the need for this hypothesis. Let $A \subset R^2$ consist of three points p, q, r equally spaced around the unit circle S^1 in R^2 and let

$$h = \{0, \{p, q\}, \{q, r\}, \{p, r\}\} \subset H_0(A) .$$

Then the proper surface S with boundary $\supset h$ of least length is unique, and consists of the cone OA over A with vertex at the origin O in R^2. The surface S, even though of codimension one, has an interior singularity, viz., the origin O where three line segments meet. Similar examples of surfaces of least area having singularities of codimension one can be constructed in higher dimensions.

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(Received December 9, 1965)