Math. Proc. Camb. Phil. Soc. (1996), **119**, 729
Printed in Great Britain

Beurling's projection theorem via one-dimensional Brownian motion

BY WENDELIN WERNER†
C.N.R.S. and University of Cambridge Statistical Laboratory, D.P.M.M.S., 16 Mill Lane, Cambridge CB2 1SB

(*Received* 7 *July* 1994; *revised* 19 *April* 1995)

Abstract

We prove some elementary intuitive estimates on moving boundaries hitting times by one-dimensional Brownian motion (in $\mathbb{R}$ and on the circle). These results give an alternative approach to Beurling's radial projection theorem on harmonic measure in a disc.

1. *Introduction*

A leading idea in mathematics and physics has always been to see that the configurations which maximize certain functionals are the most symmetric ones (e.g. 'the shortest path between two points is the straight line'). All the results this paper will deal with, can be regarded as consequences of this general principle: We are going to derive one-dimensional moving boundaries hitting times estimates, which, very loosely speaking, state that, among a certain class of boundaries, a linear Brownian motion is least likely to hit the most symmetric one. We shall then also point out some consequences of these results concerning harmonic measure.

More precisely, if $B = (B_t, t \geqslant 0)$ denotes a linear Brownian motion started from 0, and if we fix an open set I in $[0, \infty[$ (not necessarily bounded nor connected) and a continuous function $a: I \to]0, \infty[$, we put

$$U_I^a(f) = P(\forall t \in I, -a(t) < B_t - f(t) < a(t))$$

for every function $f: I \to \mathbb{R}$. It is a natural question to ask, which function f maximizes U_I^a. The following answer is not surprising:

PROPOSITION 1. *For any function* $f: I \to \mathbb{R}$,

$$U_I^a(f) \leqslant U_I^a(0).$$

Similar results hold, e.g. for the symmetric stable processes. The analogous result for Brownian motion on the unit circle C can be stated as follows:

PROPOSITION 2. *If* $M: I \to C$ *is a continuous function in the unit circle, then*

$$P(\forall t \in I, \exp(iB_t) \neq M(t)) \leqslant P(\forall t \in I, \exp(iB_t) \neq -1).$$

Let us now make the link with harmonic measure estimates: the deep connection between complex analysis (harmonic measure, conformal invariance, ...) and planar

† The author acknowledges the support of an HCM EEC 'Leibniz' fellowship.

Brownian motion has been a constant source of inspiration for both fields (see, e.g. Davis [6] and among the most recent works, Burdzy [3], using McMillan's Theorem, Burdzy-Lawler [4] using Beurling's Theorem, Oksendal [15], [16] giving stochastic proofs of complex analysis results, Carne [5], Makarov [13] or Lyons [12]). It is worth noticing that historically, the complex analysis results have often preceded their probabilistic counterparts; a lot was already known about harmonic measure before planar Brownian motion aroused interest (see, e.g. Nevanlinna's book [14]). Our (modest) purpose is now to shed a new light on one of these 'old' results: if K is a compact set in the unit disc Δ (in the complex plane), let Π_K be its radial projection on the negative axis:

$$\Pi_K = \{-|z|; z \in K\}.$$

If for all $z \in \Delta \backslash K$ (respectively $\Delta \backslash \Pi_K$), $w(z, \Delta, K)$ (resp. $w(z, \Delta, \Pi_K)$) denotes the harmonic measure of K (resp. Π_K) in Δ at z, then Beurling's Theorem can be stated as follows:

THEOREM (Beurling, [2]). *For all* $z \in \Delta$,

$$w(z, \Delta, K) \geqslant w(|z|, \Delta, \Pi_K). \tag{1}$$

In probabilistic terms, if $Z = (Z_t, t \geqslant 0)$ is a planar Brownian motion started from $z \in \Delta$ under the probability measure P_z and if for all compact sets $A, T(A) = \inf\{t \geqslant 0, Z_t \in A\}$ denotes the hitting time of A by Z, (1) can be reformulated as follows:

$$P_z(T(K) < T(C)) \geqslant P_{|z|}(T(\Pi_K) < T(C)), \tag{2}$$

where C denotes the unit circle.

See Ahlfors [1] for a complex analysis approach making use of Green's formula, or Oksendal [15] for a clever and short probabilistic proof, using reflection arguments. This theorem has turned out to be a basic tool for estimating non-intersecting exponents of planar Brownian motion and random walks and consequently also to derive bounds of Hausdorff dimensions of random fractals such as the 'self-avoiding planar Brownian motion' (see Burdzy–Lawler [4], Lawler [10], Duplantier *et al.* [7]). For some applications of Beurling's Theorem in geometric function theory, see, e.g. Ahlfors [1].

Let us briefly explain how Proposition 2 implies Beurling's Theorem and shows that (2) holds in fact independently of the radial behaviour of Z (i.e. independently of the process $(|Z_t|, t \geqslant 0)$). The radial projection in Beurling's Theorem suggests heavily that the skew-product representation of planar Brownian motion plays an important role: if $Z_0 = z \in]0, 1[$,

$$Z_t = \exp(R_{A(t)} + i\theta_{A(t)}),$$

where R and θ are independent linear Brownian motions respectively started from $R_0 = \log z$ and $\theta_0 = 0$, and where $A(t) = \int_0^t R_s^{-2}\, ds$ (this well-known representation (see, e.g. Itô–McKean [9], p. 265) is a straightforward consequence of the conformal invariance of Z and the analyticity of the exponential mapping). If $\tau = \inf\{u \geqslant 0, R_u = 0\}$ (so that $T(C) = A(\tau)$) and if $I = \{u \in [0, \tau], \exists z \in K, |z| = \exp(R_u)\}$, it is easy to notice that

$$\{T(K) \geqslant T(C)\} = \{\forall u \in I, \exp(i\theta_u)\exp(R_u) \notin K\}.$$

Therefore, for 'nice' compact sets K (for instance if K is a countable union of connected-by-paths sets), we will see that Proposition 2 shows that independently of $R = (R_u, u \geqslant 0)$,

$$P(T(K) > T(C) \mid R) \leqslant P(T(\Pi_K) > T(C) \mid R),$$

which implies (2).

Our paper is constructed as follows: in the next section, we prove Proposition 1; in Section 3, we derive Proposition 2 and the last section is devoted to Beurling's Theorem.

2. *The linear problem*

This section is essentially devoted to the proof of Proposition 1. We are going to derive this proposition as a consequence of its following discrete analog:

PROPOSITION 3. *For any fixed* $N \in \mathbb{N}\backslash\{0\}$ *and* $0 < t_1 < \ldots < t_N$, *for all* $(a_1, \ldots, a_N) \in (]0, \infty[)^N$ *and* $(f_1, \ldots, f_N) \in \mathbb{R}^N$,

$$P(\forall i \in \{1, \ldots, N\}, f_i - a_i < B_{t_i} < f_i + a_i) \leqslant P(\forall i \in \{1, \ldots, N\}, |B_{t_i}| < a_i\}.$$

We will prove Proposition 3 by induction. Let us first state a useful lemma, for which we need to introduce some further notation. For any measure ρ in $\mathbb{R}$, we will say that ρ is a 1-measure if its total mass $|\rho| \in]0, 1]$. We define a linear Brownian motion B starting with initial distribution $\rho/|\rho|$ under the probability measure P^*_ρ, and we will use the usual notation $P_\rho = |\rho| P^*_\rho$ and $P_x = P_{\delta_x}$.

LEMMA 1. *Let* μ *and* ν *be two* 1*-measures on* $[-\alpha, \alpha]$ *such that, for all* $x \in [-\alpha, \alpha]$ *and* $\beta > 0$,

$$\nu(]a - x, x + \beta[) \leqslant \mu(] -\beta, \beta[), \tag{3}$$

then for all $x_0 \in \mathbb{R}, \gamma > 0, t > 0$,

$$P_\nu(x_0 - \gamma < B_t < x_0 + \gamma) \leqslant P_\mu(-\gamma < B_t < \gamma).$$

Proof. We fix $x_0 \in \mathbb{R}, \beta > 0, t > 0$ and put

$$F(x) = \int_{-\gamma}^{\gamma} p_t(x, v)\, dv,$$

where $p_t(x, v) = (2\pi t)^{-1} \exp((v - x)^2/2t)$ is the usual Gaussian transition density; in other words,

$$F(x) = P_x(B_t \in] -\gamma, \gamma[).$$

Note that F is even and decreasing on $[0, \infty[$. We now define for all $\epsilon > 0$ and for all $n > 0, x^\epsilon_n = \inf\{x > 0, F(x) > n\epsilon\}$, so that

$$F_\epsilon = \epsilon \sum_{n > 0} 1_{]-x^\epsilon_n, x^\epsilon_n[}$$

approximates F uniformly: for all $x \in \mathbb{R}, |F(x) - F_\epsilon(x)| \leqslant \epsilon$. The definition of F implies that

$$P_\nu(x_0 - \gamma < B_t < x_0 + \gamma) = \int_{-\alpha}^{\alpha} F(x - x_0)\, \nu(dx);$$

but for all $\epsilon > 0$, (3) yields

$$\begin{aligned}\int_{-\alpha}^{\alpha} F_\epsilon(x-x_0)\,\nu(dx) &= \sum_{n>0} \epsilon \int_{-\alpha}^{\alpha} 1_{]-x_n^\epsilon,\, x_n^\epsilon[}(x-x_0)\,\nu(dx) \\ &\leqslant \sum_{n>0} \epsilon \int_{-\inf(\alpha,\, x_n^\epsilon)}^{\inf(\alpha,\, x_n^\epsilon)} \mu(dx) \\ &= \int_{-\alpha}^{\alpha} F_\epsilon(x)\,\mu(dx)\end{aligned}$$

and the lemma follows.

Proof of Proposition 3. It is a straightforward induction, using Lemma 1. We fix $N > 0$, $(f_1, \ldots, f_N) \in \mathbb{R}^N$, $(a_1, \ldots, a_N) \in]0, \infty[^N$, $0 < t_1 < \ldots < t_N$. For $n \leqslant N$, we define the 1-measures ν_n and μ_n by

$$\nu_n(U) = P(B_{t_n} - f_n \in U, \forall i \leqslant n, B_{t_i} \in]f_i - a_i, f_i + a_i[)$$

and
$$\mu_n(U) = P(B_{t_n} \in U, \forall i \leqslant n, B_{t_i} \in]-a_i, a_i[)$$

for all Borel sets U. For $1 \leqslant n < N$, one has

$$\nu_{n+1}(U) = P_{\nu_n}(B_{(t_{n+1}-t_n)} - (f_{n+1} - f_n) \in U, |B_{(t_{n+1}-t_n)} - (f_{n+1} - f_n)| < a_{n+1})$$

and
$$\mu_{n+1}(U) = P_{\mu_n}(B_{(t_{n+1}-t_n)} \in U, |B_{(t_{n+1}-t_n)}| < a_{n+1})$$

for all Borel sets U.

If for all $x \in [-a_n, a_n]$ and all $\beta > 0$,

$$\nu_n(]x-\beta, x+\beta[) \leqslant \mu_n(]-\beta, \beta[), \tag{4}$$

then Lemma 1 shows that, for all $x \in [-a_{n+1}, a_{n+1}]$, for all $\beta > 0$,

$$\nu_{n+1}(]x-\beta, x+\beta[) \leqslant \mu_{n+1}(]-\beta, \beta[).$$

By induction (the case $n = 1$ is very easy), (4) holds for $n = N$, and consequently $|\nu_N| \leqslant |\mu_N|$, which completes the proof of Proposition 3.

Proof of Proposition 1. If $(s_i, i \geqslant 1)$ is a dense sequence in I, Proposition 1 is a straightforward consequence of Proposition 3 and of the following two facts:

$$U_I^a(f) \leqslant P(\forall i \in \{1, \ldots, n\}, f(s_i) - a(s_i) < B_{s_i} < f(s_i) + a(s_i))$$

for any $n > 0$ and (as a and B are continuous on I, and as I is an open set)

$$U_I^a(0) = \lim_{n \to \infty} P(\forall i \in \{1, \ldots, n\}, -a(s_i) < B_{s_i} < a(s_i)).$$

This proof can be generalized without any single problem, if we replace B by a stable symmetric process (the only important feature we used was the fact that $p_t(0, .)$ is an even function which decreases on $[0, \infty[$).

Let us notice that the asymptotic behaviour of $U_{[0,T]}^a(f)$ as $T \to \infty$ has aroused some interest (see Lai–Wijsman [**10**] and the references therein). We recall that the explicit value of $U_{[0,T]}^a(f)$ is known if both a and f are constant (see Port–Stone [**17**], para. 2·8).

We now briefly point out that it is possible to derive directly Beurling's Theorem in the special case, where K is a continuous path joining the circles of radius $r < 1$

with centre at 0 to C (which is the version used to estimate non-intersection exponents) from Proposition 1. The outline of the proof would be as follows. Using the conformal one-to-one map $y \to (r-y)/(1-ry)$ mapping Δ in itself and changing r (respectively 0) into 0 (resp. r), we can show that it is possible to restrict ourselves to the case $r = 0$.

For any $T > 0$, for every fixed function $(r_u, u \leqslant T)$ such that $r_0 = \log z$, $r_T = 0$ and $r_u < 0$ for all $u < T$, there exists an open set $I \subset [0, T]$ of Lebesgue measure T and a continuous function $f: I \to \mathbb{R}$, such that

$$\{\forall u \in [0, T], \exp(r_u + i\theta_u) \notin K\} \subset \{\forall u \in I, f(u) - \pi < \theta_u < f(u) + \pi\}.$$

This part is not straightforward since r is not necessarily increasing.

Now, as briefly explained in the introduction, the skew-product representation and Proposition 1 imply the result.

We do not develop this proof any further, since we are going to derive Beurling's Theorem in a more general pattern later.

3. *The problem on the circle*

In this section, we are going to prove Proposition 2. Even if the estimates are more involved, this proof has many similarities with that of Proposition 1: We will derive counterparts of Lemma 1 in Paragraphs 3·1 and 3·2, and we will then use them in Paragraph 3·3 to deduce Proposition 2.

3.1. *Decreasing rearrangements of functions*

We first recall a useful result of F. Riesz [**18**] (see also Hardy *et al.* [**8**]) on decreasing rearrangements of functions. Let f be a non-negative measurable function in $\mathbb{R}$. The symmetrically decreasing rearrangement f^* of f is the only even non-negative function defined on $\mathbb{R}$ such that f^* is non-increasing and right-continuous on $(0, \infty)$, and such that for all $y > 0$, the sets $\{x, f(x) \geqslant y\}$ and $\{x, f^*(x) \geqslant y\}$ have the same Lebesgue measure (see, e.g. in Riesz [**18**]). Loosely speaking f^* is the smallest symmetrically decreasing function such that, for all open set A in $\mathbb{R}$ with Lebesgue measure $2a$, one has $\int_A f(x)\,dx \leqslant \int_a^{-a} f^*(x)\,dx$. Inequality (3) in Riesz [**18**] says the following (see also theorem 379 in Hardy *et al.* [**8**]):

LEMMA 2. *For all positive measurable functions f, g and h on $\mathbb{R}$,*

$$\int_{\mathbb{R}\times\mathbb{R}} f(x)\,g(y)\,h(y-x)\,dx\,dy \leqslant \int_{\mathbb{R}\times\mathbb{R}} f^*(x)\,g^*(y)\,h^*(y-x)\,dx\,dy.$$

Here is an immediate consequence of this result we shall use. Let $A \subset \mathbb{R}$ be an open set of Lebesgue measure $2a$ and, for all $x \in \mathbb{R}$, put $G(x) = P_x(B_T \in A)$ and $F(x) = P_x(|B_T| < a)$.

COROLLARY 1. *Suppose f and g are two measurable bounded functions on $[-1/2, 1/2]$, such that f is even, and such that for all $u \in [0, 1/2]$,*

$$g^*(u) \leqslant f(u), \tag{5}$$

then
$$\int_{-1/2}^{1/2} g(x)\,G(x)\,dx \leqslant \int_{-1/2}^{1/2} f(x)\,F(x)\,dx. \tag{6}$$

Proof. One just needs to apply Lemma 2:

$$\int_{-1/2}^{1/2} g(x)\,G(x)\,dx = \int_{\mathbb{R}^2} 1_{\{|x|\leqslant 1/2\}}\, g(x)\, 1_{\{y\in A\}}\, p_T(y-x)\,dx\,dy$$
$$\leqslant \int_{[-1/2,\,1/2]\times[-a,\,a]} g^*(x)\,p_T(y-x)\,dx\,dy. \tag{7}$$

As $F(x) = \int_{-a}^{a} p_T(x,y)\,dy$ is a symmetrically decreasing function, (5) and (7) easily imply (6).

3.2. *Periodic rearrangements*

We will also need an extension of the previous results. Let A and B be open sets in $[-1/2, 1/2]$, of respective Lebesgue measure $2a$ and $2b$. We put $A_0 = (-a, a)$, $B_0 = (-b, b)$, $\tilde{A} = A + \mathbb{Z}$, $\tilde{B} = B + \mathbb{Z}$, $\tilde{A}_0 = A_0 + \mathbb{Z}$ and $\tilde{B}_0 = B_0 + \mathbb{Z}$. Then:

LEMMA 3. *For all* 1*-periodic even bounded positive function g which decreases on* $[0, 1/2]$, *one has*

$$\int_{A\times B} g(y-x)\,dx\,dy \leqslant \int_{A_0\times B_0} g(y-x)\,dx\,dy.$$

Outline of the proof. We note that we can in fact restrict ourselves to the case where both A and B are finite unions of disjoint intervals. Moreover, as in the proof of Lemma 1, it suffices to derive the Lemma for $g(x) = 1_{\{x\in\tilde{C}\}}$, where $\tilde{C} = (-c, c) + \mathbb{Z}$ with $c \in (0, 1/2)$. We put

$$h(x) = \int_{-c}^{c} 1_{\{y-x\in\tilde{B}\}}\,dy \quad \text{for} \quad x \in [-1/2, 1/2].$$

h is continuous, piecewise linear, $h(-1/2) = h(1/2)$ and its growth rate is -1, 0 or 1. Moreover $\int_{-1/2}^{1/2} h(x)\,dx = 4cb$ and $\sup_{[-1/2,\,1/2]} h \leqslant 2c$. Elementary considerations show that it implies

$$h^*(x) \leqslant \int_{-c}^{c} 1_{\{y-x\in\tilde{B}_0\}}\,dy. \tag{8}$$

We are now ready to complete the proof of the lemma. The periodicity of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, Lemma 2, (8) and finally the periodicity of $\tilde{A}_0$, $\tilde{B}_0$, $\tilde{C}_0$ show that

$$\int_{A\times B} 1_{\{y-x\in\tilde{C}\}}\,dx\,dy = \int_{A\times C} 1_{\{y-x\in\tilde{B}\}}\,dx\,dy = \int_{A} h(x)\,dx$$
$$\leqslant \int_{-a}^{a} h^*(x)\,dx$$
$$\leqslant \int_{-a}^{a}\int_{-c}^{c} 1_{\{y-x\in\tilde{B}_0\}}\,dx\,dy = \int_{A_0\times B_0} 1_{\{y-x\in\tilde{C}\}}\,dx\,dy.$$

Again, we formulate a consequence of this lemma that we will use later. We define for all x and y in $\mathbb{R}$,

$$\tilde{p}_T(x,y) = \sum_{p\in\mathbb{Z}} p_T(x, y+p),$$

which corresponds to the transition density of Brownian motion on the circle. $\tilde{p}_T(0,.)$ is even, 1-periodic.

If $B = (B_t, t \geqslant 0)$ is a linear Brownian motion started from 0, if $0 < x < x' < 1/2$, we put

$$\sigma = \inf\{t \geqslant 0, B_t = (x+x')/2 \text{ or } B_t = (x+x'-1)/2\}.$$

The strong Markov property and a reflexion argument at time σ show immediately that $\tilde{p}_T(0,x) > \tilde{p}_T(0,x')$. Hence $\tilde{p}_T(0,.)$ is decreasing on $[0,1/2]$.

Let now $A \subset [-1/2, 1/2]$ be an open set of Lebesgue measure $2a$, and put

$$\tilde{G}(x) = \int_A \tilde{p}_T(x,y)\,dy$$

and

$$\tilde{F}(x) = \int_{-a}^{a} \tilde{p}_T(x,y)\,dy.$$

Lemma 3 (with $g(.) = \tilde{p}_T(0,.)$) implies immediately the following analogue of Corollary 1.

Corollary 2. *Let f and g be two bounded positive measureable functions on $[-1/2,1/2]$ such that f is even and for all $u \in [0,1/2]$, $g^*(u) \leqslant f(u)$. Then*

$$\int_{-1/2}^{1/2} g(x)\,\tilde{G}(x)\,dx \leqslant \int_{-1/2}^{1/2} f(x)\,\tilde{F}(x)\,dx. \tag{9}$$

3.3. *Proof of Proposition* 2

Before writing down consequences of these estimates, let us again introduce some notation. For any measure ν on $[-1/2,1/2]$, and any $\alpha \in \mathbb{R}$, the *α-shifted measure* $\Pi^1_\alpha(\nu)$ is the measure on $[-1/2,1/2]$ such that for all Borel sets U in $[-1/2,1/2]$,

$$\Pi^1_\alpha(\nu)(U) = \nu(\{x \in [-1/2,1/2], \exists p \in \mathbb{Z}, x+\alpha-p \in U\}).$$

For any $t > 0$, we put

$$\Pi^2_t(\nu)(U) = \sum_{p \in \mathbb{Z}} P_\nu(B_t - p \in U)$$

for all Borel sets $U \subset [-1/2,1/2]$.

For any continuous function $h\colon [0,t] \to \mathbb{R}$ with $h(0) = 0$,

$$\Pi^3_{h,t}(\nu)(U) = P_\nu(B_t - h(t) \in U, \forall s \leqslant t, B_s - h(s) \in\,]-1/2,1/2[)$$

for all Borel sets $U \subset [-1/2,1/2]$.

We will say that (ν,μ) satisfies (P) if ν and μ are two 1-measures on $[-1/2,1/2]$ with measurable bounded density functions f and g, if f is even and decreasing on $[0,1/2]$, and if $\nu(I) \leqslant \mu(]-a,a[)$ for all open sets $I \subset [-1/2,1/2]$ of Lebesgue measure $2a$.

We can now state

Lemma 4. *For all (ν,μ) satisfying (P),*

(i) *for all $\alpha \in \mathbb{R}$, $\Phi^1_\alpha(\nu,\mu) = (\Pi^1_\alpha(\nu),\mu)$ satisfy (P),*

(ii) *for all $t > 0$, $\Phi^2_t(\nu,\mu) = (\Pi^2_t(\nu), \Pi^2_t(\mu))$ satisfy (P),*

(iii) *for all $t > 0$ and all continuous functions $h\colon [0,t] \to \mathbb{R}$ with $h(0) = 0$, $\Phi^3_{h,t}(\nu,\mu) = (\Pi^3_{h,t}(\nu), \Pi^3_{0,t}(\mu))$ satisfy (P).*

Proof. (i) is a trivial consequence of the definition of (P); (ii) is a straightforward consequence of Corollary 2; (iii) follows from Corollary 1 exactly as Proposition 1 does from Lemma 1.

Proof of Proposition 2. We are first going to restrict ourselves to the case where I is a finite union of disjoint bounded and bounded-away-from-0 intervals. We fix $0 < m < M$ such that for all $t \in I, m < t < M$. We define the following 1-measures on $[-1/2, 1/2]$: for all Borel sets $U \subset [-1/2, 1/2]$,

$$\mu_m(U) = P_0(\exists p \in \mathbb{Z}, B_m - p \in U),$$
$$\mu_M(U) = P_0(\forall t \in I, |B_t| < a(t) \text{ and } \exists p \in \mathbb{Z}, B_M - p \in U),$$
$$\nu_M(U) = P_0(\forall t \in I, |B_t - f(t)| < a(t) \text{ and } \exists p \in \mathbb{Z}, B_M - p \in U).$$

It is straightforward to notice that

$$(\nu_M, \mu_M) = \Psi_1 \circ \ldots \circ \Psi_q(\mu_m, \mu_m)$$

where $q > 0$ and for all $i \in \{1, \ldots, q\}$, $\Psi_i = \Phi^1_\alpha$ or Φ^2_t or $\Phi^3_{g,t}$ for some suitable α, t, g. Therefore, Lemma 4 implies that (ν_M, μ_M) satisfies (P) and in particular $|\nu_M| \leqslant |\mu_M|$; Proposition 2 follows.

We now derive Proposition 2 for general time-sets I: let us first assume that I is a bounded and bounded-away-from-0 open set. In that case, $I = \bigcup_{n \geqslant 1} I_n$, where $(I_n)_{n \geqslant 1}$ is a sequence of open bounded and bounded-away-from-0 disjoint intervals (the connected components of I). Proposition 2 holds for $\bigcup_{1 \leqslant n \leqslant p} I_n$ and it is easy to see that

$$P(\forall t \in I, \exp(iB_t) \neq -1) = \lim_{p \to \infty} P(\forall t \in \bigcup_{1 \leqslant n \leqslant p} I_n, \exp(iB_t) \neq -1)$$

so that Proposition 2 also holds for I. The case where I is not bounded and (or) not-bounded-away-from-0 follows immediately, considering the approximations $I \cap]1/n, n[$, and letting $n \to \infty$.

4. *Beurling's Theorem*

We are now going to derive Beurling's Theorem, when K is a countable union of closed and connected-by-paths sets. In other words $K = \bigcup_{n \geqslant 1} K_n$ and for all $n \geqslant 1$, for all $(x, y) \in K_n^2$, there exists a continuous path joining x to y in K_n. Let us mention that trying to derive the full version of Beurling's Theorem (that is, for any compact set K, including fractal-type sets...) this way would lead to technical trouble: one would need a generalization of Proposition 2, where the continuity hypothesis on M is removed, and for which our proof via discrete approximation fails.

The notation in this section are the same as in the introduction and we first assume that $(Z_t, t \geqslant 0)$ starts from $Z_0 = z \in]0, 1[$. We now reduce this problem.

Fatou's Lemma implies that

$$P(T(K) < T(C)) = \lim_{N \to \infty} P\left(T\left(\bigcup_{1 \leqslant n \leqslant N} K_n\right) < T(C)\right)$$

and the similar result for the radial projections; hence, we can restrict ourselves to the case where $K = \bigcup_{1 \leqslant n \leqslant N} K_n$.

For all $n \in \{1, \ldots, N\}$, as K_n is compact, one can find $(x_n, y_n) \in K_n^2$, such that, for all $x \in K_n$, $|x_n| \leqslant |x| \leqslant |y_n|$. As K_n is connected by paths, there exists a continuous path $L_n: [0,1] \to K_n$, such that $L(0) = x_n, L(1) = y_n$. Obviously, $\Pi_{K_n} = \Pi_{L_n}$ and $\{T(L_n) < T(C)\} \subset \{T(K_n) < T(C)\}$, so that we can restrict ourselves to the case where $K = \bigcup_{1 \leqslant n \leqslant N} L_n$.

For all $n \in \{1, \dots, N\}$, and $p > 1$, we define the p-polygonal approximation of L_n as follows:

$$L_n^p((i+u)/p) = L_n(i/p) + u(L_n((i+1)/p) - L_n(i/p))$$

for all $u \in [0,1]$ and $i \in \{0, \dots, p-1\}$.

As L_n is uniformly continuous,

$$\lim_{p\to\infty}\left(\sup_{n\in\{1,\dots,N\}}\sup_{s\in[0,1]}|L_n^p(s) - L_n(s)|\right) = 0.$$

Therefore, for all $\epsilon > 0$,

$$P(\inf\{|x - Z_t|, x \in K, t \in [0, T(C)]\} < \epsilon) \geqslant \limsup_{p\to\infty} P(T(K^p) < T(C)),$$

where $K^p = \bigcup_{1 \leqslant n \leqslant N} L_n^p$. As K is compact,

$$\lim_{\epsilon\to 0} P(\inf\{|x - Z_t|, x \in K, t \in [0, T(C)]\} < \epsilon) = P(T(K) < T(C));$$

hence,
$$P(T(K) < T(C)) \geqslant \limsup_{p\to\infty} P(T(K^p) < T(C)).$$

On the other hand, it is straightforward to show that

$$P(T(\Pi_K) < T(C)) = \lim_{p\to\infty} P(T(\Pi_{K^p}) < T(C)).$$

Hence, we can restrict ourselves to the case where $K = K^p$. In other words, $K = \bigcup_{1 \leqslant i \leqslant M}[x_i', y_i']$.

We can again restrict ourselves to the case where for all $i \in \{1, \dots, M-1\}$,

$$|x_i'| \leqslant |y_i'| \leqslant |x_{i+1}'| \leqslant |y_{i+1}'|$$

in just reducing K without reducing Π_K.

We are now ready for the proof itself. We first assume that $z \neq 0$. We put

$$J = \bigcup_{1\leqslant i\leqslant M}]|x_i'|, |y_i'|[.$$

For all $\epsilon > 0$, we define the time-set I_ϵ corresponding to the excursions of the process $\exp(R)$ in J, which are longer than ϵ:

$$I_\epsilon = \{s \leqslant \tau, \exists \lambda \in]0, \tau - \epsilon[, \lambda < s < \lambda + \epsilon, \forall u \in]\lambda, \lambda + \epsilon[, \exp(R_u) \in J\}.$$

I_ϵ is a finite union of disjoint intervals. Proposition 2 shows readily that

$$P(\forall u \in I_\epsilon, \exp(R_u + i\theta_u) \notin K) \leqslant P(\forall u \in I_\epsilon, \exp(R_u + i\theta_u) \notin \Pi_K).$$

But (recall that $I = \{u > 0, -\exp(R_u) \in \Pi_K\}$),

$$P(\forall u \in I, \exp(R_u + i\theta_u) \notin K) \leqslant P(\forall u \in I_\epsilon, \exp(R_u + i\theta_u) \notin K)$$

and
$$P(\forall u \in I, \exp(R_u + i\theta_u) \notin \Pi_K) = \lim_{\epsilon\to 0} P(\forall u \in I_\epsilon, \exp(R_u + i\theta_u) \notin \Pi_K)$$

so that Beurling's Theorem follows.

If $z = 0$, (2) and (1) follow immediately, using for instance the continuity of the harmonic measure. For $z \notin [0, 1]$, one just has to use a rotation argument.

Acknowledgements. I would like to thank Chris Burdzy and Keith Carne for their encouragement and for useful discussions. I also thank David Hobson, John Liechty, James Norris and Chris Shortland for nice informal discussions on this subject. I am also grateful to the referee, in particular for suggesting reference [**8**] (in which I also found [**18**]).

REFERENCES

[**1**] L. V. AHLFORS. *Conformal invariants, topics in geometric function theory* (McGraw-Hill, 1973).
[**2**] A. BEURLING. *Etudes sur un problème de majoration* (Thèse; Uppsala, 1933).
[**3**] K. BURDZY. Geometric properties of two-dimensional Brownian paths. *Probab. Th. rel. Fields* **81** (1989), 485–505.
[**4**] K. BURDZY and G. L. LAWLER. Non-intersection exponents for Brownian paths. Part II: Estimates and application to a random fractal. *Ann. Probab.* **18** (1990), 981–1009.
[**5**] T. K. CARNE. Brownian motion and Nevanlinna Theory. *Proc. London Math. Soc.* (3) **52** (1986), 349–368.
[**6**] B. DAVIES. Brownian motion and analytic functions. *Ann. Probab.* **7** (1979), 913–932.
[**7**] B. DUPLANTIER, G. F. LAWLER, J. F. LE GALL and T. J. LYONS. The geometry of the Brownian curve, in: Probabilités et Analyse stochastique, Tables rondes de St-Chéron Janvier 1992. *Bull. Sc. Math.* (2) **117** (1993).
[**8**] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA. *Inequalities* (Cambridge University Press, 1934).
[**9**] K. ITÔ and H. P. MCKEAN. *Diffusion processes and their sample paths* (Springer, 1965).
[**10**] T. Y. LAI and R. A. WIJSMAN. First exit time of a random walk from the boundary $f(n) \pm cg(n)$ with applications. *Ann. Probab.* **7** (1979), 672–692.
[**11**] G. L. LAWLER. *Intersection of random walks* (Birkhäuser, 1991).
[**12**] T. J. LYONS. A synthetic proof of Makarov's law of the iterated logarithm. *Bull. London Math. Soc.* **22** (1990), 159–162.
[**13**] N. G. MAKAROV. Probability Methods in the theory of conformal mappings. *Algebra i Analiz* **1** (1989), 3–59 (Russian), English transl.: *Leningrad Math. J.* **1** (1990), 1–56.
[**14**] R. NEVANLINNA. *Eindeutige analytische Funktionen* (Springer, 1936).
[**15**] B. OKSENDAL. Projection estimates for harmonic measure. *Ark. Math.* **21** (1983), 191–203.
[**16**] B. OKSENDAL. A stochastic proof of an extension of a Theorem of Rado. *Proc. Edinburgh Math. Soc.* **26** (1983), 333–336.
[**17**] S. C. PORT and C. J. STONE. Brownian motion and classical potential theory (Academic Press, 1979).
[**18**] F. RIESZ. Sur une inégalité intégrale. *J. London Math. Soc.* **5** (1930), 162–168.