Random planar curves and Schramm-Loewner evolutions

Lecture Notes from the 2002 Saint-Flour summer school
(final version)

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Foreword and summary

The goal of these lectures is to review some of the mathematical results that have been derived in the last years on conformal invariance, scaling limits and properties of some two-dimensional random curves. The (distinguished) audience of the Saint-Flour summer school consists mainly of probabilists and I therefore assume knowledge in stochastic calculus (Itô's formula etc.), but no special background in basic complex analysis.

These lecture notes are neither a book nor a compilation of research papers. While preparing them, I realized that it was hopeless to present all the recent results on this subject, or even to give the complete detailed proofs of a selected portion of them. Maybe this will disappoint part of the audience but the main goal of these lectures will be to try to transmit some ideas and heuristics. As a reader/part of an audience, I often think that omitting details is dangerous, and that ideas are sometimes better understood when the complete proofs are given, but in the present case, partly because the technicalities often use complex analysis tools that the audience might not be so familiar with, partly also because of the limited number of lectures, I chose to focus on some selected results and on the main ideas of their proofs, sometimes omitting technical details, and giving references for those interested in full proofs or more results. In the final chapter, I will briefly review what I omitted in these lectures, as well as work in progress or open questions.

Of course, I would like to thank my coauthors Greg Lawler and Oded Schramm without which I would not have been lecturing on this subject in Saint-Flour. Collaborating with them during these last years was a great pleasure and privilege. Also, I would like to stress the fact that (almost) none of the pictures in these notes are mine. Many thanks to their authors Vincent Beffara, Tom Kennedy and Oded Schramm. I also take this opportunity to thank Stas Smirnov, Rick Kenyon, as well as all my Orsay colleagues and students who have directly or indirectly contributed to these lecture notes through their work, comments and discussions.

Finally, I owe many thanks to all participants of the summer school, as well as to all colleagues who have sent me their comments and remarks on the first draft of these notes that was distributed during the summer school and posted on the web at that time.

It has been a pleasure and a very rewarding experience to lecture in the studious, relaxed and enjoyable atmosphere of the 2002 St-Flour school. I express my gratitude to all who have contributed to it, my co-lecturers Jim Pitman and Boris Tsirelson, the Maison des Planchettes' staff, and last but not least, Jean Picard, whose outstanding organization has been both efficient and discreet.

Here is a short description of these notes: In the first introductory chapter, I will briefly describe two discrete models (loop-erased random walks and critical percolation interfaces) that have now been proved to converge in their scaling limit to SLE (Oded Schramm used these letters as shorthand for "stochastic Loewner Evolution", but I will stick to Schramm-Loewner Evolution). Using these models, I will try to show why it is natural to define this one-parameter family of random continuously growing processes based on Loewner's equation, and to introduce the difference between their chordal and radial versions.

The second chapter is a review of the necessary background on deterministic aspects of Loewner's equation in the upper half-plane, which is then used in Chapter 3 to define chordal SLE. Some first properties of this process are studied. In particular, some hitting probabilities are computed.

The fourth chapter is devoted to some special properties of SLE that hold for some special values of the parameter κ: The locality property for SLE_6, and the restriction property for $SLE_{8/3}$. These are not surprising if one thinks of these processes as the respective scaling limits of critical percolation interfaces and self-avoiding walks, but somewhat surprising if one starts from the definition of SLE itself. These properties are then used in Chapter 5, to make the link between the geometry of $SLE_{8/3}$, that of the outer boundary of a planar Brownian motion and that of the outer boundary of SLE_6.

In Chapter 6, we define radial SLE which are processes defined in a similar way as chordal SLE except that they are growing towards an interior point of the domain and not to a boundary point. We show in that chapter that radial and chordal SLE are very closely related, especially in the case $\kappa = 6$.

In Chapter 7, we show how to compute critical exponents associated to SLE that describe the asymptotic decay of certain probabilities (non-disconnection, non-intersection). Using the relation between radial SLE_6, chordal SLE_6 and planar Brownian motion, we then use these computations in Chapter 8 to determine the values of the critical exponents that describe the decay of disconnection or non-intersection probabilities for planar Brownian motions, which is one of the main goals of these lectures. As already mentioned, it will not be possible to describe all proofs in detail, but I hope that all the main ideas and steps (that are spread over the first seven chapters of these notes) are explained in sufficient detail so that the reader can get an overview of the proof. For simplicity, I will mainly focus on derivation of the disconnection exponent i.e. the proof of the fact that the probability that a complex Brownian curve $Z[0,t]$ started from $Z_0 = 1$ disconnects the origin from infinity decays like $t^{-1/8}$ when $t \to \infty$.

In Chapters 9 and 10, another important aspect of SLE is discussed: The proofs that some curves arising in discrete models from statistical physics converge to SLE in their scaling limit. The case of loop-erased random walks and uniform spanning trees is treated in Chapter 9. Chapter 10 is devoted to critical site percolation on the triangular lattice, including a brief discussion of Stas Smirnov's proof of conformal invariance and of its consequences.

A concluding chapter contains a list of other results, work in progress and open questions.

1 Introduction

1.1 General motivation

One of the main aims of both statistical physics and probability theory is to study macroscopic systems consisting of many (i.e. in the limit when this number grows to infinity) small microscopic random inputs. One may classify the results into two categories: In the limit, the behaviour of the macroscopic system becomes deterministic (these are "law of large number" type of results, and large deviations can to some extent been used in this framework), or random. The archetype for continuous random objects that appear as scaling limit of finite systems is Brownian motion. Note that it is the scaling limit of a large class of simple random walks, so that one might argue that Brownian motion is more universal than the discrete model (simple random walk) because there is no need to specify a lattice or a jump-distribution: it only captures the phenomenological properties of the walks (mean zero, stationary increments etc.).

In two dimensions, Brownian motion has an important property which was first observed by Paul Lévy ([102], see e.g. [100, 117] for "modern" proofs based on Itô's formula) and that can be heuristically related to the fact that it is the scaling limit of simple random walks on different lattices (which implies for instance invariance under rotations and under scaling): It is invariant under conformal transformations. Here is one way to state this property: Take a simply connected open planar domain D that contains the origin and is not equal to $\mathbb{C}$. Consider planar Brownian motion $(B_t, t \in [0, \tau])$ started from $B_0 = 0$ up to its exit time $\tau = \tau_D$ of the domain D. Suppose that Φ is a conformal map (that is, a one-to-one smooth map that preserves angles) from D onto some other domain D' with $\Phi(0) = 0$. Then, there exists a (random) time change $A : [0, \sigma] \to [0, \tau]$ so that $(\Phi(B_{A(s)}), s \in [0, \sigma])$ is planar Brownian motion started from 0 and killed at its first exit time σ of D'. In other words, if we forget about time-parametrization, the law of $\Phi(B)$ is again a Brownian motion. As we shall see in these lectures, conformal invariance will turn out to be instrumental in the understanding of curves arising in more complicated setups.

Actually, there exist only few known examples of probabilistic continuous models that are not directly related to Brownian motion. For instance, under mild regularity conditions, continuous finite-dimensional Markov processes

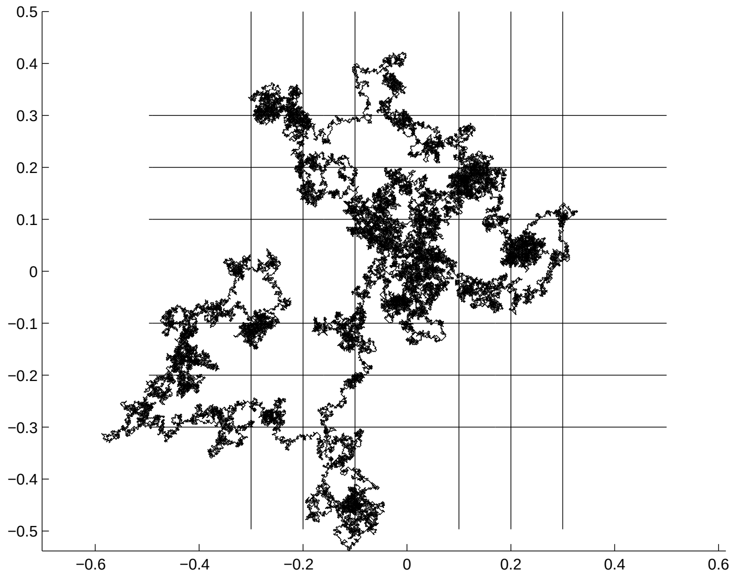

Fig. 1.1. Sample of a long simple random walk.

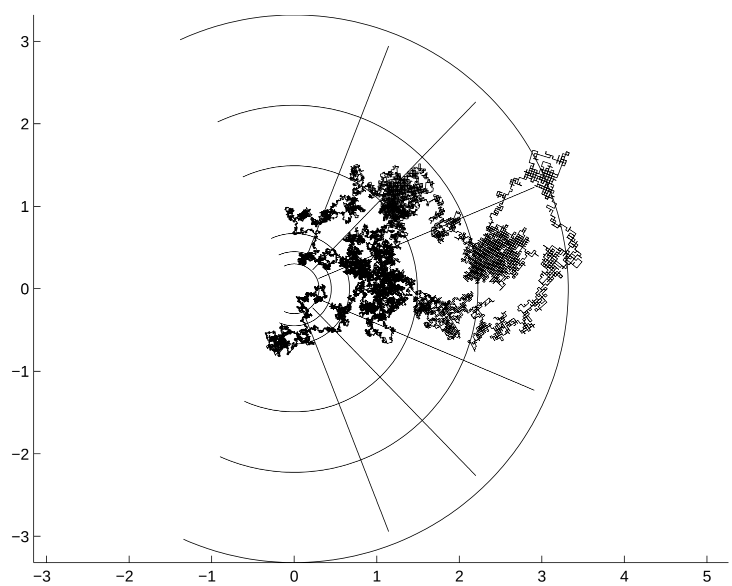

Fig. 1.2. The image of the previous sample under an exponential mapping.

are solutions of stochastic differential equations and therefore constructed using Brownian motions. If one looks for other types of continuous processes, one has therefore to give up the Markov property or the finite-dimensionality. In many complex systems that we see around us and for which probability theory seems a priori a well-suited tool (the shape of clouds, say), it is not possible to explain the phenomena via Brownian motions, and there is still a long way to go for probabilists to understand their macroscopic behaviour.

In the present lectures, we shall focus on random planar curves. In two dimensions, (random) curves appear naturally as boundaries of domains, interfaces between two phases, level lines of random surfaces etc. In all these cases, at least on microscopic level, the definition of the curve (say, as an interface) implies that it is a self-avoiding curve (or a simple closed loop). On the macroscopic scale, the continuous curves that we will be considering may have double-points (in the scaling limit, simple curves may converge to curves with multiple points), but self-crossings are forbidden. Of course, if $(\gamma_t, t \in [0,T])$ is such a random curve, we see that in general, this condition implies a strong correlation between $\gamma[0,t]$ and $\gamma[t,T]$, so that the Markov property is lost (if we look at these curves as living in the two-dimensional space). As we shall see, there is a way to recover a Markov property for the random curves, using a coding of the curve in an infinite-dimensional space of conformal maps.

1.2 Loop-erased random walks

In order to guide the intuition about the family of random curves that we will be considering, it is helpful to have some discrete models in mind, for which one expects or can prove that they converge to this continuous object. We therefore start these lectures with the description of one measure on discrete random curves that turns out to converge in the scaling limit. This is actually the model that Oded Schramm considered when he invented these random curves that he called SLE (for Stochastic Loewner Evolution, but we will replace this by Schramm-Loewner Evolution in these lectures).

For any $\mathbf{x} = (x_0, \dots, x_m)$, we define the loop-erasure $L(\mathbf{x})$ of $\mathbf{x}$ inductively as follows: $L_0 = x_0$, and for all $j \geq 0$, we define inductively $n_j = \max\{n \leq m : x_n = L_j\}$ and

$$L_{j+1} = X_{1+n_j}$$

until $j = \sigma$ where $L_\sigma := x_m$. In other words, we have erased the loops of $\mathbf{x}$ in chronological order. The number of steps σ of L is not fixed.

Suppose that $(X_n, n \geq 0)$ is a recurrent Markov chain on a discrete state-space $\mathcal{S}$ started from $X_0 = x$. Suppose that $A \subset \mathcal{S}$ is non-empty, and let τ_A denote the hitting time of A by X. Let $p(x,y)$ denote the transition probabilities for the Markov chain X. We define the loop-erasure $L = L(X[0,\tau_A]) = L^A$ of X up to its hitting time of A. We call σ the number of steps of L^A. For $y \in A$ such that with positive probability $L^A(\sigma) = X(\tau_A) = y$, we call $\mathcal{L}(x, y; A)$ the law of L^A conditioned on the event $\{L^A(\sigma) = y\}$. In other words, it is the law of the loop-erasure of the Markov chain X conditioned to hit A at y.

Lemma 1.1 (Markovian property of LERW). *Consider $y_0, \dots, y_j \in \mathcal{S}$ so that with positive probability for $\mathcal{L}(x, y_0; A)$,*

$$\{L_\sigma = y_0, L_{\sigma-1} = y_1, \dots, L_{\sigma-j} = y_j\}.$$

The conditional law of $L[0, \sigma - j]$ given this event is $\mathcal{L}(x, y_j; A \cup \{y_1, \dots, y_j\})$.

Proof. For each A and $x \in \mathcal{A}$, we denote by $G(x, A)$ the expected number of visits by the Markov chain X before τ_A if $X_0 = x$. Then, it is a simple exercise to check that for all $n \geq 1$, $\mathbf{w} = (w_0, \dots, w_n)$ with $w_0 = x$, $w_n \in A$ and $w_1, \dots, w_{n-1} \in \mathcal{S} \setminus A$,

$$\begin{aligned}
\mathbf{P}[L^A = \mathbf{w}] &= \sum_{\mathbf{x}\,:\,L(\mathbf{x})=\mathbf{w}} \mathbf{P}[X[0, \tau_A] = \mathbf{x}] \\
&= G(w_0, A) p(w_0, w_1) G(w_1, A \cup \{w_0\}) p(w_1, w_2) \cdots \\
&\quad \times G(w_{n-1}, A \cup \{w_0, w_1, \dots, w_{n-2}\}) p(w_{n-1}, w_n).
\end{aligned}$$

It is therefore natural to define the function

$$F(w_0, \dots, w_{n-1}; A) = \prod_{j=0}^{n-1} G(w_j, A \cup \{w_0, \dots, w_{j-1}\}).$$

Again, it is a simple exercise on Markov chains to check that for all A', y and y',

$$G(y, A') G(y', A' \cup \{y\}) = G(y', A') G(y, A' \cup \{y'\}).$$

It follows immediately that F is in fact a symmetric function of its arguments. Hence,

$$\begin{aligned}
&\mathbf{P}[L^A_0 = w_0, \dots, L^A_\sigma = w_n | L_\sigma = w_n, L_{\sigma-1} = w_{n-1}] \\
&= \frac{p(w_{n-1}, w_n) G(w_{n-1}, A)}{\mathbf{P}[L_\sigma = w_n, L_{\sigma-1} = w_{n-1}]} \\
&\qquad \times \prod_{j=0}^{n-2} p(w_j, w_{j+1}) G(w_j, (A \cup \{w_{n-1}\}) \cup \{w_0, \dots, w_{j-1}\}).
\end{aligned}$$

This readily implies the Lemma when $j = 1$. Iterating this j times shows the Lemma. □

This Lemma shows that it is in fact fairly natural to index the loop-erased path backwards (define $\gamma_j = L^A_{\sigma-j}$, so that γ starts on A and goes back to

$\gamma_\sigma = x$). Then, the time-reversal of loop-erased (conditioned and stopped) Markov chains have themselves a Markovian-type property.

Let us now come back to our two-dimensional setting: Suppose that X is a simple random walk on the grid $\delta\mathbb{Z}^2$ (we will then let the mesh δ of the lattice go to 0) that is started from 0. Let D denote some simply connected domain D with $0 \in D$ and $D \neq \mathbb{C}$, and let $D_\delta = \delta\mathbb{Z}^2 \cap D$, $A = A_\delta = \delta Z^2 \setminus D$. We are interested in the behaviour when $\delta \to 0$ of the law of γ^δ which is defined as before as the time-reversed loop-erasure of $X[0, \tau_A]$. We now think on a heuristic level: First, note that the law of X_{τ_A} converges to the harmonic measure on ∂D from 0, so that it is possible to study the behaviour of γ^δ conditional on the value of $\{\gamma^\delta = y_0^\delta\}$ where $y_0^\delta \to y \in \partial D$ as $\delta \to 0$. Second, one might argue that on the one hand, simple random walk converges to planar Brownian motion which is conformally invariant, and that on the other hand the chronological loop-erasing procedure is purely geometrical to conclude that when $\delta \to 0$, the law of γ^δ should converge to a conformal invariant curve that should be the loop-erasure of planar Brownian motion.

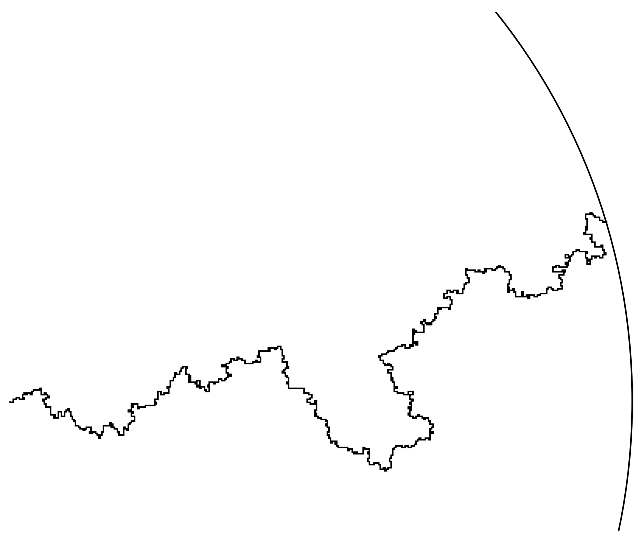

Fig. 1.3. A sample of the loop-erased random walk.

Unfortunately (or fortunately!), the geometry of planar Brownian curves is very complicated: It has points of any (even infinite) multiplicity (see e.g. [100]), loops at any scale, so that there is no “first” loop to erase, and decisions about what small microscopic loops to erase first may propagate to the decisions about what macroscopic loops one should erase. In other words, there is no simple (even random) algorithm to loop-erase a Brownian path in chronological order. Yet, the previous heuristic strongly suggests the law of γ^δ should converge, and that the limiting law is invariant under conformal transformations: The scaling limit of LERW in D should be (modulo time-change) identical to the conformal image of the scaling limit of LERW in D'. Furthermore, Lemma 1.1 should still be valid in the scaling limit. We now show that the combinations of these two properties in fact greatly reduce the family of possible scaling limits for LERW.

1.3 Iterations of conformal maps and SLE

We are therefore looking for the law of a random continuous curve $(\gamma_t, t \geq 0)$ with no self-crossings in the unit disc $\mathbb{U}$, with $\gamma_0 = 1$, $\lim_{t\to\infty} \gamma_t = 0$ that could be the scaling limit of (time-reversed) loop-erased random walk on a grid approximation of $\mathbb{U}$ (conditioned to exit $\mathbb{U}$ near 1). Define for each $t \geq 0$, the conformal map f_t from $\mathbb{U} \setminus \gamma[0,t]$ onto $\mathbb{U}$ which is normalized by $f_t(0) = 0$ and $f_t(\gamma_t) = 1$ (actually, if γ would have double-points, the domain of definition would be the connected component of $\mathbb{U}\setminus\gamma[0,t]$ that contains the origin, but let us a priori assume for convenience that γ is a simple curve).

It is easy to check that $t \mapsto |f_t'(0)|$ is an increasing continuous function that goes to ∞ as $t \to \infty$ (see for instance [2]). Hence, it is possible to reparametrize γ in such a way that

$$|f_t'(0)| = e^t. \tag{1.1}$$

This is the natural parametrization in our context. Indeed, let us now study the conditional law of $\gamma[t,\infty)$ given $\gamma[0,t]$. Lemma 1.1 suggests that this law is the scaling limit of (time-reversed) LERW in the slit domain $\mathbb{U} \setminus \gamma[0,t]$ conditioned to exit at γ_t, and conformal invariance then says that this is the same (modulo time-reparametrization) as the image under $z \mapsto f_t^{-1}(z)$ of an independent copy $\tilde\gamma$ of γ. Note that if one composes conformal maps that preserve the origin, then the derivative at the origin multiply: This shows that in fact, no time-change is necessary if we parametrize γ (and $\tilde\gamma$) by (1.1), in order for the conditional law of $(\gamma_{t+s}, s \geq 0)$ given $\gamma[0,t]$ to be identical to that of $(f_t^{-1}(\tilde\gamma_s), s \geq 0)$. In other words, for all fixed $t \geq 0$,

$$(f_{t+s}, s \geq 0) = (\tilde f_s \circ f_t,\ s \geq 0) \quad \text{in law}$$

where $(\tilde f_s, s \geq 0)$ is an independent copy of $(f_s, s \geq 0)$. In particular, $f_{2t} = \tilde f_t \circ f_t$ in law. Repeating this procedure, we see that for all $t \geq 0$ and all integer $n \geq 1$, f_{nt} is the iteration of n independent copies of f_t, and that f_t itself can be viewed as the iteration of n independent copies of $f_{t/n}$. In other words, $(f_t, t \geq 0)$ is an "infinitely divisible" process of conformal maps, and f_t is obtained by iterating infinitely many independent conformal maps that are infinitesimally close to the identity.

Back in the 1920's, Loewner observed that if $\gamma[0,\infty)$ is a simple continuous curve starting from 1 in the unit disc, then it is naturally encoded via a continuous function ζ_t taking its values on the unit circle. Let us now describe briefly how it goes. Suppose, as in the previous section, that $\gamma(0) = 1$, $\lim_{t\to\infty} \gamma_t = 0$ and that γ is parametrized in such a way that the modulus of the derivative at 0 of the conformal map f_t from $U_t := \mathbb{U} \setminus \gamma[0,t]$ into $\mathbb{U}$ that preserves the origin is e^t. Define $\zeta_t = (f_t'(0)/|f_t'(0)|)^{-1}$. In other words, if g_t denotes the conformal map from U_t onto $\mathbb{U}$ such that $g_t(0) = 0$ and $g_t'(0) = e^t \in (0,\infty)$, then

$$\zeta_t = g_t(\gamma_t)$$

and $g_t(z) = \zeta_t f_t(z)$. One can note (see e.g. [2, 49]) that for all $z \notin \gamma[0,t]$,

$$\partial_t g_t(z) = -g_t(z)\frac{g_t(z) + \zeta_t}{g_t(z) - \zeta_t}. \tag{1.2}$$

Hence, it is possible to recover γ from ζ as follows: For all $z \in \mathbb{U}$, define $g_t(z)$ as the unique solution to (1.2) starting from z. In case $g_t(z) = \zeta_t$ for some time t, then define $\gamma_t = g_t^{-1}(\zeta_t)$ (we know already a priori that since γ is a simple curve, the map g_t^{-1} extends continuously to the boundary). Note that if $g_t(z) = \zeta_t$, then $g_s(z)$ is not well-defined for $s \geq t$.

Hence, in order to define the random curve γ that should be the scaling limit of loop-erased random walks, it suffices to define the random function $\zeta_t = \exp(iW_t)$, where $(W_t, t \geq 0)$ is real-valued. Our previous considerations suggest that the following conditions should be satisfied:

- The process W is almost surely continuous.
- The process W has stationary increments (this is because g_t is obtained by iterations of identically distributed conformal maps)
- The laws of the processes W and $-W$ are identical (this is because the law of L and the law of the complex conjugate $\overline{L}$ are identical).

The theory of Markov processes tells us that the only possible choices are: $W_t = \beta_{\kappa t}$ where β is standard Brownian motion and $\kappa \geq 0$ a fixed constant. In order to simplify some future notations, we will usually write

$$W_t = \sqrt{\kappa} B_t, t \geq 0$$

where $(B_t, t \geq 0)$ is standard (one-dimensional) Brownian motion.

In summary, we have just seen that on a heuristic level, if the scaling limit of loop-erased random walk exists and is conformally invariant, then the scaling limit in the unit disk should be described as follows: For some fixed constant $\kappa = \kappa_{LERW}$, define $\zeta_t = \exp(i\sqrt{\kappa} B_t), t \geq 0$, solve for each $z \in \mathbb{U}$, the equation (1.2) with $g_0(z) = z$. This defines a conformal map g_t from the subset U_t of the unit disk onto $\mathbb{U}$. Then, one can construct γ because

$$U_t = \mathbb{U} \setminus \gamma[0,t]$$

and

$$\gamma_t = g_t^{-1}(\zeta_t).$$

As we shall see later on in the lectures, this heuristic arguments can be made rigorous, and it will turn out that $\kappa_{LERW} = 2$.

1.4 The critical percolation exploration process

In the context of LERW, the random curve joins a point in the inside of the domain to a point on the boundary of the domain. In statistical physics models, one is often interested in “interfaces”. Some of these interfaces appear to be random curves from one point on the boundary to another point on the boundary. A natural setup is to study curves from 0 to infinity in the upper half-plane $\mathbb{H} := \{x+iy \ : \ y > 0\}$. Then, we look for random non-self-crossing curves γ such that the law of $\gamma[t,\infty)$ given $\gamma[0,t]$ has the same law than the conformal image of an independent copy $\tilde{\gamma}$ of γ under a conformal map from $\mathbb{H}$ onto $\mathbb{H} \setminus \gamma[0,t]$ that maps ∞ onto itself and 0 onto γ_t.

We now very briefly describe an important discrete model for which it has now also been proved that it behaves in a conformally invariant way in the scaling limit (more details on the model and its conformal covariance will be given in Chapter 10): Critical site percolation on the triangular lattice. Actually, it is more convenient to describe this in terms of cell-colouring of the honeycombe lattice. Suppose that a simply connected domain D is fixed, as well as two distinct points a and b on ∂D. Let D_δ denote a suitably chosen approximation of D by a simply connected union of hexagonal cells of size δ. Let a_δ (resp. b_δ) denote a vertex of the honeycombe lattice on ∂D_δ that is close to a (resp. to b). Then, the cells on ∂D_δ can be divided into two “arcs” B_δ and W_δ in such a way that a_δ, B_δ, b_δ and W_δ are oriented clockwise “around” D_δ. Decide that all hexagons in B_δ are colored in black and that all hexagons in W_δ are colored in white. On the other hand, all other cells in D_δ are chosen to be black or white with probability 1/2 independently of each other. Consider now the (random) path γ_δ from a_δ to b_δ that separates the cluster of black hexagons containing B_δ from the cluster of white hexagons containing W_δ.

For deep reasons that will be discussed later in these lectures, it will turn out that when $\delta \to 0$, the law of γ_δ converges towards that of a random curve γ from a to b in D, and that the law of that curve is conformally invariant: The law of $\Phi(\gamma)$ when Φ is a conformal map from D onto $\Phi(D)$ is that of the corresponding path (i.e. of the scaling limit of percolation cluster interfaces) from $\Phi(a)$ to $\Phi(b)$ in $\Phi(D)$.

Again, on the discrete level, it is easy to see that γ_δ has the same type of Markovian property that LERW. More precisely, conditioning on the first steps of γ is equivalent to condition the percolation process to have black hexagons on the left-boundary of these steps and white hexagons on the right side. Hence, the conditional law of the remaining steps is that of the percolation interface in the new domain obtained by slitting D_δ along the first steps of γ_δ. Figure 1.4 shows the beginning of the interface γ_δ in the case where D is the upper half-plane.

Another equivalent way to define the interface γ_δ goes as follows: It is a myopic self-avoiding walk. At each step γ_δ looks at its three neighbours (on the honeycomb lattice) and chooses at random one of the sites that it has not

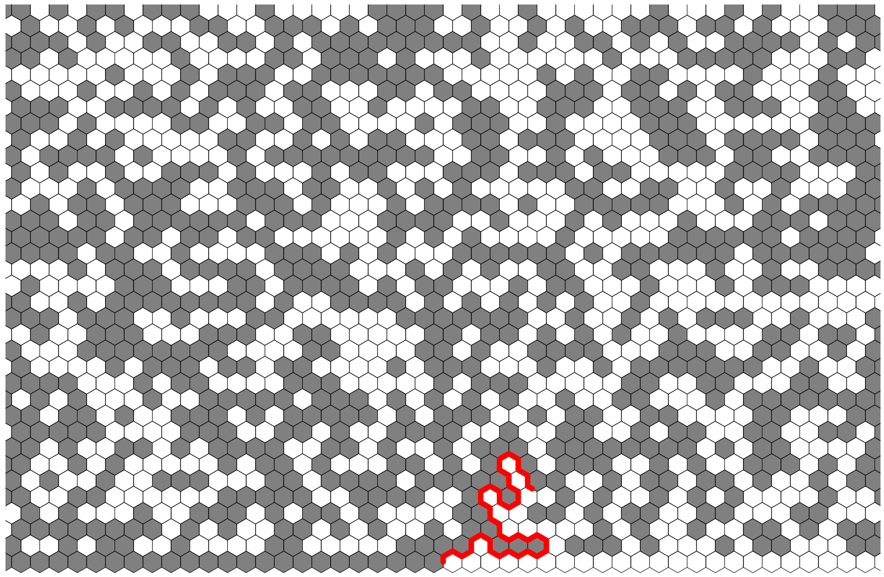

Fig. 1.4. The beginning of the discrete exploration process.

visited yet (there are one or two such sites since one site is anyway forbidden because it was the previous location of the walk).

This discrete walk in the upper half-plane is a very special discrete model that will turn out to converge to an SLE. The corresponding value of κ is 6. Here, the starting point $a = 0$ and the end-point $b = \infty$ are both on the boundary of the domain, so that the previous definition of radial SLE is not well-suited anymore.

1.5 Chordal versus Radial

The natural time-parametrization in the previous setup goes as follows: Let g_t denote the conformal map from $\mathbb{H} \setminus \gamma[0,t]$ onto $\mathbb{H}$ that is normalized at infinity in the sense that when $z \to \infty$,

$$g_t(z) = z + \frac{a_t}{z} + o(1/z).$$

It is easy to see that a_t is positive, increasing and that it is natural to parametrize g_t in such a way that a_t is a multiple of t (since the a_t terms add up when one composes two such conformal maps). It is natural to choose $a_t = 2t$ (this is consistent with the chosen parametrization in the radial case). Then, define $w_t = g_t(\gamma_t)$, and observe that

$$\partial_t g_t(z) = \frac{2}{g_t(z) - w_t}. \tag{1.3}$$

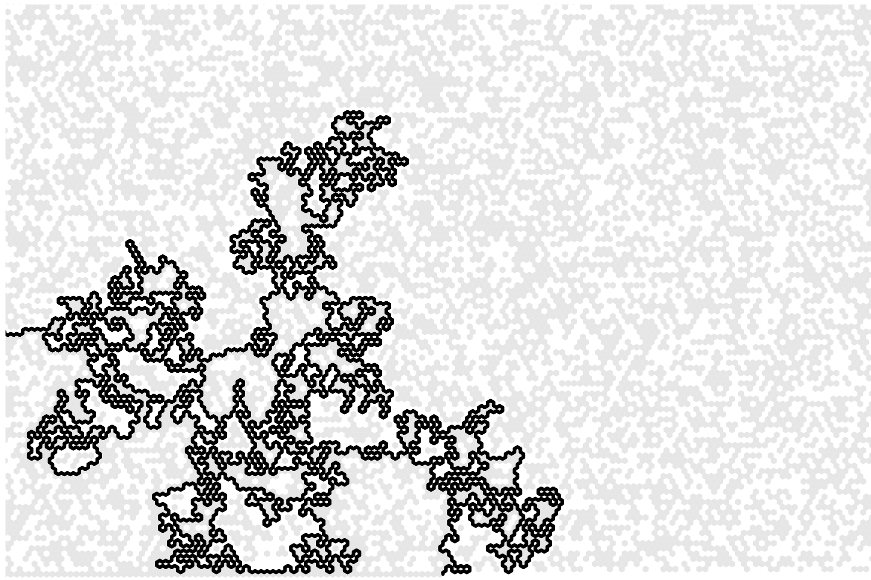

Fig. 1.5. The exploration process, proved to converge to SLE_6 (see Chapter 10)

Hence, just as in the radial case, we observe that it is possible to recover γ using w, and that the only choice for w that is consistent with the "Markovian property" is to take $w_t = \sqrt{\kappa} B_t$, where B is ordinary one-dimensional Brownian motion.

Hence, one is lead to the following definition: Let $w_t = \sqrt{\kappa} B_t$, and define for all $z \in \overline{\mathbb{H}}$, the solution $g_t(z)$ of (1.3) up to the (possibly infinite) time $T(z)$ at which $g_t(z)$ hits w_t. Then, define

$$H_t = \{z \in \mathbb{H} \ : \ T(z) > t\}$$

and

$$K_t = \{z \in \overline{\mathbb{H}} \ : \ T(z) \le t\}.$$

Then, g_t is the normalized conformal map from H_t onto $\mathbb{H}$. We call $(K_t, t \ge 0)$ the chordal SLE_κ in the upper half-plane.

It turns out that radial and chordal SLE's are rather closely related: Consider for instance, the conformal image of radial SLE_κ under the map that maps $\mathbb{U}$ onto $\mathbb{H}$, 1 to 0 and 0 to i. Consider both this process and chordal SLE_κ up to their first hitting of the circle of radius 1/2 around zero say. Then, the laws of these two processes are absolutely continuous with respect to each other [87]. This justifies a posteriori the choice of time-parametrization in the chordal case.

1.6 Conclusion

We have seen that if one considers a discrete model of random curves (or interfaces) that combine the two important features:

– The Markovian type property in the discrete setting,
– Conformal invariance in the limit when the mesh of the lattice goes to zero,

then the good way to construct the possible candidates for the scaling limit of these curves is to encode them via the corresponding conformal mappings. Then, these (random) conformal mappings are themselves obtained by iterations of identically distributed random conformal maps. Loewner's theory shows that such families of conformal maps are themselves encoded by a one-dimensional function. If one knows this one-dimensional function, one can recover the family of conformal maps, and therefore also the two-dimensional curve. The one-dimensional random function that generates the scaling limits of the discrete models must necessarily be a one-dimensional Brownian motion. The corresponding random two-dimensional curves are SLE processes.

Bibliographical comments

Most of the intuition about how to define radial and chordal SLE (with LERW as a guide) was already present in the introduction of Oded Schramm's first paper [123] on SLE that he released in March 1999. Our presentation of Lemma 1.1 is borrowed from Lawler [81], but there are other proofs of it (it is for instance closely related to Wilson's algorithm [136] that will be discussed in Chapter 9).

2 Loewner chains

This chapter does contain background material on conformal maps and on Loewner's equation (no really new results will be presented here). The setup is deterministic in this Chapter. SLE will be introduced in the next Chapter.

2.1 Measuring the size of subsets of the half-plane

We study increasing "continuously growing" compact subsets $(K_t, t \geq 0)$ of the upper half-plane. It will turn out to be important to choose the good time-parametrization. We want to find the natural way to measure the size $a(K)$ of a compact set K and we will then choose the time-parametrization in such a way that $a(K_t) = t$. We will use the following definition throughout the paper.

Definition. We say that a compact subset K of the closed upper half-plane $\overline{\mathbb{H}}$, such that $H := \mathbb{H} \setminus K$ is simply connected, is a hull.

Riemann's mapping theorem asserts that there exist conformal maps Φ from H onto $\mathbb{H}$ with $\Phi(\infty) = \infty$. Actually, if Φ is such a map, the family of maps $b\Phi + b'$ for real b' and positive b is exactly the family of conformal maps from H onto $\mathbb{H}$ that fix infinity.

Note that since K is compact, the mapping $\Psi : z \mapsto 1/\Phi(1/z)$ is well-defined on a neighbourhood of 0 in $\mathbb{H}$. It is possible to extend this map Ψ to a whole neighbourhood of 0 in the plane by reflection along the real axis (this is usually called Schwarz reflection) and to check that this extension is analytic. This implies that Φ can be expanded near infinity: There exist $b_1, b_0, b_{-1}, \ldots$, such that

$$\Phi(z) = b_1 z + b_0 + b_{-1} z^{-1} + \cdots + b_{-n} z^{-n} + o(z^{-n})$$

when $z \to \infty$ in $\mathbb{H}$. Furthermore, since Φ preserves the real axis near infinity, all coefficients b_j are real.

Hence, for each K, there exists a unique conformal map $\Phi = \Phi_K$ from $H = \mathbb{H} \setminus K$ onto $\mathbb{H}$ such that:

$$\Phi(z) = z + 0 + o(1/z) \text{ when } z \to \infty.$$

This is sometimes called the hydrodynamical normalization. In particular, there exists a real $a = a(K)$ such that

$$\Phi(z) = z + \frac{2a}{z} + o(1/z) \text{ when } z \to \infty.$$

This number $a(K)$ is a way to measure the size of K. In a way, it tells "how big K is in $\mathbb{H}$, seen from infinity". It may a priori not be clear that a is a non-negative increasing function of the set K. There is a simple probabilistic interpretation of $a(K)$ that immediately implies these facts: Suppose that $Z = X + iY$ is a complex Brownian motion started from $Z_0 = iy$ (for some large y, so that $Z_0 \notin K$) and stopped at its first exit time τ of H. The expansion $\Phi(z) = z + o(1)$ near infinity shows that $\Im(\Phi(z) - z)$ is a bounded harmonic function in H. The martingale stopping theorem therefore shows that

$$\mathbf{E}[\Im(\Phi(Z_\tau)) - Y_\tau] = \Im(\Phi(iy) - iy) = \frac{2a}{iy} + o(1/y).$$

But $\Phi(Z_\tau)$ is real because of the definition of τ. Therefore

$$2a = \lim_{y \to +\infty} y \ \mathbf{E}[\Im(Y_\tau)].$$

In particular, $a \geq 0$.

One can also view a as a function of the normalized conformal map Φ_K instead of K. The chain rule for Taylor expansions then immediately shows that

$$a(\Phi^1 \circ \Phi^2) = a(\Phi^1) + a(\Phi^2)$$

for any two normalized maps Φ^1 and Φ^2. In particular, this readily implies that $a(K) \leq a(K')$ if $K \subset K'$ (because there exists a normalized conformal map from $\mathbb{H} \setminus \Phi_K(K' \setminus K)$ onto $\mathbb{H}$).

Let us now observe two simple facts:

- If $\lambda > 0$, then $a(\lambda K) = \lambda^2 a(K)$. This is simply due to the fact that

  $$\Phi(z/\lambda) = \frac{z}{\lambda} + \frac{2a(K)\lambda}{z} + o(\lambda/z)$$

  so that

  $$\Phi_{\lambda K}(z) = \lambda \Phi_K(z/\lambda) = z + \frac{2a(K)\lambda^2}{z} + o\lambda(\lambda/z) \tag{2.1}$$

  when $z \to \infty$.
- When K is the vertical slit $[0, iy]$, then

  $$\Phi_K(z) = \sqrt{z^2 + y^2}.$$

  In particular, we see that $a([0, iy]) = y^2/4$. Note that if y is very small, the actual diameter of the vertical slit $[0, iy]$ is much larger than $a([0, iy])$.

Equation (2.1) shows that for all K such that $a(K) = 1$, one has $a(\sqrt{\lambda}K) = \lambda$ and

$$\lim_{\lambda \to 0} \frac{\Phi_{\sqrt{\lambda}K}(z) - \Phi_{\{0\}}(z)}{\lambda} = \lim_{\lambda \to 0} \frac{\Phi_{\sqrt{\lambda}K}(z) - z}{\lambda} = \frac{2}{z}. \tag{2.2}$$

Actually, it is not very difficult to prove that for all given r, there exists $C > 0$ such that this convergence takes place uniformly over all K of radius smaller than r and $|z| > Cr$. See Lemma 2.7 in [86].

2.2 Loewner chains

Suppose that a continuous real function w_t with $w_0 = 0$ is given. For each $z \in \overline{\mathbb{H}}$, define the function $g_t(z)$ as the solution to the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - w_t} \tag{2.3}$$

with $g_0(z) = z$. This is well-defined as long as $g_t(z) - w_t$ does not hit 0, i.e., for all $t < T(z)$, where

$$T(z) := \sup\{t \geq 0 \ : \ \min_{s \in [0,t]} |g_s(z) - w_s| > 0\}.$$

We define

$$\begin{aligned} K_t &:= \{z \in \overline{\mathbb{H}} \ : \ T(z) \leq t\} \\ H_t &:= \mathbb{H} \setminus K_t. \end{aligned}$$

Note for instance that if $w_t = 0$ for all t, then

$$g_t(z) = \sqrt{z^2 + 4t}$$

and $K_t = [0, 2i\sqrt{t}]$.

It is very easy to check that g_t is a bijection from H_t onto $\mathbb{H}$ (in order to see that it is surjective, one can just look at the ODE "backwards in time" to find which point z is such that $g_t(z) = y$). Moreover K_t is bounded (because w is continuous and bounded on $[0, t]$) and H_t has a unique connected component (because g_t^{-1} is continuous). Standard arguments from the theory of ordinary differential equations can be applied to check that g_t is analytic and that one can formally differentiate the ODE with respect to z, so that

$$\partial_t g_t'(z) = \frac{-2g_t'(z)}{(g_t(z) - w_t)^2}.$$

So, g_t is a conformal map from H_t onto $\mathbb{H}$.

Note also that $|\partial_t g_t(z)|$ is uniformly bounded when z is large and t belongs to a given finite interval $[0, t_0]$. Hence, it follows that $g_t(z) = z + O(1)$ near infinity and uniformly over $t \in [0, t_0]$. Hence (using the ODE yet again), $\partial_t g_t(z) = 2/z + o(1/z)$ uniformly over $t \in [0, t_0]$ so that finally, for each t,

$$g_t(z) = z + \frac{2t}{z} + o(1/z)$$

when $z \to \infty$. In other words, $a(K_t) = t$. The family $(K_t, t \geq 0)$ is called the Loewner chain associated to the driving function $(w_t, t \geq 0)$.

Loewner's original motivation was to control the behaviour of the coefficients of the Taylor expansion of conformal maps and for this goal, it is sufficient to consider smooth slit domains (see e.g., [2, 49]). For this reason, the following question was only addressed later (see [72]): If the continuous function $(w_t, t \geq 0)$ is given, what can be said about the family of compact sets $(K_t, t \geq 0)$?

In the introduction, we started with a continuous curve γ, then using γ, we defined H_t, the conformal maps g_t, the function w_t and argued that one could recover γ from w_t, using the fact that we a priori knew that g_t^{-1} extends continuously to $w_t \in \partial\mathbb{H}$ and that $g_t^{-1}(w_t)$ was well-defined (and equal to γ_t) because γ is a continuous curve. But if one starts with a general continuous function w_t, then it can in fact happen that g_t^{-1} does not extend continuously to w_t.

Before making general considerations, let us exhibit a simple example to show that $(K_t, t \geq 0)$ does not need to be a simple curve. For $\theta \in [0, \pi)$, let $\eta(\theta) = \exp(i\theta) - 1$. Define $t(\theta) = a(\eta[0, \theta])$ the "size" of the arc $\eta[0, \theta]$. Finally, define the reparametrization γ of η in such a way that $a(\gamma[0, t]) = t$. γ is defined for all $t < T := \lim_{\theta \to \pi-} a(\eta[0, \theta])$. It is simple to see that there exists a continuous function $(w_t, t < T)$ such that the normalized conformal maps g_t from $\mathbb{H} \setminus \gamma[0, t]$ onto $\mathbb{H}$ satisfy the equation (2.3). Furthermore, when $t \to T-$, w_t converges to a finite limit w_T. At time T, the curve $\gamma[0, T]$ disconnects the inside of the semi-circle from the outside. Just before T, because g_t is normalized "from infinity", the inside of the semi-circle is mapped onto a small region which is very close to $w_t = g_t(\gamma_t)$. When $t \to T-$, all points inside the semi-circle are hitting w_T. In other words, K_T is the whole semi-disc, H_T is the complement of the semi-disc, and g_T is the normalized map from the simply connected domain H_T onto $\mathbb{H}$.

Let us now give a couple of general definitions:

- We say that $(K_t, t \geq 0)$ is a simple curve if there exists a simple continuous curve γ such that $K_t = \gamma[0, t]$.
- We say that $(K_t, t \geq 0)$ is generated by a curve if there exists a continuous curve γ with no self-crossings, such that for all $t \geq 0$, $H_t = \mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma[0, t]$. In other words, K_t is the union of γ and of the inside of the loops that γ creates.
- We say that $(K_t, t \geq 0)$ is pathological if it is not generated by a curve.

In each of these three cases, one can find (deterministic) continuous functions w_t such that the family $(K_t, t \geq 0)$ that it constructs falls into this category: For the first case, consider for instance $w_t = 0$ as before, for the second case, one can use the example with the semi-circle. For the more intricate third case, let us mention the following example (due to Don Marshal and Steffen Rohde, see [108]): Let γ denote a simple curve in $\mathbb{H}$ started from $\gamma_0 = 0$ that spirals clockwise around the segment $[i, 2i]$ an infinite number of times, and then unwinds itself. Then at the "time" at which it winds around the segment an infinite number of times, γ is not continuous i.e. $K_t \setminus K_{t-}$ is the whole segment. However, this Loewner chain corresponds to a continuous function w_t. Such pathologies could arise at any scale.

We now characterize the families $(K_t, t \geq 0)$ of compact sets that are Loewner chains:

Proposition 2.1 *The following two conditions are equivalent:*

1. *$(K_t, t \geq 0)$ is a Loewner chain associated to a continuous driving function $(w_t, t \geq 0)$.*
2. *For all $t \geq 0$, $a(K_t) = t$, and for all $T > 0$, and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \leq T$, there exists a bounded connected set $S \subset \mathbb{H} \setminus K_t$ with diameter not larger than ε such that S disconnects $K_{t+\delta} \setminus K_t$ from infinity in $\mathbb{H} \setminus K_t$.*

Sketch of the proof. Let us now prove that 2. implies 1. (the fact that 1. implies 2. is very easy): 2. implies that for all $t \geq 0$, the diameter of the sets $g_t(K_{t+\delta} \setminus K_t)$ decrease towards 0 when $\delta \to 0$. Hence, one can simply define w_t by

$$\{w_t\} = \lim_{\delta \to 0} \overline{g_t(K_{t+\delta} \setminus K_t)}.$$

Then, one uses 2. to show that $t \mapsto w_t$ is uniformly continuous. It then only remains to check that indeed

$$\lim_{\delta \to 0} \frac{g_{t+\delta}(z) - g_t(z)}{\delta} = \frac{2}{g_t(z) - w_t}.$$

This is achieved by applying the uniform version of (2.2). $\square$

Suppose now that K_t is the Loewner chain

$$K_t = [0, c\sqrt{t}]$$

for some $c = c(\theta) \exp(i\theta) \in \mathbb{H}$. Here, $\theta \neq 0$ is given, and then the positive real $c(\theta)$ is chosen in such a way that $a(K_1) = 1$. Scaling immediately shows that $a(K_t) = t$ for all $t > 0$, so that there exists therefore a continuous driving function w that generates these slits. Again, scaling (because $K_{\lambda t} = \sqrt{\lambda} K_t$) shows that necessarily, this function w must be of the type

$$w_t = c_1 \sqrt{t}$$

for some real constant $c_1 = c_1(\theta)$. Let g_t^θ denote the corresponding family of conformal maps.

Let us now choose a new driving function w as follows: $w_t = 0$ when $t < 1$ and for $t \geq 1$:

$$w_t = c_1\sqrt{t-1}.$$

When $t < 1$, then K_t is just the straight slit. In particular, $g_1(z) = \sqrt{z^2+4}$. When $t > 1$, then K_t is obtained by mapping the angled slit K_{t-1}^θ back by g_1^{-1}. In particular, we see that the curve γ generated by this function w is not differentiable at $t = 1$. This is one simple hint to the fact that Hölder-1/2 regularity may be critical (note that at $t = 1$, w is just Hölder 1/2).

The general relation between smoothness of the driving function and regularity of the slit has also recently been investigated (in the deterministic setting) by Marshall-Rohde [108]. In this paper, it is shown that Hölder-1/2 is in a sense a "critical regularity" for the driving function w_t: Loosely speaking (their results are more precise than that), if w is better than Hölder-1/2, then it defines a "smooth" (in some appropriate sense) slit, but nasty "pathological" phenomena can occur for Hölder-1/2 driving functions. See [108] and the references therein.

Bibliographical comments

For general background on complex analysis, Riemann's mapping theorem, there are plenty of good references, see for instance [1, 119]. Loewner introduced his equation (in the radial setting) in 1923 [103]. For general information about Loewner's equation, and in particular how Loewner used it to prove that $|a_3| \leq 3$ for univalent functions $z + \sum_{n\geq 2} a_n z^n$ on $\mathbb{U}$ as well as other applications, see for instance [2, 49]. For how it is used in de Branges' proof of the Bieberbach conjecture, a good self-contained reference is Hayman's book [58]. For basics on hypergeometric functions, see e.g., [99].

Proposition 2.1 is derived in [86], see also [114]. Carleson and Makarov [35, 36] have used Loewner's (radial) equation in the context of Diffusion Limited Aggregation.

3 Chordal SLE

3.1 Definition

Chordal SLE_κ is the Loewner chain $(K_t, t \geq 0)$ that is obtained when the driving function

$$w_t = W_t := \sqrt{\kappa} B_t$$

is $\sqrt{\kappa}$ times a standard real-valued Brownian motion $(B_t, t \geq 0)$ with $B_0 = 0$. Let us now list a couple of consequences of the simple properties of Brownian motion:

– Brownian motion is a strong Markov process with independent increments. This implies that for any stopping time T (with respect to the natural filtration $(\mathcal{F}_t, t \geq 0)$ of B), the process

$$(g_{T+t}(K_{T+t} \setminus K_T) - W_T, t \geq 0)$$

  is independent of $\mathcal{F}_T$ and that its law is identical to that of $(K_t, t \geq 0)$. Note that one has to shift by W_T in order to obtain a process starting at the origin.

– Brownian motion is scale-invariant: For each $\lambda > 0$, the process $W_t^\lambda := W_{\lambda t}/\sqrt{\lambda}, t \geq 0$ has the same law than W. But

$$\partial_t (g_{\lambda t}(\sqrt{\lambda} z)) = \frac{2\lambda}{g_t(\sqrt{\lambda} z) - W_{\lambda t}}.$$

  In particular, if

$$g_t^\lambda(z) := g_{\lambda t}(z\sqrt{\lambda})/\sqrt{\lambda},$$

  then

$$\partial_t g_t^\lambda(z) = \frac{2}{g_t^\lambda(z) - W_t^\lambda}$$

  and $g_0^\lambda(z) = z$. In other words, $(K_{\lambda t}, t \geq 0)$ and $(\sqrt{\lambda} K_t, t \geq 0)$ have the same law: Chordal SLE_κ is scale-invariant.

– Brownian motion is symmetric (W and $-W$ have the same law). Hence, the law of $(K_t, t \geq 0)$ is symmetric with respect to the imaginary axis.

It is actually possible to prove the following result:

Proposition 3.1 *For all $\kappa \geq 0$, chordal SLE_κ is almost surely not pathological. When $\kappa \leq 4$, it is a.s. a simple curve γ, when $\kappa > 4$, it is a.s. generated by a (non-simple) curve γ.*

This result is due to Rohde-Schramm [118] (see [93] for the critical case $\kappa = 8$). It is not an easy result, especially for the values $\kappa > 4$. Actually, while this fact is important and useful in order to understand heuristically the behaviour and the properties of SLE_κ, it turns out that one can derive many of them without knowing that the SLE_κ is generated by a continuous curve. We therefore omit the proof in these lectures, and we will call $(K_t, t \geq 0)$ the SLE. In some cases that we will focus on ($\kappa = 2, 8/3, 6, 8$), the fact that SLE_κ is a.s. generated by a curve will actually follow from other considerations.

It is however easy to see that $\kappa = 4$ is a critical value: Consider chordal SLE_κ, and define

$$X_t = \frac{g_t(1) - W_t}{\sqrt{\kappa}}.$$

Note that X hits zero if and only if the chordal SLE absorbs the boundary point 1. But X satisfies

$$dX_t = dB_t + \frac{2}{\kappa X_t} dt. \tag{3.1}$$

It is a $1+(4/\kappa)$ dimensional Bessel process, and it is well-known (see e.g. [117]) that such a process a.s. hits zero if and only if $\kappa > 4$. This can for instance be viewed as a consequence of the fact that if X is a Bessel process of dimension d started away from zero, then if $d \neq 2$, X^{2-d} is a local martingale, and when $d = 2$, $\log X$ is a local martingale.

It follows that:

Proposition 3.2 – *If $\kappa \leq 4$, then almost surely $\cup_{t \geq 0} K_t \cap \mathbb{R} = \{0\}$.*
– *If $\kappa > 4$, then almost surely, $\mathbb{R} \subset \cup_{t \geq 0} K_t$.*

Assuming that the SLE is generated by a curve, this readily shows that the SLE curve is simple if and only if $\kappa \leq 4$.

If one defines, for all $z \in \mathbb{H}$, the solution X_t^z to (3.1) started from $X_0^z = z/\sqrt{\kappa}$ (up to the stopping time $T(z)$). Then, we see that SLE_κ can be interpreted in terms of the flow of a complex Bessel process: For each $t > 0$, K_t is the set of starting points such that X_t^z has hit 0 before time t.

3.2 A first computation

We now compute the probability of some simple events involving the chordal Schramm-Loewner evolution. Suppose that $a < 0 < c$. Let $\kappa > 0$ be fixed. Define the event $E_{a,c}$ that the chordal SLE_κ hits $[c, \infty)$ before $(-\infty, a]$. For the reasons that we just discussed, this makes sense only if $\kappa > 4$ (otherwise, it never hits these intervals). The goal of this section is to compute the probability of $E_{a,c}$. The scaling property of chordal SLE shows that this is a function of the ration c/a only. We can therefore define $F = F_\kappa$ on the interval $(0,1)$ by

$$\mathbf{P}[E_{a,c}] = F(-a/(c-a)).$$

Proposition 3.3 *For all $\kappa > 4$ and $z \in (0,1)$,*

$$F(z) = c(\kappa) \int_0^z \frac{du}{u^{4/\kappa}(1-u)^{4/\kappa}}$$

where $c(\kappa) = (\int_0^1 u^{-4/\kappa}(1-u)^{-4/\kappa} du)^{-1}$ is chosen so that $F(1) = 1$.

Note that this Proposition is in fact a property of the real Bessel flow: $E_{a,c}$ is the event that X^c hits 0 before X^a does.

Proof. Suppose that $\mathcal{F}_t = \sigma(B_s, s \leq t)$ is the natural filtration associated to the Brownian motion, and define $T_a = T(a)$ and $T_c = T(c)$ as before (the times at which a and c are respectively absorbed by K_t). For $t < T_a$ and $t < T_c$ respectively, define

$$A_t := g_t(a) \text{ and } C_t := g_t(c).$$

Suppose that $t < \min(T_a, T_c)$, and define

$$K_{t,s} = g_t(K_{t+s} \setminus K_t) - W_t.$$

The strong Markov property shows that $(K_{t,s}, s \geq 0)$ is also chordal SLE_κ, and that it is independent from $\mathcal{F}_t$. Also, if $t < \min(T_a, T_c)$, the event $E_{a,c}$ corresponds to the event that $(K_{t,s}, s \geq 0)$ hits $[C_t - W_t, \infty)$ before $(-\infty, A_t - W_t]$. Hence, if $t < \min(T_a, T_c)$,

$$\mathbf{P}[E_{a,c} \mid \mathcal{F}_t] = F\left(\frac{W_t - A_t}{C_t - A_t}\right).$$

In particular, this shows that the right-hand side of the previous identity is a (bounded) martingale. We know that $W_t = \sqrt{\kappa} B_t$, and that

$$\partial_t A_t = \frac{2}{A_t - W_t}, \ \partial_t C_t = \frac{2}{C_t - W_t}.$$

Hence, if we put $Z_t := (W_t - A_t)/(C_t - A_t)$, stochastic calculus yields

$$dZ_t = \frac{\sqrt{\kappa} dB_t}{C_t - A_t} + \frac{2dt}{(C_t - A_t)^2}\left(\frac{1}{Z_t} - \frac{1}{1 - Z_t}\right).$$

One can now also introduce the natural time-change

$$s = s(t) := \int_0^t \frac{du}{(C_u - A_u)^2}$$

and define $\tilde{Z}$ in such a way that $\tilde{Z}_{s(t)} = Z_t$. Then,

$$\tilde{Z}_s = \sqrt{\kappa} d\tilde{B}_s + 2\left(\frac{1}{Z_s} - \frac{1}{1 - Z_s}\right) ds$$

where $(\tilde{B}_s, s \geq 0)$ is a standard Brownian motion.

But K_t hits $(-\infty, a)$ if and only Z_t hits 0, and K_t hits (c, ∞) if and only if Z_t hits 1. Hence, $F(z)$ is the probability that the diffusion $\tilde{Z}$ started from $\tilde{Z}_0 = z$ hits 1 before 0. One can invoke (for instance) the general theory of diffusions to argue that the function F is therefore smooth on $(0, 1)$. Itô's formula (since $F(\tilde{Z}_s)$ is a martingale) then implies that

$$\frac{\kappa}{4} F''(z) + \left(\frac{1}{z} - \frac{1}{1 - z}\right) F'(z) = 0. \tag{3.2}$$

Furthermore, the boundary values of F are simple to work out: When $\kappa > 4$, one can see (for instance comparing $\tilde{Z}$ with a Bessel process) that

$$\lim_{z \to 0} F(z) = 0 \text{ and } \lim_{z \to 1} F(z) = 1.$$

Hence, F is the only solution to the ODE (3.2) with boundary values $F(0) = 0$ and $F(1) = 1$. This immediately proves the Proposition. $\square$

Note that when $z \to 0$,

$$F(z) \sim \frac{c(\kappa)}{1 - 4/\kappa} z^{1 - 4/\kappa}.$$

In particular, for $\kappa = 6$, we get the exponent $1/3$.

Exactly in the same way, it is possible (for $\kappa > 4$) to compute the probability that chordal SLE_κ (started from 0) hits the interval $[a, c]$ before $[c, \infty)$ when $0 < a < c$. This is a function $\tilde{F}$ of the ratio a/c, satisfying a linear second-order differential equation, with the boundary conditions

$$\tilde{F}(1) = 0 \text{ and } \tilde{F}(0) = 1.$$

3.3 Chordal SLE_κ in other domains

Suppose that D is some given non-empty open simply connected subset of the complex plane with $D \neq \mathbb{C}$. We do not impose any regularity condition on ∂D. Riemann's mapping theorem shows that there exist (many) conformal maps Φ from the upper half-plane $\mathbb{H}$ onto D. Even if the boundary of ∂D is not smooth, one can define a general notion that coincides with that of boundary points when it is smooth: For each $x \in \overline{\mathbb{R}}$, we say that (if some map Φ is given) $\Phi(x)$ is a prime end of D (see e.g. [116] for a more precise and correct definition).

Suppose that O and U are two distinct prime ends in D. Then, there exists a conformal map Φ from $\mathbb{H}$ onto D such that $\Phi(0) = O$ and $\Phi(\infty) = U$. Actually, this only characterizes $\Phi(\cdot)$ up to a multiplicative factor (because $\Phi(\lambda \cdot)$ would then also do).

Suppose that $(K_t, t \geq 0)$ is chordal SLE_κ in $\mathbb{H}$ as defined before. We define SLE_κ in D from O to U as the image of the process $(K_t, t \geq 0)$ under Φ. Recall that Φ is defined up to a multiplicative constant. However, the scaling property of SLE_κ in $\mathbb{H}$ shows that the law of $(\Phi(K_t), t \geq 0)$ is invariant (modulo linear time-change) if we replace $\Phi(\cdot)$ by $\Phi(\lambda \cdot)$.

To illustrate this definition, consider the following setup: Suppose that $\kappa = 6$ and that OAC is an equilateral triangle. Let Φ denote the conformal map from $\mathbb{H}$ onto the triangle defined in such a way that

$$\Phi(a) = A, \Phi(0) = O, \Phi(c) = C$$

where $a < 0 < c$ are given. This conformal map can be easily described explicitly using the Schwarz-Christoffel transformations [1, 119]. Note that $U = \Phi(\infty)$ is on the interval AC. It turns out that

$$\frac{AU}{AC} = F(z)$$

where $z = -a/(c-a)$ and $F = F_{\kappa=6}$ is precisely the same hypergeometric function as in Proposition 3.3. Hence, the probability that chordal SLE_6 from O to U in the equilateral triangle OAC hits AU before UC is simply the ratio AU/AC.

Suppose now that $\kappa \in (4, 8)$. Just as for the hypergeometric function F, the functions $\tilde{F}$ that were defined at the end of the last subsection have a nice interpretation in terms of conformal mappings onto triangles: Consider an isocele triangle $\mathcal{T} = OAU$ with $OA = AU = 1$ and angle $\pi(1 - 4/\kappa)$ at the vertices O and U. The angle at the vertex A is therefore $\pi(8/\kappa - 1)$. Consider now a chordal SLE_κ from O to U in the triangle $\mathcal{T}$. Let X denote the random point at which it first hits the segment AU.

Proposition 3.4 *The law of X is the uniform distribution on AU.*

This is a direct consequence of the explicit computation of $\tilde{F}$ and of the explicit Schwarz-Christoffel mapping from the upper half-plane onto $\mathcal{T}$: For each $C \in AU$, one can compute the probability that $X \in [AC]$ via the function $\tilde{F}$. $\square$

This gives a first justification to the fact that the only possible conformally invariant scaling limit of the critical percolation exploration process is SLE_6 (see more on this in Chapter 10). Indeed, suppose that the critical percolation exploration process is conformally invariant. We have argued in the first chapter that the scaling limit is one of the SLEs. Suppose that it is SLE_κ for a given value of κ, and consider the corresponding triangle $\mathcal{T}$.

Clearly in the discrete case (for a fixed small meshsize), up to the first time at which it hits the edge AU, the critical exploration process from O to U and the critical exploration process from O to A in $\mathcal{T}$ coincide. Hence, the hitting distributions on AU for chordal SLE_κ from O to U and for chordal SLE_κ from O to A coincide. In particular, the uniform distribution on AU must be invariant under the anti-conformal map from $\mathcal{T}$ onto itself that maps O onto itself and interchanges the vertices A and U. This is only true when the triangle is symmetric (i.e. the angles at U and A are identical), in other words when $\alpha = \pi/3$ or $\kappa = 6$.

We shall see in the next chapter that indeed, for SLE_6, the whole paths from O to A and from O to U coincide up to their first hitting of AU. This is the so-called locality property of SLE_6.

3.4 Transience

We conclude this chapter with the following fact (assuming the fact that the SLE is a.s. a simple curve $\gamma_t = K_t \setminus K_{t-}$ for $\kappa < 4$). This is also to illustrate the type of techniques that is used to derive such properties of SLE:

Proposition 3.5 *For $\kappa < 4$, almost surely,* $\lim_{t\to\infty} \gamma_t = \infty$.

Loosely speaking, the SLE is transient. Actually (see [118]), this result is in fact valid for all κ, but the proof is (a little bit) more involved.

Proof. Let $\delta \in (0, 1/4)$, $x > 1$, and suppose that

$$t_\delta := \inf\{t > 0 \ : \ d(\gamma_t, [1, x]) \le \delta\}$$

is finite. Let $z_\delta = \gamma_{t_\delta}$. Clearly, $g_{t_\delta}(z_\delta) = W_{t_\delta}$. Note that $g_{t_\delta}(1/2) - W_{t_\delta}$ is (up to a multiplicative constant) the limit when $y \to +\infty$ of y times the probability that a planar Brownian motion started from iy exits $\mathbb{H}$ in the interval $[W_{t_\delta}, g_{t_\delta}(1/2)]$. By conformal invariance, this is the same as the limit of y times the probability that a planar Brownian motion started from iy exits H_{t_δ} through the boundary of H_{t_δ} which is "between" z_δ and $1/2$. But in order to achieve this, the planar Brownian motion has in particular to hit the vertical segment joining z_δ to the real line before exiting $\mathbb{H}$. This segment has length at most δ. Hence,

$$|g_{t_\delta}(1/2) - W_{t_\delta}| \le O(\delta).$$

On the other hand, $\lim_{t\to\infty}(g_t(1/2) - W_t) = \infty$ because $\kappa < 4$ (and the corresponding Bessel process is transient). It follows that a.s.,

$$d(\gamma[0, \infty], [1, x]) > 0.$$

By the scaling property and monotonicity, it follows that almost surely, for all $0 < x_1 < x_2$, the distance $d(\gamma[0, \infty], [x_1, x_2])$ is almost surely strictly positive.

Let τ denote the hitting time of the unit circle by the SLE. Since $\mathbb{R} \cap \gamma[0,\tau] = \{0\}$, it follows that $0 \in \partial H_\tau$. For all $\varepsilon > 0$, there exists $0 < x_1 < x_2$ such that with probability at least $1-\varepsilon$ the two images of 0 under g_τ are in $[W_\tau - x_2, W_\tau - x_1] \cup [W_\tau + x_1, W_\tau + x_2]$. It follows from the strong Markov property and from the previous result that with probability at least $1-\varepsilon$,

$$d(g_\tau(\gamma[\tau,\infty)) - W_\tau, [-x_2,-x_1] \cup [x_1,x_2]) > 0.$$

Hence, it follows that in fact, almost surely

$$d(0, \gamma[\tau,\infty)) > 0$$

and the Lemma readily follows (for instance using the scaling property once again). □

Bibliographical comments

Again, many of the ideas in this chapter were contained or follow readily from Schramm's first paper [123]. Rohde-Schramm [118] have derived various almost sure properties of SLE (Hölder boundary, generated by a continuous path, transience). Proposition 3.3 is derived (in a more general setting) in [86]. It was Carleson who first noted that Cardy's formula (which Cardy predicted for crossing probabilities for critical percolation) has a simple interpretation in an equilateral triangle. The interpretation of the functions $\tilde{F}$ in terms of isocele triangles was pointed out by Dubédat [39]. Another justification to the fact that $\kappa = 6$ is the unique possible scaling limit of critical percolation exploration processes (for site percolation on the triangular lattice, or for bond percolation on the square lattice) uses the fact that for these models the probability of existence of a left-right crossing of a square must be $1/2$ (see [123]). For references on Bessel processes, stochastic calculus, see e.g. [59, 117].

4 Chordal SLE and restriction

4.1 Image of SLE under conformal maps

Suppose now that $(K_t, t \geq 0)$ is chordal SLE_κ in the upper half-plane $\mathbb{H}$.

Definition. We say that a hull A that is at positive distance of the origin is a Hull (with capital H). When A is such a Hull, we define Φ_A the normalized conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ as before. We also define Ψ_A the conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ such that $\Psi(z) \sim z$ when $z \to \infty$ and $\Psi(0) = 0$. Note that $\Psi(z) = \Phi(z) - \Phi(0)$.

Let $A \subset \overline{\mathbb{H}}$ denote a Hull. Define $T = \inf\{t \;:\; K_t \cap A \neq \emptyset\}$ and for all $t < T$,

$$\tilde{K}_t := \Phi(K_t).$$

Let us immediately emphasize that the time-parametrization of K_t and therefore also of $\tilde{K}_t$ is given in terms of the "size" of $K_t = \Phi^{-1}(\tilde{K}_t)$ in $\mathbb{H}$ and not in terms of the "size" of $\tilde{K}_t$ itself in $\mathbb{H}$. One of the goals of this section is to study the evolution of $\tilde{K}_t$ and to compare it with that of K_t.

For $t < T$, we also define the conformal map h_t from $g_t(H_t \cap H)$ onto $\mathbb{H}$ (where $H = \mathbb{H} \setminus A$). Note that $h_0 = \Phi$. Since $g_t(A)$ is at positive distance of W_t for $t < T$, we can define

$$\tilde{W}_t = h_t(W_t).$$

Define finally also the normalized conformal map $\tilde{g}_t$ from $\Phi(H_t \cap H)$ onto $\mathbb{H}$. Note that (as long as $t < T$),

$$h_t \circ g_t = \tilde{g}_t \circ h_0.$$

In short, all these maps are normalized, $h_0 = \Phi$ removes A and $\tilde{g}_t$ removes $\tilde{K}_t$, while g_t removes K_t and h_t removes $g_t(A)$.

The family $(\tilde{K}_t, t < T)$ is a "continuously" growing family of subsets of $\mathbb{H}$ satisfying Proposition 2.1 except that a time-change is required in order to parametrize it as a Loewner chain. We therefore define the function

$$a(t) := a(A \cup K_t) = a(A) + a(\tilde{K}_t).$$

A simple time-change shows that

$$\partial \tilde{g}_t(z) = \frac{2\partial_t a}{\tilde{g}_t(z) - \tilde{W}_t}.$$

Hence, in order to understand the evolution of $\tilde{K}_t$, we have to understand the evolutions of $\tilde{W}_t$ and of $a(t)$.

The scaling rule $a(\lambda \cdot) = \sqrt{\lambda} a(\cdot)$ shows that

$$\partial_t a(t) = h_t'(W_t)^2.$$

On the other hand,

$$h_t = \tilde{g}_t \circ \Phi \circ g_t^{-1}$$

and

$$\partial_t (g_t^{-1}(z)) = -2 \frac{(g_t^{-1})'(z)}{z - W_t}$$

so that putting the pieces together, we see that

$$\partial_t h_t(z) = \frac{2h_t'(W_t)^2}{h_t(z) - \tilde{W}_t} - \frac{2h_t'(z)}{z - W_t}. \tag{4.1}$$

Recall that $\tilde{W}_t = h_t(W_t)$. The previous formula is valid for all $z \in \mathbb{H} \setminus g_t(A)$. In fact, one can even extend it to $z = W_t$:

$$(\partial_t h_t)(W_t) = \lim_{z \to W_t} \left(\frac{2h_t'(W_t)^2}{h_t(z) - \tilde{W}_t} - \frac{2h_t'(z)}{z - W_t} \right) = -3h_t''(W_t)$$

(note that h_t is smooth near W_t because of Schwarz reflection). Itô's formula (this is not the classical formula since h_t is random, but it is adapted with respect to the filtration of W_t, it is C^1 with respect to t, so that Itô's formula still holds, see e.g., exercise IV.3.12 in [117]) can be applied:

$$d\tilde{W}_t = (\partial_t h_t)(W_t)dt + h_t'(W_t)dW_t + \frac{\kappa}{2} h_t''(W_t)dt.$$

Hence,

$$d\tilde{W}_t = h_t'(W_t)dW_t + [(\kappa/2) - 3]h_t''(W_t).$$

Clearly, the value $\kappa = 6$ will play a special role here. The next section is devoted to this case.

4.2 Locality for SLE_6

Throughout this section, we will assume that $\kappa = 6$. Then,

$$\tilde{W}_t = \int_0^t h_s'(W_s)dW_s.$$

Recall also that $a_t - a_0 = \int_0^t h_s'(W_s)^2 ds = \langle \tilde{W} \rangle_t$. Hence, if we define $(\hat{W}_a, a \geq 0)$ in such a way that

$$\tilde{W}_t = \hat{W}_{a(t)-a(0)},$$

then $\hat{W} - \hat{W}_0$ and W have the same law. If we define $\hat{g}_a$ in such a way that $\tilde{g}_t = \hat{g}_{a(t)}$, then

$$\partial_a \hat{g}_a(z) = \frac{2}{\hat{g}_a(z) - \hat{W}_a}.$$

Hence, modulo time-change, the evolution of $\tilde{K}_t - \hat{W}_0$ up to $t = T$ is that of chordal SLE_6. Suppose that $\tilde{T}$ is the first time at which K_t hits $\Phi(\partial A)$. We have just proved SLE_6's locality property:

Theorem 4.1. *Modulo time-reparametrization, the processes $(\tilde{K}_t - \Phi(0), t < T)$ and $(K_t, t < \tilde{T})$ have the same law.*

We now discuss some consequences of this result. Suppose first that

$$A = A_\varepsilon = \{e^{i\theta} \ : \ \theta \in [0, \pi - \varepsilon]\}.$$

Recall that $\Phi = \Phi_\varepsilon$ is the normalized map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$. Let

$$\psi_\varepsilon(z) = \frac{\Phi_\varepsilon(z)}{\Phi_\varepsilon'(0)}.$$

It is easy to see that when $\varepsilon \to 0$, the mappings ψ_ε converge uniformly on any set $V_\delta := \{z \in \mathbb{H} \ : \ |z| < 1 - \delta\}$ towards the conformal map ψ from $V := \{z \in \mathbb{H} \ : \ |z| < 1\}$ onto $\mathbb{H}$ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi(-1) = \infty$. Theorem 4.1 shows that for each $\varepsilon > 0$, the law of the process $\psi_\varepsilon(K_t)$ up to its hitting time of $\psi_\varepsilon(A_\varepsilon)$ is a time-change of chordal SLE_6. In particular, letting $\varepsilon \to 0$ for each fixed $\delta > 0$ shows readily that:

Corollary 4.1. *Let $(K_t, t \geq 0)$ denote the law of chordal SLE_6 from 0 to -1 in V. Let T the first time at which K_t hits the unit circle. Then, the law of $(K_t, t < T)$ is identical (modulo time-change) to that of chordal SLE_6 in $\mathbb{H}$ (from 0 to ∞) up to its first hitting time of the unit circle.*

The same reasoning can be applied to $\{e^{i\theta} \ : \ \theta \in [\varepsilon, \pi]\}$ instead of A_ε. It shows that the law described in the corollary is also identical to that of chordal SLE_6 from 0 to $+1$ in V (up to the hitting time of the unit circle). By mapping the set V onto any other simply connected domain, we get the following splitting property:

Corollary 4.2. *Let $D \subset \mathbb{H}$ denote a simply connected subset of $\mathbb{H}$ such that the boundary of ∂D is a continuous Jordan curve. Let a, b, b' denote three distinct points on ∂D and call ∂ the connected component of $\partial D \setminus \{b, b'\}$ that does not contain a. Then: up to their first hitting times of ∂ and modulo time-change, the laws of chordal SLE_6 from a to b and from a to b' in D are identical.*

Note that these properties of chordal SLE_6 are not surprising if one thinks of SLE_6 as the scaling limit of critical percolation interfaces. They generalize the properties of hitting probabilities for SLE_6 that we derived in the previous chapter.

4.3 Restriction for $SLE_{8/3}$

We now apply the same technique as in the first subsection to understand how $h_t'(W_t)$ evolves. Recall that h_t is smooth in the neighbourhood of W_t by Schwarz reflection. Hence $h_t'(W_t)$ is a positive real (as long as $t < T$). Differentiating Equation (4.1) with respect to z (this is licit as long as $t < T$) gives

$$\partial_t h_t'(z) = \frac{-2h_t'(W_t)^2 h_t'(z)}{(h_t(z) - \tilde{W}_t)^2} + \frac{2h_t'(z)}{(z - W_t)^2} - \frac{2h_t''(z)}{z - W_t}.$$

If we take the limit when $z \to W_t$, we get that

$$(\partial_t h_t')(W_t) = \frac{h_t''(W_t)^2}{2h_t'(W_t)} - \frac{4}{3} h_t'''(W_t).$$

Hence, Itô's formula (in its random version as before) shows that

$$d[h_t'(W_t)] = h_t''(W_t) dW_t + \left[\frac{h_t''(W_t)^2}{2h_t'(W_t)} + (\kappa/2 - 4/3) h_t'''(W_t) \right] dt.$$

This time, it is the value $\kappa = 8/3$ that plays a special role. Let us in this section from now on suppose that $\kappa = 8/3$. Then, we see that

$$d[h_t'(W_t)^{5/8}] = \frac{5h_t''(W_t)}{8h_t'(W_t)^{3/8}} dW_t.$$

The important feature is that the drift term disappear so that: $(h_t'(W_t)^{5/8}, t < T)$ is a local martingale. This has the following important consequence:

Proposition 4.2 *Consider chordal $SLE_{8/3}$ in $\mathbb{H}$. Then, for any Hull A,*

$$\mathbf{P}[\forall t \geq 0,\ K_t \cap A = \emptyset] = \Phi_A'(0)^{5/8}.$$

Proof. The quantity $M_t := h_t'(W_t)^{5/8}$ is a local martingale. Recall that h_t is a normalized map from a subset of $\mathbb{H}$ onto $\mathbb{H}$. Hence, for all $t < T$, $M_t \leq 1$ and M is a bounded martingale. We have to understand the behaviour of M_t when $t \to T$ in the two cases $T < \infty$ and $T = \infty$. When $T = \infty$, one can use the transience of the SLE: Define for each R, the hitting time τ_R of the circle of radius R. Then, simple considerations using harmonic measure for instance show that

$$\lim_{R \to \infty} h_{\tau_R}'(W_{\tau_R}) = 1.$$

In the case where $T < \infty$, one can for instance first approximate A by a Hull with a smooth boundary, and show that in this case, $\lim_{t \to T} h'_t(W_t) = 0$ for any path γ in the upper half-plane that hits A away from the real line. See [95] for details.

Finally, since M_t converges in L^1 and almost surely when $t \to T$, we get that $\mathbf{P}[T = \infty] = \mathbf{E}[M_T] = E[M_0] = \Phi'(0)^{5/8}$. $\square$

Let us now define the random set

$$K_\infty = \cup_{t>0} K_t.$$

Corollary 4.3. *Suppose that A_0 is a Hull, then the conditional law of K_∞ given $K_\infty \cap A_0 = \emptyset$ is identical to the law of $\Psi_{A_0}^{-1}(K_\infty)$.*

Proof. Note that K_∞ is a closed set because of the transience of $(K_t, t \geq 0)$. The law of such a random set is characterized by the value of $\mathbf{P}[K_\infty \cap A = \emptyset]$ for all Hulls A (this set of events is a generating π-system of the σ-field on which we define K_∞). Suppose now that the Hull A_0 is fixed. By Proposition 4.2, K_∞ avoids A_0 with positive probability. Suppose that A is another Hull. Then

$$\begin{aligned}
&\mathbf{P}[\Psi_{A_0}(K_\infty) \cap A = \emptyset | K_\infty \cap A_0 = \emptyset] \\
&= \frac{\mathbf{P}[K_\infty \cap (\mathbb{H} \setminus (\Psi_{A_0}^{-1} \circ \Psi_A^{-1}(\mathbb{H})) = \emptyset]}{\mathbf{P}[K_\infty \cap A_0 = \emptyset]} \\
&= \left(\frac{\Psi'_{A_0}(0) \Psi'_A(0)}{\Psi'_{A_0}(0)} \right)^{5/8} \\
&= \mathbf{P}[K_\infty \cap A = \emptyset].
\end{aligned}$$

Since this is true for all Hull A, it follows that the the law of $\Psi_{A_0}(K_\infty)$ given $\{K_\infty \cap A_0 = \emptyset\}$ is identical to the law of K_∞. $\square$

This striking property of $SLE_{8/3}$ has many nice consequences. It will enable us to relate it to the Brownian frontier in the next chapter. It also shows that it is the natural candidate for the scaling limit of planar self-avoiding walks. More precisely, one can show that when $n \to \infty$, the uniform measure on self-avoiding walks of length n in the upper half-plane $\mathbb{N} \times \mathbb{Z}$ started from the origin converges to a law of infinite self-avoiding walks. The conjecture is that the scaling limit of this infinite self-avoiding walk is $SLE_{8/3}$. See [94] for more on this. Note that there exist algorithms to simulate half-plane self-avoiding walks (see [60, 105]; Figure 4.1 is due to Tom Kennedy). The conjecture that the half-plane SAW scaling limit is chordal $SLE_{8/3}$ has recently been comforted by simulations [61].

Let us briefly conclude this chapter by mentioning the following characterization of $SLE_{8/3}$ that does not use explicitly Loewner's equation (even though its proof does):

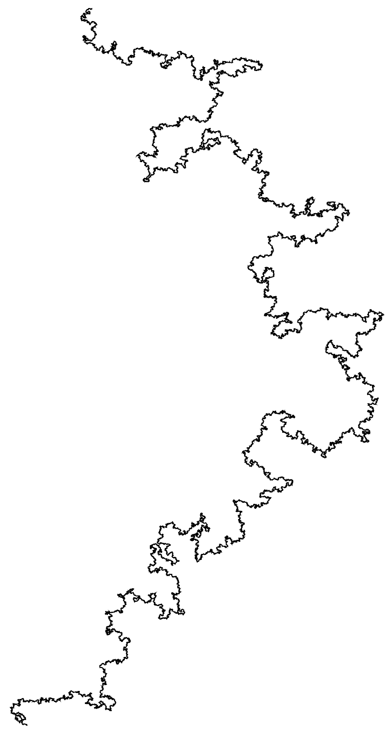

Fig. 4.1. Sample of the beginning of a half-plane walk (conjectured to converge to chordal $SLE_{8/3}$).

Theorem 4.3. *Chordal $SLE_{8/3}$ is the unique measure on continuous simple curves γ from 0 to ∞ in $\mathbb{H}$ such that for all Hull, the law of γ conditioned to avoid A is identical to the law of $\Psi^{-1}(\gamma)$.*

The proof of this Theorem uses the complete description of all measures on simply connected closed sets (not necessarily curves) joining 0 to ∞ in $\mathbb{H}$ that satisfy this condition. These measures (called restriction measures in [95]) are constructed using SLE_κ (in fact, by adding Brownian bubbles to the SLE_κ paths) for other values of κ (in fact for $\kappa \in (0, 8/3]$) and it turns out that the only measure with these properties that is supported on simple curves is $SLE_{8/3}$.

Bibliographical comments

All the material of this chapter is borrowed from [95], to which we refer for further details. The locality property for SLE_6 was first proved in [87], using a different method. Restriction properties are closely related to conformal field theory [17, 18, 30, 31, 32, 34], as pointed out in [52, 53]. They have also interpretations in terms of highest-weight representations of the Lie algebra of polynomial vector fields on the unit circle. In fact, Theorem 4.3 corresponds to the fact that the unique such representation that is degenerate at level 2 has its highest weight equal to $5/8$. See [52, 53].

5 SLE and the Brownian frontier

5.1 A reflected Brownian motion

In this section, we introduce a two-dimensional Brownian motion with a certain oblique reflection on the boundary of a domain, and we will relate its outer boundary to that of SLE_6.

Let us first define this reflected Brownian motion in the upper half-plane $\mathbb{H}$. Define for any $x \in \mathbb{R}$, the vector $u(x) = \exp(i\pi/3)$ if $x \geq 0$ and $u(x) = \exp(2i\pi/3)$ if $x < 0$. It is the vector field with angle $2\pi/3$ pointing "away from the origin". Suppose that $Z_t^* = X_t^* + iY_t^*$ is an ordinary planar Brownian path started from 0. Then, there exists a unique pair (Z_t, ℓ_t) of continuous processes such that Z_t takes its values in $\overline{\mathbb{H}}$, ℓ_t is a non-decreasing real-valued continuous function with $\ell_0 = 0$ that increases only when $Z_t \in \mathbb{R}$, and

$$Z_t = Z_t^* + \int_0^t u(Z_s) d\ell_s.$$

The process $(Z_t, t \geq 0)$ is called the reflected Brownian motion in $\mathbb{H}$ with reflection vector field $u(\cdot)$. Note that the process Z in fact only depends on the direction of $u(\cdot)$ and not on its modulus. For instance Z is also the reflected Brownian motion in $\mathbb{H}$ with reflection vector field $2u(\cdot)$ (just change ℓ into $\ell/2$).

An equivalent way to define this process is to first define the reflected (one-dimensional) Brownian motion

$$Y_t = Y_t^* - \min_{s \in [0,t]} Y_s^*.$$

The local time at 0 of Y is simply $l_t = -\min_{[0,t]} Y^*$. Then, define X in such a way that

$$X_t = X_t^* + \int_0^t \operatorname{sgn}(X_s) \frac{1}{\sqrt{3}} dl_s$$

and verify that $Z_t = X_t + iY_t$ satisfy the required conditions.

Brownian motion with oblique reflection on domains have been extensively studied, and this is not the proper place to review all results. We just mention that the general theory of such processes (e.g., [130]) ensures that the previously defined process Z^* exists.

Reflected planar Brownian motion (even with oblique reflection) are also invariant under conformal transformations. Suppose for instance that ϕ is a conformal transformation from a smooth subset V (such that $[-1,1] \subset \partial V$) of $\mathbb{H}$ onto a smooth domain D. Recall that

$$Z_t = Z_t^* + \int_0^t u(Z_s) d\ell_s.$$

Define

$$\sigma_V := \inf\{t > 0 \ : \ \partial V \setminus (-1,1)\}.$$

Taylor-expanding each term in the sum

$$\phi(Z_t) - \phi(0) = \sum_{j=1}^{n} (\phi(Z_{jt/n}) - \phi(Z_{(j-1)t/n}))$$

just as in the proof of Itô's formula (letting $n \to \infty$), it follows (using the fact that the real and imaginary parts of ϕ are harmonic) that for all $t \leq \sigma_V$,

$$\phi(Z_t) = \int_0^t \phi'(Z_s) dZ_s^* + + \int_0^t u(Z_s)\phi'(Z_s) d\ell_s.$$

Hence, if one time-changes $\phi(Z)$ using the clock $u(t) = \int_0^t |\phi'(Z_s)|^2 ds$, we see that $\phi(Z_u)$ is also a stopped reflected Brownian motion in D with the reflection vector field $(\phi'(\phi^{-1}(\cdot)) \times u(\phi^{-1}(\cdot))$ on ∂D.

This has the following useful consequences: Suppose that $V \subset \mathbb{H}$ and σ_V are as before. Note that $\sigma_{\mathbb{H}}$ is the first time at which Z_t hits $\mathbb{R} \setminus (-1,1)$. There exists a unique conformal map ϕ from V onto $\mathbb{H}$ such that $\phi(-1) = -1$, $\phi(0) = 0$ and $\phi(1) = 1$.

Lemma 5.1. *Modulo time-change, the laws of $(\phi(Z_t), t \leq \sigma_V)$ and of $(Z_t, t \leq \sigma_{\mathbb{H}})$ are identical.*

In other words, The reflected Brownian motion Z satisfies the same locality property as SLE_6.

A slight modification of the above proof of conformal invariance for reflected Brownian motions shows that the image of Z under the conformal map $z \mapsto z^{1/3}$ from $\mathbb{H}$ onto the wedge

$$\mathcal{W} := \{re^{i\theta} \ : \ r > 0, \theta \in (0, \pi/3)\}$$

is reflected Brownian motion in that wedge, started from the origin, with reflection vector field $u(x) = e^{i\pi/3}$ on $\mathbb{R}_+$ and $u(x) = 1$ on $e^{i\pi/3}\mathbb{R}_+$. We use this observation to give a simple proof of the following fact on hitting probabilities for Z:

Lemma 5.2. *Suppose that Φ is the conformal transformation from $\mathbb{H}$ onto an equilateral triangle OAC such that $\Phi(0) = O$, $\Phi(-1) = A$ and $\Phi(1) = C$. Then, the law of $\Phi(Z_{\sigma_{\mathbb{H}}})$ is uniform on AC.*

Proof. One elementary convincing proof uses discrete approximations. Here is a brief outline of this proof: Define $\omega = \exp(i\pi/3)$. Consider a triangular grid in the wedge $\mathcal{W}$ i.e. $\{m + m'\omega \ : \ m, m' \geq 0\}$. Let $(S_n, n \geq 0)$ denote simple random walk on this grid that is started from 0. In the inside of $\mathcal{W}$, its transition probabilities are that of simple random walk (with probability 1/6 to jump to each of its neighbours). When S hits the (positive) real line at x, it has the following transition probabilities: $p(x, x+1) = 1/3$ and

$$p(x, x-1) = p(x, x+\omega) = p(x, x+\omega^2) = p(x, x) = \frac{1}{6}.$$

and the symmetric ones on $\omega\mathbb{N}$: $p(x, x+\omega) = 1/3$ and

$$p(x, x+1) = p(x, x+1/\omega) = p(x, x+1/\omega^2) = p(x, x) = \frac{1}{6}.$$

Finally, at the origin, $p(0, 1) = p(0, \omega) = 1/2$. It is not difficult to see that in the scaling limit, such a random walk converges to reflected Brownian motion in $\mathcal{W}$ with the reflection vector field $u(\cdot)$ on $\partial\mathcal{W}$. This is due to the fact that the bias of the simple random walk when it hits $\partial\mathcal{W}$ is proportional to u. Moreover, it is easy to check that if $S_0 = 0$, then if one writes $S_n = e^{i\pi/6} r_n + \omega^2 s_n$, then the conditional law of s_n given $(r_j, j \leq n)$ is the uniform distribution among the permitted values of s given r_n. In other words, the "uniform distribution of s is preserved, independently from r". In particular, the hitting distribution of the simple random walk S on the segment $N + \omega^2[0, N]$, is simply the uniform distribution on $\{N, N+\omega^2, N+2\omega^2, \ldots, N+N\omega^2\}$. The Lemma follows, letting $N \to \infty$. $\square$

We are now ready to state and prove the following result:

Theorem 5.1. *Define the following two sets:*

- *Consider chordal SLE_6 $(K_t, t \geq 0)$ in $\mathbb{H}$ (or in V) up to its first hitting time T of $\mathbb{R} \setminus (-1, 1)$. Let e denote the point at which the SLE hits $\mathbb{R} \setminus (-1, 1)$, and let $E := \{e\} \cup \cup_{t<T} K_t$.*
- *Consider the set of points F in $\overline{\mathbb{H}}$ that are disconnected (in $\mathbb{H}$) from $\mathbb{R} \setminus (-1, 1)$ by $Z[0, \sigma_{\mathbb{H}}]$.*

Then, the laws of E and of F are identical.

Proof. Note that Lemma 5.2, Lemma 5.1, Theorem 4.1 and Proposition 3.4 show that E and F both have the following properties:

- They are random compact sets that intersect $\mathbb{R} \setminus (-1, 1)$ at just one point x and the law of $\Phi(x)$ is uniform on AC.

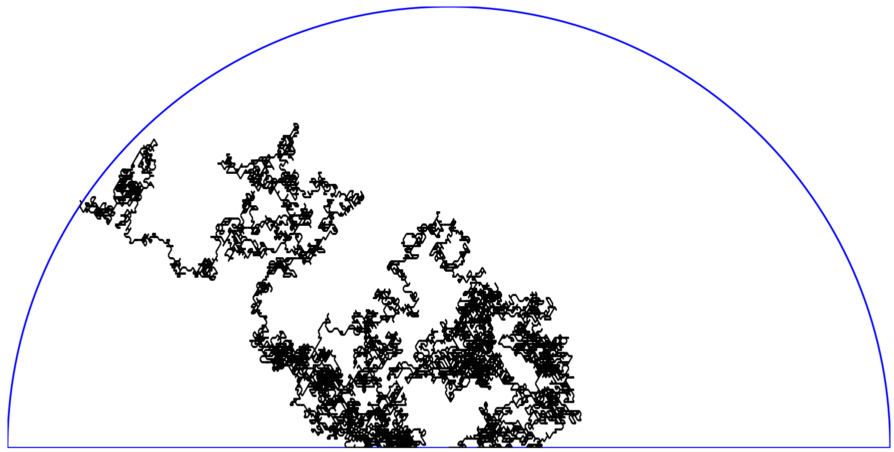

Fig. 5.1. The reflected Brownian motion stopped at its hitting time of the unit circle

– Their complement in $\overline{\mathbb{H}}$ consists of two connected components (one unbounded, one bounded).
– For all V as before, the probability that $E \subset V$ is identical to the probability that $\sigma_V = \sigma_{\mathbb{H}}$ (and the corresponding result for F).

If we combine these two properties, we see that for all such V,

$$\mathbf{P}[E \subset V] = \mathbf{P}[F \subset V] = \frac{\text{length}(\Phi \circ \phi(\partial V \setminus \mathbb{R}))}{AC}$$

(this is because the law of the image under $\Phi \circ \phi$ of the "hitting point" of $\partial V \setminus (-1, 1)$ is uniform on AC. But this determines completely the laws of E and of F and therefore implies that they are equal. $\square$

Using conformal invariance, the previous result can be adapted in any domain. For instance, Figure 5.2 could represent both the filling of a reflected Brownian motion (or of a SLE_6 curve), started at the bottom of the triangle stopped at their first hitting of the top segment. Recall that the law of this hitting point is uniformly distributed.

5.2 Brownian excursions and $SLE_{8/3}$

We now describe a probability measure on Brownian excursions from 0 to infinity in $\mathbb{H}$ (which is closely related to the measures on excursions that were considered in [97]). One can view this measure on paths as the law of planar Brownian motion W (not to be confused with the $\sqrt{\kappa}B$ in the previous chapters) started from 0 and conditioned to stay in $\mathbb{H}$ at all positive times.

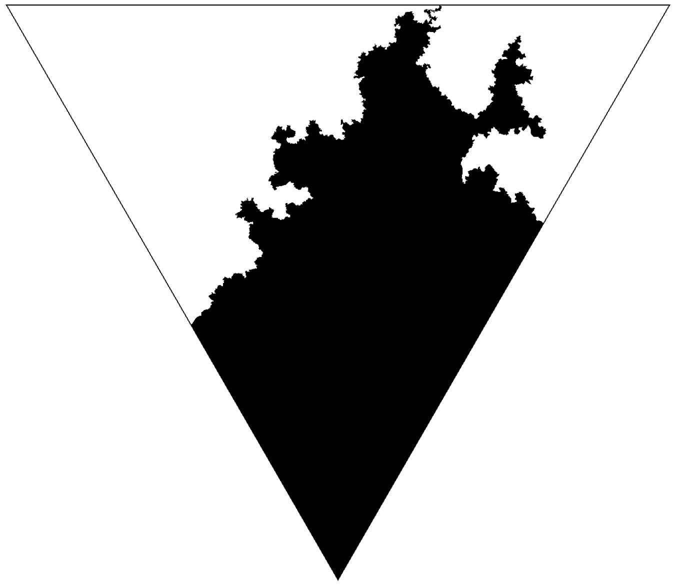

Fig. 5.2. The filling of RBM (or of the SLE_6 curve) in a triangle

Let X and Y denote two independent processes such that X is standard one-dimensional Brownian motion and Y is a three-dimensional Bessel process (see e.g., [117] for background on three-dimensional Bessel processes, its relation to Brownian motion conditioned to stay positive and stochastic differential equations) that are both started from 0. Let us briefly recall that a three-dimensional Bessel process is the modulus of a three-dimensional Brownian motion, and that it can be defined as the solution to the stochastic differential equation

$$dY_t = dw_t + \frac{1}{Y_t} dt$$

(where w is one-dimensional standard Brownian motion). It is very easy to see that $(1/Y, t \geq t_0)$ is a local martingale for all $t_0 > 0$, and that if T_r denotes the hitting time of r by Y, then the law of $(Y_{T_r+t}, t < T_R - T_r)$ is identical to that of a Brownian motion started from r and conditioned to hit R before 0 (if $0 < r < R$). Loosely speaking Y is a Brownian motion started from 0 and conditioned to stay forever positive. Note that almost surely $\lim_{t\to\infty} Y_t = \infty$.

We now define $W = X + iY$. In other words, W has the same law as the solution to the following stochastic differential equation:

$$dW_t = d\beta_t + i \frac{1}{\Im(W_t)} dt \tag{5.1}$$

with $W_0 = 0$, where β is a complex-valued Brownian motion. Note that W is a strong Markov process. Let T_r denote the hitting time of the line $\mathbb{R} + ir$

by this process W (i.e., the hitting time of r by X). Let S denote a random variable with the same law as W_{T_1}. Then, scaling and the relation between one-dimensional Brownian motion conditioned to stay positive and the three-dimensional Bessel process shows immediately that for all $0 < r < R$, the law of $W[T_r, T_R]$ is the law of a Brownian motion started with the same law as rS, stopped at its first hitting of $iR + \mathbb{R}$, and conditioned to stay in the upper half-plane up to that time. Note that the probability of this event is r/R.

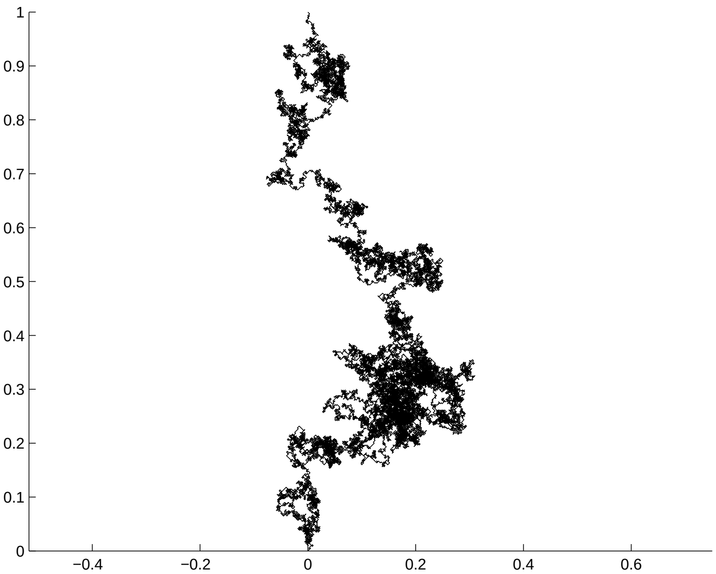

Fig. 5.3. An excursion from 0 to i in the strip $\mathbb{R} \times [0, 1]$

By mapping conformally $\mathbb{H}$ onto any other simply connected domain D ($D \neq \mathbb{C}$), and looking at the image of the Brownian excursion in $\mathbb{H}$ under this map, one gets the law of a Brownian excursion in D from the image of 0 to the image of ∞. As for SLE, this law is well-defined up to linear time-change. One can also directly define this excursion in D as the solution to a stochastic differential equation "forcing the Brownian motion to hit ∂D at the image of infinity."

The following result was observed by Bálint Virág [129] (see also [97, 95]):

Lemma 5.3. *Suppose A is a Hull and W is a Brownian excursion in $\mathbb{H}$ from 0 to ∞. Then $\mathbf{P}[W[0, \infty) \cap A = \emptyset] = \Phi'_A(0)$.*

Proof. Suppose that W is a solution to (5.1) started from $z \in \Phi^{-1}(\mathbb{H})$. Let Z denote a planar Brownian motion started from z. Let $\tau_R(V)$ denote the hitting time of $iR + \mathbb{R}$ by a process V. When $\Im(z) \to \infty$, $\Im(\Phi(z)) =$

$\Im(z) + o(1)$, and it therefore follows easily from the strong Markov property of planar Brownian motion that when $R \to \infty$,

$$\mathbf{P}[\Phi(Z)[0, \tau_R(Z)] \subset \mathbb{H}] \sim \mathbf{P}[\Phi(Z)[0, \tau_R(\Phi(Z))] \subset \mathbb{H}].$$

But since $\Phi(Z)$ is a time-changed Brownian motion, the right-hand probability is equal to $\Im(\Phi(z))/R$, so that

$$\mathbf{P}[W[0, \tau_R(W)] \subset \Phi^{-1}(\mathbb{H})] = \frac{\mathbf{P}[Z[0, \tau_R(Z)] \subset \Phi^{-1}(\mathbb{H})]}{\mathbf{P}[Z[0, \tau_R(Z)] \subset \mathbb{H}]} = \frac{\Im\Phi(z)}{\Im(z)} + o(1)$$

when $R \to \infty$. In the limit $R \to \infty$, we get

$$\mathbf{P}[W[0, \infty) \subset \Phi^{-1}(\mathbb{H}) \mid W_0 = z] = \frac{\Im\Phi(z)}{\Im(z)}. \tag{5.2}$$

When $z \to 0$, $\Phi(z) = z\Phi'(0) + O(|z|^2)$ so that

$$\begin{aligned} \mathbf{P}[W[0, \infty) \subset \Phi^{-1}(\mathbb{H})] &= \lim_{r \to 0} \mathbf{P}[W[T_r, \infty) \subset \Phi^{-1}(\mathbb{H})] \\ &= \lim_{r \to 0} \mathbf{E}[\Im(\Phi(rA))/\Im(rA)] \\ &= \Phi'(0) \end{aligned}$$

(one can use dominated convergence here since $\Im(\Phi(z)) \leq \Im(z)$ for all z). □

We now define the filling $\mathcal{H}$ of $W[0, \infty)$ as the set of points in $\overline{\mathbb{H}}$ that are disconnected from $\mathbb{R}$ by $W[0, \infty)$. This set is obtained by filling in all the bounded connected components of the complement of the curve W. Then, $\mathcal{H}$ is a closed unbounded set and $\mathbb{H} \setminus \mathcal{H}$ consists of two open connected components (with $[0, \infty)$ and $(-\infty, 0]$ on their respective boundaries). The law of such a random set is characterized by the values of $\mathbf{P}[\mathcal{H} \cap A = \emptyset]$, where A spans all Hulls, because this family of events turn out to generate the σ-field on which $\mathcal{H}$ is defined, and to be stable under finite intersections. Hence, as in the case of K_∞ for $SLE_{8/3}$, the fact that

$$\mathbf{P}[\mathcal{H} \cap A = \emptyset] = \Phi'(0) \tag{5.3}$$

characterizes the law of $\mathcal{H}$ and yields that $\mathcal{H}$ also satisfies Corollary 4.3.

Theorem 5.2. *Suppose that $\mathcal{H}_8$ denotes the filling of the union of 8 independent chordal $SLE_{8/3}$'s. Suppose that $\mathcal{H}_5$ denotes the filling of the union of 5 independent Brownian excursions. Then, $\mathcal{H}_5$ and $\mathcal{H}_8$ have the same law.*

Proof. This is simply due to the fact that for all Hull A

$$\mathbf{P}[\mathcal{H}_5 \cap A = \emptyset] = \mathbf{P}[\mathcal{H}_8 \cap A = \emptyset] = \Phi_A'(0)^5$$

and that this characterizes these laws. □

This has various nice consequences (see [95]), some of which we now heuristically describe: First, since the boundary of $\mathcal{H}_8$ consists of the union of some parts of the $SLE_{8/3}$ curves, it follows that "locally", the outer boundary of a Brownian excursion (and therefore also of a Brownian motion) looks like one $SLE_{8/3}$ path. In the previous section, we did see that the outer boundaries of reflected Brownian motion and of SLE_6 are the same. Hence, "locally", the outer frontiers of SLE_6 and of planar Brownian motion look like an $SLE_{8/3}$ curve. Furthermore, since $SLE_{8/3}$ is symmetric, this shows that one cannot distinguish the inside from the outside of a planar Brownian curve by only seeing a part of its frontier. Since $SLE_{8/3}$ is conjectured to be the scaling limit of self-avoiding walks, this would also show that the Brownian frontier looks locally like the scaling limit of long self-avoiding curves (see [94]).

Bibliographical comments

The idea that conformal invariance and restriction defines measures on random sets and makes it possible to understand the Brownian frontier in terms of other models (or the corresponding exponents) first appears in [97]. Most of the material of this chapter is borrowed from [95].

A discussion of the conjectured relation between $SLE_{8/3}$ and planar self-avoiding walks is discussed in [94]; one can in particular recover the predictions of Nienhuis [110] on the critical exponents for self-avoiding walks using SLE arguments.

The fact that the Brownian frontier had the same dimension as the scaling limit of self-avoiding walks was first observed visually by Mandelbrot [107].

6 Radial SLE

6.1 Definitions

Motivated by the example of LERW (among others) given in the introductory chapter, we now want to find a nice way to encode growing families of compact subsets $(K_t, t \geq 0)$ of the closed unit disk that are growing from the boundary point 1 towards 0. As in the chordal case, we are in fact going to focus on the conformal geometry of the complement H_t of K_t in the unit disc $\mathbb{U}$. One first has to find a natural time-parametrization. It turns out to be convenient to define the conformal map g_t from H_t onto $\mathbb{U}$ that is normalised by

$$g_t(0) = 0 \text{ and } g_t'(0) > 0.$$

Note that $g_t'(0) \geq 1$. This can be for instance derived using the fact that $\log g_t'(0)$ is the limit when $\varepsilon \to 0$ of $\log(1/\varepsilon)$ times the probability that a planar Brownian motion started from ε hits the circle of radius ε^2 before exiting H_t (an analyst would find this justification very strange, for sure).

Then (and this is simply because with obvious notation, $(\tilde{g}_s \circ g_t)(0) = \tilde{g}_s(0) \circ g_t'(0))$, one measures the "size" $a(K_t)$ of K_t via the derivative of g_t at the origin:

$$g_t'(0) = \exp(a(t)).$$

Hence, we will consider growing families of compact sets such that $a(K_t) = t$.

Suppose now that $(\zeta_t, t \geq 0)$ is a continuous function on the unit circle $\partial\mathbb{U}$. Define for all $z \in \overline{\mathbb{U}}$, the solution $g_t(z)$ to the ODE

$$\partial_t g_t(z) = -g_t(z) \frac{g_t(z) + \zeta_t}{g_t(z) - \zeta_t} \tag{6.1}$$

such that $g_0(z) = z$. This solution is well-defined up to the (possibly infinite) time $T(z)$ defined by

$$T(z) = \sup\{t > 0 \ : \ \min_{s \in [0,t)} |g_s(z) - \zeta_s| > 0\}.$$

We then define

$$K_t := \{z \in \overline{\mathbb{U}} \ : \ T(z) \leq t\}$$

and

$$U_t := \mathbb{U} \setminus K_t.$$

The family $(K_t, t \geq 0)$ is called the (radial) Loewner chain associated to the driving function ζ.

The general statements that we described in the chordal case are also valid in this radial case. One can add one feature that has no analog in the chordal case: It is possible to estimate the Euclidean distance d_t from 0 to K_t in terms of $a(t) = t$. Indeed, since U_t contains the disc $d_t \times \mathbb{U}$, it is clear that $g_t'(0) \leq 1/d_t$. On the other hand, a classical result of the theory of conformal mappings known as Koebe's $1/4$ Theorem states that (if $a(K_t) = t$) $1/d_t \leq 4g_t'(0)$. This is loosely speaking due to the fact that the best K_t can do to get as close to 0 in "time t" is to shoot straight i.e. to choose $\zeta = 1$. Hence, for all $t \geq 0$,

$$e^{-t}/4 \leq d(0, K_t) \leq e^{-t}. \tag{6.2}$$

This will be quite useful later on.

Radial SLE_κ is then simply the random family of sets $(K_t, t \geq 0)$ that is obtained when

$$\zeta_t = \exp(i\sqrt{\kappa}B_t)$$

where $\kappa > 0$ is fixed and $(B_t, t \geq 0)$ is standard one-dimensional Brownian motion.

As in the chordal case, one can then define radial SLE from $a \in \partial D$ to $b \in D$ in any open simply connected domain D by taking the image of radial SLE in $\mathbb{U}$ under the conformal map Φ from $\mathbb{U}$ onto D such that $\Phi(1) = a$ and $\Phi(0) = b$. Note that this time, the time-parametrization is also well-defined since there exists only one such conformal map (recall that in the chordal case, one had to invoke the scaling property to make sure that chordal SLE in other domains than the half-space was properly defined).

6.2 Relation between radial and chordal SLE

In this section, we show that chordal SLE and radial SLE are very closely related. Let us start with the special case $\kappa = 6$.

Theorem 6.1. *Suppose that $x \in (0, 2\pi)$. Let $(K_t, t \geq 0)$ be a radial SLE_6 process. Set*

$$T := \inf\{t \geq 0 \; : \; \exp(ix) \in K_t\}.$$

Let $(\tilde{K}_u, u \geq 0)$ be a chordal SLE_6 process in $\mathbb{U}$ starting also at 1 and growing towards $\exp(ix)$, and let

$$\tilde{T} := \inf\{u \geq 0 \; : \; 0 \in \tilde{K}_u\}.$$

Then, up to a random time change, the process $t \mapsto K_t$ restricted to $[0, T)$ has the same law as the process $u \mapsto \tilde{K}_u$ restricted to $[0, \tilde{T})$.

Note that T (resp. $\tilde{T}$) is the first time where K_t (resp. $\tilde{K}_u$) disconnects 0 from 1.

When $\kappa \neq 6$, a weaker form of equivalence holds:

Proposition 6.2 *Let $(K_t, t \geq 0)$, $(\tilde{K}_u, u \geq 0)$, T and $\tilde{T}$ be defined just as in Theorem 6.1, except that they are SLE with general $\kappa > 0$. There exist two nondecreasing families of stopping times $(T_n, n \geq 1)$ and $(\tilde{T}_n, n \geq 1)$ such that almost surely, $T_n \to T$ and $\tilde{T}_n \to \tilde{T}$ when $n \to \infty$, and such that for each $n \geq 1$, the laws of $(K_t, t \in [0, T_n])$ and $(\tilde{K}_u, u \in [0, \tilde{T}_n])$ are equivalent (in the sense that they have a positive density with respect to each other) modulo increasing time change.*

These results imply that the properties of chordal SLE such as "being generated by a continuous curve" are also valid for radial SLE.

We prove both results simultaneously:

Proof. Let us first briefly recall how $\tilde{K}_u$ is defined. For convenience, we will restrict ourselves to $x = \pi$ (the proof in the general case is almost identical). Define the conformal map

$$\psi(z) = i\frac{1-z}{1+z}$$

from $\mathbb{U}$ onto $\mathbb{H}$ that satisfies $\psi(-1) = \infty$, $\psi(1) = 0$, and $\psi(0) = i$. Suppose that $u \mapsto \tilde{B}_u$ is a real-valued Brownian motion such that $\tilde{B}_0 = 0$. For all $z \in \mathbb{U}$, define the function $\tilde{g}_u = \tilde{g}_u(z)$ such that $\tilde{g}_0(z) = \psi(z)$ and

$$\partial_u \tilde{g}_u = \frac{2}{\tilde{g}_u - \sqrt{\kappa}\tilde{B}_u}.$$

This function is defined up to the (possibly infinite) time $\tilde{T}_z$ where $\tilde{g}_u(z)$ hits $\sqrt{\kappa}\tilde{B}_u$. Then, $\tilde{K}_u$ is defined by $\tilde{K}_u = \{z \in \mathbb{U} \ : \ \tilde{T}_z \leq u\}$, so that $\tilde{g}_u$ is a conformal map from $\mathbb{U} \setminus \tilde{K}_u$ onto the upper half-plane. This defines the process $(\tilde{K}_u, u \geq 0)$.

We are now going to compare it to radial SLE. Let $g_t : \mathbb{U} \setminus K_t \to \mathbb{U}$ be the conformal map normalized by $g_t(0) = 0$ and $g_t'(0) > 0$. Recall that

$$\partial_t g_t(z) = g_t(z) \frac{\zeta_t + g_t(z)}{\zeta_t - g_t(z)}, \tag{6.3}$$

where $\zeta_t = \exp(i\sqrt{\kappa}B_t)$, and B is Brownian motion on $\mathbb{R}$ with $B_0 = 0$. Let ψ be the same conformal map as before, and define

$$\begin{aligned} e_t &:= g_t(-1), \\ f_t(z) &:= \psi\big(g_t(z)/e_t\big), \\ r_t &:= \psi\big(\zeta_t/e_t\big). \end{aligned}$$

These are well defined, as long as $t < T$. Note that f_t is a conformal map from $\mathbb{U} \setminus K_t$ onto the upper half-plane, $f_t(1) = \infty$, and $r_t \in \mathbb{R}$. From (6.3) it follows that

$$\partial_t f = -\frac{(1+r^2)(1+f^2)}{2(r-f)}.$$

Let

$$\phi_t(z) = a(t)z + b(t)$$

where

$$a(0) = 1, \qquad \partial_t a = -(1+r^2)a/2$$

and

$$b(0) = 0, \qquad \partial_t b = -(1+r^2)ar/2.$$

Set

$$h_t := \phi_t \circ f_t \,,$$
$$\beta_t := \phi_t\big(r(t)\big).$$

Then (and this is the reason for the choice of the functions a and b)

$$\partial_t h = -(a/2)\frac{(1+r^2)^2}{r-f} = -\frac{(1+r^2)^2a^2/2}{\beta - h}\,.$$

h_t is also a conformal map from $\mathbb{U}\backslash K_t$ onto the upper half-plane with $h_t(1) = \infty$. Note also that $h_0(z) = \psi(z)$. We introduce a new time parameter $u = u(t)$ by setting

$$\partial_t u = (1+r^2)^2a^2/4, \qquad u(0) = 0\,.$$

Then

$$\frac{\partial h}{\partial u} = \frac{-2}{\beta - h}.$$

Since this is the equation defining the chordal SLE process, it remains to show that $u \mapsto \beta_{t(u)}/\sqrt{\kappa}$ is related to Brownian motion (stopped at some random time). This is a direct but tedious application of Itô's formula:

$$dr_t = \frac{(1+r^2)\sqrt{\kappa}}{2}\,dB_t + \frac{r(1+r^2)}{2}\Big(\frac{\kappa}{2} - 1\Big)\,dt$$

and

$$d\beta_t = \frac{(1+r^2)a}{2}\Big(\sqrt{\kappa}\,dB_t + (-3+\frac{\kappa}{2})r\,dt\Big)\,.$$

When $\kappa = 6$, the drift term disappears and this proves Theorem 6.1. When $\kappa \neq 6$, the drift term does not disappear. However, the law of $u \mapsto \beta_{t(u)}$ is absolutely continuous with respect to that of $\sqrt{\kappa}$ times a Brownian motion, as long as r and u remain bounded. More precisely: It suffices to take

$$T_n = \min\Big\{n,\, \inf\{t > 0 \;:\; |\zeta_t - e_t| < 1/n\}\Big\}.$$

Before T_n, $|r|$ remains bounded, a is bounded away from 0 (note also that $a \leq 1$ always), so that t/u is bounded and bounded away from 0. Hence, $u(T_n)$ is also bounded (since $T_n \leq n$).

It now follows directly from Girsanov's Theorem (see e.g., [117]) that the law of $\big(\beta(u)/\sqrt{\kappa}\big)_{u \leq u(T_n)}$ is equivalent to that of Brownian motion up to some (bounded) stopping time, and Proposition 6.2 follows. $\square$

6.3 Radial SLE_6 and reflected Brownian motion

If one combines the radial-chordal equivalence for SLE_6 with the locality property for chordal SLE_6, one gets immediately a locality property for radial SLE_6, and the relation between fillings of radial SLE_6 and of reflected Brownian motion. We do not state the locality property here (and leave it to the interested reader), but we state the relation between fillings of radial SLE and of reflected Brownian motions that we will use in the next chapters.

Before that, we have to say some words about how this reflected Brownian motion is defined in the unit disc. Suppose that $(Z_t, t \geq 0)$ is the reflected Brownian motion in the upper half-plane with reflection angle $2\pi/3$ away from the origin as in the previous chapter. Let us now define

$$\tilde{Z}_t := \exp(-iZ_t)$$

so that $\tilde{Z}$ takes its values in the unit disk and is started from $\tilde{Z}_0 = 1$. Clearly, since $\tilde{Z}_t \neq 0$ for all t, one can define the continuous version of its argument $(\theta_t, t \geq 0)$. Conformal invariance of planar Brownian motion shows that $\tilde{Z}_t$ behaves like (time-changed) Brownian motion as long as it stays away from the unit circle, and when it hits the unit circle, then it is reflected with angle $2\pi/3$ in the direction that "increases" $|\theta|$. Define

$$\tilde{\sigma}(r) := \inf\{t > 0 \ : \ |\tilde{Z}_t| = r\}$$

which is also the first time at which the imaginary part of Z hits $\log(1/r)$.

Theorem 6.3. *Suppose that $r < 1$. Define the two following random hulls:*

- *Suppose that $(K_t, t \geq 0)$ is radial SLE_6 as before. Let τ_r denote the first time at which radial K_t intersects the circle $\{|z| = r\}$. Define the event $\mathcal{H}(x, \tau_r)$ that K_{τ_r} does not disconnect 0 from $\exp(ix)$.*
- *On the event $\tilde{\mathcal{H}}(x, \tilde{\sigma}_r)$ that $\tilde{Z}[0, \tilde{\sigma}_r]$ does not disconnect 0 from $\exp(ix)$, define the connected component H of $\mathbb{U} \setminus \tilde{Z}$ that contains 0, and the hull $\tilde{K}_{\sigma_r} = \overline{\mathbb{U} \setminus H}$.*

Then, the two random sets $1_{\mathcal{H}(x,\tau_r)} K_{\tau_r}$ and $1_{\tilde{\mathcal{H}}_r(x,\tilde{\sigma}_r)} \tilde{K}_{\tilde{\sigma}_r}$ have the same law.

In particular,

$$\mathbf{P}[\mathcal{H}(x, \tau_r)] = \mathbf{P}[\tilde{\mathcal{H}}(x, \tilde{\sigma}_r))].$$

This shows that one can compute non-disconnection probabilities for reflecting Brownian motions using radial SLE_6.

Bibliographical comments

For basic results on Loewner's equation, and basic complex analysis, we refer again to [1, 2, 49, 58]. The radial-chordal equivalence for SLE_6 has been derived in [87].

7 Some critical exponents for SLE

7.1 Disconnection exponents

In this section, we fix $\kappa > 4$, and we consider radial SLE_κ in the unit disc started from 1. Our goal will be to estimate probabilities of events like

$$\mathcal{H}(x,t) = \{\exp(ix) \in \partial H_t\}$$

that K_t has not swallowed the point $\exp(ix) \in \partial\mathbb{U}$ from 0 at time t. Let us define the numbers

$$q_0 = q_0(\kappa) := 1 - \frac{4}{\kappa}$$

and

$$\lambda_0 = \lambda_0(\kappa) := \frac{\kappa}{8} - \frac{1}{2}.$$

Proposition 7.1 *There exists a constant c such that for all $t \geq 1$ and for all $x \in (0, 2\pi)$,*

$$e^{-\lambda_0 t}(\sin(x/2))^{q_0} \leq \mathbf{P}[\mathcal{H}(x,t)] \leq c e^{-\lambda_0 t}(\sin(x/2))^{q_0}.$$

Proof. We will use the notation

$$f(x,t) = \mathbf{P}[\mathcal{H}(x,t)].$$

Let $\zeta_t = \exp(i\sqrt{\kappa}B_t)$ be the driving process of the radial SLE_κ, with $B_0 = 0$. For all $x \in (0, 2\pi)$, let Y_t^x be the continuous real-valued function of t which satisfies

$$g_t(e^{ix}) = \zeta_t \exp(iY_t^x)$$

and $Y_0^x = x$. The function Y_t^x is defined on the set of pairs (x,t) such that $\mathcal{H}(x,t)$ holds. Since g_t satisfies Loewner's differential equation

$$\partial_t g_t(z) = g_t(z)\frac{\zeta_t + g_t(z)}{\zeta_t - g_t(z)}, \tag{7.1}$$

we find that

$$dY_t^x = \sqrt{\kappa}\, dB_t + \cot(Y_t^x/2)\, dt. \tag{7.2}$$

Let

$$\tau^x := \inf\Big\{t \geq 0 \ : \ Y_t^x \in \{0, 2\pi\}\Big\}$$

denote the time at which $\exp(ix)$ is absorbed by K_t, so that

$$f(x,t) = \mathbf{P}[\tau_x > t].$$

We therefore want to estimate the probability that the diffusion Y^x (started from x) has not hit $\{0, 2\pi\}$ before time t as $t \to \infty$. This is a standard problem. The general theory of diffusion processes can be used to argue that $f(x,t)$ is smooth on $(0, 2\pi) \times \mathbb{R}_+$, and Itô's formula immediately shows that

$$\frac{\kappa}{2}\partial_x^2 f + \cot(x/2)\partial_x f = \partial_t f. \tag{7.3}$$

Moreover, for instance comparing Y with Bessel processes when Y is small, one can easily see that (here we use that $\kappa > 4$) for all $t > 0$,

$$\lim_{x \to 0+} f(x,t) = \lim_{x \to 2\pi-} f(x,t) = 0. \tag{7.4}$$

Hence, f is solution to (7.3) with boundary values (7.4) and $f(x, 0) = 1$. This in fact characterizes f, and its long-time behaviour is described in terms of the first eigenvalue of the operator $\kappa\partial_x^2/2 + \cot(x/2)\partial_x$. More precisely, define

$$F(x,t) = \mathbf{E}[1_{\mathcal{H}(x,t)} \sin(Y_t^x/2)^{q_0}].$$

Then, it is easy to see that F also solves (7.3) with boundary values (7.4) but this time with initial data $F(x,0) = \sin(x/2)^{q_0}$. One can for instance invoke the maximum principle to construct a handcraft proof (as in [86]) of the fact that this characterizes F. Since $e^{-\lambda_0 t}\sin(x/2)^{q_0}$ also satisfies these conditions, it follows that

$$F(x,t) = e^{-\lambda_0 t}\sin(x/2)^{q_0}.$$

Hence,

$$f(x,t) = \mathbf{P}[\mathcal{H}(x,t)] \geq \mathbf{E}[1_{\mathcal{H}(x,t)} \sin(Y_t^x/2)^{q_0}] = e^{-\lambda_0 t}\sin(x/2)^{q_0}.$$

To prove the other inequality, one can for instance use an argument based on Harnack-type considerations: For instance, one can see that (uniformly in x) a positive fraction of the paths $(Y_t^x, t \in [0,1])$ such that $\tau_x > 1$ satisfy $Y_1^x \in [\pi/2, 3\pi/2]$. This then implies readily (using the Markov property at time $t-1$) that for all $t \geq 1$,

$$f(x,t) \leq c_0\mathbf{P}[\tau_x > t \text{ and } Y_t^x \in [\pi/2, 3\pi/2]] \leq c_1 F(x,t) = c_1 e^{-\lambda_0 t}\sin(x/2)^{q_0}.$$

□

7.2 Derivative exponents

The previous argument can be generalized in order to derive the value of other exponents that will be very useful later on: We will focus on the moments of the derivative of g_t at $\exp(ix)$ on the event $\mathcal{H}(x,t)$. Note that on a heuristic level, $|g_t'(e^{ix})|$ measures how "far" e^{ix} is from the origin in H_t.

More precisely, we fix $b \geq 0$, and we define

$$f(x,t) := \mathbf{E}\Big[\big|g_t'\big(\exp(ix)\big)\big|^b \, 1_{\mathcal{H}(x,t)}\Big].$$

We also define the numbers

$$q = q(\kappa,b) := \frac{\kappa - 4 + \sqrt{(\kappa-4)^2 + 16b\kappa}}{2\kappa}$$
$$\lambda = \lambda(\kappa,b) := \frac{8b + \kappa - 4 + \sqrt{(\kappa-4)^2 + 16b\kappa}}{16}.$$

The main result of this Section is the following generalization of Proposition 7.1:

Proposition 7.2 *There is a constant $c > 0$ such that for all $t \geq 1$, for all $x \in (0,2\pi)$,*

$$e^{-\lambda t}\big(\sin(x/2)\big)^q \leq f(x,t) \leq c e^{-\lambda t}\big(\sin(x/2)\big)^q$$

Proof. We can assume that $b > 0$ since the case $b = 0$ was treated in the previous section. Let Y_t^x be as before and define for all $t < \tau^x$

$$\Phi_t^x := \big|g_t'\big(\exp(ix)\big)\big| \ .$$

On $t \geq \tau^x$ set $\Phi_t^x := 0$. Note that on $t < \tau^x$

$$\Phi_t^x = \partial_x Y_t^x.$$

By differentiating (7.1) with respect to z, we find that for $t < \tau^x$

$$\partial_t \log \Phi_t^x = -\frac{1}{2\sin^2(Y_t^x/2)} \tag{7.5}$$

and hence (since $\Phi_0^x = 1$),

$$(\Phi_t^x)^b = \exp\left(-\frac{b}{2}\int_0^t \frac{ds}{\sin^2(Y_s^x/2)}\right) \ , \tag{7.6}$$

for $t < \tau^x$. So, we can rewrite

$$f(x,t) = \mathbf{E}\left[1_{\mathcal{H}(x,t)} \exp\left(-\frac{b}{2}\int_0^t \frac{ds}{\sin^2(Y_s^x/2)}\right)\right].$$

Again, it is not difficult to see that the right hand side of (7.6) is 0 when $t = \tau^x$ and that

$$\lim_{x\to 0} f(x,t) = \lim_{x\to 2\pi} f(x,t) = 0 \tag{7.7}$$

holds for all fixed $t > 0$.

Let $F : [0, 2\pi] \to \mathbb{R}$ be a continuous function with $F(0) = F(2\pi) = 0$, which is smooth in $(0, 2\pi)$, and set

$$h(x,t) = h_F(x,t) := \mathbf{E}\Big[(\Phi_t^x)^b \, F(Y_t^x)\Big].$$

By (7.6) and the general theory of diffusion Markov processes, we know that h is smooth in $(0, 2\pi) \times \mathbb{R}_+$. From the Markov property for Y_t^x and (7.6), it follows that $h(Y_t^x, t' - t)(\Phi_t^x)^b$ is a local martingale on $t < \min\{\tau^x, t'\}$. Consequently, the drift term of the stochastic differential $d\big(h(Y_t^x, t'-t)(\Phi_t^x)^b\big)$ is zero at $t = 0$. By Itô's formula, this means

$$\partial_t h = \Lambda h\,, \tag{7.8}$$

where

$$\Lambda h := \frac{\kappa}{2}\,\partial_x^2 h + \cot(x/2)\,\partial_x h - \frac{b}{2\sin^2(x/2)}\,h\,.$$

We therefore choose

$$F(x) := \big(\sin(x/2)\big)^q,$$

and note that $F(x)e^{-\lambda t} = h_F$ because both satisfy (7.8) on $(0, 2\pi) \times [0, \infty)$, and have the same boundary values. Finally, one can conclude using the same type of argument as in Proposition 7.1. □

7.3 First consequences

Recall that for all $t \geq 0$, $d(0, K_t)e^t \in [1/4, 1]$. Hence, if τ_r denotes the hitting time of the circle of radius $r < 1$ by the radial SLE_κ, then $re^{\tau_r} \in [1/4, 1]$. Combining this with Propositions 7.1 and 7.2 then implies that for all fixed $\kappa > 4$, all $b \geq 0$, if λ, q are defined as before, there exists two positive finite constants c_1 and c_2 such that for all $r < r_0$,

$$c_1 r^\lambda (\sin(x/2))^q \leq \mathbf{E}\left[1_{\mathcal{H}(x,\tau_r)} |g'_{\tau_r}(\exp(ix))|^b\right] \leq c_2 r^\lambda (\sin(x/2))^q \tag{7.9}$$

(we used also the fact that $|g'_t(\exp(ix)|$ is an decreasing function of t).

When $b = 1$, one can note that

$$l_t := \int_0^{2\pi} dx |g'_t(e^{ix})| 1_{\mathcal{H}(x,t)}$$

is simply the length of the image under g_t of the arc $A_t := \partial H_t \cap \partial\mathbb{U}$ on the unit circle that have not yet been swallowed by K_t. In particular, if one starts a planar Brownian motion from 0, it has a probability $l_t/2\pi$ to hit the unit circle on the arc $g_t(A_t)$. By conformal invariance of planar Brownian motion, we see that $l_t/2\pi$ is also the probability that a planar Brownian motion started from 0 hits the unit circle before hitting K_t. Let Z denote planar Brownian motion, stopped at its hitting time σ of the unit circle. Integrating Proposition 7.2 for $x \in [0, 2\pi]$ therefore shows that there exist constants c_1' and c_2' such that (if K_t is radial SLE_6)

$$c_1' r^{5/4} \leq \mathbf{P}[Z[0,\sigma] \cap K_{\tau_r} = \emptyset] \leq c_2' r^{5/4}. \tag{7.10}$$

Combining these results with Theorem 6.3, we see that these estimates are also valid for reflected Brownian motions. In particular, let us now define a reflected Brownian motion $\tilde{Z}$ in the unit disc as in Theorem 6.3 (reflected on $\partial\mathbb{U}$ with angle $2\pi/3$ "away" from $\tilde{Z}_0 = 1$). Let $\tilde{\sigma}_r$ denote its hitting time of the circle $r\partial\mathbb{U}$. Then there exist constants c_1 and c_2 such that for all $r < 1/2$,

$$c_1 r^{1/4} \leq \mathbf{P}[\tilde{Z}[0,\tilde{\sigma}_r] \text{does not disconnect 0 from } -1] \leq c_2 r^{1/4}. \tag{7.11}$$

Similarly, (7.10) holds if one replaces K_{τ_r} by $\tilde{Z}[0,\tilde{\sigma}_r]$.

We will see in the next chapter that this also yields the corresponding estimates for (non-reflected) Brownian motions.

Bibliographical comments

The material of this chapter is borrowed from [87], in which the reader can find more detailed proofs. It is possible to compute analogous exponents for chordal SLE. These "half-plane exponents" are determined in [86, 88].

Other important exponents are derived in [118, 92, 15]. As in this chapter, the exponents appear always as leading eigenvalues of some differential operators.

8 Brownian exponents

8.1 Introduction

The goal of this chapter is to relate the previous computations to the exponents associated to planar Brownian motion itself (not only to reflected Brownian motion).

Suppose that a planar Brownian motion Z is started from 1. Let σ_r denote its hitting time of the circle of radius $r > 0$, and let

$$p_r := \mathbf{P}[\mathcal{D}(Z[0, \sigma_r])],$$

where $\mathcal{D}(K)$ denotes the event that K does not disconnect the origin from infinity. Note that by inversion, $p_R = p_{1/R}$ for all $R > 1$ (one can map the disk $\{|z| < R\}$ conformally on $\{|z| > 1/R\}$ by $z \mapsto 1/z$ and use conformal invariance of planar Brownian motion).

The strong Markov property and the scaling property of planar Brownian motion imply readily that for all $R, R' > 1$,

$$p_{RR'} \leq \mathbf{P}[\mathcal{D}(Z[0, \sigma_R]) \text{ and } \mathcal{D}(Z[\sigma_R, \sigma_{RR'}])] \leq p_R p_{R'}.$$

On the other hand, it is not difficult to see that

$$p_R \geq \mathbf{P}[Z[0, \sigma_R] \cap [-R, 0] = \emptyset] \geq c R^{-1/2}$$

for all $R > 1$ and some constant c. Hence, a standard subadditivity argument implies that there exists a constant $\eta \leq 1/2$ such that

$$p_R \approx R^{-\eta}$$

when $R \to \infty$, where this notation means that $\log p_R \sim -\eta \log R$. It turned out that there seems to be no direct way to determine the value of this exponent η.

Similarly, if Z^1 and Z^2 denote two independent Brownian motions started uniformly on the unit circle, then subadditivity implies the existence of a positive constant ξ such that

$$\mathbf{P}[Z^1[0, \sigma_R^1] \cap Z^2[0, \sigma_R^2] = \emptyset] \approx R^{-\xi}.$$

The exponents η and ξ are respectively called the disconnection exponent and the intersection exponent for planar Brownian motion.

8.2 Brownian crossings

We now make some considerations that will help us relating the results on reflected Brownian motions derived in the previous chapter to the exponents η and ξ. For simplicity, we first focus on the disconnection exponent η.

Suppose that Z denotes a planar Brownian motion that is started from 1, and define the random times:

$$\begin{aligned}\sigma_r &:= \inf\{t > 0 \ : \ |Z_t| = r\}\\ \sigma_r^{\#} &:= \max\{t < \sigma_r \ : \ |Z_t| = 1\}\\ \sigma_r^{*} &= \inf\{t > \sigma_r^{\#} \ : \ |Z_t| = 1/2\}.\end{aligned}$$

It is a fairly standard application of the decomposition of the path $\Re(\log Z)$ into excursions away from the origin to see that

- The paths $P_r^1 := (Z_t, t \in [0, \sigma_r^{\#}])$ and $P_r^2 := (Z_{t+\sigma_r^{\#}}/Z_{\sigma_r^{\#}}, t \in [0, \sigma_r - \sigma_r^{\#}])$ are independent.
- The law of $P_r^3 := (Z_{t+\sigma_r^*}/Z_{\sigma_r^*}, t \in [0, \sigma_r - \sigma_r^*])$ is identical to the conditional law of $(Z_t, t \leq \sigma_{2r})$ on the event $E_r := \{Z_{[0,\sigma_{2r}]} \subset 2\mathbb{U}\}$.

Note also that $\mathbf{P}[E_r] = \log 2/\log r$ because $\log|Z|$ is a local martingale. We will call P_r^2 a Brownian crossing of the annulus $\mathcal{A}_r := \{1 > |z| > r\}$.

When $r' < r$, one can construct a Brownian crossing of the annulus $\mathcal{A}_{r'}$ starting from a crossing P_r^2 of the annulus $\mathcal{A}_r$ as follows: Attach to the endpoint $e_r := Z_{\sigma_r}/Z_{\sigma_r^{\#}}$ of P_r^2 a Brownian motion started from e_r, that is conditioned to hit the circle of radius r' before the unit circle, and stop it at that hitting time of the circle of radius r' (note that this event has probability $\log(1/r)/\log(1/r')$).

We now define the probability p_r^* that the crossing does not disconnect the origin from infinity:

$$p_r^* := \mathbf{P}[\mathcal{D}(P_r^2)].$$

Since a crossing is a subpath of a stopped Brownian motion, it follows from the a priori lower bound for p_r that $p_r^* \geq c r^{1/2}$ for some absolute constant c.

We now define for $\delta > 0$,

$$p_r^*(\delta) := \mathbf{P}[\mathcal{D}(P_r^2 \cup \mathcal{B}(1,\delta) \cup \mathcal{B}(e_r, \delta))],$$

where $\mathcal{B}(z, r)$ stands for the ball of radius r around z.

The following observations will be useful:

Lemma 8.1. *There exists $\delta > 0$ and $\varepsilon > 0$ such that for all integer n, then for at least 99% of the integers $j \in \{1, \ldots, n\}$, one has*

$$p_{r_j}^*(\delta) > \varepsilon p_{r_j}^*$$

where $r_j = 2^{-j}$.

Proof. We only sketch the main ideas of the proof. First, notice that $j \mapsto p^*_{r_j}$ is decreasing in j so that the a priori lower bound for $p^*_{r_j}$ implies that there exists ε such that for all n, then for at least 99% of the values of j in $\{1, \dots, n-1\}$, $2\varepsilon p^*_{r_j} \le p^*_{r_{j+1}}$ (otherwise p_{r_n} would be too small). On the other hand, it is easy to see that there exists $\delta > 0$ such that

$$p^*_{r_{j+1}} \le \varepsilon(p^*_{r_j} - p^*_{r_j}(\delta)) + p^*_{r_j}(\delta).$$

This is due to the fact that one can construct a sample of $P^2_{r_{j+1}}$ by extending the crossing $P^2_{r_j}$ into a crossing of $\{\sqrt{2} > |z| > r/\sqrt{2}\}$ by attaching conditioned Brownian motions to both ends (and then rescale this into a crossing of $\mathcal{A}_{r_{j+1}}$). And if δ is sufficiently small, then each of the attached parts disconnect the ball of radius δ around their starting point with very high probability. It therefore follows that "for 99% of the values of j",

$$p^*_{r_j}(\delta) \ge p^*_{r_{j+1}} - \varepsilon p^*_{r_j} \ge \varepsilon p^*_{r_j}.$$

□

Lemma 8.2. *For all fixed δ, for some constant $c = c(\delta)$,*

$$\mathbf{P}[P^1_r \subset \mathcal{B}(1, \delta/2)] \ge \frac{c}{\log(1/r)}.$$

Proof. With positive probability, Z hits the circle of radius $1 - \delta/4$ around 0 before $\partial\mathcal{B}(1, \delta/2)$. Then, if this is the case, with probability $\log(1/(1 - \delta/4))/\log(1/r)$ it hits the circle of radius r before going back to the unit circle. □

8.3 Disconnection exponent

We now use combine these considerations with the computation of the exponents for reflected Brownian motion to prove the following result:

Theorem 8.1. *One has $\eta = 1/4$. Furthermore, there exist two constants c_1 and c_2 such that for all $R > 1$,*

$$c_1 R^{-1/4} \le p_R \le c_2 R^{-1/4}.$$

As we shall see later, it is important to have estimates "up-to-constants" as in this Theorem (rather than $\approx$) in order to make the link with Hausdorff dimensions.

Proof. By inversion, this is equivalent to corresponding result for small r i.e., that for all $r < 1$,

$$c_1 r^{1/4} \le p_r \le c_2 r^{1/4}. \tag{8.1}$$

In order to compare p_r to $\tilde{p}_r$ (this is the non-disconnection probability for reflected Brownian motion that was defined at the end of the previous chapter where we proved that it is close to $r^{1/4}$), we will in fact compare both to p_r^*.

First, one can notice using the previous lemma that

$$\begin{aligned} p_r &\ge \mathbf{P}[\mathcal{D}(P_r^2 \cup \mathcal{B}(1,\delta)) \cap \{P_r^1 \subset \mathcal{B}(1,\delta/2)\}] \\ &\ge p_r^*(\delta) \times \frac{c}{\log(1/r)} \end{aligned}$$

for some constant c which is independent of $r < 1/2$. The same argument can be adapted to the reflected Brownian motion $\tilde{Z}$. Hence, "for 99% of j's",

$$p_{r_j} \ge \frac{cp^*_{r_j}}{j} \text{ and } \tilde{p}_{r_j} \ge \frac{cp^*_{r_j}}{j}$$

for some universal constant c.

On the other hand, let us now define inductively the stopping times: $\rho_0 = 0$ and for all $n \ge 0$,

$$\begin{aligned} \tau_n &:= \inf\{t > \rho_n \ : \ |Z_t| = 1/2\} \\ \rho_{n+1} &:= \inf\{t > \tau_n \ : \ |Z_t| = 1\} \end{aligned}$$

the successive times of downcrossings and upcrossings between the two circles $\{|z| = 1\}$ and $\{|z| = 1/2\}$. Let N_r denote the number of upcrossings before σ_r. In other words,

$$N = N(r) := \max\{n \ge 0 \ : \ \rho_n < \sigma_r\}.$$

Note that the probability that a Brownian motion started on the circle $\{|z| = 1/2\}$ hits $\{|z| = r\}$ before the unit circle is $c_r := \log 2/\log(1/r)$, because $\log|Z|$ is a local martingale. Hence, $\mathbf{P}[N_r \ge n] = (1-c_r)^n$. For each $n \ge 0$, the probability that $Z[\rho_n, \tau_n]$ disconnects 0 from the unit circle and does not hit the circle of radius $1/4$ is strictly positive (and independent from n). Note that if $Z[0, \sigma_r]$ does not disconnect the origin from the unit circle, then for all $n \le N$, $Z[\rho_n, \tau_n]$ does not disconnect the origin from the unit circle, and $Z[\tau_n, \sigma_{n,r}]$ doesn't either, where

$$\sigma_{n,r} = \inf\{t > \tau_n \ : \ |Z_t| = r\}.$$

It follows that for some absolute constant $c > 0$,

$$\begin{aligned} p_r &\le \sum_{n \ge 0} (1-c)^n (1-c_r)^n \mathbf{P}[\mathcal{D}(Z[\tau_n, \sigma_{n,r}])] \\ &\le \frac{p^*_{2r}}{cc_r} \\ &\le \frac{\log 2}{c} \times \frac{p^*_{2r}}{\log 1/r} \end{aligned}$$

A close inspection at the proof actually shows that the very same proof goes through if one replaces the Brownian motion Z by the reflected Brownian motion $\tilde{Z}$. Hence, for some absolute constant c',

$$p_r \leq c' \frac{p^*_{2r}}{\log(1/r)} \text{ and } \tilde{p}_r \leq c' \frac{p^*_{2r}}{\log(1/r)}.$$

Putting the pieces together, we see that "for 98% of $j's$",

$$\tilde{p}_{r_j} \leq c_1 \frac{p^*_{2r_j}}{\log(1/r_j)} \leq c_2 p_{2r_j} \leq c_3 \frac{p^*_{2r_j}}{\log(1/r_j)} \leq c_4 \tilde{p}_{4r_j}.$$

But we know that $r^{-1/4}\tilde{p}_r$ is bounded and bounded away from zero. It therefore follows that for some absolute constants c_1 and c_2 and at least 98% of the j's,

$$c_1 r_j^{1/4} \leq p_{r_j} \leq c_2 r_j^{1/4}.$$

It then remains to get rid of the last 2% of "bad" values of j. This can be done by pasting together "good" configurations that are "well-separated at the end" of the annuli $\{1 > |z| > r_{j_1}\}$ and $\{r_{j_1} > |z| > r_{j_1+j_2}\}$, where j_1 and j_2 are "good" values such that $j_1 + j_2 = j$. See for instance [91] for more details. □

8.4 Other exponents

The previous proofs need to be somewhat adjusted to show the corresponding result for the intersection exponent ξ (things are more complicated due to the fact that there are two Brownian motions to take care of, but no really new ideas are needed):

Theorem 8.2. *One has $\xi = 5/4$. Furthermore, there exist two constants c_1 and c_2 such that for all $R > 1$,*

$$c_1 R^{-5/4} \leq \mathbf{P}[Z^1[0, \sigma^1_R] \cap Z^2[0, \sigma^2_R] = \emptyset] \leq c_2 R^{-5/4}.$$

Actually, it is possible to derive the value of many other exponents. For instance, suppose that $Z^1, \dots, Z^k, \dots$ are independent planar Brownian motions started uniformly on the unit circle, and denote by $\sigma^1_R, \sigma^2_R, \dots$ their respective hitting times of the circle $R\partial\mathbb{U}$, then:

Theorem 8.3. *For all $k \geq 1$, there exist constants c_1, c_2 such that for all $R > 1$,*

$$c_1 R^{-\eta_k} \leq \mathbf{P}[\mathcal{D}(Z^1[0, \sigma^1_R] \cup \dots \cup Z^k[0, \sigma^k_R])] \leq c_2 R^{-\eta_k}$$

and

$$c_1 R^{-\xi_k} \leq \mathbf{P}[\textit{The sets } Z^1[0, \sigma^1_R], \dots, Z^k[0, \sigma^k_R] \textit{ are disjoint}] \leq c_2 R^{-\xi_k},$$

where

$$\eta_k = \frac{(\sqrt{24k+1}-1)^2-4}{48}$$

and

$$\xi_k = \frac{4k^2-1}{12}.$$

The proof of these results is however more involved. For other results and generalizations, see [86, 87, 88]. For instance, one can make sense of a continuum of exponents, or study intersection exponents for Brownian motion in a half-plane.

Let us mention that an instrumental role is also played in the definition and determination of the exponents in Theorem 8.3 by the critical exponents associated to non-intersection events in a half-space. For instance, the half-space analog of the intersection exponent ξ is:

Theorem 8.4. *If Z^1 and Z^2 are defined as before. Define*

$$q_R := \mathbf{P}[Z^1[0,\sigma_R^1] \cap Z^2[0,\sigma_R^2] = \emptyset \text{ and } Z^1[0,\sigma_R^1] \cup Z^2[0,\sigma_R^2] \subset \mathbb{H}].$$

There exist two constants c_1 and c_2 such that for all $R > 1$,

$$c_1 R^{-10/3} \le q_R \le c_2 R^{-10/3}.$$

There is a close relation between all these exponents (disconnection, in the whole space, in the half-space), see [96]. The critical exponents in the half-space can be determined in a similar way than the the whole-space exponents: First one computes the "derivative" exponents associated to chordal SLE. Then, using the identification between chordal SLE_6 and reflected Brownian motion, one transfers the SLE results into Brownian motion results. For the statements and proofs of all these "half-space exponents", see [86, 88]. In order to get the value of all η_k exponents, one then uses the fact that a family of generalized exponents is analytic, see [89] for more on this.

It has also been proved (using strong approximation of simple planar random walks by Brownian motions) that these exponents describe the probabilities of the corresponding events for planar simple random walks (see [27, 37, 84, 85]). For instance, if S^1 and S^2 denote two independent simple random walks starting from neighbouring points, then

$$\mathbf{P}[S^1[0,n] \cap S^2[0,n] = \emptyset] \approx n^{-\xi/2} = n^{-5/8}$$

when $n \to \infty$ (up-to-constants hold as well). The exponent is here $\xi/2$ because we used here the parametrization in time and not in space. It is worthwhile stressing that it seems that to prove this result that seems of combinatorial nature, one has to understand and use conformal invariance of planar Brownian motion, its relation to SLE_6 as well as the properties of SLE_6.

8.5 Hausdorff dimensions

In series of papers [76, 77, 78, 79] (before the mathematical determination of the exponents in [86, 87, 88]), Lawler showed how to use such up-to-constants estimates to estimate the Hausdorff dimension of various interesting random subsets of the planar Brownian curve in terms of the corresponding exponents.

More precisely, let $(Z_t, t \geq 0)$ denote a planar Brownian motion. Then, we say that

– The point $z = Z_t$ is a cut-point if $Z[0,t] \cap Z(t,1] = \emptyset$.
– The point $z = Z_t$ is a boundary point if $\mathcal{D}(Z[0,1] - z)$ i.e. if $Z[0,1]$ does not disconnect z from infinity.
– The point $z = Z_t$ is a pioneer point if $\mathcal{D}(Z[0,t] - z)$.

Note that, loosely speaking, near $z = Z_t$, there are two independent Brownian paths starting at z: The future $Z^1 := (Z_{t+s}, s \in [0, 1-t])$ and the past $Z^2 := (Z_{t-s}, s \in [0,t])$. Furthermore, $z = Z_t$ is a cut-point if $Z^1 \cap Z^2 = \{z\}$, z is a boundary point if $Z^1 \cup Z^2$ do not disconnect z from infinity and z is a pioneer point if Z^2 does not disconnect z from infinity. Hence, the previous theorems enable us to estimate the probability that a given point $x \in \mathbb{C}$ is in the ε-neighbourhood of a cut-point (resp. boundary point, pioneer point). Independence properties of planar Brownian paths then make it also possible to derive second moment estimates (i.e. the probability that two given points x and x' are both in the ε-neighbourhood of such points) and to obtain the following result:

Theorem 8.5.

– *The Hausdorff dimension of the set of cut-points is almost surely* $2 - \xi$*.*
– *The Hausdorff dimension of the set of boundary points is almost surely* $2 - \eta_2$*.*
– *The Hausdorff dimension of the set of pioneer points is almost surely* $2 - \eta$*.*

Recall that $2 - \xi = 3/4$, $2 - \eta_2 = 4/3$, $2 - \eta = 7/4$. Similar results hold for various other random subsets of the planar curve. We choose not to give the proof of this theorems in these lectures since they are more using features of planar Brownian motion rather than SLE_6, but here is a brief sketch in the case of the pioneer points.

Sketch of the proof. Let $\mathcal{P}$ denote the set of pioneer points on $Z[0,1]$. Theorem 8.1 roughly shows that for each z, the probability that Z comes ε-close to z without disconnecting z from infinity is comparable to $\varepsilon^{1/4}$. It follows that the expectation of the number N_ε of ε-balls that are needed in order to cover $\mathcal{P}$ is comparable to (i.e. up-to-constants away from) $\varepsilon^{-2+1/4} = \varepsilon^{-7/4}$. This in fact already shows that the Hausdorff dimension of $\mathcal{P}$ can a.s. not be larger than $7/4$.

On the other hand, one has good bounds on the second moment of N_ε: This is due to the fact that for two points x and x' with $|x - x'| = r$ to be

ε-close to pioneer points, then the following three events must occur before time one:

– Z reaches $\mathcal{B}(x, 2r)$ without disconnecting x
– Z crosses the annulus $\{z : \varepsilon < |z - x| < r/2\}$ without disconnecting x
– Z crosses the annulus $\{z : \varepsilon < |z - x'| < r/2\}$ without disconnecting x'.

Hence, it follows that $\mathbf{E}[N_\varepsilon^2] \le cst \times \varepsilon^{-7/2} \le cst E[N_\varepsilon]^2$. Standard arguments can then be used to deduce from this that with positive probability, the dimension of $\mathcal{P}$ is not smaller than 7/4. A zero-one law can finally be used to conclude that the dimension is a.s. equal to 7/4. See e.g. [82] for details. □

Bibliographical comments

The fact that one probably had to compute the value of the Brownian exponents via an universality argument using another model (that should be closely related to critical percolation scaling limits) first appeared in [97]. The mathematical derivation of the value of the exponents was performed in the series of papers [86, 87, 88, 89]. The properties of SLE that were later derived in [95] enable to shorten some parts of some proofs, but it seems that analyticity of the family of generalized exponents derived in [89] can not be by-passed for all exponents (for instance, it seems that it is needed to determine the exponent describing the probability that the union of three Brownian motions does not disconnect a given point). It can however be by-passed for those exponents that we have to focus on i.e., η, η_2, ξ.

Lemma 8.1 is a "separation Lemma" of the type that had been derived by Lawler in the series of papers relating the Hausdorff dimensions to the exponents [75, 76, 77, 78, 79]. The proof presented here is adapted from the proof of the analogous but more general results for the other exponents in [91]. A good reference for the relation between Brownian exponents and Hausdorff dimensions is Lawler's review paper [82]. See also, Beffara [13, 14].

Determining the Hausdorff dimensions of subsets of the SLE processes is a difficult question. Rohde-Schramm [118] have shown that the dimension of the SLE generating curve is not larger than $1 + \kappa/8$. It was conjectured to be a.s. equal to that value (for $\kappa \le 8$). This has been proved to hold for the special values $\kappa = 8/3$ and $\kappa = 6$, making use of the locality and restriction properties (see [95], Beffara [14]). It now seems that Beffara [15] managed to prove the general conjecture.

The value of most of these exponents had been predicted/conjectured before: Duplantier-Kwon [48] had predicted the values of ξ_k using non-rigorous conformal field theory considerations, Duplantier [44] more recently used also the so-called "quantum gravity" to predict the values of all exponents. The fact that the dimension of the Brownian boundary was 4/3 was first observed visually and conjectured by Mandelbrot [107]. Before the proof of this conjecture, some rigorous bounds had been derived, for instance that the dimension of the Brownian boundary is strictly larger than 1 and strictly smaller than 3/2 (see [24, 28, 132]).

9 SLE, UST and LERW

9.1 Introduction, LERW

In the next two chapters, we will survey the rigorous results that show that for some values of κ, SLE_κ is indeed the scaling limit of discrete models. There are at present only three values of κ for which this is the case: $\kappa = 2$ is the scaling limit of LERW, $\kappa = 6$ is the scaling limit of percolation cluster interfaces, and $\kappa = 8$ is the scaling limit of the uniform spanning tree contour.

In all three cases, the convergence to SLE is derived as a consequence of three facts:

– The "Markovian" property holds in the discrete case (this is usually a trivial consequence of the definition of the microscopic model).
– Some macroscopic functionals of the model converge to conformally invariant quantities in the scaling limit (for a wide class of domains).
– One has "a priori" bounds on the regularity of the discrete paths.

Before going into more details, let us state the convergence theorem in the case of LERW that was presented in the introductory chapter: Consider γ^δ the (time-reversal of the) loop-erasure of a simple random walk in $D \cap \delta\mathbb{Z}^2$, started from 0 and stopped at the first exit time of the simply connected (say, bounded) domain D. Let γ denote a radial SLE_2 in the unit disc started uniformly on the unit circle (and aiming at 0). Let Φ denote a conformal map from $\mathbb{U}$ onto D that preserves 0. We endow the set of paths with the metric of uniform convergence modulo time-reparametrization:

$$d(\Gamma, \Gamma') = \inf_{\varphi} \sup_{t \geq 0} |\Gamma(t) - \Gamma'(\varphi(t))|$$

where the inf is over all increasing bijections φ from $[0, \infty)$ into itself. Then,

Theorem 9.1. *The law of γ^δ converges weakly when $\delta \to 0$ to the law of $\Phi(\gamma)$.*

Actually, one can also use the convergence result to justify the fact that SLE_2 is a simple path. Instead of giving the basic ideas of the proof of this theorem, we will focus on a closely related problem: The uniform spanning trees scaling limit.

9.2 Uniform spanning trees, Wilson's algorithm

Suppose that a connected finite graph $G = (V, E)$ is given (V is the set of vertices and E is the set of edges). We say that the subgraph $T \subset E$ is a spanning tree if it contains no loop, and if it has only one connected component. We then define the uniform spanning tree as the uniform measure on the set of spanning trees. For any two fixed points a and b in G, and any spanning tree T, there exists a unique simple path in T that joins a to b (it exists because T has one connected component, it is unique because T has no loops). Hence, if T is picked according to the UST measure, this defines a random path γ from a to b. The following result had first been observed by Pemantle [113]:

Proposition 9.2 *The law of γ is that of the loop-erasure of simple random walk on G started at a and stopped at its first hitting of b.*

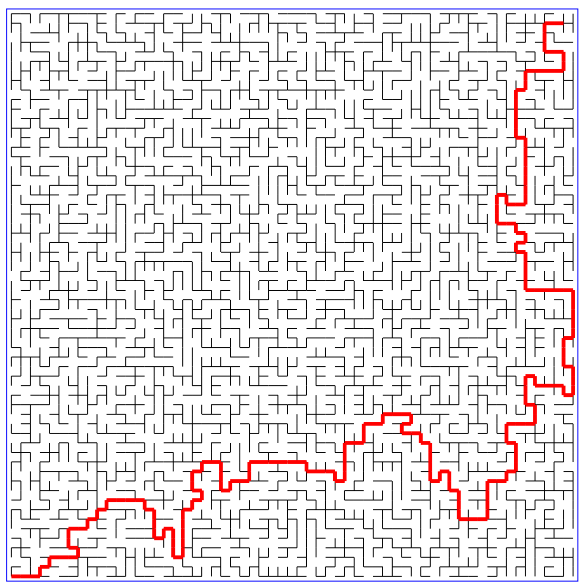

Fig. 9.1. A loop-erased walk as a subpath of the UST

This shows that LERW and UST are very closely related. Actually, it turns out that an even stronger relationship hold: Suppose that an ordering of the vertices $v_0, v_1, \ldots, v_m$ of G is given. Define inductively the sets A_m as follows: $A_0 = \{v_0\}$, and for all $j \leq m$, $A_j = A_{j-1} \cup \gamma_j$ where γ_j is the loop-erasure of a random walk started from v_j and stopped at its first hitting of A_{j-1}. Clearly, in this way, A_m is a (random) tree that contains all vertices: It is a spanning tree.

Proposition 9.3 (Wilson's algorithm) *The law of A_m is the uniform spanning tree measure.*

Note that this algorithm yields a natural extension of uniform spanning trees (or forests) in infinite graphs (see e.g. [20] and the references therein for more on this subject).

Proof. One can derive this result using the explicit formulas that we derived in the introductory chapter for loop-erased random walks: Indeed, it follows readily from the definition and the symmetry of the function F that was defined there, and the fact that (since we are considering simple random walks), the transition probabilities $p(x,y)$ are simply equal to $1/d_x$ where d_x is the number of neighbours of x), that for any possible spanning tree T,

$$\mathbf{P}[A_m = T] = F(v_1, v_2, \dots, v_m; \{v_0\}) \prod_{j=1}^{m} (1/d_{v_j}).$$

This quantity is the same for all T: The law of A_m is uniform. □

Hence, if LERW has a conformally invariant scaling limit then UST also has a conformally invariant scaling limit (in a rather weak sense though, such as: for all k given fixed points, the "finite subtree that go through these points" converges in the scaling limit).

There is another way to encode planar trees that goes as follows. Suppose for instance that we are looking at a spanning tree of a bounded "simply connected" graph $G \subset \mathbb{Z}^2$. Then, one can associate to each tree the contour of the tree which is a simple closed curve living on a subset $G^{\#}$ of the lattice $(1/4 + \mathbb{Z}/2)^2$. It is easy to see that (under mild assumptions on the domain), this curve visits every point of $(1/4 + \mathbb{Z}/2)^2$ that is close to the vertices of G. If the tree is chosen according to the uniform measure on spanning trees, then the contour is chosen according to the uniform measure on space-filling simple closed curves in this graph $G^{\#}$.

Hence, it is natural to study the behaviour of this space-filling curve in the scaling limit. In order to obtain SLE (and not a closely related object that we would have to define first) it is (slightly) more convenient to consider a variant of the previously defined space-filling curve.

More precisely, suppose that a certain connected graph of $(1/4 + \mathbb{Z}/2)^2$ is given together with two distinct "boundary points" a and b. Then (for a suitable class of "admissible" graphs), one is interested in the uniform measure on simple space-filling curves η from a to b in the graph (i.e. paths from a to b that visit all vertices exactly once). An example of "admissible" graphs is given by the graph obtained from removing from $G^{\#}$ a part of a simple closed space-filling curve γ. This time, there is a one-to-one correspondence between the family of simple space-filling curve η (from a to b) and the set of spanning trees in a certain subgraph G of $\mathbb{Z}^2$ obtained by wiring one part of the boundary between a and b (i.e. by conditioning the tree to contain

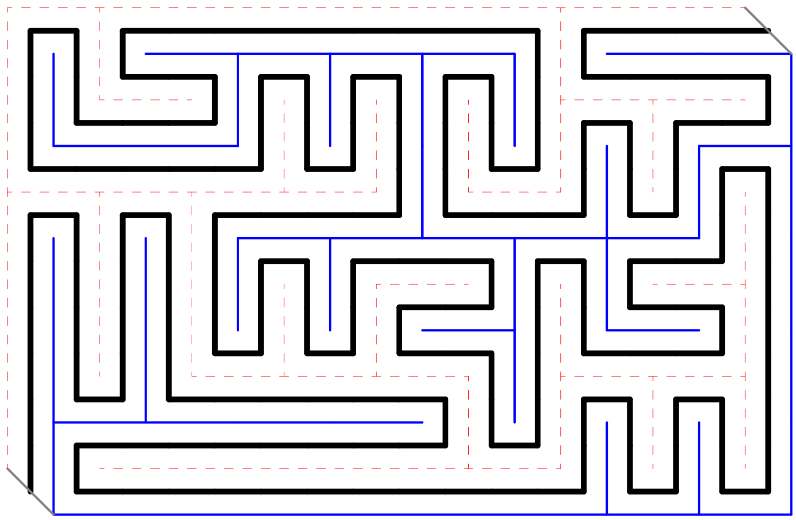

Fig. 9.2. The wired tree, the dual tree, the Peano curve.

this part of the boundary). This is best seen on pictures, and not difficult to understand heuristically, but it is somewhat messy to formulate precisely, so we will omit the precise statements here (see e.g., [93] for more details).

Note that in this set-up, the Markovian type property for η is immediate: If one conditions on the first step $\eta(1)$ of η, then the law of $\eta(1), \dots, \eta(n) = b$ is simply the uniform measure on the space filling curves from $\eta(1)$ to b in the remaining graph.

9.3 Convergence to chordal SLE_8

Suppose that D is a simply connected bounded planar domain with C^1 boundary and let a, b denote two distinct points on ∂D. For each δ, we associate in a "suitable approximation" of $D \cap \delta Z^2$, denoted by D_δ, and the two boundary points a_δ and b_δ close to a and b. We define η^δ, a uniformly chosen space-filling curve from a_δ to b_δ in D_δ.

Theorem 9.4. *When $\delta \to 0$, the law of η^δ converges weakly to that of a space-filling continuous path η, such that the law of $(\eta[0,t], t \geq 0)$ is (up to time-change) that of chordal SLE_8 in D from a to b.*

Some rough ideas from the proof. A first step is to obtain regularity estimates on the (discrete) random space-filling curve. This shows that the families of probability measures defining η^δ is tight in an appropriate sense

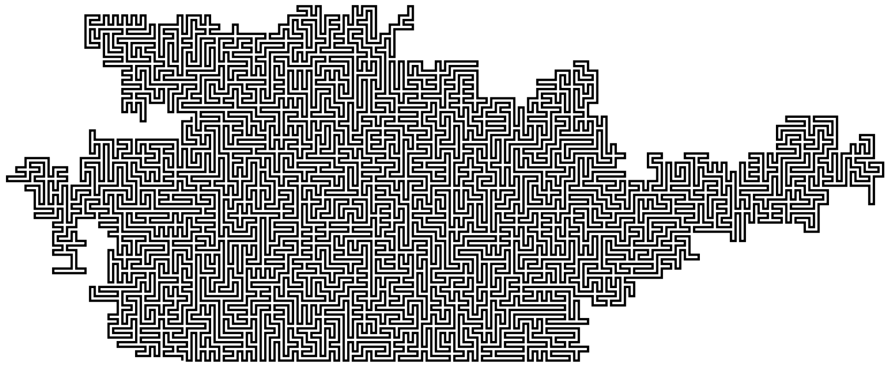

Fig. 9.3. A sample of the beginning of the Peano curve.

and therefore has subsequential limits. These estimates have been derived in [123] (see also [5, 6]) and the basic tools are Wilson's algorithm and estimates for simple random walks. This is not easy, and we refer to [123] for details. Hence, one can work with a given decreasing sequence $\delta_n \to 0$ such that the law of η_{δ_n} converges towards that of a random curve η, and one has to show that η is in fact chordal SLE_8.

Let us first work on the discrete level. Suppose that z_δ is some discrete lattice approximation of $z \in D$ and that c_δ is some discrete lattice approximation of $c \in \partial D$ that is on the wired part of the boundary of D. Let P_1^δ denote the part of the wired boundary of D_δ which is between a_δ and c_δ, and let P_2^δ denote the part of the wired boundary which is between c_δ and b_δ (and P_1, P_2 are defined similarly in D).

We consider the event E^δ that there exists a path in the corresponding tree that goes from z_δ to P_2^δ without touching P_1^δ. By Wilson's algorithm, we see that $\mathbf{P}[E^\delta(c,z)]$ is the probability that simple random walk on D_δ hits P_2^δ before P_1^δ. One first key-observation is that when δ goes to 0, the probability of this event can be controlled in a rather uniform way: Uniformly over some suitable choices of z, c, a, b and D, it converges towards the probability that a Brownian motion in D that is orthogonally reflected on the 'free' part of ∂D, hits P_2 before P_1. This is a conformally invariant quantity. Mapping D onto the upper half-plane by some given fixed mapping g in such a way that $g(b) = \infty$, we see that

$$\lim_{\delta \to 0} \mathbf{P}[E^\delta] := h(A, C, Z) = F\left(\frac{Z - A}{C - A}\right),$$

where

$$F(re^{i\theta}) = \frac{1}{\pi} \tan^{-1}\left(\frac{1 - r}{2\sqrt{r} \sin \theta / 2}\right)$$

and $A = g(a), Z = g(z), C = g(c)$. This function F can be computed for instance by first using reflection so that this probability is the probability that (non-reflected) Brownian motion in the complex plane, started from $g(z)$ hits $[g(c), \infty)$ before $[g(a), g(z)]$, then to use the map $x \mapsto (\sqrt{x} - \sqrt{z})/(\sqrt{x} + \sqrt{z})$ from $\mathbb{C} \setminus [g(a), +\infty)$ onto the unit disk and to look at the length of the image of $[g(c), +\infty)$ on the unit circle).

At each step n, define the conformal map ϕ_n^δ from a continuous approximation D_n^δ of $D^\delta \setminus \eta[0, n]$ onto the upper half-plane that is characterized by $\phi_n^\delta(x) - \phi_0^\delta(b) = o(1)$ when $x \to b$. We then define t_n^δ to be the "size" $a(\phi_0^\delta(\eta^\delta[0, n]))$ of $\phi_0^\delta(\eta[0, n])$ and we put

$$A_n^\delta = \phi_n^\delta(\eta_n), \; C_n^\delta = \phi_n^\delta(c_\delta) \text{ and } Z_n^\delta = \phi_n^\delta(z_\delta).$$

Suppose now that $\varepsilon > 0$ is small but fixed. If one stops the uniform Peano curve at the first step N, at which either $|A_n^\delta - A_0^\delta|$ reaches ε or t_n^δ reaches ε^2, (if c and z are not close to a), then one does not yet know whether E^δ holds or not. In fact the conditional probability is just equal to

$$\mathbf{P}[E^\delta(c_\delta, z_\delta, \eta_N, b_\delta, D_N^\delta)].$$

Hence,

$$\mathbf{E}[\mathbf{P}[E^\delta(c_\delta, z_\delta, \eta_N, b_\delta, D_N^\delta)] = \mathbf{P}[E^\delta].$$

The right-hand side is close to $h(A, C, Z)$ and the right-hand side is close to $\mathbf{E}[h(A_N^\delta, C_N^\delta, Z_N^\delta)]$ (in a uniform way as δ goes to 0). In fact, one can prove that

$$\mathbf{E}[h(A_N^\delta, C_N^\delta, Z_N^\delta)] = \mathbf{E}[h(A_0, C_0, Z_0)] + O(\varepsilon^3).$$

It turns in fact out, that the conformal map Φ_N^δ is very close to the (properly normalized) conformal map from $D \setminus \eta[0, N]$ onto $\mathbb{H}$ (i.e. removing the slit or the "tube" does not make much difference when δ is small). In particular, when ε is small (and δ very small), Loewner's equation shows that

$$(Z_N^\delta - Z_0) = \frac{2t_N^\delta}{Z_0 - A_0} + O(\varepsilon^3) \text{ and } (C_N^\delta - C_0) = \frac{2t_N^\delta}{C_0 - A_0} + O(\varepsilon^3).$$

Hence, one can Taylor-expand h in the previous estimate, so that

$$\frac{1}{2}\mathbf{E}[(A_N^\delta - A_0)^2]\partial_A^2 h(A_0, C_0, Z_0) + \mathbf{E}[A_N^\delta - A_0]\partial_A h(A_0, C_0, Z_0)$$
$$+2\mathbf{E}[t_N^\delta]\left(\frac{\partial_C h(A_0, C_0, Z_0)}{C_0 - A_0} + \frac{\partial_Z h(A_0, C_0, Z_0)}{Z_0 - A_0}\right) = O(\varepsilon^3).$$

Using the explicit expression of h as well as the fact that this holds for various values c and z yields that in fact:

$$\mathbf{E}[A_N^\delta - A_0] = O(\varepsilon^3) \text{ and } \mathbf{E}[(A_N^\delta - A_0)^2] = 8\mathbf{E}[t_N^\delta] + O(\varepsilon^3).$$

One can iterate this procedure using inductively defined stopping times $N_2, N_3, \ldots$, and one can then use this as a seed to show that it is possible to find a Brownian motion B such that A_n^δ remains close to $B_{8t_n^\delta}$, and then, after some additional work can be improved into the convergence theorem. $\square$

As the reader can see, this is only a very sketchy outline of a fairly long and technical proof. For details, see [93].

9.4 The loop-erased random walk

The strategy of the proof of Theorem 9.1 follows roughly the same lines. One has to identify a conformal invariant quantity that appears in the scaling limit of LERW and that plays the role of the probability of the events E in the case of the uniform Peano curve. The macroscopic quantities that are used are related to the mean number of visits to a given point z by the simple random walk started from 0 and conditioned to leave the domain at the same point as the LERW. See [93] for details.

Bibliographical comments

The convergence results presented in this chapter are proved in [93], where the reader can find more details. For an introduction to LERW and UST, see for instance [104, 81]. Rick Kenyon [62, 64] had proved that LERW (and UST's) have conformally invariant features exploiting the relation between UST and dimer models (and some explicit computations). He also managed to determine directly (without using SLE_2 or SLE_8) [65, 66] the value of various critical exponents related to LERW and UST that had been conjectured by Majumdar and Duplantier [106, 43]. For instance, he showed that the expected length of a LERW from 0 to the boundary of the unit disc on the lattice $\delta\mathbb{Z}^2$ is of the order $\delta^{-5/4}$. See also, Fomin's paper [50] for another approach to some of these exponents.

In the recent preprint [71], Gady Kozma gives a completely different approach and justification to the existence of a scaling limit of LERW (that does not seem to use conformal invariance or SLE).

10 SLE and critical percolation

10.1 Introduction

Consider a planar "periodic" lattice such that simple random walk on that lattice converges to planar Brownian motion. For convenience, let us limit our discussion to the square lattice and to the triangular lattice. Fix $p \in [0,1]$, and for each site of the lattice, decide that with probability p, the site is open (with probability $1-p$, it is therefore closed), and do that independently for all sites of the lattice. One is interested in the properties of the connected components (or "clusters") of open sites. It is now classical (see e.g., [55] for an introduction to percolation) that there exists a critical value $p_c \in (0,1)$ such that:

– If $p \le p_c$, there exists a.s. no infinite open cluster (note that in dimension greater than 2, the non-existence of an infinite open cluster at p_c is still an open problem).
– If $p < p_c$, there exists a positive $\xi(p)$ such that when $n \to \infty$, the probability that 0 is in the same connected component than $(n,0)$ decays exponentially fast, like $\approx \exp(-n/\xi(p))$ (the positive quantity $\xi(p)$ is called the correlation length).
– If $p > p_c$, there exists almost surely no infinite open cluster.

The value of p_c is lattice-dependent. In the case of the square lattice, it has been shown to be larger than .556 [22] (it is not expected to be any special number), while for the triangular lattice, it has been shown by Kesten and Wierman to be equal to $1/2$ (see e.g. [67]). This is not surprising because the triangular lattice has a self-matching property: It is equivalent to say that the origin is in a finite open cluster or to say that it is surrounded by a circuit (on the same lattice, this is what makes the triangular lattice so special) of closed sites. This property shows also that if $p = 1/2$ on the triangular lattice, the probability that there exists a left-to-right crossing of open sites of a square is exactly $1/2$ (otherwise, there is a top-to-bottom crossing of closed sites). Russo, Seymour and Welsh [120, 125] have shown (this is sometimes known as the RSW theory) that this in fact implies that for any fixed a and b, there exists a constant $c > 0$, such that the probability $q(aN, bN)$ of a left-to-right crossing of the $aN \times bN$ rectangle satisfies

$$1 - c > q(aN, bN) > c$$

for all large N. This strongly suggests that when $N \to \infty$, $q(aN, bN)$ converges to a limit $F(b/a)$. A renormalizing group argument (loosely speaking, the rectangle $2aN \times 2bN$ can be divided into four rectangles of size $aN \times bN$, which themselves can be divided into four rectangles etc.) also heuristically suggests that not only the crossing probabilities converge but that in some sense, the information about "macroscopic connectivity properties" should converge. Note however that things are rather subtle. Benjamini, Kalai and Schramm [19] have for instance proved that if $A[N]$ denotes the event that there is a left-to-right crossing of a $N \times N$ square say, and if one changes the status of a fixed proportion ε of the N^2 sites and looks at the event $\tilde{A}[N]$ that there exists a left-to-right crossing for the new configuration, then the events $A[N]$ and $\tilde{A}[N]$ are asymptotically independent when $N \to \infty$. These events are "sensitive to noise". When N is large, it is not easy to "see" whether the crossing events occur or not (in the Figure 10.1, each occupied site on the triangular lattice is represented by a white hexagon).

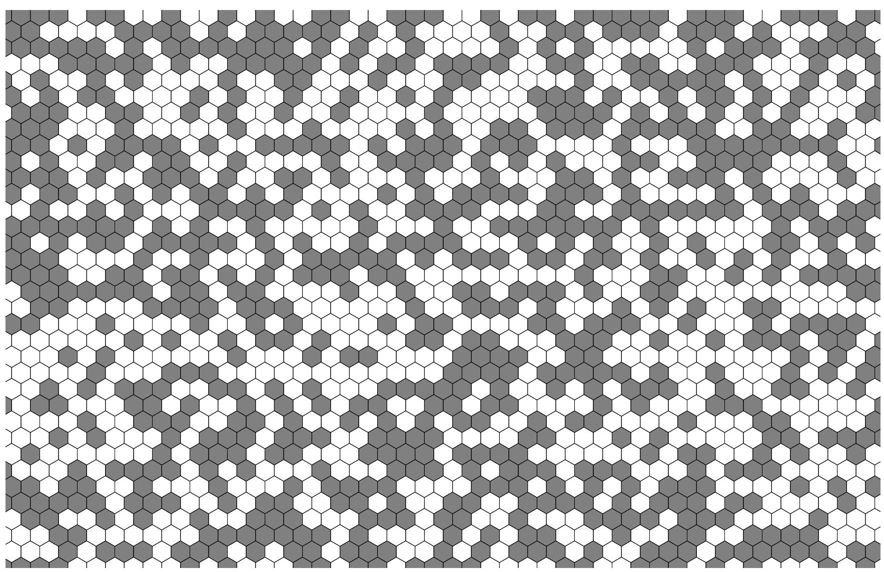

Fig. 10.1. Is there a left to right crossing of white hexagons?

In fact, the renormalization argument suggests that even though the value of p_c is lattice-dependent, on large scale, what one sees at the value p_c becomes lattice-independent. In other words, in the scaling limit, the behaviour of critical percolation should become lattice-independent (just as simple random walk converges to Brownian motion, for all "regular" lattices). Hence,

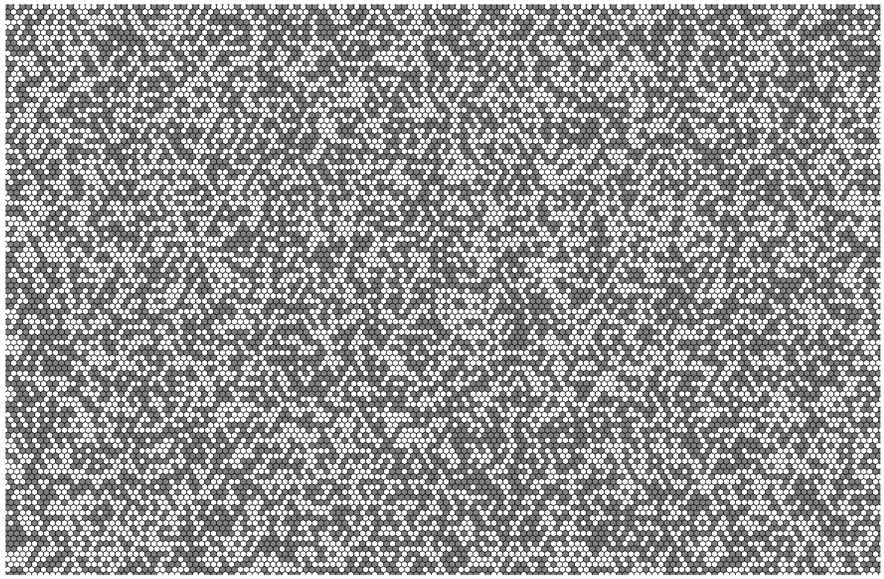

Fig. 10.2. And now?

the function $F(b/a)$ should be a universal function describing the crossing-probabilities of a "continuous percolation process." In fact, this continuous percolation should be scale-invariant (it is a scaling limit) as well as rotationally invariant (which would follow from lattice-independence). This leads to the stronger conjecture that it should be conformally invariant: The connections in a domain D and those in a domain D' should have the same law, modulo a conformal map from D onto D'.

10.2 The Cardy-Smirnov formula

Using the conformal field theory ideas developed in [18, 30], John Cardy [31] gave an exact prediction for the function F. Extensive numerical work (e.g., [73]) did comfort these predictions. Carleson noted that Cardy's function F is closely related with the conformal maps from rectangles onto equilateral triangles, and that Cardy's prediction could be rephrased as follows:

Conjecture 10.1 (Cardy's formula). If D is conformally equivalent to the equilateral triangle OAC, and if the four boundary points a, o, c, x are respectively mapped onto $A, O, C, X \in [CA]$, then (in the scaling limit when the mesh of the lattice goes to zero), the probability that there exists a crossing in D from the part (ao) of ∂D to (cx) is equal to CX/CA.

We have seen that the SLE approach did provide a new justification to this formula. Indeed, if the percolation exploration path has a conformally invariant scaling limit, it must be one of the chordal SLEs, as argued it the first Chapter. Also, as the hitting probabilities computations in Chapter 3 show, SLE_6 is the unique SLE such that for all $X \in [CA]$, the two following probabilities are identical:

– The SLE from O to A in the equilateral triangle hits AX before XC
– The SLE from O to X in the equilateral triangle hits AX before XC

This has to hold for the scaling limit of the critical exploration process. Hence, the unique possible conformally invariant scaling limit of the critical exploration process is SLE_6. Another way to justify this is that this scaling limit has to satisfy locality and (cf. Chapter 4) that SLE_6 is the unique SLE that satisfies locality. Yet another (simpler) justification is that SLE_6 is the unique SLE for which the probability of the event corresponding to a left-right crossing of a square (or a rhombus) is 1/2 (for an SLE starting from one corner and aiming at a neighbouring corner).

We have also seen that an SLE_6 from O to C in the equilateral triangle hits XC before AX with probability CX/CA. Also, for the discrete exploration process, the corresponding event is precisely the event that there exists a crossing from AO to CX. Hence, we get a conditional result of the following type: If the scaling limit of critical percolation exists and is conformally invariant, then the scaling limit of the exploration process is SLE_6 and Cardy's formula holds.

But in order to prove conformal invariance of critical percolation, one has to work with discrete percolation itself. In 2001, Stas Smirnov, proved that:

Theorem 10.1. *Cardy's prediction is true in the case of critical site percolation on the triangular lattice.*

In fact, Smirnov's proof is a direct proof of Cardy's formula that does not rely at all on SLE. Then, with Smirnov's result, one can show that indeed the scaling limit of the percolation exploration process is SLE_6.

Sketch of the proof. Suppose first for convenience that AOC is an equilateral triangle and that the sides of the triangle have unit length and are parallel to the axis of the triangular grid (as we will see, this has in fact no other influence on the proof than simplifying the notations). For all $\delta = 1/n$, consider critical site percolation in AOC on the triangular grid with mesh-size $1/n$. For convenience, put $\tau = \exp(2i\pi/3)$ and write $A_1 = A$, $A_\tau = A_2 = O$ and $A_{\tau^2} = A_3 = C$. For each face z of the triangular grid (i.e. for each site of the dual hexagonal lattice), let $E_1(z)$ denote the event that there exists a simple open (i.e. white) path from A_1A_τ to $A_1A_{\tau^2}$ that separates z from $A_\tau A_{\tau^2}$. Similarly, define the events $E_\tau(z)$ and $E_{\tau_2}(z)$ corresponding to the existence of simple open paths separating z from $A_1A_{\tau^2}$ and A_1A_τ respectively. Define finally for $j = 1, \tau, \tau^2$,

$$H_j(z) = H_j^\delta(z) := \mathbf{P}[E_j(z)].$$

The Russo-Seymour-Welsh theory ensures that the functions H_j^δ are uniformly "Hölder" (actually, one first has to smooth out their discontinuities for instance in a linear way keeping only the values of H_j^δ at the center of the triangles). In particular, it shows that any for any sequence $\delta_n \to 0$, the triplet of functions $(H_1^\delta, H_\tau^\delta, H_{\tau^2}^\delta)$ has a subsequential limit. Our goal is now to identify the only possible such subsequential limit.

The Russo-Seymour-Welsh estimates also show that when $z \to A_{j\tau}A_{j\tau^2}$, the functions H_j^δ go uniformly to zero, and that when $z \to A_j$, the functions H_j^δ go uniformly to one. Hence, for any subsequential limit $(H_1, H_\tau, H_{\tau^2})$, one has $H_j(z) \to 0$ when $z \to A_{j\tau}A_{j\tau^2}$, and $H_j(z) \to 1$ when $z \to A_j$.

Now comes the key-observation of combinatorial nature: Suppose that z is the center of a triangular face. Let z_1, z_2, z_3 denote the three (centers of the) neighbouring faces (with the same orientation as the triangle $A_1A_2A_3$) and s_1, s_2, s_3 the three corners of the face containing z chosen in such a way that s_j is the corner "opposite" to z_j. We focus on the event $E_1(z_1) \setminus E_1(z)$. This is the event that there exists three disjoint paths l_1, l_2, l_3 such that

– The two paths l_2 and l_3 are open and join the two sites s_2 and s_3 to A_1A_3 and A_1A_2 respectively.
– The path l_1 is closed (i.e., it consists only of closed sites), and joins s_1 to A_2A_3.

One way to check whether this event holds is to start an exploration process from the corner A_3, say (leaving the open sites on the side of A_1 and the closed sites on the side of A_2). If the event $E_1(z_1)\setminus E_1(z)$ is true, then the exploration process has to go through the face z, arriving into z through the edge dual to s_1s_2. In this way, one has "discovered" the simple paths l_2 and l_1 that are "closest" to A_3. Then, in the remaining (unexplored domain), there must exist a simple open path from s_3 to A_1A_3. But, the conditional probability of this event is the same as that of the existence of a simple closed path from s_3 to A_1A_3 (interchanging open and closed in the unexplored domain does not change the probability measure). Changing all the colors once again, shows finally that $E_1(z_1) \setminus E_1(z)$ has the same probability as the event that there exist three disjoint paths l_1, l_2, l_3 such that

– The paths l_1 and l_3 are open and join the two sites s_1, s_3 to A_2A_3 and A_1A_2 respectively.
– The path l_2 is closed, and joins s_2 to A_1A_3.

This event is exactly $E_\tau(z_2) \setminus E_\tau(z)$. Hence, we get that,

$$\mathbf{P}[E_1(z_1) \setminus E_1(z)] = \mathbf{P}[E_\tau(z_2) \setminus E_\tau(z)] = \mathbf{P}[E_{\tau^2}(z_3) \setminus E_{\tau^2}(z)].$$

These identities can then be used to show that for any equilateral contour Γ (inside the equilateral triangle), the contour integrals of H_j^δ for $j = 1, \tau, \tau^2$ are very closely related:

$$\int_\Gamma dz H_1^\delta(z) = \int_\Gamma dz H_\tau^\delta(z)/\tau + O(\delta^\varepsilon) = \int_\Gamma dz H_{\tau^2}^\delta(z)/\tau^2 + O(\delta^\varepsilon)$$

when $\delta \to 0$ for some $\varepsilon > 0$. To see this, one has to expand the contour integrals as the sum of all properly oriented contour integrals along all small triangles inside Γ. Then, the previous identities ensure that almost all terms cancel out. The remaining "boundary" terms are controlled with the help of RSW estimates.

This result then shows that for any subsequential limit $(H_1, H_\tau, H_{\tau^2})$, the contour integrals of H_1, H_τ/τ and of H_{τ^2}/τ^2 coincide. It readily follows that the contour integrals of the functions

$$H_j + \frac{i}{\sqrt{3}}(H_{j\tau} - H_{j\tau^2})$$

for $j = 1, \tau, \tau^2$ vanish. By Morera's theorem (see e.g. [1]), this ensures that these functions are analytic. In particular, H_1 is harmonic. The boundary conditions $H_j = 0$ on $A_{j\tau}A_{j\tau^2}$ for $j = 1, \tau, \tau^2$ then ensure that $H_1 = 0$ on A_2A_3 and that the horizontal derivative of H_1 on $A_1A_3 \cup A_2A_3$ vanishes. Also, $H_1(A_1) = 1$. The only harmonic function in the equilateral triangle with these boundary conditions is the height

$$H_1(z) = \frac{d(z, BC)}{d(A, BC)}.$$

This completes the proof of the Theorem when the domain is an equilateral triangle.

If D now any simply connected domain, and $a = a_1$, $o = a_\tau$, $c = a_{\tau^2}$ are boundary points, the proof is almost identical. In its first part, the only difference is that one replaces the straight boundaries $A_jA_{j\tau}$ by approximations of the boundary of D on the triangular lattice that is between the points $a_ja_{j\tau}$. In exactly the same way, one obtains tightness and boundary estimates for the discrete functions H_j^δ. Also, the argument leading to the fact that the contour integrals on equilateral triangles of $H_j + i(H_{j\tau} - H_{j\tau^2})/\sqrt{3}$ for any subsequential limit vanish, remains unchanged. Hence, for any subsequential limit, one obtains a triplet of functions $(H_1, H_\tau, H_{\tau^2})$ such that for $j = 1, \tau, \tau^2$:

- The function $H_j + i(H_{j\tau} - H_{j\tau^2})/\sqrt{3}$ is analytic
- The function $H_j(x)$ tends to zero when x approaches the part of the boundary between $a_{j\tau}$ and $a_{j\tau^2}$.
- The function $H_j(x)$ tends to one when $x \to a_j$.

The important feature is that this problem is conformally invariant: If Φ denotes a conformal map from D onto the equilateral triangle such that $\Phi(a_j) = A_j$, and if $(H_1, H_\tau, H_{\tau^2})$ is such a triplet of functions, then the triplet $(H_1 \circ \Phi^{-1}, H_\tau \circ \Phi^{-1}, H_{\tau^2} \circ \Phi^{-1})$ solves the same problem in the equilateral triangle. In the latter case, we have seen that the unique solution is given by $H_j(x) = d(x, A_{j\tau}A_{j\tau^2})/d(A_j, A_{j\tau}A_{j\tau^2})$. Hence, the Theorem follows. $\square$

One should stress that this proves much more than just the asymptotic behaviour of the crossing probabilities. It yields the asymptotic probability of the events $E_j(x)$ for x inside the domain D (and not only on its boundary).

10.3 Convergence to SLE_6 and consequences

One can use the previous result to prove that the discrete exploration process described in the introductory chapter indeed converges to chordal SLE_6.

The regularity estimates are provided by the RSW theory and the discrete Markovian property is immediate. It remains to show that some macroscopic quantities converge to a conformally invariant quantity in the scaling limit, but this is precisely what Smirnov's theorem shows. Hence, the method described in the previous chapter can be applied. Some adjustments are needed to take care of domains with rough boundary, though. In particular, one can use the a priori bounds on the probability of having 5 arms joining the vicinity of the origin to a large circle (the exponent α_5 below) derived in [69].

Exploiting this, one can therefore use the computations of critical exponents for SLE_6, to deduce asymptotic probabilities for discrete critical percolation on the triangular lattice: For instance [128, 92], let $A_n[N]$ denote the event that there exists n disjoint open clusters joining the vicinity of the origin to the circle of radius N. Then:

Theorem 10.2. *When $N \to \infty$, one has $\mathbf{P}[A_n[N]] \approx N^{-\alpha_n}$, where $\alpha_1 = 5/48$ and for all $n \geq 2$, $\alpha_n = (4n^2 - 1)/12$.*

Note that the exponents α_n for $n \geq 2$ are the same than the Brownian intersection exponents ξ_n in Chapter 8. This is not surprising because of the close relation between SLE_6 and planar Brownian motion. The exponent α_1 corresponds to the event that radial SLE_6 winds only "in one direction" around 0 (see [92].

Actually, Harry Kesten [68] had shown that the previous result (for $n = 1$ and $n = 2$) would imply the following description of the behaviour of percolation when the probability is near to the critical probability:

Theorem 10.3. *If one performs site percolation on the triangular lattice with probability p, then when $p \to 1/2+$, the probability that the origin belongs to the infinite cluster behaves like $(p - 1/2)^{5/36+o(1)}$. When $p \to 1/2-$, the correlation length explodes like $(1/2 - p)^{-4/3+o(1)}$.*

See [68, 128] for more results as well as for the proofs...

Let us conclude with the following combination of results that we have mentioned in these lectures: The following three curves are (locally) the same:

- The outer boundary of the scaling limit of a large critical percolation cluster.
- The outer boundary of a planar Brownian motion.

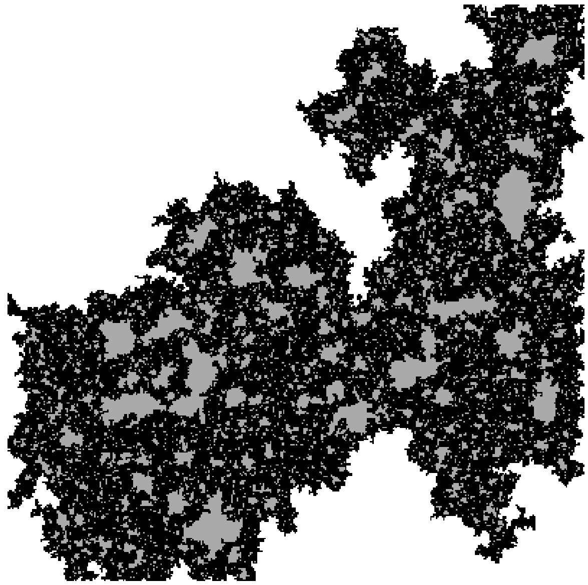

Fig. 10.3. Part of a (big) critical percolation cluster on the square lattice

Fig. 10.4. A critical percolation cluster on the triangular lattice

– The scaling limit of long self-avoiding walks, provided this scaling limit exists and is conformally invariant.

Bibliographical comments

The value of critical exponents for percolation had been predicted by theoretical physicists [38, 112, 109, 110, 122, 121, 54, 33, 7]. The conformal invariance conjecture for critical percolation had been discussed by Aizenman [3, 4].

Smirnov's complete detailed proof of Cardy's formula is contained in [126, 127]. The actual detailed proof of the convergence of the discrete exploration process to SLE_6 (announced in [126]) should be written up [127] soon. For the derivation of formulas and exponents for critical percolation using SLE_6, see [124, 92, 128].

11 What is missing

11.1 A list of ideas

We have listed at the end of each chapter a list of references to papers that develop ideas that are related to those presented in the corresponding chapter. One aspect of SLE that we could have spent more time on is the actual computation of critical exponents. For simplicity, we have shown how to derive the Brownian exponents using radial SLE_6, but in general (for instance to derive the Hausdorff dimension of the SLE), one might as well work with chordal SLE. Various exponents are derived in for instance in [86, 87, 88, 92, 118, 14, 15].

Before very briefly reviewing the results related to restriction properties, we would like to stress that the important ideas underlying Rohde-Schramm's [118] proof of the existence and transience of the SLE paths have not been presented in these lectures. The arguments [118] require some non-trivial background in complex geometry. In two cases, the existence and/or transience of the SLE path is especially difficult to establish: For $\kappa = 4$, because the domains generated by the SLE curve are not Hölder (see [118]). For $\kappa = 8$, the only proof uses the fact that it is the scaling limit of the discrete uniform Peano curves [93] described in Chapter 9.

One can also study geometric questions such as: Does the SLE have (local) cut points? The answer is positive if and only if $\kappa < 8$ (see [14]).

I plan to discuss the following restriction properties in forthcoming lecture notes. The main reference is the long recent paper [95].

- The full classification of the measures satisfying the restriction properties is one of the main goals of [95]. These measures form a one-dimensional family indexed by a positive real-valued parameter N, that can be interpreted as the number of Brownian excursions that the measure is equivalent to. There exist two other important ways to describe this one-dimensional family: The first one is via a variant of the $SLE_{8/3}$ process called $SLE(8/3, \rho)$. Loosely speaking, one replaces the driving Brownian motion by a Bessel process (see [95] for all this), and the obtained simple random curve describes the outer boundary of the set satisfying the restriction property. The second description goes as follows: Consider an SLE_κ with $\kappa \leq 8/3$ and add to this path a certain cloud Brownian loops (this Poisson cloud of loops is also studied in [98]). For a well-tuned density $d(\kappa)$ of the loops, one constructs the restriction measure corresponding to $N(\kappa)$ Brownian excursions. See also [40].

– This last description makes it possible to tie a link [52, 53] with representation theory, and more precisely with highest-weight representations of the Lie Algebra of polynomial vector fields on the unit circle (the number $N(\kappa)$ is the highest-weight). This is related to considerations from conformal field theory. See also [10, 11, 12] for the relation of SLE with ideas from conformal field theory.
– The $SLE(\kappa, \rho)$ processes shed also some light on the computation of the (chordal) critical exponents. It turns out that they can be understood via the absolute continuity relations between Bessel processes (following from Girsanov's Theorem); see [135].

11.2 A list of open problems

Here is a list of open problems. Some of these were already mentioned in the previous chapters:

11.2.1 Conformal invariance of discrete models

So far, convergence of natural discrete models towards SLE in the scaling limit has been proved only in the two very special cases that we described in the last two chapters (LERW-UST, and critical site percolation on the triangular lattice). It is believed to hold for many other models:

– The interface for a critical FK-percolation (see e.g. [56] for an introduction to this dependent percolation model introduced by Fortuin and Kasteleyn) model for $q \leq 4$ is conjectured to converge to chordal SLE_κ. Recall that the probability of a given realization is proportional to

$$p^{\#\text{open edges}}(1-p)^{\#\text{closed edges}}q^{\#\text{connected components}}.$$

The relation between q and κ should be

$$\cos\frac{4\pi}{\kappa} = -\frac{\sqrt{q}}{2},$$

where $q \in [0,4]$ and $\kappa \in [4,8]$. Here (as in the UST case and in some sense in the percolation case), the boundary conditions have to be mixed (free on one part of the boundary, wired on the other – this influences the way of counting the connected components). See [118] for a more precise statement of this conjecture. Recall that for critical FK-percolation with parameter q on the square lattice, the self-dual point is $p = \sqrt{q}/(\sqrt{q}+1)$ (proving that this self-dual point is the critical point is another open question, but it is not directly related to the SLE question; the question on the square grid is to prove that for this value of p, the interface converges to SLE). Here self-dual means that the law of the dual graph of an FK percolation sample is also an FK percolation sample (in the dual lattice) with the same parameters (see [56]).

Recall that when $q > 4$, the FK percolation phase transition is conjectured to be a first-order transition (i.e. there can exist an infinite open cluster at the critical probability). The critical value $q = 4$ corresponds to the special case $\kappa = 4$. Recall also (see e.g. [56]) that the correlation functions of the critical q-Potts models are the same as those of the critical FK-percolation model. Recall also that the usual percolation is the $q = 1$ FK percolation model, and that the UST can be viewed as the $q = 0$ critical FK percolation model (see e.g. [57]). For the critical FK percolation models, the Markovian property is clearly valid in the discrete case. The missing step is therefore the proof of conformal invariance.

It is interesting (and encouraging) to note that the integer values of q correspond to the "nice" values of the angles $\alpha = \pi(1 - \kappa/4)$ of the isocele triangles for which hitting distributions are uniform (Dubédat's observations [39] mentioned at the end of Chapter 2): $\cos\alpha = \sqrt{q}/2$. For $q = 1$, it is the equilateral triangle, for $q = 2$ (Ising), it is the isocele-rectangular triangle, and for $q = 3$, $\alpha = \pi/6$.

- Among all the critical percolation interfaces that are conjectured to converge to SLE_6 (this is the special case $q = 1$ in the previous conjectures), it is worth stressing two cases, for which one has self-duality (and therefore some little hope to be able to prove something): The first one is bond-percolation on the square grid, and the second one is percolation on a Voronoi tessellation (see e.g. [21]).
- There exists a special model for which (as for the Ising model and for the uniform spanning tree model), the tools and arguments developed by Kenyon seem promising: It is the so-called double-domino path, that is conjectured to converge to the special curve SLE_4 in the scaling limit.
- Note also that the Ising model itself (on the triangular lattice) has some self-duality properties (this is due to the fact that for the Ising model, there are exactly two possible states for each site). Hence, Ising cluster interfaces (for appropriate boundary conditions, and on the triangular lattice) might converge to an SLE in the scaling limit.
- For $\kappa < 4$, the relation with discrete models from statistical physics is not so clear. One relation is via the duality conjectures that we will discuss below. The main open question is the convergence of the self-avoiding walk towards the $SLE_{8/3}$ curve. Again, the main problem is to derive its conformal invariance. See [94] for a discussion. Let us insist that basically nothing is known rigorously on the asymptotic behaviour of the self-avoiding walk. For instance, to our knowledge, it has not even been disproved that the curve becomes space-filling or a straight line in the scaling limit!

– It is likely that some discrete dynamic models can be shown to converge to SLE (but their relation to models from statistical physics is unclear). For instance, variations on the Laplacian random walk description of LERW that have some conformally invariant features built in the model should in principle converge to SLEs.

11.2.2 Duality

Another approach to the SLE curves when $\kappa < 4$ goes as follows: It was conjectured (based on the computation of the dimensions) that in the scaling limit, the outer boundary of an $SLE_{\kappa'}$ hull for $\kappa' > 4$ at a given time looks (locally) like an $SLE_{16/\kappa'}$ curve. Hence, the SLE_κ curves for $\kappa < 4$ correspond to the outer boundary of the scaling limit of critical FK-percolation clusters. The duality has been proved to hold in two cases: $\kappa = 2$ (because of the relation between LERW and UST that respectively converge to SLE_2 and SLE_8) and $\kappa = 8/3$ (because of the restriction property considerations that allow to describe the outer boundary of conditioned SLE_6 processes in terms of $SLE_{8/3}$ processes (see [95]). In the general case, a weak form of duality has been identified by Dubédat [40], that leads to conjecture a precise identity in law between the outer boundary of an $SLE(\kappa', \rho_{\kappa'})$ process and the $SLE(16/\kappa', \rho'_{\kappa'})$ curve for well-chosen values of ρ and ρ'.

Proving this duality relation would be one way to settle the following open problem (it is only proved when $\kappa = 6$ and $\kappa = 8$): Prove that the Hausdorff dimension of the boundary of K_t is almost surely $1 + 2/\kappa$ when K_t is the hull of an SLE_κ (chordal or radial) for $\kappa > 4$. One would then combine duality with the computation of the dimension of the SLE curves in [15]. There should however also exist a direct proof of this fact that does not rely on duality.

11.2.3 Reversibility

The following conjecture follows very naturally from the fact that the SLEs are believed to be scaling limit of the previously described lattice models: Suppose that $\kappa \le 8$ is given, and consider the chordal SLE_κ curve γ from a to b in a domain D (where a and b are two boundary points). One can time-reverse γ, and view it as a curve from b to a in D. Then, the law of this time-reversal should be (modulo time-change) the law of an SLE_κ curve from b to a in D. Another equivalent way of phrasing this is that if γ is the chordal SLE path in the upper half-plane, the path $-1/\gamma$ has the same law as γ (modulo time-change).

This conjecture is very natural in terms of the lattice models, but on the other hand, it is not natural at all if one thinks of the actual definition of the SLE in terms of the Loewner chain (this is very non-reversible!). In the special cases $\kappa = 6$, $\kappa = 8$ and $\kappa = 2$, the result is a consequence of the convergence of the discrete reversible models to the SLEs. So far, the reversibility of $\kappa = 8/3$ is the only one that can be proved without reference to a reversible discrete model, and the tool here is the characterization of $SLE_{8/3}$ as the unique simple random curve that satisfies the restriction property. In all other cases, the problem is to our knowledge open. This problem does not seem as out of reach as some of those that we just discussed.

Note that (as shown to me by Oded Schramm), it is possible to show that reversibility of SLE_κ fails to be true when $\kappa > 8$. This can seem surprising; more generally, the interpretation of SLE_κ when $\kappa > 8$ in terms of models from statistical physics is not well-understood. Note that the asymptotic behaviour of SLE_κ when $\kappa \to \infty$ is studied in [16].

11.2.4 Quantum gravity and conformal field theory

The arguments developed in conformal field theory under the name of quantum gravity suggest that some very interesting critical phenomena also occur for systems on certain random lattices. In particular, Duplantier [44, 46, 47] showed that the value of the critical exponents in the plane (those exponents that can now be understood thanks to the SLE) can be predicted using the formula proposed by Knizhnik, Polyakov and Zamolodchikov in [70], that should relate the value of the critical exponents in the plane to the corresponding exponents on random lattices.

Recent progress has been made in the rigorous understanding of some of these random systems on these random graphs; see e.g. [9, 8, 25, 26] and the references therein. It seems that (as opposed to the rigid lattice case), the behaviour of some of these systems on random lattices might be accessible by ingenious combinatorial methods.

Note [135] that the KPZ formula seems to have a simple interpretation in terms of the ρ in the $SLE(\kappa, \rho)$ processes. Maybe the combination of the determination of the exponents for SLE, and the results on random graphs will provide in the end the rigorous justification to the KPZ relation.

More generally, the relation between SLE and conformal field theory (that has started to be investigated in [10, 11, 12, 51, 52, 53]) and with the mathematical concepts used in conformal field theory needs further understanding. It is not so clear whether this will be helpful to improve the knowledge on these critical two-dimensional systems (which was after all probably the initial motivation for the conformal field framework). One related issue is to manage to define SLE on general Riemann surfaces, see [51, 137, 42].

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