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The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets

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Dedicated to Professor John W. Milnor on the occasion of his sixtieth birthday

Abstract

It is shown that the boundary of the Mandelbrot set M has Hausdorff dimension two and that for a generic $c \in \partial M$, the Julia set of $z \mapsto z^2 + c$ also has Hausdorff dimension two. The proof is based on the study of the bifurcation of parabolic periodic points.

Introduction

The dynamics of complex quadratic polynomials $P_c(z) = z^2 + c$ has been studied extensively in recent years (e.g., see [DH]). The main interest in this subject is the nature of the Julia sets J_c in the dynamical plane and the Mandelbrot set M in the parameter space. The boundary of M also has a meaning as "the locus of bifurcation", or more precisely (by Mañé-Sad-Sullivan [MSS] or by Lyubich [Ly1]) $\partial M = \{c \in \mathbb{C} \mid P_c \text{ is not J-stable}\}$ (see §1 for the definition). In this paper, we are concerned with the Hausdorff dimension (denoted by H-dim$(\cdot)$) of these sets. Some of the consequences are:

THEOREM A.

$$\text{H-dim}(\partial M) = 2.$$

Moreover for any open set U which intersects ∂M, $\text{H-dim}(\partial M \cap U) = 2$.

THEOREM B. *For a generic $c \in \partial M$,*

$$\text{H-dim}\, J_c = 2.$$

In other words, there exists a residual (hence dense) subset $\mathcal{R}$ of ∂M such that if $c \in \mathcal{R}$, then $\text{H-dim}\, J_c = 2$.

*This paper was originally written in 1991.

THEOREM C. *There exists a residual set $\mathcal{R}'$ of $\mathbb{R}/\mathbb{Z}$ such that if P_c has a periodic point with multiplier* $\exp(2\pi i\alpha)$ *with* $\alpha \in \mathcal{R}'$, *then* H-dim $J_c = 2$.

Theorem A was conjectured by many people (for example, Mandelbrot [Ma], Milnor [Mi2, Conjecture 1.7]). It means that the bifurcation locus is large in dimension, and this explains the complexity of M, demonstrated by many numerical experiments. As to the Julia sets, if P_c is hyperbolic, or if 0 is strictly preperiodic, H-dim J_c is less than 2 (see §1, Property (1.4)). However, it was conjectured that there exists a sequence of parameters such that H-dim J_c tends to 2. Theorem B gives a stronger solution to this conjecture. We will see that the method in this paper applies to other families under certain condition.

The above theorems are obtained as consequences of Theorems 1 and 2 stated in Section 1. Theorem 1 amounts to comparing the Hausdorff dimension of the set of J-unstable parameters with that of a certain subset ("hyperbolic subset") of the Julia set. It reflects the similarity between the Mandelbrot set and some Julia sets (cf. Tan Lei [T]). Theorem 2 is the most important result in this paper, and it assures that one can obtain maps whose Julia sets have Hausdorff dimension (or "hyperbolic dimension") arbitrarily close to 2, from "the secondary bifurcation" of a parabolic periodic point. See Figures 1, 2 and the remark after Theorem 2 in Section 2.

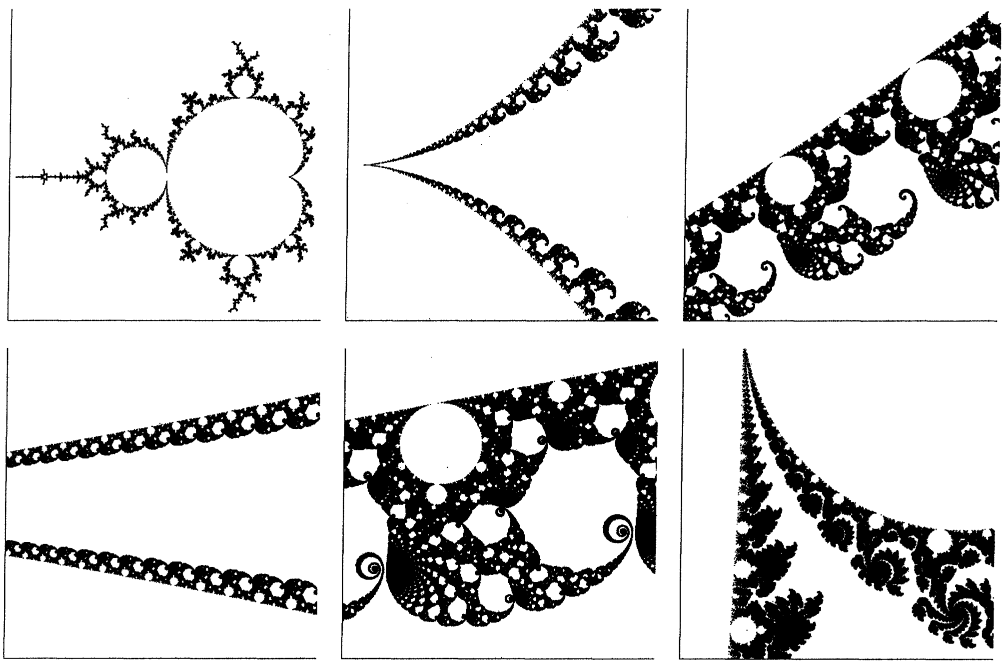

FIGURE 1. The boundary of the Mandelbrot set (top left) and its blow-ups near the cusp $c = \frac{1}{4}$.

It has been observed that after a small perturbation of a parabolic periodic point, the Julia set may inflate drastically. In fact, the proof of Theorem

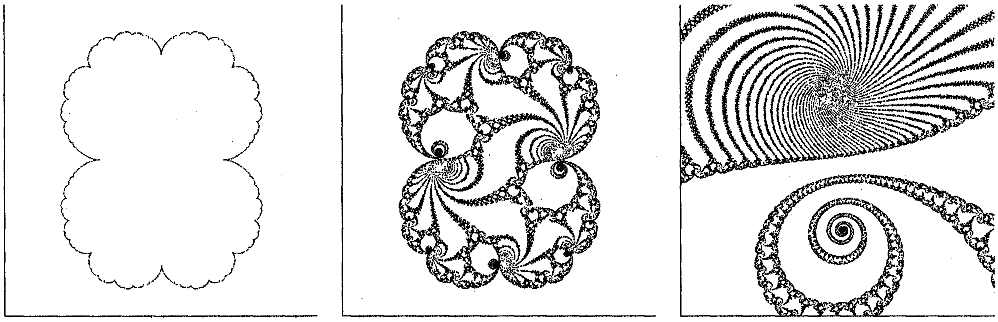

FIGURE 2. The Julia sets for $c = \frac{1}{4}$ (left) and for $c = 0.25393 + 0.00048i$ (middle) and its blow-up near the fixed points (right).

2 shows that such an inflated part of the Julia set can have Hausdorff dimension close to 2. The main tool in the study of such a bifurcation is the theory of Ecalle cylinders, which was introduced by Douady-Hubbard [DH] and developed by Lavaurs [L]. Using the Ecalle cylinder, we introduce a new renormalization procedure associated with parabolic fixed points (see Remark (ii) in §6). Our result can be interpreted as follows: The renormalization induces a map between old and new dynamical planes, which resembles an exponential map. Comparing this observation with McMullen's result [Mc] which claims that the Julia set of an exponential map always has Hausdorff dimension 2, we can conceive that a certain subset of the Julia set can have Hausdorff dimension close to 2. The proof in this paper will justify this argument, or even more, twice renormalization is enough to attain dimension two.

One can compare the above theorems with Jakobson's result for the family of unimodal interval maps [J], M. Rees' result for a certain family of rational maps [Re] and Benedicks-Carleson's result on the family of Hénon maps [BC]. These results show the existence of a "chaotic dynamics" for a set of positive Lebesgue measure of parameters. For example, M. Rees' result shows that there exists a set of positive Lebesgue measure of parameters for which the Julia sets are the whole Riemann sphere. Such parameters are found near a special parameter for which all critical points are strictly preperiodic. For this parameter, the map has good ergodic theoretical properties and the Julia set is the whole sphere.

On the other hand, for a polynomial P_c acting on $\mathbb{C}$, the Julia set J_c or the filled-in Julia set K_c can never be the entire plane, since there is always the basin of ∞. So there always exist some orbits which escape to ∞ from the neighborhood of K_c. Moreover, as remarked above, if the critical point 0 is strictly preperiodic, H-dim $J_c < 2$. For example if $c = -2$, $J_{-2} = [-2, 2] \subset \mathbb{R}$ and H-dim $J_{-2} = 1$. Therefore one can hardly expect an analogous result or approach for the family P_c as for those of Jakobson and Rees. Instead, in this paper, we use the perturbation of parabolic periodic points.

As for the area (the 2-dimensional Lebesgue measure), it is conjectured that ∂M and J_c (for any c) have area zero. There are partial results: the set of parameters in ∂M for which P_c are not infinitely renormalizable has area zero [Sh1]. If P_c has no irrationally indifferent periodic point and is not infinitely renormalizable, then the Julia set J_c has area zero [Ly2] and [Sh1].

This paper is organized as follows: In Section 1, we define the notion of hyperbolic subsets and the hyperbolic dimension, state our main results (Theorems 1 and 2, Corollary 3). Assuming these results, we prove Theorems A, B and C. In Section 2 we prove basic properties of hyperbolic subsets and hyperbolic dimension. Theorem 1 is proved in Section 3, using holomorphic motions. The rest of the paper is devoted to the proof of Theorem 2. The theory of the parabolic bifurcation and Ecalle cylinders is reviewed in Section 4. Further properties of the Ecalle transformation are studied in Section 5. After these preparations, Theorem 2 is proved in Section 6 (the case with multiplier 1) and in Section 7 (the other cases). In the appendix, we give proofs of the facts stated in Sections 4 and 7.

Acknowledgement. I would like especially to thank Curt McMullen for introducing me to these problems and for having inspired me throughout their investigation; A. Douady for his lectures which introduced me to the theory of Ecalle cylinders; and also A. Hinkkanen, M. Lyubich, M. Rees, D. Sullivan, Tan Lei, S. Ushiki and J. Milnor for helpful discussions and comments. This paper was written during my visit to the Institute for Mathematical Sciences, State University of New York at Stony Brook, to which I am grateful for its hospitality. I also would like to thank the editors for their patience with my slow revision of this paper.

Computer pictures have been produced using J. Milnor's program.

1. Some definitions and main results

In this section, we define the hyperbolic subset and the hyperbolic dimension for a rational map, and state our main results, Theorems 1 and 2. Assuming these results, we give the proofs of Theorems A, B and C from the introduction.

Definitions. Let f be a rational map. A closed subset X of $\overline{\mathbb{C}}$ is called a *hyperbolic subset* for f, if $f(X) \subset X$ and there exist positive constants c and $\kappa > 1$ such that

$$\|(f^n)'\| \geq c\kappa^n \text{ on } X \quad \text{for } n \geq 0,$$

where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric of $\overline{\mathbb{C}}$. (A similar notion was discussed in [Ru] and [Ly1]. The condition requires that f be expanding on X; however, we call such a set "hyperbolic" following the standard terminology for dynamical systems.)

The *hyperbolic dimension* of f is

$$\text{hyp-dim}(f) = \sup\{\text{H-dim}(X) \mid X \text{ is a hyperbolic subset of } f\}.$$

PROPERTIES OF HYPERBOLIC SUBSETS AND HYPERBOLIC DIMENSION. *Let X be a hyperbolic subset of f.*

(1.0) *There are no critical points of f in X.*

(1.1) *X is a subset of the Julia set $J(f)$ of f. Hence*

$$\text{H-dim}\, J(f) \geq \text{hyp-dim}(f).$$

(1.2) *X is "stable" under a perturbation, i.e. there exists a neighborhood $\mathcal{N}$ of f in the space of rational maps of the same degree, such that if $g \in \mathcal{N}$ then g has a hyperbolic subset X_g and there is a homeomorphism ι_g: $X \to X_g$ which conjugates f to g. Moreover, for each $z \in X$, $\iota_g(z)$ is a complex analytic function in g, and $\iota_f = \text{id}_X$. ($\{\iota_g\}$ is a holomorphic motion in the sense of Section* 3.)

(1.3) *$f \mapsto \text{hyp-dim}(f)$ is lower semi-continuous, or equivalently, for any number k, the set $\{f \mid \text{hyp-dim}(f) > k\}$ is open.*

(1.4) *Suppose that f is a hyperbolic rational map (i.e., all critical points are attracted to attracting periodic orbits) or a subhyperbolic polynomial (every critical point is either attracted to an attracting periodic orbit or a pre-periodic orbit; see* [DH]). *Then*

$$\text{H-dim}\, J(f) = \text{hyp-dim}(f).$$

Moreover $J(f)$ has positive and finite δ-dimensional Hausdorff measure if $\delta =$ $\text{H-dim}\, J(f)$, and $\text{H-dim}\, J(f) < 2$.

For the proof and remarks, see Section 2.

Problem. When does $\text{hyp-dim}(f)$ coincide with $\text{H-dim}\, J(f)$?

Definition. A family $\{f_\lambda \mid \lambda \in \Lambda\}$ of rational maps is *J-stable at* $\lambda_0 \in \Lambda$, if there exists a continuous map h: $\Lambda' \times J(f_{\lambda_0}) \to \overline{\mathbb{C}}$, such that Λ' is a neighborhood of λ_0 in Λ, $h_\lambda \equiv h(\lambda, \cdot)$ is a conjugacy from $(J(f_{\lambda_0}), f_{\lambda_0})$ to $(J(f_\lambda), f_\lambda)$ and $h_{\lambda_0} = \text{id}_{J(f_{\lambda_0})}$. We also say that f_{λ_0} is J-stable in this family, if there is no confusion.

THEOREM 1. *Let $\{f_\lambda \mid \lambda \in \Lambda\}$ be a complex analytic family of rational maps of degree d (> 1), where Λ is an open set in $\mathbb{C}$. Suppose f_{λ_0} ($\lambda_0 \in \Lambda$) is not J-stable in this family. Then*

$$\text{H-dim}\left\{\lambda \in \Lambda \;\middle|\; \begin{array}{l} f_\lambda \text{ is not J-stable and has a hyperbolic} \\ \text{subset containing a forward orbit of} \\ \text{a critical point} \end{array}\right\} \geq \text{hyp-dim}(f_{\lambda_0}).$$

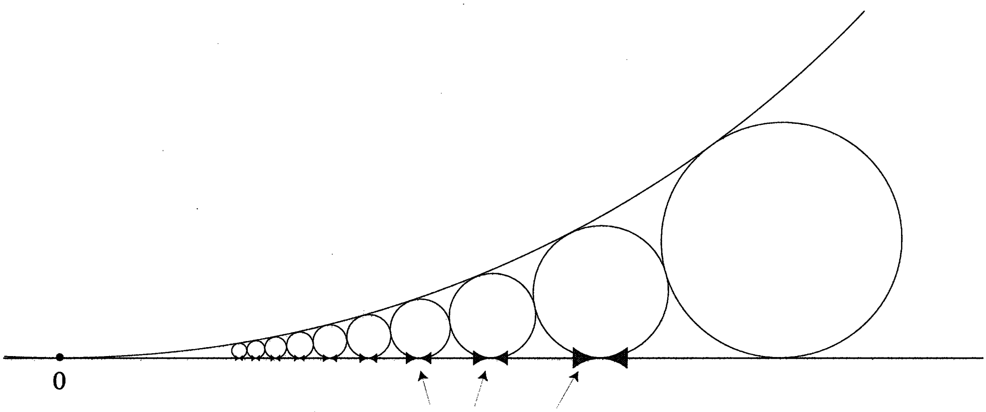

FIGURE 3. The region for α

The proof will be given in Section 3. There we will make use of a "similarity" between a hyperbolic subset and a subset of the set on the left-hand side of the above inequality.

Definition. A periodic point is called *parabolic* if its multiplier is a root of unity. The *parabolic basin* of a parabolic periodic point ζ of period k is

$$\{z \in \overline{\mathbb{C}} \mid f^{nk} \to \zeta \ (n \to \infty) \ \text{ in a neighborhood of } z\},$$

and the *immediate parabolic basin* of ζ is the union of periodic connected components of the parabolic basin.

THEOREM 2. *Suppose that a rational map f_0 of degree d (> 1) has a parabolic fixed point ζ with multiplier* $\exp(2\pi i p/q)$ $(p,q \in \mathbb{Z},\ (p,q) = 1)$ *and that the immediate parabolic basin of ζ contains only one critical point of f_0. Then: For any $\varepsilon > 0$ and $b > 0$, there exist a neighborhood $\mathcal{N}$ of f_0 in the space of rational maps of degree d, a neighborhood V of ζ in $\overline{\mathbb{C}}$, positive integers N_1 and N_2 such that if $f \in \mathcal{N}$, and if f has a fixed point in V with multiplier* $\exp(2\pi i\alpha)$, *where*

$$q\alpha = p \pm \cfrac{1}{a_1 \pm \frac{1}{a_2+\beta}}$$

with integers $a_1 \geq N_1$, $a_2 \geq N_2$ and $\beta \in \mathbb{C}$, $0 \leq \operatorname{Re}\beta < 1, |\operatorname{Im}\beta| \leq b$, then

$$\text{hyp-dim}(f) > 2 - \varepsilon.$$

The proof will be given in Sections 6 and 7, after preparation in Sections 4 and 5.

Figure 3 shows the region for α described in Theorem 2.

The condition $a_1 \geq N_1$ is in fact unnecessary, since a_1 must be large when f is close to f_0. However if we take a family $\{f_\alpha\}$ such that $f_\alpha(0) = 0$,

$f'_\alpha(0) = \exp(2\pi i\alpha)$, then the condition in Theorem 2 can be expessed in terms of α (without $\mathcal{N}$ and V).

The condition on α has a meaning that f is obtained by a "secondary bifurcation", in the following sense: The primary bifurcation of f_0 produces a sequence of maps having parabolic fixed points with multipliers $\exp(2\pi i\alpha_n)$, where $q\alpha_n = p \pm \frac{1}{n}$ $(n = 1, 2, \ldots)$. Then the bifurcation from these maps gives rise to the maps as in Theorem 2.

There are immediate consequences of Theorems 1 and 2 for the family P_c. Before stating them, let us recall the definition of J_c and M:

$$\begin{aligned} K_c &= \{z \in \mathbb{C} \mid P_c^n(z) \not\to \infty \text{ as } n \to \infty\} \quad (\textit{the filled-in Julia set of } P_c), \\ J_c &= \partial K_c = \text{Closure of } \{\text{repelling periodic points of } P_c\} \\ &\quad\ (\textit{the Julia set of } P_c), \\ M &= \{c \in \mathbb{C} \mid 0 \in K_c\} = \{c \in \mathbb{C} \mid K_c \text{ is connected}\} \quad (\textit{the Mandelbrot set}). \end{aligned}$$

COROLLARY 3. (i) *If U is an open set containing $c \in \partial M$, then*

$$\begin{aligned} \text{H-dim}(\partial M \cap U) &\geq \text{H-dim}\{c \in \partial M \cap U \mid 0 \textit{ is non-recurrent under } P_c\} \\ &\geq \text{hyp-dim}(P_c). \end{aligned}$$

(ii) *If P_c has a parabolic periodic point, then there exists a sequence $\{c_n\}$ in ∂M such that $c_n \to c$ and* $\text{hyp-dim}(P_{c_n}) \to 2$, *as $n \to \infty$.*

Proof of Corollary 3. (i) This is immediate from Theorem 1 and the fact that $c \in \partial M$ if and only if P_c is J-unstable ([MSS],[Ly1]).

(ii) It is known that if P_c has a parabolic periodic point ζ of order k, then $f_0 = P_c^k$ satisfies the hypothesis of Theorem 2, since there is only one critical point (in $\mathbb{C}$) for P_c. Any parabolic periodic point of P_c is not persistent (otherwise all P_c would have parabolic periodic points), and it can be perturbed into a periodic point whose multiplier is as in Theorem 2. If $\operatorname{Im}\beta = 0$, then the new periodic point is indifferent, and the perturbed polynomial is also J-unstable. One can thus obtain the sequence $\{c_n\}$ $(\subset \partial M)$. Furthermore, it is also possible to choose c_n such that 0 is strictly preperiodic under P_{c_n}, since such parameters are dense in ∂M by [MSS]. □

Now we can prove Theorems A, B and C.

Proof of Theorem A. By Mañé-Sad-Sullivan [MSS] or Lyubich [Ly1], parameters for which P_c has (non-persistent) parabolic periodic points are dense in ∂M. The assertion follows immediately from Corollary 3. □

Proof of Theorem B. Let $\mathcal{R}_n = \{c \in \partial M \mid \text{hyp-dim}(P_c) > 2 - \frac{1}{n}\}$ $(n = 1, 2 \cdots)$. Then $\mathcal{R} = \bigcap_{n \geq 0} \mathcal{R}_n = \{c \in \partial M \mid \text{hyp-dim}(P_c) = 2\} \subset \{c \in \partial M \mid \text{H-dim } J(P_c) = 2\}$. By Property (1.3), the $\mathbb{R}_n$ are open in ∂M. Moreover $\mathbb{R}$ are dense in ∂M, by Corollary 3(ii) and the above remark (the density of parabolic parameters in ∂M). Hence $\mathcal{R}$ is residual. (A residual set is a set containing the intersection of a countable collection of open dense subsets of ∂M.) Such an $\mathcal{R}$ is dense in ∂M by Baire's theorem. □

Proof of Theorem C. Let W be a hyperbolic component, i.e. a connected component of the set of c's such that P_c has an attracting periodic point. Then as shown by Douady-Hubbard [DH], there exists a homeomorphism from $\overline{W}$ onto the closed unit disc (conformal in W) defined by the multiplier of a non-repelling periodic orbit. So in ∂W, the parameters with parabolic periodic points are dense. By the proof of Corollary 3(ii) and a similar argument to the proof of Theorem B, we can prove that for generic $\alpha \in \mathbb{R}/\mathbb{Z}$, if $c \in \partial W$ and P_c has a periodic point with multiplier $\exp(2\pi i\alpha)$, then H-dim $J_c = 2$. However, there are only countably many hyperbolic components. Hence the assertion follows. □

Remark 1.1. (i) It also follows easily from Theorem 2 that

$$\sup_{c\in W_0} \text{H-dim}(J_c) = \sup_{c\in\mathbb{C}\setminus M} \text{H-dim}(J_c) = 2 ,$$

where $W_0 = \{c \mid P_c \text{ has an attracting fixed point}\}$ (*the cardioid*).

(ii) It is easy to see that a similar result holds for other families of rational maps which have "only one critical point" that can be involved in the parabolic basin. For example, $f_a(z) = z^3 + az^2$, $g_b(z) = (z^2 + b)/(z^2 - 1)$.

(iii) There are immediate consequences on the continuity of functions $c \mapsto \text{H-dim}(J_c)$ and $c \mapsto \text{hyp-dim}(P_c)$. If $c_0 \in \mathbb{C} - \partial M$, then both functions are continuous at $c = c_0$ because of J-stability (see Section 3). Now suppose $c_0 \in \partial M$. If $\text{hyp-dim}(P_{c_0}) = 2$, then $\text{H-dim}(J_{c_0}) = 2$ and both dimension functions are still continuous at $c = c_0$ by (1.1) and (1.3). If $\text{H-dim}(J_{c_0}) < 2$, then $\text{hyp-dim}(P_{c_0}) < 2$ by (1.1) and both dimension functions are *discontinuous* at $c = c_0$ by the proof of Theorem B. If $\text{H-dim}(J_{c_0}) = 2$ and $\text{hyp-dim}(P_{c_0}) < 2$, then the hyperbolic dimension is *discontinuous* at $c = c_0$ again by the proof of Theorem B, but the continuity of the Hausdorff dimension is not known. In fact, it is not known whether this case can happen.

2. Hyperbolic subsets and hyperbolic dimension

In this section, we give proof of Properties (1.0)–(1.3). We also give an example of the hyperbolic subset and an estimate of its Hausdorff dimension, which will be used later.

Proof of the properties of hyperbolic subsets and hyperbolic dimension. (1.0) is obvious.

(1.1) The family $\{f^n\}$ cannot be normal in any open set intersecting X, since the derivatives grow exponentially. Hence the assertion follows.

(1.2) This is a well-known fact. The outline of the proof is as follows. There exists a neighborhood V of X such that for g near f, g is expanding on V. Hence for $x \in X$, the orbit $\{f^n(x)\}_{n=0}^{\infty}$ with respect to f is a pseudo-orbit for g, which can be traced by an orbit (a real orbit) of a point y for g (the Pseudo-Orbit Tracing Property; see [Bo1]). Then let $y = \iota_g(x)$. In fact, y can be expressed as

$$y = \lim_{n\to\infty} g_{x_0}^{-1} \circ g_{x_1}^{-1} \circ \cdots \circ g_{x_n}^{-1}(x_{n+1}),$$

where $x_j = f^j(x)$ and g_z^{-1} is the branch of g^{-1} defined in a neighborhood of $f(z)$ such that $g_z^{-1}(f(z))$ is near z. By the expanding property for g near f, ι_g is well-defined and conjugates $f|_X$ to $g|_{X_g}$ with $X_g = \iota_g(X)$. Moreover, $\iota_g(x)$ depends analytically on g, since the convergence in the above is uniform. (In [Ly1], the analyticity is proved under the assumption that periodic points are dense in X, which is in fact unnecessary by the above.)

(1.3) Let $\mathcal{N}$, X_g and ι_g be as in (1.2). (Suppose $\mathcal{N}$ is open.) It is enough to prove that $\mathcal{N} \ni g \mapsto \text{H-dim}\, X_g$ is continuous. We will prove this fact in Section 3, using a result on holomorphic motions. It is also possible to prove it directly. In fact, one can estimate the exponent of the Hölder continuity of ι_g, in the proof of (1.2).

(1.4) We do not use this fact for the proof of our main theorems. If f is hyperbolic, then $J(f)$ itself is a hyperbolic subset; hence $\text{H-dim}\, J(f) = \text{hyp-dim}(f)$; the second assertion follows from Bowen's formula ([Bo], [Ru]), and the fact that $\text{H-dim}\, J(f) < 2$ can be shown by a standard argument using the expanding property of f and Lebesgue's density theorem. (See [Su].) The case with preperiodic critical points will be discussed in another paper, but the fact that $\text{H-dim}\, J(f) < 2$ seems already to be known. $\square$

In the proof of Theorem 2 (§6), we will construct special kinds of hyperbolic subsets which are described as follows.

Suppose that U is a simply connected open set of $\overline{\mathbb{C}}$ (with $\sharp(\overline{\mathbb{C}} - U) \geq 2$); $U_1, \ldots, U_N$ are disjoint simply connected open subsets of U with $\overline{U}_i \subset U$; $n_1, \ldots, n_N$ are positive integers such that f^{n_i} maps U_i onto U bijectively ($i = 1, \ldots, N$). It follows from Schwarz' lemma that $\tau_i \equiv (f^{n_i}|_{U_i})^{-1}\colon U \to U_i$ is a contraction with respect to the Poincaré metric of U (at least on $\cup_i \overline{U}_i$). So there exists a Cantor set X_0 generated by $\{\tau_i\}$; that is, X_0 is the minimal nonempty closed set satisfying

$$X_0 = \tau_1(X_0) \cup \ldots \cup \tau_N(X_0).$$

LEMMA 2.1. *The set* $X = X_0 \cup f(X_0) \cup \ldots \cup f^{M-1}(X_0)$ (*where* $M = \max n_i$) *is a hyperbolic subset of* f.

Proof. Obviously X is closed. We have $f(X) \subset X$, since $f(f^{M-1}(X_0)) = f^M(\tau_1(X_0) \cup \cdots \cup \tau_N(X_0)) = f^{M-n_1}(X_0) \cup \cdots \cup f^{M-n_N}(X_0) \subset X$.

Note that $f^{n_i}|_{U_i}$ is expanding on $X_0 \cap U_i$ with respect to the Poincaré metric of U which is equivalent to the spherical metric on X_0. In order to see that f is expanding on X, it suffices to factorize the iterate f^n at z ($\in X$) to

$$f^{j_1} \circ (f^{n_{i_1}}|_{U_{i_1}}) \circ \cdots \circ (f^{n_{i_k}}|_{U_{i_k}}) \circ f^{j_2}$$

where $i_1, \dots, i_k \in \{1, \dots, N\}$ and $0 \leq j_1, j_2 < M$ are determined by $z' = f^{j_2}(z) \in X_0$, $z' \in \tau_{i_k} \circ \cdots \circ \tau_{i_1}(X_0)$ and $n = j_1 + n_{i_1} + \cdots + n_{i_k} + j_2$. □

We only use the simplest estimate for the Hausdorff dimension of X_0. Suppose $\infty \notin U$.

LEMMA 2.2. *Let $\delta = \text{H-dim}\, X_0$. Then*

$$1 \geq \sum_{i=1}^{N} \inf_U |\tau_i'|^\delta \geq N \cdot \left(\max_i \sup_{U_i} |(f^{n_i})'| \right)^{-\delta} ;$$

hence

$$\delta \geq \frac{\log N}{\log \left(\max_i \sup_{U_i} |(f^{n_i})'| \right)} .$$

Proof. The first inequality is well-known; it can be proved, for example, from Bowen's formula ([Bo2] and [Ru]). The rest is immediate. □

3. Holomorphic motions

Definition. Let X be a subset of $\overline{\mathbb{C}}$ and Λ a complex manifold with a base point $\lambda_0 \in \Lambda$. A family of maps $i_\lambda\colon X \to \overline{\mathbb{C}}$ ($\lambda \in \Lambda$) is called a *holomorphic motion* of X, if each i_λ is injective, $i_{\lambda_0} = \mathrm{id}_X$ and for each $z \in X$, $i_\lambda(z)$ is analytic in λ. We also say that $X_\lambda \equiv i_\lambda(X)$ is a holomorphic motion of X. We are mostly interested in the case $\Lambda = \{\lambda \in \mathbb{C} \,\big|\, |\lambda| < R\}$ ($R > 0$) with the base point $\lambda_0 = 0$.

LEMMA 3.1. *If $i_\lambda\colon X \to \overline{\mathbb{C}}$ ($|\lambda| < R$) is a holomorphic motion, then both i_λ and i_λ^{-1} are Hölder continuous with exponent $\alpha(|\lambda|/R)$, where $\alpha\colon (0,1) \to \mathbb{R}_+$ is a function (independent of the motion) satisfying $\alpha(t) \nearrow 1$, as $t \searrow 0$.*

Proof. The improved λ-lemma in [ST] (see also [MSS], [BR]) implies that i_λ can be extended to a $K(|\lambda|/R)$-quasiconformal mapping and $K(t) \searrow 1$ (as $t \searrow 0$). Since a K-quasiconformal mapping is $1/K$-Hölder continuous (Mori's inequality, see [A]), the assertion holds with $\alpha(t) = 1/K(t)$. For example, by [BR], one can have

$$1 \geq \alpha(t) \geq (1-3t)/(1+3t), \text{ for } 0 < t < 1/3.$$

As an application, we can now complete the proof of (1.3) (see §2), by showing that: For $\mathcal{N}$ (an open neighborhood of f) and X_g in (1.2), $\mathcal{N} \ni g \mapsto$ H-dim X_g is continuous.

Proof. For any $g_0 \in \mathcal{N}$, $\iota_g \circ \iota_{g_0}^{-1}(z)$ is a holomorphic motion of X_{g_0}. Lemma 3.1 implies that for g near g_0, $\iota_g \circ \iota_{g_0}^{-1}$ is α'-bi-Hölder $(0 < \alpha' = \alpha'(g_0, g) \leq 1)$ and $\alpha' \to 1$ when $g \to g_0$. Then we have

$$\alpha' \cdot \text{H-dim}\, X_{g_0} \leq \text{H-dim}\, X_g \leq \alpha'^{-1} \cdot \text{H-dim}\, X_{g_0}.$$

Hence H-dim X_g is continuous in g. □

LEMMA 3.2. *Let i_λ: $X \to \overline{\mathbb{C}}$ $(\lambda \in \Delta \equiv \{\lambda \in \mathbb{C}: \mid |\lambda| < 1\})$ be a holomorphic motion. Suppose v: $\Delta \to \overline{\mathbb{C}}$ is an analytic map such that $v(0) = z_0 \in X$ and $v(\lambda) \not\equiv i_\lambda(z_0)$. Then*

$$\text{H-dim}\{\lambda \in \Delta \mid v(\lambda) \in i_\lambda(X)\} \geq \lim_{r\to 0} \text{H-dim}(X \cap D_r(z_0)),$$

where $D_r(z_0)$ denotes the disc of radius r centered at z_0 with respect to the spherical metric.

Proof. Changing the coordinate by Möbius transformations depending analytically on λ, we may assume that $z_0 = 0$ and $i_\lambda(0) \equiv 0$. First suppose that $v'(0) \neq 0$. There exist positive constants ρ (< 1) and a such that in $\{\lambda \mid |\lambda| < \rho\}$, $v(\lambda)$ is injective and $a|\lambda| \leq |v(\lambda)| < \infty$. Let

$$b_r = \sup\{|i_\lambda(z)| \mid z \in X \cap D_r(0),\ |\lambda| \leq \rho\}.$$

It follows from the λ-lemma [MSS] (or Montel's theorem) that $b_r \to 0$ as $r \to 0$. So there is $r_0 > 0$ such that $a\rho > b_r$ for $0 < r < r_0$. Take such an r.

For $z \in X \cap D_r(0)$ and $|\mu| < R_r \equiv a\rho/b_r$, let us consider the equation

$$(3.3) \qquad v(\lambda) - i_{\lambda\mu}(z) = 0 \quad (\lambda \in \Delta_\mu \equiv \{\lambda \mid |\lambda| < \min\{\rho, \rho/|\mu|\}\}).$$

Both $v(\lambda)$ and $i_{\lambda\mu}(z)$ are analytic in Δ_μ, and for $\lambda \in \partial\Delta_\mu$ we have

$$|v(\lambda)| \geq a \cdot \min\{\rho, \rho/|\mu|\} > b_r \geq |i_{\lambda\mu}(z)|.$$

Since $v = 0$ has the unique solution 0 in Δ_μ, the equation (3.3) also has a unique solution by Rouché's theorem, and it depends analytically on μ. Moreover, for the same μ and a different z, the equation gives a different solution, because of the injectivity of i_λ.

Now define

$$Y_\mu^r = \{\lambda \in \Delta_\mu \mid v(\lambda) = i_{\lambda\mu}(z) \text{ for some } z \in X \cap D_r(0)\}.$$

Then by the above, Y_μ^r $(|\mu| < R_r)$ is a holomorphic motion of Y_0^r, and the injections j_μ^r: $Y_0^r \to Y_\mu^r$ are given by the following: $\lambda = j_\mu^r(v^{-1}(z))$ is the unique solution of the equation (3.3). Note that $Y_0^r = v^{-1}(X \cap D_r(0))$, hence

H-dim Y_0^r = H-dim$(X \cap D_r(0))$, and that $Y_1^r \subset \{\lambda \in \Delta |\ v(\lambda) \in i_\lambda(X)\}$. By Lemma 3.1, j_μ^r: $Y_0^r \to Y_\mu^r$ is $\alpha(|\mu|/R_r)$-bi-Hölder; therefore we have

$$\begin{aligned}\text{H-dim}\{\lambda \mid v(\lambda) \in i_\lambda(X)\} &\geq \text{H-dim}\, Y_1^r \geq \alpha(1/R_r) \cdot \text{H-dim}\, Y_0^r \\ &= \alpha(1/R_r) \cdot \text{H-dim}(X \cap D_r(0)).\end{aligned}$$

Letting $r \to 0$, we obtain the desired inequality.

Let us consider the case $v'(0) = 0$. By the assumption, $v \not\equiv 0$. By a coordinate change, we may assume that $\infty \in X$ and $i_\lambda(\infty) \equiv \infty$. Let $m = \text{order}(v, 0)$ and $G(z) = z^m$. Define $\tilde{X}_\lambda = G^{-1}(X_\lambda)$ and $\tilde{X} = \tilde{X}_0$. By lifting v and i_λ by G which is a branched covering branched over 0 and ∞, one obtains an analytic map $\tilde{v}$: $\Lambda \to \overline{\mathbb{C}}$ satisfying $v = G \circ \tilde{v}$, $\tilde{v}'(0) \neq 0$, and a holomorphic motion $\tilde{i}_\lambda$: $\tilde{X} \to \tilde{X}_\lambda$ satisfying $i_\lambda \circ G = G \circ \tilde{i}_\lambda$. Hence the inequality holds for $\tilde{i}_\lambda$ and $\tilde{v}$. On the other hand,

$$\begin{aligned}\{\lambda \mid v(\lambda) \in X_\lambda\} &= \{\lambda \mid \tilde{v}(\lambda) \in \tilde{X}_\lambda\} \text{ and} \\ \text{H-dim}(X \cap D_r(0)) &= \text{H-dim}(\tilde{X} \cap G^{-1}(D_r(0))),\end{aligned}$$

since G is locally Lipschitz except at 0 and ∞. Thus we obtain the inequality for i_λ and v. □

Now we can give:

Proof of Theorem 1. For any $\varepsilon > 0$, there exists a hyperbolic subset X for f_{λ_0} such that H-dim $X >$ hyp-dim$(f_{\lambda_0}) - \varepsilon$. By the compactness of X, there exists a point $z_0 \in X$ such that $\lim_{r\to 0}$ H-dim$(X \cap D_r(z_0))$ = H-dim X.

By Property (1.2) of the hyperbolic subset, there exist a neighborhood $\Lambda'(\subset \Lambda)$ of λ_0 and a holomorphic motion i_λ: $X \to X_\lambda$ for $\lambda \in \Lambda'$ such that $i_\lambda \circ f_{\lambda_0} = f_\lambda \circ i_\lambda$, X_λ is a hyperbolic subset of f_λ and $i_{\lambda_0} = \text{id}_X$. Moreover Λ' can be chosen smaller so that $\lim_{r\to 0}$ H-dim$(X_\lambda \cap D_r(i_\lambda(z_0)))$ > H-dim $X - \varepsilon$ for $\lambda \in \Lambda'$, by Lemma 3.1, and so that the critical points of f_λ do not bifurcate in Λ' except at λ_0.

It follows from Mañé-Sad-Sullivan's theory (Lemma III.2 [MSS]) that there exist $\lambda_1 \in \Lambda' - \{\lambda_0\}$, an integer $N > 0$ and a critical point c of f_{λ_1} such that $f_{\lambda_1}^N(c) = i_{\lambda_1}(z_0)$. Then there exists a branch of critical points c_λ of f_λ in a neighborhood $\Lambda''(\subset \Lambda')$ of λ_1 with $c_{\lambda_1} = c$, (hence c_λ is a meromorphic function). Note that in the above, λ_1 and c can be chosen so that $f_\lambda^N(c_\lambda) \not\equiv i_\lambda(z_0)$ in Λ''. Applying Lemma 3.2 to i_λ ($\lambda \in \Lambda''$) and $v(\lambda) = f_\lambda^N(c_\lambda)$, (after a suitable affine change of parameter), one obtains

$$\begin{aligned}\text{H-dim}\{\lambda \in \Lambda'' \mid f_\lambda^N(c_\lambda) \in X_\lambda\} &\geq \lim_{r\to 0} \text{H-dim}(X_{\lambda_1} \cap D_r(i_{\lambda_1}(z_0))) \\ &> \text{hyp-dim}(f_{\lambda_0}) - 2\varepsilon.\end{aligned}$$

It is easy to see that if $f_\lambda^N(c_\lambda) \in X_\lambda$, f_λ is not J-stable in the family, since $f_\lambda^N(c_\lambda) \not\equiv i_\lambda(z)$ for any z. As $\varepsilon > 0$ was arbitrary, the theorem is proved. □

Remark 3.1. (i) If we assume a certain transversality condition for the motion of the critical value $v(\lambda)$ relative to the hyperbolic subset (this condition corresponds to $v'(0) \neq 0$ in the proof of Lemma 3.2), it is also possible to prove a similar result (to Theorem 1 or to Lemma 3.2) without assuming the analytic dependence of i_λ and v on the parameter.

(ii) Under the transversality assumption, the proof of Lemma 3.2 gives rise to the map

$$j_1^r \circ v\colon\ X \cap D_r(0) \to Y_1^r \subset \{\lambda \in \Lambda'' \mid f_\lambda^N(c_\lambda) \in X_\lambda\},$$

which means a "similarity" between the hyperbolic subset X and a certain part of the "J-unstable set" in the parameter space.

4. Parabolic bifurcation and Ecalle cylinders

4.0. *Overview.* Let us consider a holomorphic mapping

$$f_0(z) = z + a_2 z^2 + \cdots$$

defined near 0 with $a_2 \neq 0$. The origin $z = 0$ is a parabolic fixed point of f_0. If we perturb f_0, this fixed point bifurcates into two fixed points near 0 in general.

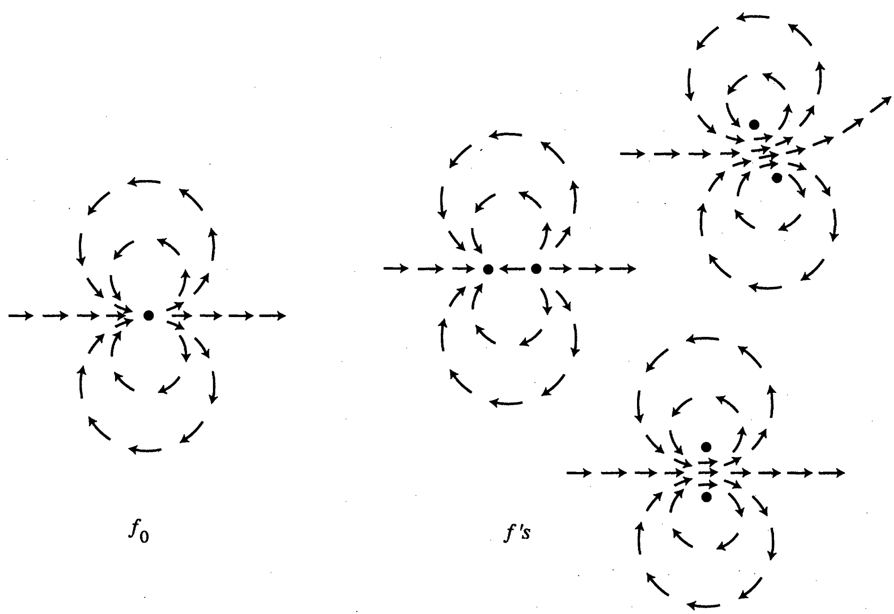

FIGURE 4.

Figure 4 indicates the phase portraits of f_0 and some of its perturbations. Observe that a perturbation can create new types of orbits which go through

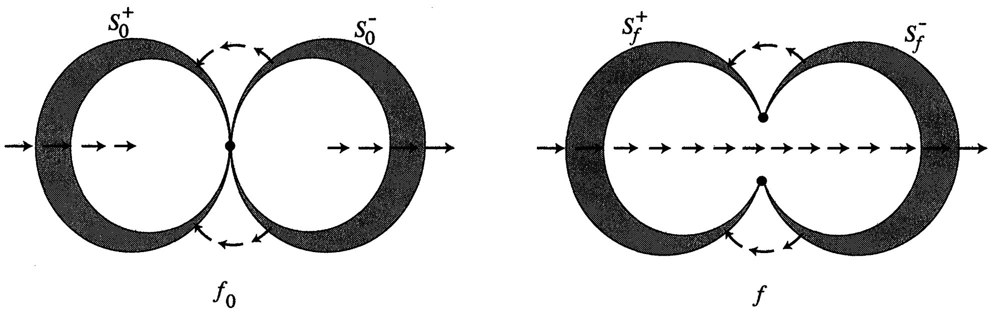

FIGURE 5.

"the gate" between the two fixed points. Such orbits can give rise to a drastic change of the global dynamics (such as the inflation of the Julia set). However it takes an extremely long time for these orbits to go through the gate, so we need to consider a large number of iterates of the map in order to see the phenomenon.

We analyze such a bifurcation using the theory of Ecalle cylinders, and the principle can be summarized as follows. For f_0, one can find "fundamental regions" S_0^+ and S_0^-, each of which has a boundary consisting of two curves joining the fixed point 0 such that one curve is mapped to the other. See Figure 5.

Gluing these two curves of S_0^- (resp. S_0^+), one obtains a topological cylinder $\mathcal{C}_0^-$ (resp. $\mathcal{C}_0^+$), called the *outgoing* (resp. *incoming*) *Ecalle cylinder*, which turns out to be conformally isomorphic to the bi-infinite cylinder $\mathbb{C}^*$ (or $\mathbb{C}/\mathbb{Z}$). The orbits going from the ends ("horns") of S_0^- to S_0^+ induce a continuous and analytic mapping $\mathcal{E}_{f_0}$ from a neighborhood of the ends of $\mathcal{C}_0^-$ to $\mathcal{C}_0^+$ (*the Ecalle transformation*). The identification $\mathbb{C}/\mathbb{Z} \to \mathcal{C}_0^-$ can be lifted and extended to a map φ_0, defined on a subset of $\mathbb{C}$ (which is considered to be the universal cover of $\mathbb{C}/\mathbb{Z}$) into the dynamical plane of f_0; similarly the identification $\mathcal{C}_0^+ \to \mathbb{C}/\mathbb{Z}$ can be lifted to a map Φ_0: $\mathcal{B} \to \mathbb{C}$, where $\mathcal{B}$ is the parabolic basin of 0. Note that these functions can have critical points after extension to the maximal domain of definition.

We consider perturbation of the form $f(z) = e^{2\pi i\alpha}z + O(z^2)$ with $\alpha \neq 0$ satisfying $|\arg\alpha| < \pi/4$. Then it can be shown that fundamental regions S_f^-, S_f^+ continue to exist, with boundary curves joining two fixed points. See Figure 5. The quotient cylinders $\mathcal{C}_f^-$, $\mathcal{C}_f^+$ are still isomorphic to $\mathbb{C}^*$. So we can define functions φ_f, Φ_f, $\mathcal{E}_f$ which are similar to φ_0, Φ_0, $\mathcal{E}_{f_0}$. (The domains of definition may be smaller.)

Now there is "a gate" open between the fixed points, and any orbit starting from S_f^+ passes through the gate and eventually falls into S_f^-. This induces a new map χ_f from $\mathcal{C}_f^+$ to $\mathcal{C}_f^-$, which is a conformal isomorphism. Thus we define *the return map* $\mathcal{R}_f = \chi_f \circ \mathcal{E}_f$, which corresponds near the ends of $\mathcal{C}_f^-$ to the map of S_f^- sending $z \in S_f^-$ to the first return point to S_f^- along its forward orbit. Therefore the return map corresponds to a high iterate of the map f. Now one can study orbits of f which return many times to the neighborhood of 0 using the return map.

Moreover it will be shown that when f tends to f_0 with the above restriction on $\arg \alpha$, the limit behavior of the return map is determined by α and $\mathcal{E}_{f_0}$.

These facts are formulated in the following subsections, and the proof will be given in the appendix, although most of the results can be found in [Mi] and [DH].

Note. The Ecalle cylinders were first studied and applied to some problems by Douady-Hubbard and Lavaurs ([DH], [L]). The aim of this section is to state some notions and facts in this theory; actually, we state only the facts about φ_0, Φ_0, $\mathcal{E}_{f_0}$, φ_f and $\mathcal{R}_f$. The formulation presented here is somewhat different from [DH], [L]. In this paper, we focus more on the return map, the quantitative aspect of the theory and the renormalization arising from the Ecalle transformation.

Notation. In the following, a function f is always associated with its domain of definition $\mathrm{Dom}(f)$ (an open set of $\overline{\mathbb{C}}$ or $\mathbb{C}/\mathbb{Z}$, etc.); that is, two functions are considered as distinct if they have distinct domains, even if one is an extension of the other. A neighborhood of an analytic map $f\colon \mathrm{Dom}(f) \to \overline{\mathbb{C}}$ is a set containing

$$\{g\colon \mathrm{Dom}(g) \to \overline{\mathbb{C}} \mid g \text{ is analytic}, \mathrm{Dom}(g) \supset K \ \text{ and } \ \sup_{z\in K} d(g(z), f(z)) < \varepsilon\},$$

where K is a compact set in $\mathrm{Dom}(f)$, $\varepsilon > 0$ and $d(\cdot,\cdot)$ is the spherical metric. (If the map is to some other space, then d should be replaced by an appropriate metric.) The system of these neighborhoods defines "the compact-open topology together with the domain of definition", which is unfortunately not Hausdorff, since an extension of f is contained in any neighborhood of f.

Let

$$\mathcal{F} = \{f\colon \mathrm{Dom}(f) \to \overline{\mathbb{C}} \mid \ f \text{ is analytic }, 0 \in \mathrm{Dom}(f) \subset \overline{\mathbb{C}} \text{ and } f(0) = 0\}.$$

We use the following notation:

$\pi\colon \mathbb{C} \to \mathbb{C}^* \equiv \mathbb{C} - \{0\}$, $\pi(z) = \exp(2\pi i z)$;

$\pi_1\colon \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ (the natural projection); $\pi_2\colon \mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$, $\pi = \pi_2 \circ \pi_1$;

Note that π_2 sends "the upper end" of $\mathbb{C}/\mathbb{Z}$ ($\mathrm{Im}\, z \to +\infty$) to 0, and "the lower end" ($\mathrm{Im}\, z \to -\infty$) to ∞.

$T\colon \mathbb{C} \to \mathbb{C}$, $T(z) = z + 1$;

$\tau_0(z) = -\frac{1}{z}$.

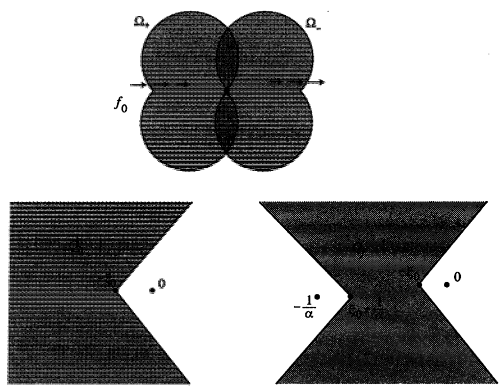

FIGURE 6.

4.1. *Parabolic fixed point.* Let $f_0 \in \mathcal{F}$ be such that $f_0'(0) = 1$ and $f_0''(0) \neq 0$. By a linear coordinate change, we may suppose that $f_0''(0) = 1$. We assume this throughout this section.

FACT. *There exist objects* $\Omega_\pm$,φ_0,Φ_0 *and* $\mathcal{E}_{f_0}$ *satisfying* (4.1.1)–(4.1.4).

(4.1.1) Ω_+ and Ω_- are simply connected domains and $\overline{\Omega}_+, \overline{\Omega}_- \subset \mathrm{Dom}(f_0)$; the boundaries $\partial\Omega_+$, $\partial\Omega_-$ are Jordan curves containing 0; $\Omega_+ \cup \Omega_- \cup \{0\}$ is a neighborhood of 0; $f_0(\overline{\Omega}_+) \subset \Omega_+ \cup \{0\}$ and $f_0(\Omega_- \cup \{0\}) \supset \overline{\Omega}_-$; f is injective on $\Omega_+ \cup \Omega_- \cup \{0\}$; $\Omega_+ \cap \Omega_-$ consists of two components; $f_0^n \to 0$ as $n \to \infty$ uniformly on Ω_+; a point z belongs to the parabolic basin $\mathcal{B}$ of 0 for f_0, if and only if for some $n \geq 0$, $f_0^n(z)$ is defined and belongs to Ω_+. (See Figure 6.)

Here, the *parabolic basin* of a parabolic fixed point ζ of an analytic function f is

$$\mathcal{B} = \left\{ z \,\middle|\, \begin{array}{c} z \text{ has a neighborhood on which } f^n \ (n = 1, 2, \dots) \text{ are defined} \\ \text{and } f^n \to \zeta \ \text{ uniformly as } n \to \infty \end{array} \right\}.$$

Note that ζ itself is not in the parabolic basin.

(4.1.2) φ_0: $\mathrm{Dom}(\varphi_0) \to \overline{\mathbb{C}}$ is an analytic function satisfying:

$$\varphi_0(w+1) = f_0 \circ \varphi_0(w) \quad \text{if both sides are defined,}$$

and in fact the left-hand side is defined if and only if the right-hand side is;

$\mathrm{Dom}(\varphi_0)$ contains $\mathcal{Q}_0 = \{w \in \mathbb{C} \mid \pi/3 < \arg(w + \xi_0) < 5\pi/3\}$ (see Figure 6) and $\{w \mid |\mathrm{Im}\, w| > \eta_0\}$ for large $\xi_0, \eta_0 > 0$;

$\varphi_0(\mathcal{Q}_0) = \Omega_-$; φ_0 is injective on $\mathcal{Q}_0$;

let $\varphi_0^* = \tau_0^{-1} \circ \varphi_0$, then

$$\varphi_0^*(w) = w + a \log w + b + o(1) \quad \text{as } \mathcal{Q}_0 \ni w \to \infty,$$

where a, b are some constants.

It follows from the above condition that if f_0 is a rational map or an entire function, then $\mathrm{Dom}(\varphi_0) = \mathbb{C}$.

(4.1.3) Φ_0: $\mathcal{B} \to \mathbb{C}$ is an analytic function satisfying

$$\Phi_0 \circ f_0(z) = \Phi_0(z) + 1 \quad \text{for } z \in \mathcal{B},$$

and Φ_0 is injective on Ω_+ $(\subset \mathcal{B})$.

(4.1.4) Let $\tilde{\mathcal{B}} = \varphi_0^{-1}(\mathcal{B})$; then $T(\tilde{\mathcal{B}}) = \tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}$ contains $\{w \mid |\operatorname{Im} w| > \eta_0\}$ for some $\eta_0 > 0$.

Now define $\tilde{\mathcal{E}}_{f_0}$: $\tilde{\mathcal{B}} \to \mathbb{C}$ by

$$\tilde{\mathcal{E}}_{f_0} = \Phi_0 \circ \varphi_0.$$

It satisfies

$$\tilde{\mathcal{E}}_{f_0}(w+1) = \tilde{\mathcal{E}}_{f_0}(w) + 1 \quad \text{for } w \in \tilde{\mathcal{B}}.$$

Hence $\mathcal{E}_{f_0} = \pi \circ \tilde{\mathcal{E}}_{f_0} \circ \pi^{-1} : \pi(\tilde{\mathcal{B}}) \to \mathbb{C}^*$ is well-defined. Moreover it extends to 0 and ∞ analytically by $\mathcal{E}_{f_0}(0) = 0$ and $\mathcal{E}_{f_0}(\infty) = \infty$, and $\mathcal{E}'_{f_0}(0) \neq 0$, $\mathcal{E}'_{f_0}(\infty) \neq 0$. So $\mathrm{Dom}(\mathcal{E}_{f_0}) = \pi(\tilde{\mathcal{B}}) \cup \{0, \infty\}$.

The map $\mathcal{E}_{f_0}$ is called the *Ecalle transformation*, or the *horn map* (since it is defined near the ends of the cylinder).

(4.1.5) Normalization. Note that $\varphi_0(w + c)$ and $\Phi_0(z) + c'$ also satisfy (4.1.2) and (4.1.3). So we adjust Φ_0 by adding a constant so that

$$\mathcal{E}'_{f_0}(0) = 1.$$

4.2. *Perturbation.* Let

$$\mathcal{F}_1 = \{f \in \mathcal{F} \mid f'(0) = \exp(2\pi i \alpha) \text{ with } \alpha \neq 0 \text{ and } |\arg \alpha| < \pi/4\}.$$

In the following, we consider only the perturbations $f \in \mathcal{F}_1$.

Notation. If $f \in \mathcal{F}$ and $f'(0) \neq 0$, we express the derivative $f'(0)$ as

$$f'(0) = \exp(2\pi i \alpha(f))$$

where $\alpha(f) \in \mathbb{C}$ and $-\frac{1}{2} < \operatorname{Re}\alpha(f) \leq \frac{1}{2}$.

If $f \in \mathcal{F}$ is close to f_0, then f has two fixed points near 0 counted with multiplicity; one of them is 0 and the other is denoted by $\sigma(f)$. Note that

$$\sigma(f) = -2\pi i \alpha(f)(1 + o(1)) \text{ as } f \to f_0.$$

FACT. *There exist a neighborhood* $\mathcal{N}_0$ *of* f_0 *and a constant* $\xi_0 > 0$ (*which can be the same as that in the definition of* $\mathcal{Q}_0$ (4.1.2)) *such that for* $f \in \mathcal{N}_0 \cap \mathcal{F}_1$, *there exist analytic maps* $\varphi_f \colon \operatorname{Dom}(\varphi_f) \to \mathbb{C}$ *and* $\mathcal{R}_f \colon \operatorname{Dom}(\mathcal{R}_f) \to \overline{\mathbb{C}}$ *satisfying* (4.2.1)–(4.2.4).

(4.2.1) If $w, w+1 \in \operatorname{Dom}(\varphi_f)$, then $\varphi_f(w) \in \operatorname{Dom}(f)$ and

$$\varphi_f(w+1) = f \circ \varphi_f(w);$$

$\operatorname{Dom}(\varphi_f)$ contains $\mathcal{Q}_f$ (see Figure 6), where

$$\mathcal{Q}_f = \left\{ w \in \mathbb{C} \;\middle|\; \pi/3 < \arg(w + \xi_0) < 5\pi/3 \text{ and } \left|\arg\left(w + \frac{1}{\alpha(f)} - \xi_0\right)\right| < 2\pi/3 \right\};$$

$\varphi_f(w) \to 0$ when $w \in \mathcal{Q}_f$ and $\operatorname{Im} w \to +\infty$; $\varphi_f(w) \to \sigma(f)$ when $w \in \mathcal{Q}_f$ and $\operatorname{Im} w \to -\infty$.

(4.2.2) $0, \infty \in \operatorname{Dom}(\mathcal{R}_f)$, $\mathcal{R}_f(0) = 0$, $\mathcal{R}_f(\infty) = \infty$; $\mathcal{R}_f(\operatorname{Dom}(\mathcal{R}_f) \cap \mathbb{C}^*) \subset \mathbb{C}^*$;

$$\mathcal{R}_f'(0) = \exp\left(-2\pi i \frac{1}{\alpha(f)}\right), \text{ hence } \alpha(\mathcal{R}_f) \equiv -\frac{1}{\alpha(f)} \pmod{\mathbb{Z}}.$$

(4.2.3) If $w, w' \in \operatorname{Dom}(\varphi_f)$, $\mathcal{R}_f(\pi(w)) = \pi(w')$ and $|\arg(w' + \frac{1}{2\alpha(f)} - w)| < 2\pi/3$, then $f^n(\varphi_f(w)) = \varphi_f(w')$ for some $n \geq 1$.

Moreover if U, U' are connected subsets of $\operatorname{Dom}(\varphi_f)$ such that $\mathcal{R}_f^m(\pi(U)) \subset \pi(U')$ for some $m \geq 1$, $\varphi_f|_U$, $\pi|_{U'}$ are injective, and $|\arg(w' + \frac{1}{2\alpha(f)} - w)| < 2\pi/3$ for $w \in U$, $w' \in U'$, then there exists an $n > m$ such that

$$f^n = \varphi_f \circ (\pi|_{U'})^{-1} \circ \mathcal{R}_f^m \circ \pi \circ (\varphi_f|_U)^{-1} \text{ on } \varphi_f(U).$$

(4.2.4) With respect to the topology defined in Section 4.0,

$$\varphi_f \to \varphi_0 \text{ and } e^{2\pi i/\alpha(f)}\mathcal{R}_f \to \mathcal{E}_{f_0} \text{ when } f \in \mathcal{N}_0 \cap \mathcal{F}_1 \text{ and } f \to f_0.$$

Denote $\hat{\mathcal{E}}_{f_0} = \pi_2^{-1} \circ \mathcal{E}_{f_0} \circ \pi_2$ and $\hat{\mathcal{R}}_f = \pi_2^{-1} \circ \mathcal{R}_f \circ \pi_2$. Then

$$\hat{\mathcal{R}}_f + \frac{1}{\alpha(f)} \to \hat{\mathcal{E}}_{f_0} \text{ when } f \in \mathcal{N}_0 \cap \mathcal{F}_1 \text{ and } f \to f_0.$$

(4.2.5) To study the quantitative aspect, it is often more convenient to work with $\varphi_f^* = \tau_0^{-1} \circ \varphi_f$.

For $\eta \in \mathbb{R}$, define $D(\eta) = \{w \in \mathbb{C} \mid |w - i\eta| \leq |\eta|/4\}$ and $D'(\eta) = \{w \in \mathbb{C} \mid |w - i\eta| \leq |\eta|/2\}$.

As a corollary of the above facts, we have:

For large $\eta > 0$, there exists a neighborhood $\mathcal{N}_1(\eta)(\subset \mathcal{N}_0)$ of f_0, depending on η, such that if $f \in \mathcal{N}_1(\eta) \cap \mathcal{F}_1$, then φ_f is defined and injective on $D(\pm\eta)$, the image $\varphi_f^*(D(\pm\eta))$ is contained in $D'(\pm\eta)$ and $\frac{1}{2} < |(\varphi_f^*)'| < 2$ on $D(\pm\eta)$.

4.3. *Remarks.*

(4.3.1) For $f \in \mathcal{F}$ with $3\pi/4 < \arg \alpha(f) < 5\pi/4$, instead of being in $\mathcal{F}_1$, we can obtain a similar result with the following changes:

The lower end of $\mathbb{C}/\mathbb{Z}$ corresponds to the fixed point 0;

In (4.2.1), $\varphi_f(w) \to \sigma(f)$ ($w \in \mathcal{Q}_f$ and $\operatorname{Im} w \to +\infty$), $\varphi_f(w) \to 0$ ($w \in \mathcal{Q}_f$ and $\operatorname{Im} w \to -\infty$);

In (4.2.1), (4.2.3) and (4.2.4), use $-\frac{1}{\alpha(f)}$ instead of $\frac{1}{\alpha(f)}$;

In (4.2.2), $\mathcal{R}_f'(\infty) = \exp(-2\pi i \frac{1}{\alpha(f)})$.

(4.3.2) The above facts suggest the following factorization:

$$\hat{\mathcal{R}}_f = \hat{\mathcal{T}}_{-1/\alpha} \circ \left(\hat{\mathcal{R}}_f + \frac{1}{\alpha}\right),$$

where $\hat{\mathcal{T}}_{-1/\alpha}(w) = w - 1/\alpha$ on $\mathbb{C}/\mathbb{Z}$, $\alpha = \alpha(f)$. Here $(\hat{\mathcal{R}}_f + \frac{1}{\alpha})$ is a nonlinear map defined only in a subset of $\mathbb{C}/\mathbb{Z}$, but approaches a fixed map $\hat{\mathcal{E}}_{f_0}$, whereas, $\hat{\mathcal{T}}_{-1/\alpha}$ is an isomorphism of $\mathbb{C}/\mathbb{Z}$, but depends sensitively on f, since $\alpha(f) \to 0$ as $f \to f_0$. Therefore if $\{f_n\}$ is a sequence in $\mathcal{N}_0 \cap \mathcal{F}_1$ such that $f_n \to f_0$ and $1/\alpha(f_n) - k_n \to -c$ as $n \to \infty$, where k_n are integers, then there exists a limit

$$\lim_{n\to\infty} \hat{\mathcal{R}}_{f_n} = \hat{\mathcal{E}}_{f_0} + c,$$

since we are considering the maps in $\mathbb{C}/\mathbb{Z}$.

The Ecalle transformation $\mathcal{E}_{f_0}$ was originally defined as a map between two different spaces— (a part of) $\mathcal{C}_0^-$ and $\mathcal{C}_0^+$. So there is, *a priori*, no meaning as a dynamical system. However the above observation implies that for any $c \in \mathbb{C}$, the map

$$w \longmapsto \hat{\mathcal{E}}_{f_0}(w) + c$$

on $\mathbb{C}/\mathbb{Z}$ can be realized as the limit of return maps of f_n.

(4.3.3) In [DH] and [L], they use $\varphi_0 \circ \mathcal{T}_c \circ \Phi_0$, where $\mathcal{T}_c(z) = z + c$ ($c \in \mathbb{C}$), instead of $\tilde{\mathcal{E}}_{f_0} + c = \mathcal{T}_c \circ \Phi_0 \circ \varphi_0$. Then for a sequence $\{f_n\}$ as above, there is a sequence of integers k_n such that $\varphi_0 \circ \mathcal{T}_c \circ \Phi_0$ is the limit of $f_n^{k_n}$ on the parabolic basin $\mathcal{B}$.

(4.3.4) A more accurate statement of (4.2.3) is as follows: There exists a particular lift of $\mathcal{R}_f$ to the universal cover, $\tilde{\mathcal{R}}_f$: $\pi^{-1}(\operatorname{Dom}(\mathcal{R}_f)) \to \mathbb{C}$, such that if $w, w' \in \mathrm{Dom}(\varphi_f)$ and $\tilde{\mathcal{R}}_f^m(w) + n = w'$, where $n, m \geq 0$ are integers, then $f^n(\varphi_f(w)) = \varphi_f(w')$.

5. Global behaviour of the Ecalle transformation

In this section, we study the property of the maximal extension of the Ecalle transformation for a certain class of maps $\mathcal{F}_0$. As in the previous section, we suppose that $f_0 \in \mathcal{F}$ is a function satisfying $f_0'(0) = 1$ and $f_0''(0) = 1$. So we have φ_0 and $\mathcal{E}_{f_0}$ as in the previous section. Let us denote $g_0 = \mathcal{E}_{f_0}$.

5.1. *Inverse orbits and* φ_0.

Definition. For a mapping f, a sequence of points $\{z_j\}_{j=0}^{\infty}$ is called *an inverse orbit* (of z_0) for f, if $z_j \in \mathrm{Dom}(f)$ and $f(z_j) = z_{j-1}$ for $j \geq 1$.

LEMMA 5.1. *For* $w \in \mathrm{Dom}(\varphi_0)$, *let* $z_j = \varphi_0(w - j)$ $(j = 0,1,\ldots)$. *Then* $\{z_j\}_{j=0}^{\infty}$ *is an inverse orbit for* f_0 *converging to* 0. *This gives a one-to-one correspondence between* $\mathrm{Dom}(\varphi_0)$ *and the set of inverse orbits converging to* 0, *except the orbit* $z_j = 0$ $(j = 0,1,\ldots)$.

Moreover if z_j $(j \geq 1)$ *are not critical points, then* $\varphi_0'(w) \neq 0$.

Proof. If $w \in \mathrm{Dom}(\varphi_0)$, then $w - j \in \mathrm{Dom}(\varphi_0)$ $(j = 0, 1, \ldots)$ and $\varphi_0(w - j) \to 0$ by (4.1.2); so the first statement is obvious. Let $\{z_j\}$ be an inverse orbit converging to 0 and suppose $z_j \neq 0$ for some j. For large j, say for $j \geq j_0$, z_j belongs to $\Omega_+ \cup \Omega_-$ (see (4.1.1)). But f_0^n tends to 0 uniformly on Ω_+, so there exists j_0 such that $z_j \in \Omega_-$ for $j \geq j_0$. Let $w = (\varphi_0|_{\mathcal{Q}_0})^{-1}(z_j) + j$ $(j \geq j_0)$. It is easy to see that w does not depend on $j \geq j_0$ and corresponds to the inverse sequence $\{z_j\}$. If w and w' give the same inverse sequence, take $j \geq 0$ such that $w - j, w' - j \in \mathcal{Q}_0$, then we have $w = w'$ by the injectivity of φ_0 on $\mathcal{Q}_0$.

The last statement follows from the facts that $\varphi_0(w) = f_0^n \circ \varphi_0(w - n)$ for $w \in \mathrm{Dom}(\varphi_0)$ and that $\varphi_0' \neq 0$ in $\mathcal{Q}_0$. □

5.2. *Immediate parabolic basin.*

Definition. Let ζ be a parabolic fixed point of an analytic function f. A connected open set $B \subset \mathrm{Dom}(f)$ is called *an immediate parabolic basin* of ζ for f, if B is a connected component of the parabolic basin of ζ for f (see (4.1.1)) such that $f(B) = B$ and $f\colon B \to B$ is proper (hence a branched covering).

For a rational map, a parabolic periodic point always has an immediate parabolic basin and it coincides with the previous definition (§1). However, a parabolic fixed point for analytic maps in general may not have any immediate parabolic basin in this sense.

LEMMA 5.2. *Suppose that $f \in \mathcal{F}$ and $f'(0) = 1$, $f^{(k)}(0) = 0$ $(1 < k \leq q)$, $f^{(q+1)}(0) \neq 0$ $(q \geq 1)$, i.e. $q+1$ is the order of $f(z) - z$ at 0. Then f has at most q immediate parabolic basins of 0, each of which contains at least one critical value of f, except for a parabolic Möbius transformation. Moreover if an immediate parabolic basin contains only one critical point (or critical value), then it is simply connected.*

Proof. This is a well-known argument for rational maps. Let us examine it briefly. By local analysis as in Section 4.1 (case $q = 1$) (see also §7 or [Mi]), it can be shown that there exist q disjoint simply connected open sets $V^{(1)}, \ldots, V^{(q)}$ $(\subset \mathrm{Dom}(f))$ (called "attracting petals") such that $0 \in \partial V^{(i)}$; $f(\overline{V}^{(i)}) \subset V^{(i)} \cup \{0\}$; f is injective on $V^{(i)}$; a point $z \in \mathrm{Dom}(f)$ belongs to the parabolic basin of 0 if and only if $f^n(z) \in \cup_i V^{(i)}$ for some $n \geq 0$. The orbit spaces $V^{(i)}/_\sim$ (where $z \sim f(z)$ if $z, f(z) \in V^{(i)}$) are isomorphic to $\mathbb{C}/\mathbb{Z}$.

Let B be an immediate basin of 0, and B' the parabolic basin. By the above, $B \cap \cup_i V^{(i)} \neq \emptyset$ and $\cup_i V^{(i)} \subset B'$. Since B is a component of B', B contains one of the $V^{(i)}$'s. So there are at most q immediate basins.

Define $V_n^{(i)}$ $(n = 0, 1, \ldots)$ inductively, as follows: $V_0^{(i)} = V^{(i)}$; if $f(V_n^{(i)}) \subset V_n^{(i)}$, $V_{n+1}^{(i)}$ is the component of $f^{-1}(V_n^{(i)})$ containing $V_n^{(i)}$, which exists and $f(V_{n+1}^{(i)}) = V_n^{(i)} \subset V_{n+1}^{(i)}$. It is easy to see that for each i, $\cup_{n\geq 0} V_n^{(i)}$ is a component of B'. Now suppose $B = \cup_{n\geq 0} V_n^{(i)}$ is an immediate parabolic basin; hence by definition, $f\colon B \to B$ is a branched covering. If B contains no critical value or no critical point, then $f\colon V_{n+1}^{(i)} \to V_n^{(i)}$; hence $f^n\colon V_n^{(i)} \to V^{(i)}$ are covering maps; therefore they induce a covering map $B \to V^{(i)}/_\sim$. Moreover the $V_n^{(i)}$'s hence B are simply connected. Therefore B must be isomorphic to $\mathbb{C}$ and f is a Möbius transformation.

Similarly, if B contains only one critical point or critical value, then one can show inductively that $V_n^{(i)}$ are simply connected; hence so is B. □

Let us go back to the f_0 defined at the beginning of this section. Suppose that f_0 has an immediate parabolic basin B of 0. The above argument applies to $V^{(1)} = \Omega_+$, hence B is unique and contains Ω_+ (see (4.1.1)). Let B' be the (whole) parabolic basin of 0, and define $\tilde{B} = \varphi_0^{-1}(B)$ and $\tilde{B}' = \varphi_0^{-1}(B')$. By (4.1.4), $\{w \mid |\mathrm{Im}\, w| > \eta_0\} \subset \tilde{B}'$. Since for any $w \in \tilde{B}'$ there is $n \geq 0$ such that $T^n(w) \in \tilde{B}$, we have

$$\{w \mid |\mathrm{Im}\, w| > \eta_0\} \subset \tilde{B}.$$

Denote by $\tilde{B}^u$ (resp. by $\tilde{B}^\ell$) the component of $\tilde{B}$ containing $\{w \mid \mathrm{Im}\, w > \eta\}$ (resp. $\{w \mid \mathrm{Im}\, w < -\eta\}$). (The superscript "$u$" stands for upper, and "ℓ" for lower.) In general, $\tilde{B}^u$ and $\tilde{B}^\ell$ may coincide. Obviously $T\tilde{B}^u = \tilde{B}^u$, $T\tilde{B}^\ell = \tilde{B}^\ell$. Then define $B^u = \pi(\tilde{B}^u)$, $B^\ell = \pi(\tilde{B}^\ell)$.

5.3. *Covering property of* g_0.

Definition. Let $\mathcal{F}_0$ be the set of functions $f \in \mathcal{F}$ such that $f'(0) = 1$, $f''(0) = 1$ and f has an immediate parabolic basin which contains only one critical point of f. Then the basin is automatically simply connected by Lemma 5.2.

PROPOSITION 5.3. *Let* $f_0 \in \mathcal{F}_0$ *and* B, $\tilde{B}^u$, $\tilde{B}^\ell$ *as before. Then* g_0: $B^u \cup B^\ell \to \mathbb{C}^*$ *is a branched covering of infinite degree, ramified only over one point* $v \in \mathbb{C}^*$.

The sets $\tilde{B}^u$, $\tilde{B}^\ell$, $B^u \cup \{0\}$, $B^\ell \cup \{\infty\}$ *are simply connected, and* $B^u \cap B^\ell = \emptyset$.

Proof. Let us first show that Φ_0: $B \to \mathbb{C}$ is a branched covering. In fact, $\Phi_0|_{\Omega_+}$ is injective; hence Φ_0: $f_0^{-n}(\Omega_+) \cap B \to T^{-n}\Phi_0(\Omega_+)$ is a branched covering ramified only over the translations of the image of the critical value by Φ_0 ($n \geq 0$). So it follows that Φ_0 on B is a branched covering.

Now let us show that φ_0: $\tilde{B}^u \cup \tilde{B}^\ell \to B$ is a branched covering. Let z be a point in B. Take simply connected neighborhoods U, U' of z such that $\overline{U} \subset U'$ and U' contains at most one forward orbit of the critical point. Let z' be a point in $\varphi_0^{-1}(U) \cap (\tilde{B}^u \cup \tilde{B}^\ell)$. Let U'_n be the component of $f_0^{-n}(U')$ containing $\varphi_0(z'-n)$ ($n = 0, 1, \ldots$). Then for some $m \geq 0$, U'_n ($n \geq m$) do not contain the critical point. Hence there exist inverse branches $f_{0,U'_m}^{(-k)}$: $U'_m \to U'_{m+k}$ of f_0^k. The family $\{f_{0,U'_m}^{(-k)}\}$ is normal, since it avoids at least three values (0 and the orbit of the critical point). Moreover it converges to 0 uniformly on compact sets, since it does so near $\varphi_0(z' - m)$. Hence there exists an $n \geq m$ such that $U_n = f_{0,U'_m}^{(-n+m)}(U_m) \subset \Omega_- = \varphi_0(\mathcal{Q}_0)$. Let $V = T^n \circ (\varphi_0|_{\mathcal{Q}_0})^{-1}(U_n)$. Then $z' \in V$ and $\varphi_0|_V = (f_0^n|_{U_n}) \circ (\varphi_0|_{\mathcal{Q}_0}) \circ T^{-n}$. So φ_0: $V \to U$ is a branched covering with at most one critical point, since U contains at most one critical orbit. This shows that each component of $\varphi_0^{-1}(U)$ is either unramified or ramified over a common point in U; hence φ_0: $\tilde{B}^u \cup \tilde{B}^\ell \to B$ is a branched covering.

Therefore $\tilde{\mathcal{E}}_{f_0} = \Phi_0 \circ \varphi_0$: $\tilde{B}^u \cup \tilde{B}^\ell \to \mathbb{C}$ and $\mathcal{E}_{f_0}$: $B^u \cup B^\ell \to \mathbb{C}^*$ are branched coverings. Moreover it is easy to see from the above that $\mathcal{E}_{f_0}$ is ramified only over $v = \pi \circ \Phi_0(c)$, where c is the unique critical point.

Now let U be a component of $f_0^{-1}(f_0(\Omega_+))$ contained in B, different from Ω_+. Then we have $f_0^n(U) \cap U = \emptyset$ ($n \geq 1$). Hence for any component V of $\varphi_0^{-1}(U)$, $\pi|_V$ is injective (by the functional equation for φ_0). On the other hand, $\pi \circ \Phi_0$: $U \to \mathbb{C}^*$ is infinite to one, since $\pi \circ \Phi_0|_U = \pi \circ \Phi_0|_{\Omega_+} \circ f_0|_U$ and $\pi \circ \Phi_0|_{\Omega_+}$ is infinite to one. Hence g_0 is of infinite degree.

Let us show the simple connectivity of $\tilde{B}^u$ (or $\tilde{B}^\ell$). If γ is a closed curve in $\tilde{B}^u$, $\gamma' = T^{-n}\gamma \subset \mathcal{Q}_0$ for some $n \geq 0$. Let W be a region bounded by $\varphi_0(\gamma')$, not containing 0. Since both the immediate basin B and $\Omega_- = \varphi_0(\mathcal{Q}_0)$ are simply connected and do not contain 0, we have $W \subset B \cap \Omega_-$. It follows that γ' is trivial in $\tilde{B}^u$ and so is γ. Therefore $\tilde{B}^u$ is simply connected.

Then $B^u \cup \{0\}$ (or $B^\ell \cup \{0\}$) is also simply connected, since the fundamental group of B^u is generated by a curve around 0, which is trivial in $B^u \cup \{0\}$.

Finally, let us show that $\tilde{B}^u$ and $\tilde{B}^\ell$ are different components of $\tilde{B}$. Suppose $\tilde{B}^u = \tilde{B}^\ell$. Then $\tilde{B}^u = \tilde{B}^\ell = \mathbb{C}$, since $\tilde{B}^u$ and $\tilde{B}^\ell$ are invariant under T, simply connected, and contain half planes. Therefore B contains Ω_+ and $\Omega_- = \varphi_0(\mathcal{Q}_0)$, and the union is a punctured neighborhood of 0. But B is a simply connected, punctured neighborhood of 0, so that $B = \overline{\mathbb{C}} - \{0\}$. Hence f_0 is a parabolic Möbius transformation, since it is analytic on $\overline{\mathbb{C}}$ and has no periodic point in B. Then f_0 has no critical point and this contradicts the assumption. Thus we have $\tilde{B}^u \cap \tilde{B}^\ell = \emptyset$, hence $B^u \cap B^\ell = \emptyset$. □

5.4. *Iteration of* g_0. By the normalization in (4.1.5), we have $g_0'(0) = 1$. So 0 is again a parabolic fixed point of g_0.

LEMMA 5.4. *Let* $f_0 \in \mathcal{F}_0$ *and* $g_0 = \mathcal{E}_{f_0}$. *Then* $g_0''(0) \neq 0$ *and* g_0 *has a simply connected immediate parabolic basin which contains only one critical point. In other words,* g_0 *belongs to* $\mathcal{F}_0$ *after a linear scaling of the coordinate. Moreover* $g_0^n(v)$ $(n = 0,1,\ldots)$ *are defined and* $g_0^n(v) \to 0$ $(n \to \infty)$, *where* v *is the unique critical value of* g_0.

Proof. Obviously $g_0(z) \not\equiv z$. So there exist attracting petals $V^{(i)}$ $(i = 1, \ldots, q)$ for g_0 as in the proof of Lemma 5.2, where $q+1 = \operatorname{ord}(g_0(z) - z)$. Since $g_0\colon B^u \cup B^\ell \to \mathbb{C}^*$ is a branched covering, we can also construct $V_n^{(i)}$ and $g_0\colon V_{n+1}^{(i)} \to V_n^{(i)}$ is a branched covering with at most one critical point. Then $V_n^{(i)}$ are simply connected and $\deg g_0|_{V_n^{(i)}}$ $(n = 0, 1, \ldots)$ are eventually constant. It follows that the $B_i = \cup_i V_n^{(i)}$ are simply connected and $g_0\colon B_i \to B_i$ is a branched covering with at most one critical point. Hence the B_i are immediate parabolic basins. By Lemma 5.2, we have $q = 1$, i.e., $g_0''(0) \neq 0$. The rest follows easily. □

6. The construction of a hyperbolic subset

In this section, we prove Theorem 2 in case the multiplier is 1 (hence $q = 1$, $p = 0$). We will see in Section 7 how to modify the proof in other cases.

To construct a hyperbolic subset as in Section 2, we trace certain inverse images of the fixed point 0 using φ_0, $\mathcal{E}_{f_0}$, ψ_0, $\mathcal{E}_{g_0}$, etc., then analyze the perturbed maps along these orbits.

Step 0. f_0 *and* $\{z_j\}$. Suppose f_0 is a rational map and $f_0 \in \mathcal{F}_0$. There exists an inverse orbit (see §5 for definition) $\{z_j\}_{j=0}^{\infty}$ for f_0 such that $z_0 = 0$,

$z_j \neq 0$ $(j > 0)$ and $z_j \to 0$ $(j \to \infty)$. In fact, since 0 is in the Julia set, there is an inverse image of 0 near 0 (see [Bl], [Mi]). This point must have an inverse sequence coverging to 0, otherwise it would belong to the parabolic basin (see the local analysis in §4.1).

Step 1. g_0 *and* $\{w_j\}$, $\{w'_j\}$. By the construction in the previous sections, we obtain φ_0 and $g_0 = \mathcal{E}_{f_0}$. Let v be the unique critical value of g_0.

By Lemma 5.1, there exists a unique $\tilde{w}_0 \in \mathrm{Dom}(\varphi_0)$ corresponding to the inverse orbit $\{z_j\}$ such that $\varphi_0(\tilde{w}_0 - j) = z_j$. In fact, note that $z_0 = 0$ does not belong to the parabolic basin of 0; hence $\pi(\tilde{w}_0) \notin \mathrm{Dom}(g_0)$ by the definition of g_0.

LEMMA 6.1. *For given* $w_0 \in \mathbb{C}^*$, *there exist two inverse orbits* $\{w_j\}_{j=0}^{\infty}$, $\{w'_j\}_{j=0}^{\infty}$ *of* w_0 *for the map* g_0 *such that:* $w_0 = w'_0$; $w_j, w'_j \to 0$ $(j \to \infty)$; $\{w_j\}_{j=1}^{\infty} \cap \{w'_j\}_{j=1}^{\infty} = \emptyset$; *and* w_j, w'_j $(j \geq 2)$ *are not critical points of* g_0. *Moreover if* $w_0 \notin \mathrm{Dom}(g_0)$, w_1, w'_1 *are not critical points.*

Proof. First note that $g_0^{-1}(w_0)$ can contain at most one periodic point. It is easy to see that if $g_0^{-1}(w_0)$ contains two points of $\{g_0^n(v) \mid n \geq 0\}$, then one of them must be periodic. Hence $\sharp(g_0^{-1}(w_0) \cap \{g_0^n(v) \mid n \geq 0\}) \leq 2$. So we can choose two distinct non-periodic points w_1 and w'_1 in $(g_0)^{-1}(w_0) - \{g_0^n(v) \mid n \geq 0\}$, since g_0 is a branched covering of infinite degree.

Since 0 is a parabolic fixed point of g_0, there exists an inverse orbit $\{w''_j\}_{j=1}^{\infty}$ for g_0 such that $w''_j \to 0$ as $j \to 0$, $w''_j \neq 0$ and $\{w''_j\}_{j=1}^{\infty} \cap \{(g_0)^n(v)\}_{n=0}^{\infty} = \emptyset$. By Lemma 5.4, $g_0^n(v) \to 0$ $(n \to \infty)$. So we can take a simply connected open set $D \subset \mathbb{C}^* - \{g_0^n(v)\}_{n=0}^{\infty}$ containing w_1, w'_1 and w''_1.

Since g_0 is a branched covering onto $\mathbb{C}^*$ ramified only over v, for each $j \geq 1$, there exists an inverse branch G_j: $D \to \mathbb{C}^*$ of g_0^j such that $g_0^j \circ G_j = \mathrm{id}_D$ and $G_j(w''_1) = w''_{j+1}$. The family $\{G_j\}$ is normal, since it omits at least three values 0, ∞ and v. As $w''_{j+1} = G_j(w''_1) \to 0$ $(j \to \infty)$, $G_j \to 0$ on D.

Now define $w_{j+1} = G_j(w_1)$ and $w'_{j+1} = G_j(w'_1)$. It is easy to check that these sequences have the claimed properties. □

Applying this lemma to $w_0 = \pi(\tilde{w}_0)$, we obtain $\{w_j\}$, $\{w'_j\}$ as above.

Step 2. $\hat{h}$ *and* $\{W_j\}$. As $g_0 \in \mathcal{F}_0$, one can apply the construction of Sections 4 and 5 to g_0. Denote $\psi_0 = \varphi_{0,g_0}$ ("φ_0" corresponding to g_0) and $h_0 = \mathcal{E}_{g_0}$, for simplicity. As in the previous step, by Lemma 5.1, there exist $\tilde{\zeta}_0, \tilde{\zeta}'_0 \in \mathrm{Dom}(\psi_0)$ corresponding to $\{w_j\}$, $\{w'_j\}$, respectively. Hence $w_0 = \psi_0(\tilde{\zeta}_0)$ and $w'_0 = \psi_0(\tilde{\zeta}'_0)$. Moreover $\psi'_0(\tilde{\zeta}_0) \neq 0$ and $\psi'_0(\tilde{\zeta}'_0) \neq 0$. The following is the key lemma in the proof of Theorem 2.

LEMMA 6.2. *Let* $b > 0$. *There exist: a neighborhood* $\mathcal{N}_2$ *of* h_0, *where* $0,\infty \in \mathrm{Dom}(h_0)$; *two disjoint discs* $W, W' \subset \mathbb{C}/\mathbb{Z}$ *containing* $\hat{\zeta}_0 = \pi_1(\tilde{\zeta}_0)$,

$\hat{\zeta}_0' = \pi_1(\tilde{\zeta}_0')$ *respectively; and positive constants* C_0, C_1, C_1', *with the following properties.*
If $h_1 \in \mathcal{N}_2$, $\beta \in \mathbb{C}/\mathbb{Z}$ *with* $|\operatorname{Im}\beta| \le b$ *and*

$$\hat{h}(\zeta) = \pi_2^{-1} \circ h_1 \circ \pi_2(\zeta) - \beta \quad (\textit{for } \zeta \in \operatorname{Dom}(\hat{h}) \equiv \pi_2^{-1}(\operatorname{Dom}(h_1)) \subset \mathbb{C}/\mathbb{Z}),$$

then there exists a sequence of disjoint topological discs $W_j \subset \mathbb{C}/\mathbb{Z}$ *satisfying:*
$W_0 = W$ *or* W';
$W_j \subset \operatorname{Dom}(\hat{h})$, $\hat{h}(W_j) = W_{j-1}$ *and* $\hat{h}|_{W_j}$ *is injective* $(j \ge 1)$;
for any $K > 0$, $W_j \subset \{\zeta \in \mathbb{C}/\mathbb{Z} \mid |\operatorname{Im}\zeta| > K\}$, *for large* j;
$\operatorname{diam} W_j < 1/2$, $\operatorname{dist}(W_j, W_{j+1}) < C_0$, *and*
$C_1 < |(\hat{h}^j)'| < C_1'$ *on* W_j, *for* $j \ge 0$.

Proof. Let us first consider $h = e^{2\pi i\beta}h_0$ with $|\operatorname{Im}\beta| \le b$, and $\hat{h} = \pi_2^{-1} \circ h \circ \pi_2$. We denote by $\tilde{B}^u$ and $\tilde{B}^\ell$, as in Section 5, the upper and the lower domains of definition of $\tilde{\mathcal{E}}_{g_0}$ (not for $\tilde{g}_0 = \tilde{\mathcal{E}}_{f_0}$!) and also define $B^u = \pi(\tilde{B}^u)$ and $B^\ell = \pi(\tilde{B}^\ell)$

Since B^u and B^ℓ are disjoint, at least one of them, say B^u, does not contain the unique critical value of h. Hence, by Lemma 5.3, the local inverse of h near 0 can be extended to $B^u \cup \{0\}$. So we have an analytic function H: $B^u \cup \{0\} \to B^u \cup \{0\}$ such that $h \circ H = \mathrm{id}$, $H(0) = 0$ and $|H'(0)| < 1$ by Schwarz' lemma. Then it is well known (see [Mi]) that there exists a linearizing coordinate $L(z)$ such that L is conformal near 0, $L(0) = 0$, $L'(0) = 1$ and $L \circ H(z) = H'(0) \cdot L(z)$ near 0. Passing to the $\mathbb{C}/\mathbb{Z}$ model, where we denote $\hat{H} = \pi_2^{-1} \circ H \circ \pi_2$, we obtain the following:

(a) There exist constants $y_0 \in \mathbb{R}$, $C_1'', C_2'' > 0$ and an analytic function $\hat{L}$: $Y = \{\zeta \in \mathbb{C}/\mathbb{Z} \mid \operatorname{Im}\zeta > y_0\} \to \mathbb{C}/\mathbb{Z}$ such that in Y, $\hat{H}$ is defined, $0 < \operatorname{Im}(\hat{H}(\zeta) - \zeta) < C_1''$, $\hat{L} \circ \hat{H} = \hat{L} + a$, where $a = \frac{1}{2\pi i}\log H'(0)$ and $\operatorname{Im} a > 0$, and $C_2''^{-1} < |\hat{L}'| < C_2''$.

Note that in the case where B^ℓ does not contain the critical value, we obtain a similar result with $\operatorname{Im}(\cdot)$ replaced by $-\operatorname{Im}(\cdot)$.

As for $\hat{\zeta}_0$ and $\hat{\zeta}_0'$, at least one of them, say $\hat{\zeta}_0$, is not the critical value. So pick $\hat{\zeta}_1 \in \hat{h}^{-1}(\hat{\zeta}_0) \cap B^u$ and let $\hat{\zeta}_j = H^{j-1}(\hat{\zeta}_1)$ $(j \ge 2)$. Then there exists $j_0 \ge 1$ such that $\zeta_{j_0} \in Y$.

(b) There exist $j_0 \ge 1$, a small disc neighborhood W_0 of $\hat{\zeta}_0$ (or $\hat{\zeta}_0'$) and its inverse image $W_1, \dots, W_{j_0}$ such that $\overline{W}_j$ $(j = 0, \dots, j_0)$ are disjoint, are contained in B^u, and contain no critical points of $\hat{h}$; $\hat{h}$ maps W_j onto W_{j-1} bijectively $(j = 1, \dots, j_0)$; $W_0 \cap Y = \emptyset$ and $W_{j_0} \subset Y$.

Note that the properties (a) and (b) are stable under a perturbation, i.e., (a) and (b) still hold for $h = e^{2\pi i\beta'}h_1$ with h_1 near h_0 and β' near β, and the constants are uniform in the neighborhood. Then, by the compactness of $\{\beta \in \mathbb{C}/\mathbb{Z} \mid |\operatorname{Im}\beta| \le b\}$, there exist a neighborhood $\mathcal{N}_2$ of h_0 disjoint discs W, W' in $\mathbb{C}/\mathbb{Z}$ containing $\hat{\zeta}_0$, $\hat{\zeta}_0'$, respectively, such that for $h_1 \in \mathcal{N}_2$ and

β with $|\operatorname{Im}\beta| \le b$, (a) and (b) hold with uniform constants and $W_0 = W$ or W'. Moreover we may assume that there are also uniform estimates for $(\hat{h}^j|_{W_j})'$, $\operatorname{diam} W_j$, $\operatorname{dist}(W_j, W_{j+1})$ $(0 \le j \le j_0)$. Now define $W_j = H^{j-j_0}(W_{j_0})$ $(j > j_0)$. Then we have uniform estimates on $(\hat{h}^j|_{W_j})'$ etc., since $|(\hat{h}^{j-j_0})'(\zeta)| = |\hat{L}'(\zeta)| \cdot |\hat{L}'(\hat{h}^{j-j_0}(\zeta))|^{-1}$. The rest of statements can be checked easily. □

Step 3. *Many* U_i*'s.* The results in Section 4 apply both to f_0 and to g_0. So let us denote the objects φ_f, $\mathcal{R}_f$, $\mathcal{N}_1(\eta)$, etc. in Section 4.2, by φ_f, $\mathcal{R}_f$, $\mathcal{N}_1(f_0, \eta)$ for f_0, f, and ψ_g, $\mathcal{R}_g$, $\mathcal{N}_1(g_0, \eta)$ for g_0, g.

Let W_j be the discs in Lemma 6.2. Then there exists a constant $\gamma > 0$ (depending only on C_0) such that for large $\eta > 0$, if $f \in \mathcal{N}_1(f_0, \eta) \cap \mathcal{F}_1$ and $g \in N_1(g_0, e^{2\pi\eta}) \cap \mathcal{F}_1$, then there exist disjoint topological discs $U_1, \dots, U_N$, where $N \ge \gamma\eta(e^{2\pi\eta})^2$, satisfying:

$\overline{U}_i \subset \varphi_f(D(\eta)) \subset D'(\eta)$;

$V_i = \pi \circ (\varphi_f|_{D(\eta)})^{-1}(U_i) \subset \psi_g(D(\eta')) \subset D'(\eta')$, where $\eta' = \pm e^{2\pi\eta}$;

$W_{j(i)} = \pi_1 \circ (\psi_g|_{D(\eta')})^{-1}(V_i)$ for some $j(i) \in \mathbb{N}$.

Proof. Let us denote $\varphi_f^* = \tau_0^{-1} \circ \varphi_f$, $\psi_f^* = \tau_0^{-1} \circ \psi_f$ and $\pi^* = \tau_0^{-1} \circ \pi$. By Lemma 6.2, the W_i's tend to the upper or lower end of $\mathbb{C}/\mathbb{Z}$. Suppose they tend to the upper end, and let $\eta' = e^{2\pi i\eta}$. (In the other case we take $\eta' = -e^{2\pi i\eta}$.)

Then the disc $D(\eta')$ contains entirely at least $\gamma'(\eta')^2$ components of the $\pi_1^{-1}(W_i)$'s for some constant $\gamma' > 0$, since the area of $D(\eta')$ is $(\pi/16)(\eta')^2$. By (4.2.5), ψ_g maps these components into $D'(\eta')$ injectively. The inverse image of $D'(\eta')$ by π^* consists of components, each of which is contained in a "box"

$$\left\{ z \in \mathbb{C} \mid n < \operatorname{Re} z < n+1,\ \eta + \frac{1}{2\pi}\log\frac{1}{2} < \operatorname{Im} z < \eta + \frac{1}{2\pi}\log\frac{3}{2} \right\}$$

for some $n \in \mathbb{Z}$. So $D(\eta)$ contains at least $\eta/3$ of these components, for η large. Finally φ_f^* maps $D(\eta)$ into $D'(\eta)$ injectively. As for the components of $\pi_1^{-1}(W_i)$'s, they have at least $(\gamma'/3)\eta(\eta')^2$ entire preimages by $\pi_1 \circ (\psi_g^*|_{D(\eta')})^{-1} \circ \pi^* \circ (\varphi_f|_{D(\eta)})^{-1}$. And this proves the above assertion, since $(\psi_g^*|_{D(\eta')})^{-1} \circ \pi^* = (\psi_g|_{D(\eta')})^{-1} \circ \pi$. □

Let us denote $U_i^* = \tau_0^{-1}(U_i)$, $\tilde{V}_i = (\varphi_f|_{D(\eta)})^{-1}(U_i)$, $V_i^* = \tau_0^{-1}(V_i)$. Then we have maps

$$U_i \xrightarrow{\tau_0^{-1}} U_i^* \xrightarrow{(\varphi_f^*|_{D(\eta)})^{-1}} \tilde{V}_i \xrightarrow{\pi^*} V_i^* \xrightarrow{\pi_1 \circ (\psi_g^*|_{D(\eta')})^{-1}} W_{j(i)}$$

and the estimates on the derivatives

$$\begin{aligned} \frac{1}{2} &< \left|[(\varphi_f^*|_{D(\eta)})^{-1}]'\right| < 2, \frac{1}{2} < \left|[\pi_1 \circ (\psi_g^*|_{D(\eta')})^{-1}]'\right| < 2, \\ \frac{1}{4}\eta^2 &< |(\tau_0^{-1}|_{U_i})'| < \frac{9}{4}\eta^2 \text{ and } \pi e^{2\pi\eta} < |(\pi^*|_{\tilde{V}_i})'| < 3\pi e^{2\pi\eta}. \end{aligned}$$

For the last estimate, use the fact that $\tilde{V}_i$ is contained in a box as above.

Step 4. *From* W_0 *to* U. There exists an integer $k \geq 0$ such that the z_j ($j > k$) are not critical points of f_0. Let $\pi^{(-1)}$ be the local inverse of π near w_0 such that $\pi^{(-1)}(w_0) = \tilde{w}_0 - k$. Similarly let $\pi_1^{(-1)}$ be the local inverse of π_1 near ζ_0 and ζ_0' such that $\pi_1^{(-1)}(\zeta_0) = \tilde{\zeta}_0$ and $\pi_1^{(-1)}(\zeta_0') = \tilde{\zeta}_0'$.

Note that in Lemma 6.2, we may take W, W' smaller without changing the statement. By Lemma 5.1, we have $\varphi_0'(\tilde{w}_0 - k) \neq 0$, $\psi_0'(\tilde{\zeta}_0) \neq 0$ and $\psi_0'(\tilde{\zeta}_0') \neq 0$.

Then it follows that we can change W, W' smaller, so that there exist a small neighborhood U of z_k, neighborhoods $\mathcal{N}_3(f_0)$, $\mathcal{N}_3(g_0)$ of f_0, g_0 and constants C_2, C_2' such that for $f \in \mathcal{N}_3(f_0) \cap \mathcal{F}_1$ and $g \in \mathcal{N}_3(f_0) \cap \mathcal{F}_1$,

$\varphi_f \circ \pi^{(-1)} \circ \psi_g \circ \pi_1^{(-1)}$ is defined and injective on W and on W';

both of the images of W and of W' cover U;

the derivative has a bound $C_2 < |(\varphi_f \circ \pi^{(-1)} \circ \psi_g \pi_1^{(-1)})'| < C_2'$ on $W \cup W'$.

Step 5. *Last k iterate.* Let U be as above and $\nu = \deg_{z_k} f_0^k$. Then it can be easily seen that there exists a neighborhood $\mathcal{N}_4(f_0, \eta)$ of f_0, depending on large $\eta > 0$, such that for $f \in \mathcal{N}_4(f_0, \eta)$, there exists an open set $\mathcal{U}' \subset U$ such that $f^k \colon \mathcal{U}' \to \tau_0(D'(\eta))$ is bijective and

$$C_3 \left(\frac{1}{\eta}\right)^{\frac{\nu-1}{\nu}} < |(f^k)'| < C_3' \left(\frac{1}{\eta}\right)^{\frac{\nu-1}{\nu}} \quad \text{on } \mathcal{U}',$$

where $C_3, C_3' > 0$ are constants independent of η.

Step 6. *A hyperbolic set* X_f. Let f_0 be a rational map belonging to the class $\mathcal{F}_0$, $b > 0$ and $\eta > 0$ large. First note that if an analytic function is close to f_0, then it can be conjugated to a function in $\mathcal{F}$ close to f_0 by a translation near id. So for Theorem 2, we only need to consider the functions in $\mathcal{F}$.

Suppose that $f \in \mathcal{F}$ is close to f_0 and

$$\alpha(f) = \frac{1}{a_1 - \frac{1}{a_2 + \beta}}$$

with large positive integers a_1, a_2 and $\beta \in \mathbb{C}$ satisfying $0 \leq \operatorname{Re}\beta < 1$, $|\operatorname{Im}\beta| \leq b$. Other cases with different signes in the expression of $\alpha(f)$ can be treated similarly, by using the complex conjugates or by reformulating the procedure for the lower end of $\mathbb{C}/\mathbb{Z}$ instead of the upper end. See Remark (4.3.1).

If f is close to f_0 and a_1 is large, then $|\arg \alpha(f)| < \pi/4$, $\mathcal{R}_f$ is defined and $e^{2\pi i/\alpha(f)}\mathcal{R}_f$ is close to $g_0 = \mathcal{E}_{f_0}$ (or $\hat{\mathcal{R}}_f + \frac{1}{\alpha(f)}$ is close to $\hat{g}_0 = \hat{\mathcal{E}}_{f_0}$). Let us denote $g = \mathcal{R}_f$. If, moreover, a_2 is sufficiently large, then g itself is close to g_0, $|\arg \alpha(g)| = |\arg \frac{1}{a_2+\beta}| < \pi/4$, $h = \mathcal{R}_g$ exists and $h_1 = e^{2\pi i/\alpha(g)}\mathcal{R}_g = e^{2\pi i\beta}h$ is close to $h_0 = \mathcal{E}_{g_0}$. (Note here that, by (4.2.2), $\alpha(g) \equiv -1/\alpha(f) \equiv 1/(a_2 + \beta) \pmod{\mathbb{Z}}$.) Therefore, for f close to f_0, a_1, a_2 large, we have $f \in \mathcal{N}_1(f_0, \eta) \cap N_3(f_0) \cap N_4(f_0, \eta)$, $g \in \mathcal{N}_1(g_0, |\eta'|) \cap N_3(g_0)$ and $h_1 \in \mathcal{N}_2$. In

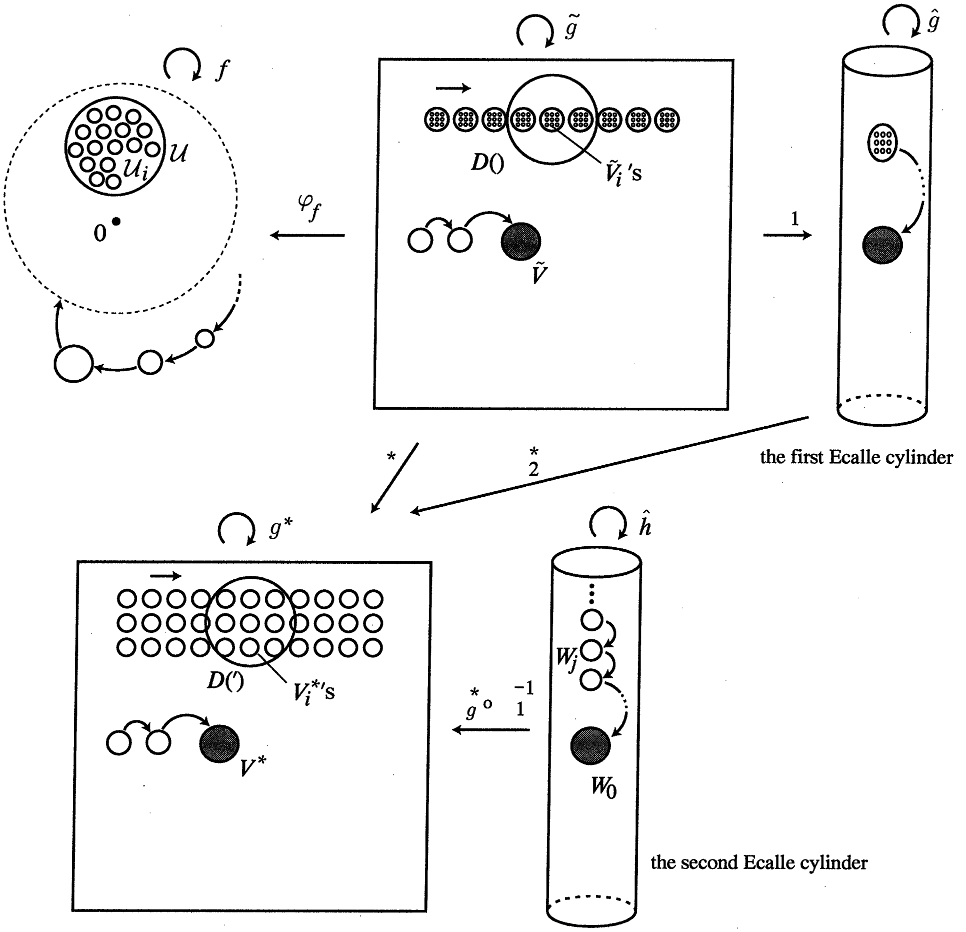

FIGURE 7.

particular, we can apply Lemma 6.2 to $\hat{h} = \pi_2^{-1} \circ h \circ \pi_2$ to obtain W_j. Then let U_i, U be as in Steps 3 and 4.

Now let us consider the following sequence of maps:

(*)

$$\begin{array}{ccccccccc} U_i & \xrightarrow{\tau_0^{-1}} & U_i^* & \xrightarrow{(\varphi_f^*|_{D(\eta)})^{-1}} & \tilde{V}_i & \xrightarrow{\pi^*} & V_i^* & \xrightarrow{\pi_1 \circ (\psi_g^*|_{D(\eta')})^{-1}} & W_{j(i)} \\ & & & & & & & & \big\downarrow \hat{h}^{j(i)} \\ f^k(U) & \xleftarrow{f^k} & U \subset U' & \xleftarrow{\varphi_f \circ \pi^{(-1)}} & V & & \xleftarrow{\psi_g \circ \pi_1^{(-1)}} & & W_0 \end{array}$$

where $V = \psi_g \circ \pi_1^{(-1)}(W_0)$ and $U' = \varphi_f \circ \pi^{(-1)}(V)$. Note that all maps except the last f^k are injective. Write $\mathcal{U} = \tau_0(D'(\eta))$ and let $\mathcal{U}' \subset U$ be the set obtained in Step 5. Then tracing the inverse image of $\mathcal{U}$ via $\mathcal{U}'$ by the maps (*), we obtain $\mathcal{U}_i \subset U_i$.

Moreover the composition of (*) on $\mathcal{U}_i$ is equal to f^{n_i} for some integer $n_i \geq 1$ $(i = 1, \ldots, N)$. In fact, by (4.2.3), $g^{m_i} = \psi_g \circ \pi_1^{(-1)} \circ \hat{h}^{j(i)} \circ \pi_1 \circ (\psi_g|_{D(\eta')})^{-1}$ on V_i, since $|\arg(z' + \frac{1}{2\alpha(g)} - z)| < 2\pi/3$ for $z \in D(\eta')$, $z' \in \pi_1^{(-1)}(W \cup W')$, if $|\eta'|$ is large. Similarly, we have $f^{n_i'} = \varphi_f \circ \pi^{(-1)} \circ g^{m_i} \circ \pi \circ (\varphi_f|_{D(\eta)})^{-1}$ on U_i, for some $n_i' \geq 1$. And recall that $\varphi_f^* = \tau_0^{-1} \circ \varphi_f$, $\pi^* = \tau_0^{-1} \circ \pi$, etc.

Thus we obtained $(f, \mathcal{U}, \mathcal{U}_i)$ as described in Section 2, i.e., $\overline{\mathcal{U}}_i \subset \mathcal{U}$ such that the $\mathcal{U}_i$'s are disjoint, simply connected subsets of $\mathcal{U}$, $f^{n_i}\colon \mathcal{U}_i \to \mathcal{U}$ is bijective $(i = 1, \ldots, N)$.

Now, combining all the above estimates, we have

$$N \geq \gamma \eta (e^{2\pi\eta})^2,$$

and

$$|(f^{n_i}|_{\mathcal{U}_i})'| \leq C \eta^{1+1/\nu} e^{2\pi\eta},$$

where γ and C are positive constants independent of η.

Hence by Lemmas 2.1 and 2.2, we have a hyperbolic subset X_f for f and an estimate for the Hausdorff dimension

$$\delta = \text{H-dim}\, X_f \geq \frac{\log \gamma + \log \eta + 4\pi\eta}{\log C + (1 + \frac{1}{\nu}) \log \eta + 2\pi\eta}.$$

The right-hand side tends to 2 as $\eta \to \infty$. Thus we have proved Theorem 2 in the case $f_0'(0) = 1$. □

Remarks. (i) Note that the maps in (*) other than $f^k|_U$, π^*, $\tau_0^{-1}|_{U_i}$ have bounded derivatives, and that the effect of π^* dominates others.

(ii) The procedure $\mathcal{F}_0 \ni f_0 \longmapsto g_0 = \mathcal{E}_{f_0} \in \mathcal{F}_0$ can be considered as a renormalization. It it related, as we have seen, to the return map of f near f_0. One can interpret $\pi \circ \varphi_0^{-1}$ as a correspondence between the phase spaces of f_0 and "its renormalization" g_0. Also, it has an exponential effect, because f moves points extremely slowly near 0 and requires a large number of iterates for the return map. In this sense, the renormalization procedure is essential in the above estimate.

(iii) It will be instructive to make a "caricature" (proposed by Curt McMullen) to understand the situation. For simplicity, assume $k = 0$ $(\nu = 1)$, and pretend that the maps in (*) other than τ_0^{-1} and π^* were affine. In particular, $\hat{h}$ is supposed to be a translation on $\mathbb{C}/\mathbb{Z}$. So it produces a one-dimensional array of discs W_j; then π_1^{-1} unwraps them to a two-dimensional array. These discs are squeezed by $(\pi^*)^{-1}$, and finally inverted by τ_0. One can show that an invariant set produced by this system has dimension two.

(iv) Note that the map f_0 need not be a rational map. In fact, it is enough to assume that $f_0 \in \mathcal{F}_0$ and 0 has an inverse orbit $\{z_j\}$ as in Step 0.

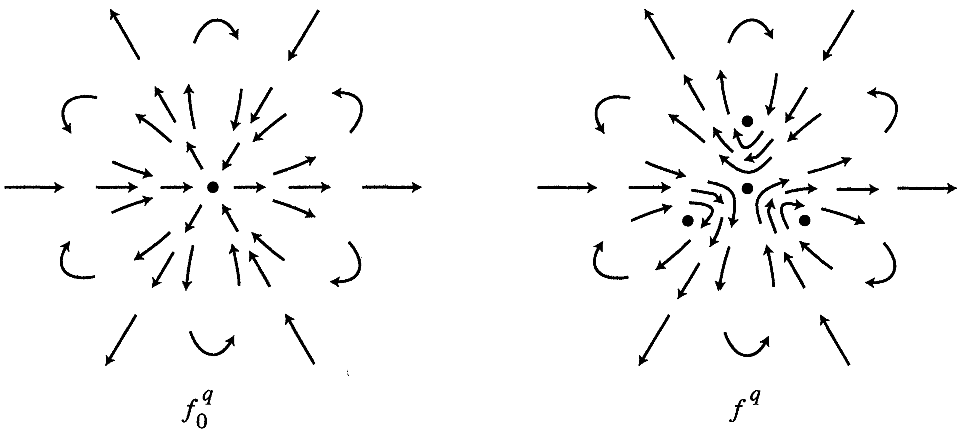

FIGURE 8.

7. Parabolic fixed points with multiplier $\neq 1$

Let us consider an analytic function $f_0(z)$ near 0 such that

$$f_0(0) = 0, f_0'(0) = \exp(2\pi i p/q),$$

where $p, q \in \mathbb{Z}$, $q > 1$ and $(p, q) = 1$. It is known that if $f_0^q(z) \not\equiv z$, then it has an expansion of the form

$$f_0^q(z) = z + a_{\nu q+1} z^{\nu q+1} + O(z^{\nu q+2}),$$

where ν is a positive integer and $a_{\nu q+1} \neq 0$. In the following, we assume that $\nu = 1$; in other words, $(f_0^q)^{(q+1)} \neq 0$. Then, as before, we may assume that $a_{q+1} = 1$. The dynamics of f_0 and its perturbation are shown in Figure 8.

To analyze the bifurcation, we need to consider q incoming and q outgoing Ecalle cylinders $\mathcal{C}_0^{k,+}, \mathcal{C}_0^{k,-}$ $(k \in \mathbb{Z}/q\mathbb{Z})$; see Figure 9. Now the Ecalle transformations map the upper end of $\mathcal{C}_0^{k,-}$ to that of $\mathcal{C}_0^{k,+}$, the lower end of $\mathcal{C}_0^{k,-}$ to that of $\mathcal{C}_0^{k-1,+}$.

Consequently, the statements of Sections 4–6 should be changed as follows. We only note the part which is to be changed.

Changes in Section 4 (4.1).

(4.1.1) There are $2q$ regions $\Omega_+^{(k)}, \Omega_-^{(k)}$ $(k \in \mathbb{Z}/q\mathbb{Z})$ instead of two regions Ω_+, Ω_-;

$\cup_k \Omega_+^{(k)} \cup \Omega_-^{(k)} \cup \{0\}$ is a neighborhood of 0, on which f_0 is injective;

$f_0(\overline{\Omega}_+^{(k)}) \subset \Omega_+^{(k+p)} \cup \{0\}$ and $f_0(\Omega_-^{(k)} \cup \{0\}) \supset \overline{\Omega}_-^{(k+p)}$;

$\Omega_+^{(j)} \cap \Omega_-^{(k)}$ is nonempty and connected, if $j = k$ or $j = k - 1$, it is empty otherwise;

$f_0^n \to 0$ as $n \to \infty$ uniformly on $\Omega_+^{(k)}$.

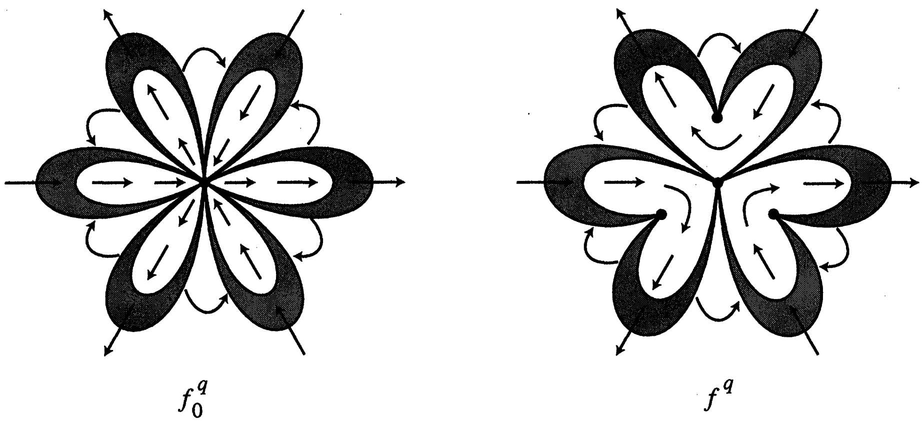

FIGURE 9.

The parabolic basins $\mathcal{B}^{(k)}$ are defined to be $\cup_{n\geq 0} f^{-nq}(\Omega_+^{(k)})$; then a point belongs to $\cup_k \mathcal{B}^{(k)}$ if and only if it has a neighborhood on which f^n $(n = 1, 2, \dots)$ are defined and $f^{nq} \to 0$ uniformly as $n \to \infty$.

Let us fix a $k \in \mathbb{Z}/q\mathbb{Z}$.

(4.1.2) φ_0, f_0, Ω_- should be replaced by $\varphi_0^{(k)}$, f_0^q, $\Omega_-^{(k)}$. As for φ_0^*, define

$$\varphi_0^{(k)*}(w) = -\frac{1}{q(\varphi_0^{(k)})^q};$$

then

$$\varphi_0^{(k)*}(w) = w + O\left(w^{\frac{q-1}{q}}\right).$$

(4.1.3) Φ_0, $\mathcal{B}$, f_0, Ω_+ are to be replaced by $\Phi_0^{(k)}$, $\mathcal{B}^{(k)}$, f_0^q, $\Omega_+^{(k)}$.

(4.1.4) Let $\tilde{\mathcal{B}}^{(k,u)} = (\varphi_0^{(k)})^{-1}(\mathcal{B}^{(k)})$ and $\tilde{\mathcal{B}}^{(k,\ell)} = (\varphi_0^{(k)})^{-1}(\mathcal{B}^{(k-1)})$. Then $\tilde{\mathcal{B}}^{(k,u)}$, $\tilde{\mathcal{B}}^{(k,\ell)}$ are invariant under T and $\tilde{\mathcal{B}}^{(k,u)} \supset \{w \mid \operatorname{Im} w > \eta_0\}$, $\tilde{\mathcal{B}}^{(k,\ell)} \supset \{w \mid \operatorname{Im} w < -\eta_0\}$ for some $\eta_0 > 0$.

Define $\tilde{\mathcal{E}}_{f_0}^{(k,u)}\colon \tilde{\mathcal{B}}^{(k,u)} \to \mathbb{C}$, $\tilde{\mathcal{E}}_{f_0}^{(k,\ell)}\colon \tilde{\mathcal{B}}^{(k,\ell)} \to \mathbb{C}$ by

$$\tilde{\mathcal{E}}_{f_0}^{(k,u)} = \Phi_0^{(k)} \circ \varphi_0^{(k)} \quad \text{and} \quad \tilde{\mathcal{E}}_{f_0}^{(k,\ell)} = \Phi_0^{(k-1)} \circ \varphi_0^{(k)}.$$

They satisfy

$$\tilde{\mathcal{E}}_{f_0}^{(k,u)}(w+1) = \tilde{\mathcal{E}}_{f_0}^{(k,u)}(w) + 1 \quad \text{for } w \in \tilde{\mathcal{B}}^{(k,u)}, \text{ etc;}$$

$\mathcal{E}_{f_0}^{(k,u)} = \pi \circ \tilde{\mathcal{E}}_{f_0}^{(k,u)} \circ \pi^{-1}\colon \pi(\tilde{\mathcal{B}}) \cup \{0\} \to \mathbb{C}$ is well-defined and analytic, and $\mathcal{E}_{f_0}^{(k,u)\prime}(0) \neq 0$. Similarly, $\mathcal{E}_{f_0}^{(k,\ell)} = \pi \circ \tilde{\mathcal{E}}_{f_0}^{(k,\ell)} \circ \pi^{-1}\colon \pi(\tilde{\mathcal{B}}) \cup \{\infty\} \to \overline{\mathbb{C}} - \{0\}$ is analytic, and $\mathcal{E}_{f_0}^{(k,\ell)\prime}(\infty) \neq 0$.

In 4.2. *Perturbation.* Let

$$\mathcal{F}_1 = \left\{ f \in \mathcal{F} \mid f'(0) = \exp\left(2\pi i \frac{p+\alpha}{q}\right) \text{ with } \alpha \neq 0 \text{ and } |\arg \alpha| < \pi/4 \right\},$$

and denote

$$f'(0) = \exp\left(2\pi i \frac{p+\alpha(f)}{q}\right).$$

The periodic points of f of period q near 0 are labelled so that

$$\sigma^{(k)}(f) = (-2\pi i q \alpha(f))^{1/q} e^{2\pi i k/q}(1 + o(1)) \ \text{ as } f \to f_0,$$

where $|\arg(-2\pi i \alpha(f))^{1/q}| < \pi/q$; then $f(\sigma^{(k)}(f)) = \sigma^{(k+p)}(f)$.

The functions φ_f, $\mathcal{R}_f$, $\mathcal{E}_{f_0}$ are to be replaced by $\varphi_f^{(k)}$, $\mathcal{R}_f^{(k,u)}$, $\mathcal{E}_{f_0}^{(k,u)}$. The functionnal equation for φ_f becomes

$$\varphi_f^{(k)}(w+1) = f^q \circ \varphi_f^{(k)}(w).$$

In (4.2.3) and (4.3.4), f^n is to be replaced by f^{nq+r}, where r is an integer such that $0 < r < n$ and $rp \equiv -1 \pmod{q}$. Finally

$$\varphi_f^{(k)*}(w) = -\frac{1}{q(\varphi_f(w))^q}$$

satisfies (4.2.5).

In 4.3. *Remarks*: (4.3.1) If $f \in \mathcal{F}$ satisfies $3\pi/4 < \arg \alpha(f) < 5\pi/4$, then we use $\mathcal{R}_f^{(k,\ell)}$, $\mathcal{E}_{f_0}^{(k,\ell)}$ instead of $\mathcal{R}_f^{(k,u)}$, $\mathcal{E}_{f_0}^{(k,u)}$.

Changes in Section 5. Let f_0 be as in Section 4. Then f_0 has q parabolic basins $\mathcal{B}^{(k)}$ ($k \in \mathbb{Z}/q\mathbb{Z}$). Let $\mathcal{F}_0$ be the set of such functions $f_0 \in \mathcal{F}$ having an immediate parabolic basin $B^{(k)}$ in each $\mathcal{B}^{(k)}$, containing only one critical point of f_0^q. Let $\tilde{B}^{(k,u)}$ be the component of $\tilde{\mathcal{B}}^{(k,u)}$ containing $\{w \mid \operatorname{Im} w > \eta_0\}$, and $B^{(k,u)} = \pi(\tilde{\mathcal{B}}^{(k,u)})$.

Then $g_0 = \mathcal{E}_{f_0}^{(k,u)}$: $B^{(k,u)} \to \mathbb{C}^*$ is a branched covering of infinite degree, ramified only over one point (Proposition 5.3).

Other statements are similar.

Changes in Section 6. Note that g_0 is in $\mathcal{F}_0$ in the sense of Section 4; that is, $g_0'(0) = 1$. So we only need to change φ_f as above and $\tau_0^{-1}|_{U_i}$ to

$$z \mapsto -\frac{1}{qz^q}.$$

Hence in Step 3, the estimate on the derivative of τ_0^{-1} should be replaced by

$$C\eta^{q+1} < (\tau_0^{-1}|_{U_i})' < C'\eta^{q+1},$$

and in Step 6, the estimate on $(f^{n_i}|_{\mathcal{U}_i})'$ becomes

$$|(f^{n_i}|_{\mathcal{U}_i})'| \le C\eta^{q+1/\nu}e^{2\pi\eta}.$$

This is enough to prove that H-dim $X_f \to 2$ as $\eta \to \infty$.

Appendix. Proof of the properties of Ecalle cylinders

In this appendix, we give the proof or comments for the facts which were stated in Sections 4 and 7. Note that the facts in Section 4.1 can be found in [Mi] and most of those in Section 4.2 in [DH]. We complete 4.2 by introducing a new coordinate, which makes clearer the relationship between the return map and the renormalization of functions with irrationally indifferent fixed points (cf. Yoccoz [Y]).

A.1. *Coordinate changes.* Let $f_0(z) = z + z^2 + \cdots$ be as in Section 4.1. We introduce a new coordinate w by $z = -1/w = \tau_0(w)$; then f_0 corresponds to the map F_0 of the form

$$F_0(w) = w + 1 + O(1/w). \tag{A.1.1}$$

For functions near f_0, we introduce a new coordinate by the following.

LEMMA A.1.2. *There exist a neighborhood $\mathcal{N}'$ of f_0 in $\mathcal{F}$ and a neighbourhood $\mathcal{V}$ of* 0 *in* $\mathbb{C}$ *such that if $f \in \mathcal{N}'$ then $\overline{\mathcal{V}} \subset \mathrm{Dom}(f)$ and $f(z)$ can be expressed as*

$$f(z) = z + z(z-\sigma)u(z), \tag{A.1.3}$$

where $\sigma = \sigma(f)$ is a point in $\mathcal{V}$ and $u(z) = u_f(z)$ is a nonzero holomorphic function defined in a neighborhood of $\overline{\mathcal{V}}$. Hence 0 *and $\sigma(f)$ are the only fixed points of f in $\mathcal{V}$. Moreover $\sigma(f_0) = 0$, $u_{f_0}(z) = (f_0(z) - z)/z^2$, $e^{2\pi i\alpha(f)} = f'(0) = 1 - \sigma(f)u_f(0)$; hence*

$$\sigma(f) = -2\pi i\alpha(f)(1 + o(1)) \quad \text{as } f \to f_0. \tag{A.1.4}$$

The correspondences $f \mapsto \sigma(f)$, $f \mapsto u_f(z)$ (with $\mathrm{Dom}(u_f) = \mathcal{V}$ fixed) are continuous (with respect to the topology defined in Section 4.0).

The proof is left to the reader.

For a function $f \in \mathcal{N}'$ with $\alpha(f) \ne 0$ (i.e. $\sigma(f) \ne 0$), let us introduce a new coordinate $w \in \mathbb{C}$ by

$$z = \tau_f(w) \equiv \frac{\sigma}{1 - e^{-2\pi i\alpha w}}, \tag{A.1.5}$$

where $\sigma = \sigma(f)$ and $\alpha = \alpha(f)$. Define the map $F_f(w)$ by

$$F_f(w) = w + \frac{1}{2\pi i\alpha} \log\left(1 - \frac{\sigma u(z)}{1 + zu(z)}\right) \quad \text{with } z = \tau_f(w) \tag{A.1.6}$$

and

$$T_f(w) = w - \frac{1}{\alpha}.$$

Here $F_f(w)$ is defined for w such that $|\sigma u(z)/(1 + zu(z))| \leq 1/2$, and the above formula defines a single-valued function using the branch of logarithm with $-\pi < \operatorname{Im}\log(\cdot) \leq \pi$.

LEMMA A.1.7 (properties of F_f, τ_f and T_f). *There exist $R_0 > 0$ and a neighborhood $\mathcal{N} \subset \mathcal{N}'$ of f_0 such that if $f \in \mathcal{N}$ and $\alpha(f) \neq 0$, then:*

(i) *The map τ_f: $\mathbb{C} \to \overline{\mathbb{C}} - \{0,\sigma(f)\}$, $w \mapsto z = \tau_f(w)$, is a universal covering, whose covering transformation group is generated by T_f; $\tau_f(w) \to 0$ as $\operatorname{Im}\alpha w \to \infty$, and $\tau_f(w) \to \sigma$ as $\operatorname{Im}\alpha w \to -\infty$;*

(ii) *If*

$$w \in \mathbb{C} - \bigcup_{n\in\mathbb{Z}} T_f^n D_{R_0}, \quad \text{where } D_{R_0} = \{w' \mid |w'| < R_0\},$$

then $\tau_f(w) \in \mathcal{V}$ and $|\sigma u(z)/(1 + zu(z))| \leq 1/2$; hence (A.1.6) *is well-defined and moreover satisfies*

$$|F_f(w) - (w + 1)| < \frac{1}{4}, \; |F_f'(w) - 1| < \frac{1}{4}.$$

Now, $F_f(w) = w + 1 + O(1/w^2)$ as $\operatorname{Im}\alpha w \to \infty$;

(iii) *$f \circ \tau_f = \tau_f \circ F_f$ and $T_f \circ F_f = F_f \circ T_f$;*

(iv) *When $f \to f_0$, $\tau_f(w) \to \tau_0(w)$ uniformly on $\{w \mid |\operatorname{Re}(\alpha w)| < \frac{3}{4}\} \setminus D_{R_0}$ and $F_f(w) \to F_0(w)$ uniformly on $\mathbb{C} - \cup_{n\in\mathbb{Z}} T_f^n D_{R_0}$.*

The proof is immediate by a computation and is left to the reader.

A.2. *General construction.* For $b_1, b_2 \in \mathbb{C}$ with $\operatorname{Re} b_1 < \operatorname{Re} b_2$, define

$$\mathcal{Q}(b_1, b_2) = \{z \in \mathbb{C} \mid \operatorname{Re}(z - b_1) > -|\operatorname{Im}(z - b_1)|, \; \operatorname{Re}(z - b_2) < |\operatorname{Im}(z - b_2)|\}.$$

If $b_1 = -\infty$ (resp. $b_2 = \infty$), the condition involving b_1 (resp. b_2) should be removed.

PROPOSITION A.2.1. *Let F be a holomorphic function defined in $\mathcal{Q} = \mathcal{Q}(b_1,b_2)$, where $\operatorname{Re} b_2 > \operatorname{Re} b_1 + 2$ (here b_1 or b_2 may be $-\infty$ or ∞). Suppose*

$$|F(z) - (z + 1)| < \frac{1}{4}, \quad \text{and } |F'(z) - 1| < \frac{1}{4} \quad \text{for } z \in \mathcal{Q}. \tag{A.2.2}$$

Then

(0) *F is univalent on $\mathcal{Q}$.*

(i) *Let* $z_0 \in \mathcal{Q}$ *be a point such that* $\operatorname{Re} b_1 < \operatorname{Re} z_0 < \operatorname{Re} b_2 - 5/4$. *Denote by* S *the closed region (a strip) bounded by the two curves* $\ell = \{z_0 + iy \mid y \in \mathbb{R}\}$ *and* $F(\ell)$. *Then for any* $z \in \mathcal{Q}$, *there exists a unique* $n \in \mathbb{Z}$ *such that* $F^n(z)$ *is defined and belongs to* $S - F(\ell)$.

(ii) *There exists a univalent function* Φ: $\mathcal{Q} \to \mathbb{C}$ *satisfying*

$$\Phi(F(z)) = \Phi(z) + 1$$

whenever both sides are defined. Moreover Φ *is unique up to addition of a constant.*

(iii) *If* Φ *is normalized by* $\Phi(z_0) = 0$, *where* $z_0 \in \mathcal{Q}$, *then the correspondence* $F \mapsto \Phi$ *is continuous with respect to the compact-open topology. (See also Section* 4.0. *Notation.)*

Proof. (0) and (i) are easy and left to the reader.

(ii) Let $z_0 \in \mathcal{Q}$ be as in (i). Define h_1: $\{z \mid 0 \le \operatorname{Re} z \le 1\} \to \mathcal{Q}$ by

$$h_1(x+iy) = (1-x)(z_0+iy) + xF(z_0+iy), \quad \text{for } 0 \le x \le 1,\ y \in \mathbb{R}.$$

Then

$$\frac{\partial h_1}{\partial x} = F(z_0+iy) - (z_0+iy),\ \frac{\partial h_1}{\partial y} = ixF'(z_0+iy) + i(1-x).$$

Hence

$$\left|\frac{\partial h_1}{\partial z} - 1\right| = \frac{1}{2}\Big|\{F(z_0+iy) - (z_0+iy+1)\} + x(F'(z_0+iy)-1)\Big| \le \frac{1}{4},$$

$$\left|\frac{\partial h_1}{\partial \bar{z}}\right| = \frac{1}{2}\Big|\{F(z_0+iy) - (z_0+iy+1)\} - x(F'(z_0+iy)-1)\Big| \le \frac{1}{4}.$$

Therefore $|\frac{\partial h_1}{\partial \bar{z}} / \frac{\partial h_1}{\partial z}| \le 1/3$ and h_1 is a quasiconformal mapping onto the strip S, and satisfies $h_1^{-1}(F(z)) = h_1^{-1}(z) + 1$ for $z \in \ell$. Let σ_0 be the standard conformal structure of $\mathbb{C}$, and take the pull-back $\sigma = h_1^* \sigma_0$ on $\{z \mid 0 \le \operatorname{Re} z \le 1\}$. Then extend σ to $\mathbb{C}$ by $\sigma = (T^n)^* \sigma$ on $\{z \mid -n \le \operatorname{Re} z \le -n+1\}$, where $T(z) = z+1$. By the Ahlfors-Bers measurable mapping theorem [A], there exists a unique quasiconformal mapping h_2: $\mathbb{C} \to \mathbb{C}$ such that $h_2^* \sigma_0 = \sigma$ and $h_2(0) = 0$, $h_2(1) = 1$. By the definition of σ, T preserves σ. Hence $h_2 \circ T \circ h_2^{-1}$ preserves the standard conformal structure σ_0; therefore it must be an affine function. Since T has no fixed point in $\mathbb{C}$, neither does $h_2 \circ T \circ h_2^{-1}$; hence it is a translation. Using $h_2 \circ T \circ h_2^{-1}(0) = 1$, we have $h_2 \circ T \circ h_2^{-1} = T$.

Now define Φ by $\Phi = h_2 \circ h_1^{-1}$ on S , and extend to the whole $\mathcal{Q}$ using the relation $\Phi(F(z)) = \Phi(z) + 1$. Then Φ is well-defined by (i), continuous and homeomorphic by the above relations on h_1^{-1} and h_2. Moreover Φ is analytic outside the orbit of ℓ, then analytic in the whole $\mathcal{Q}$ by Morera's theorem. Thus we have obtained the desired univalent function Φ.

If Φ' is another such function, then $\Phi''(z) = \Phi' \circ \Phi^{-1}$ commutes with T at least in $\Phi(\mathcal{Q})$. Hence $\Phi''(z) - z$ extends to $\mathbb{C}$ as a periodic function; then Φ'' extends to $\mathbb{C}$ as a holomorphic function commuting with T. Similarly Φ''^{-1} also has this property, and therefore Φ'' must be an affine function. However an affine function commuting with the translation T is also a translation by a constant. Hence the assertion follows.

(iii) Let us consider F and F_0 defined in the same $\mathcal{Q}$ and satisfying the condition of Proposition A.2.1. As in (ii), we can construct h_1, h_2, Φ for F and $h_{1,0}$, $h_{2,0}$, Φ_0 for F_0. It is easy to see that on any compact set $\{z \mid 0 \le \operatorname{Re} z \le 1\}$, $\frac{\partial h_1}{\partial \bar{z}} / \frac{\partial h_1}{\partial z} \to \frac{\partial h_{1,0}}{\partial \bar{z}} / \frac{\partial h_{1,0}}{\partial z}$ as $F \to F_0$. Hence $\sigma_F = h_1^* \sigma_0 \to \sigma_{F_0} = h_{1,0}^* \sigma_0$ and $h_2 \to h_{2,0}$ on any compact set as $F \to F_0$. It follows from the definition of the extension of Φ that $\Phi \to \Phi_0$ as $F \to F_0$. □

A strip S as in Proposition A.2.1 (i), is called a *fundamental region* for the map $F|_{\mathcal{Q}}$. The quotient space

$$\begin{aligned} \mathcal{C} &= S/_{\sim}, \quad \text{where } \ell \ni z \sim F(z) \in F(\ell) \\ &= \mathcal{Q}/_{\sim}, \quad \text{where } z \sim F(z) \text{ if } z \in \mathcal{Q} \cap F^{-1}(\mathcal{Q}) \end{aligned}$$

is topologically a cylinder which is called the *Ecalle cylinder*. Moreover, $\mathcal{C}$ has the natural structure of a Riemann surface, when F near ℓ is used as a coordinate patching.

LEMMA A.2.3. *Let F, $\mathcal{Q}$, S be as above. Then $\pi \circ \Phi$ induces an isomorphism*

$$\overline{\Phi}\colon \mathcal{C} = S/_{\sim} \to \mathbb{C}^* = \mathbb{C} - \{0\}.$$

Proof. It is easy to see from the construction that $\overline{\Phi}$ is a covering map and induces an isomorphism between the fundamental groups. □

LEMMA A.2.4. *Suppose that Φ and v are holomorphic functions in a region $\mathcal{U}$ satisfying:*

$$\Phi \text{ is univalent in } \mathcal{U}, \quad |v(z) - 1| < 1/4 \text{ for } z \in \mathcal{U} \text{ and}$$

$$\Phi(z + v(z)) = \Phi(z) + 1, \quad \text{if } z, z + v(z) \in \mathcal{U}.$$

(i) *There exist universal constants $R_1, C_1, C_2 > 0$ such that if $\mathcal{U} = \{z \mid |z - z_0| < R\}$ for $R \ge R_1$, then*

$$\left| \Phi'(z_0) - \frac{1}{v(z_0)} \right| \le C_1 \left(\frac{1}{R^2} + |v'(z_0)| \right) \le \frac{C_2}{R}.$$

(ii) *Suppose $\mathcal{U} = \{z \in \mathbb{C}^* \mid \theta_1 < \arg z < \theta_2\}$ $(\theta_2 < \theta_1 + 2\pi)$ and $|v'(z)| \le K/|z|^{1+\nu}$ $(z \in \mathcal{U})$ for some $K, \nu > 0$. For $z_0 \in \mathcal{U}$ and θ_1', θ_2' with $\theta_1 < \theta_1' < \theta_2' < \theta_2$, there exist $R_2, C_3 > 0$ and $\xi \in \mathbb{C}$ such that*

$$\left| \Phi(z) - \int_{z_0}^{z} \frac{d\zeta}{v(\zeta)} - \xi \right| \le C_3 \left(\frac{1}{|z|} + \frac{K}{|z|^{\nu}} \right),$$

for z satisfying $\theta_1' < \arg z < \theta_2'$, $\mathrm{dist}(z, \mathbb{C} - \mathcal{U}) > R_2$. *Moreover* C_3 *depend only on* θ_i, θ_i'.

See [Y] for a similar estimate.

Proof. In the following, C and C' denote universal constants, which may differ at each appearence.

(i) We may suppose $z_0 = 0$. We take R so that $R \gg 1$. It follows from Koebe's distortion theorem [P] that if $|z| < R - 2$, then

$$\frac{|v(z)|}{(1 + |v(z)|/2)^2} \le \left| \frac{\Phi(z + v(z)) - \Phi(z)}{\Phi'(z)} \right| \le \frac{|v(z)|}{(1 - |v(z)|/2)^2},$$

since Φ is univalent in $\{\zeta \mid |\zeta - z| < 2\}$. Hence $C \le |\Phi'(z)| \le C'$ if $|z| < R - 2$. We have $|\Phi''(z)| \le C/R$ if $|z| < R/2$. (In fact, by Cauchy's formula, $\Phi''(z)$ can be expressed in terms of an integral of $\Phi'(\zeta)/(\zeta - z)^2$ over the contour $\{\zeta \mid |\zeta - z| = R/3\}$, and use of the above estimate.) By the formula

$$\Phi(z + a) = \Phi(z) + a\Phi'(z) + a^2 \int_0^1 (1 - t)\Phi''(z + at)dt,$$

we obtain $|1 - \Phi'(z)v(z)| \le C/R$ if $|z| < R/2 - 5/4$. Again by Cauchy's formula, $|(1 - \Phi'(z)v(z))'| = |\Phi''(z)v(z) + \Phi'(z)v'(z)| \le C/R^2$ if $|z| < R/4$. It also follows from Cauchy's formula that $|v'(z)| \le C/R$ and $|v''(z)| \le C/R^2$ for $|z| < R/2$. Therefore $|v'(z)| \le |v'(0)| + C/R^2$ for $|z| < 5/4$. Hence we have $|\Phi''(z)| \le C(1/R^2 + |v'(0)|)$ for $|z| < 5/4$. By the above formula again, we have $|1 - \Phi'(0)v(0)| \le C(1/R^2 + |v'(0)|) \le C/R$. So we obtain the desired inequality.

(ii) Note that in the sector $\theta_1' < \arg z < \theta_2'$ we have $\mathrm{dist}(z, \mathbb{C} - \mathcal{U}) > C_4|z|$ for some constant $C_4 > 0$. So the result in (i) applies to the region $\{w \mid |w - z| < C_4|z|\}$, and gives

$$\left|\Phi'(z) - \frac{1}{v(z)}\right| \le C_1 \left(\frac{1}{(C_4|z|)^2} + |v'(z_0)| \right) \le C_5 \left(\frac{1}{|z|^2} + \frac{K}{|z|^{1+\nu}} \right).$$

Integrating this formula along a straight path from ∞ to z in the smaller sector, we have

$$\left| \int_\infty^z \left(\Phi'(\zeta) - \frac{1}{v(\zeta)} \right) d\zeta \right| \le C_3 \left(\frac{1}{|z|} + \frac{K}{|z|^\nu} \right),$$

where the integral does not depend on the choice of the path. Since

$$\int_\infty^z \left(\Phi'(\zeta) - \frac{1}{v(\zeta)} \right) d\zeta = \Phi(z) - \int_{z_0}^z \frac{d\zeta}{v(\zeta)} - \left(\Phi(z_0) + \int_{z_0}^\infty \left(\Phi'(\zeta) - \frac{1}{v(\zeta)} \right) d\zeta \right),$$

this completes the proof. □

A.3. *Fatou coordinates for* f_0. *Proof of* 4.1. Let $\mathcal{Q}_0^+ = \mathcal{Q}(\xi_1, \infty)$, $\mathcal{Q}_0^- = \mathcal{Q}(-\infty, -\xi_1)$ for $\xi_1 > 0$. If ξ_1 is large enough, F_0 satisfies (A.2.2) on $\mathcal{Q}_0^+$ and $\mathcal{Q}_0^-$. Hence by Proposition A.2.1, there exist univalent analytic functions $\Phi_{+,0}\colon \mathcal{Q}_0^+ \to \mathbb{C}$ and $\Phi_{-,0}\colon \mathcal{Q}_0^- \to \mathbb{C}$ satisfying $\Phi_{+,0}(F_0(w)) = \Phi_{+,0}(w) + 1$ and $\Phi_{-,0}(F_0(w)) = \Phi_{-,0}(w) + 1$. If

$$F_0(w) = w + 1 + \frac{a}{w} + O\left(\frac{1}{w^2}\right),$$

then by Lemma A.2.4 (ii),

$$\Phi_{\pm,0}(w) = w - a\log w + c_{\pm,0} + o(1)$$

as w tends to ∞ within a sector as in Lemma A.2.4 (ii) with $\mathcal{U} = \mathcal{Q}_0^\pm$, where $c_{\pm,0}$ are constants and the branches of logarithms for $\Phi_{\pm,0}$ are chosen so that they coincide in the upper component of $\mathcal{Q}_0^+ \cap \mathcal{Q}_0^-$ and differ by $2\pi i$ in the lower. Then for large $\xi_0 > 0$, $\mathcal{Q}_0$ as in (4.1.2) is contained in $\Phi_{-,0}(\mathcal{Q}_0^-)$. Define $\Omega_+ = \tau_0(T(\mathcal{Q}_0^+))$, $\Omega_- = \tau_0(\Phi_{-,0}^{-1}(\mathcal{Q}_0))$. The properties in (4.1.1) are easily verified. (If necessary, take a larger ξ_0.)

Let $\varphi_0 = \tau_0 \circ \Phi_{-,0}^{-1}$ and then $\varphi_0^* = \tau_0^{-1} \circ \varphi_0 = \Phi_{-,0}^{-1}$. Now by the above, $\mathcal{Q}_0 \subset \mathrm{Dom}(\varphi_0)$ and $\varphi_0(w+1) = f_0 \circ \varphi_0(w)$ if $w, w+1 \in \mathcal{Q}_0$. Using this relation, we extend φ_0 to the maximal domain $\{w \in \mathbb{C} \mid w - n \in \mathcal{Q}_0$ for an integer $n \geq 0$ and $f_0^j(\varphi_0(w-n)) \in \mathrm{Dom}(f_0)$ for $j = 0, \ldots, n-1\}$. Note that for a large $\eta_0 > 0$, $\{w \mid |\mathrm{Re}\, w| \leq 1/2,\ |\mathrm{Im}\, w| > \eta_0\}$ is contained in $\mathcal{Q}_0$ and its image by φ_0 is contained in $\tau_0(\mathcal{Q}_0^+) \subset \mathcal{B}$. The rest of (4.1.2) can be checked easily.

Let $\Phi_0 = \Phi_{+,0} \circ \tau_0^{-1}$ and extend it similarly to $\mathcal{B}$ using the functional equation for $\Phi_{+,0}$. Hence (4.1.3) follows.

The properties of $\tilde{\mathcal{B}}$ in (4.1.4) follow from the above. The function $\tilde{\mathcal{E}}_{f_0} = \Phi_0 \circ \varphi_0 = \Phi_{+,0} \circ \Phi_{-,0}^{-1}$ satisfies the functional equation and is univalent in $\{w \mid |\mathrm{Im}\, w| > \eta_0'\}$ for a large $\eta_0' > 0$. Moreover $\mathrm{Im}\, \tilde{\mathcal{E}}_{f_0}(w) \to \pm\infty$ as $\mathrm{Im}\, w \to \pm\infty$. Hence $\mathcal{E}_{f_0}$ can be extended to 0 and ∞ conformally. This proves (4.1.4).

By the normalization (4.1.5), we have $c_{+,0} = c_{-,0}$.

A.4. *Fatou coordinates for* $f \in \mathcal{N} \cap \mathcal{F}_1$. *Proof of* 4.2. Let $f \in \mathcal{N} \cap \mathcal{F}_1$, where $\mathcal{N}$ is as in Lemma A.1.7 and $\mathcal{F}_1$ is as in Section 4.2. Define $\mathcal{Q}_f^+ = \mathcal{Q}(\xi_1, -\xi_1 + \frac{1}{\alpha})$ and $\mathcal{Q}_f^- = \mathcal{Q}(\xi_1 - \frac{1}{\alpha}, -\xi_1) = T_f(\mathcal{Q}_f^+)$ for $\xi_1 > 0$. One can choose $\mathcal{N}$ small and ξ_1 large so that $\mathrm{Re}\, \frac{1}{\alpha} > 2\xi_1 + 2$, $\mathcal{Q}_f^\pm \subset \mathrm{Dom}(F_f)$ and F_f satisfies (A.2.2) on $\mathcal{Q}_f^\pm$. By Proposition A.2.1, there exist univalent analytic functions $\Phi_{+,f}\colon \mathcal{Q}_f^+ \to \mathbb{C}$ and $\Phi_{-,f}\colon \mathcal{Q}_f^- \to \mathbb{C}$ satisfying $\Phi_{+,f}(F_f(w)) = \Phi_{+,f}(w) + 1$ and $\Phi_{-,f}(F_f(w)) = \Phi_{-,f}(w) + 1$. We fix two points $w_+ \in \mathcal{Q}_0^+$ and $w_- \in \mathcal{Q}_0^-$, and normalize $\Phi_{\pm,f}$ by setting $\Phi_{\pm,f}(w_\pm) = \Phi_{\pm,0}(w_\pm)$.

Since $F_f(w) = w + 1 + O(1/w^2)$ as $\mathrm{Im}\, \alpha w \to \infty$ by Lemma A.1.7, it follows from Lemma A.2.4 (ii) that there exist constants $c_\pm = c_\pm(f)$ such that

$$\Phi_{\pm,f}(w) = w + c_\pm + o(1) \qquad \text{(A.4.1)}$$

when w tends to ∞ within a sector of the form $\{w \mid \theta'_1 < \arg(w-\omega_0) < \theta'_2\} \subset \mathcal{Q}_f^{\pm}$, where $\pi/3 < \theta'_1 < \theta'_2 < 3\pi/4$ and $\omega_0 \in \mathbb{C}$.

The function $\Phi_{-,f} \circ T_f$ is defined on $\mathcal{Q}_f^{+}$ and satisfies the same functional equation as $\Phi_{+,f}$ by Lemma A.1.7 (iii). It follows from the uniqueness in Proposition A.2.1 (ii) that there exists a constant $d_1 = d_1(f)$ such that

$$\text{(A.4.2)} \qquad \Phi_{-,f} \circ T_f = \Phi_{+,f} + d_1.$$

In fact, (A.4.1) determines the constant: $d_1 = c_-(f) - c_+(f) - \frac{1}{\alpha}$.

Now define $\tilde{\mathcal{E}}_f(w) = \Phi_{+,f} \circ \Phi_{-,f}^{-1}$ on $\Phi_{-,f}(\mathcal{Q}_f^{+} \cap \mathcal{Q}_f^{-})$, which contains at least the vertical strip $\{w \mid |\operatorname{Re} w| \le 1,\ |\operatorname{Im} w| \ge \eta_0\}$ for a large $\eta_0 > 0$. This η_0 can be chosen uniformly for $f \in \mathcal{N} \cap \mathcal{F}_1$ near f_0. It satisfies the functional equation $\tilde{\mathcal{E}}_f(w+1) = \tilde{\mathcal{E}}_f(w) + 1$ whenever both sides are defined. By this relation, $\tilde{\mathcal{E}}_f$ can be extended to $\{w \mid |\operatorname{Im} w| > \eta_0\}$. By (A.4.1), we have

$$\text{(A.4.3)} \qquad \tilde{\mathcal{E}}_f(w) = w + c_+(f) - c_-(f) + o(1) \ \text{ when } \operatorname{Im} w \to \infty.$$

Similarly, $\tilde{\mathcal{E}}_f(w) - w$ tends to a constant as $\operatorname{Im} w \to -\infty$.

By Proposition A.2.1, we have $\Phi_{+,f} \to \Phi_{+,0}$ and $\Phi_{-,f} \to \Phi_{-,0}$ as $f \to f_0$; therefore $\tilde{\mathcal{E}}_f \to \tilde{\mathcal{E}}_{f_0}$ uniformly on $\{w \mid 0 \le \operatorname{Re} w \le 1,\ \operatorname{Im} w = \eta_0\}$. Then

$$\text{(A.4.4)}\ c_+(f) - c_-(f) = \int_{i\eta_0}^{1+i\eta_0} (\tilde{\mathcal{E}}_f(w) - w)\, dw \to \int_{i\eta_0}^{1+i\eta_0} (\tilde{\mathcal{E}}_{f_0}(w) - w)\, dw = c_{+,0} - c_{-,0} = 0,$$

where the integrals are over the segment joining $i\eta_0$ and $1 + i\eta_0$.

Let us show that $\Phi_{-,f}(\mathcal{Q}_f^{-})$ contains $\mathcal{Q}_f$ as in Section 4.2, for large $\xi_0, \eta_0 > 0$. Combining (A.1.6), Lemma A.2.4 and Proposition A.2.1 (iii), one can show that the right boudary curve of $\mathcal{Q}_f$ is contained in $\Phi_{-,f}(\mathcal{Q}_f^{-})$, if ξ_0 and η_0 are large enough. Similarly, the left boundary curve of $T_f^{-1}(\mathcal{Q}_f)$ is contained in $\Phi_{+,f}(\mathcal{Q}_f^{+})$. Using $\mathcal{Q}_f^{-} = T_f(\mathcal{Q}_f^{+})$, (A.4.2) and (A.4.4) and increasing ξ_0, η_0 if necessary, we conclude that the left boundary curve of $\mathcal{Q}_f$ is contained in $\Phi_{-,f}(\mathcal{Q}_f^{-})$. Therefore $\mathcal{Q}_f \subset \Phi_{-,f}(\mathcal{Q}_f^{-})$.

Defining $\varphi_f = \tau_f \circ \Phi_{-,f}^{-1}$, we have $\mathcal{Q}_f \subset \operatorname{Dom}(\varphi_f)$ and $\varphi_f(w+1) = f \circ \varphi_f(w)$ if $w, w+1 \in \mathcal{Q}_f$. Using this relation, extend φ_f to the domain $\{w \in \mathbb{C} \mid w-n \in \mathcal{Q}_f$ for an integer $n \ge 0$ and $f^j(\varphi_f(w-n)) \in \operatorname{Dom}(f)$ for $j = 0, \dots, n-1\}$. Then φ_f satisfies (4.2.1).

Let $\tilde{\mathcal{R}}_f = \Phi_{-,f} \circ T_f \circ \Phi_{-,f}^{-1}$. Then $\tilde{\mathcal{R}}_f$ coincides with $\tilde{\mathcal{E}}_f + c_-(f) - c_+(f) - \frac{1}{\alpha}$ on its domain of definition and satisfies $\tilde{\mathcal{R}}_f(w+1) = \tilde{\mathcal{R}}_f(w) + 1$. So $\mathcal{R}_f = \pi \circ \tilde{\mathcal{R}}_f \circ \pi^{-1}$ is well-defined and extends analytically to 0 by (A.4.3) and similarly extends to ∞. Since $\tilde{\mathcal{R}}_f(w) = w - \frac{1}{\alpha} + o(1)$ as $\operatorname{Im} w \to \infty$, $\mathcal{R}'_f(0) = \exp(-2\pi i \frac{1}{\alpha})$. Thus (4.2.2) is proved. □

It is easy to see that $\varphi_f \to \varphi_0$ and $\tilde{\mathcal{E}}_f \to \tilde{\mathcal{E}}_{f_0}$ when $f \in \mathcal{N} \cap \mathcal{F}_1$ and $f \to f_0$. Hence

$$\tilde{\mathcal{R}}_f + \frac{1}{\alpha} = \tilde{\mathcal{E}}_f + (c_-(f) - c_+(f)) \to \tilde{\mathcal{E}}_{f_0}.$$

Using the fact that $\mathcal{R}_f$ is extended analytically to 0 and ∞, we have

$$e^{2\pi i/\alpha}\mathcal{R}_f \to \mathcal{E}_{f_0}.$$

Thus (4.2.4) is proved. □

In order to see (4.2.3), let us first verify (4.3.4). In fact the latter is immediate from the definition of φ_f and $\tilde{\mathcal{R}}_f$, $\tau_f \circ T_f = \tau_f$ and the functional equation for φ_f. To obtain (4.2.3), we restrict $\mathrm{Dom}(\tilde{\mathcal{R}}_f)$ so that

(A.4.5) $$\left|\tilde{\mathcal{R}}_f(w) + \frac{1}{\alpha} - w\right| < \frac{1}{2|\alpha|} \text{ on } \mathrm{Dom}(\tilde{\mathcal{R}}_f).$$

Note that (4.2.4) is still true. Now suppose that $w, w' \in \mathrm{Dom}(\varphi_f)$ and $\mathcal{R}_f^m(\pi(w)) = \pi(w')$ for a positive integer m. Then for the lift, there exists an integer n such that $\tilde{\mathcal{R}}_f^m(w) + n = w'$. By (A.4.5), we have $n > m$. Hence the assertions of (4.2.3) follows from (4.3.4).

The corollary (4.2.5) follows from (4.1.2) and (4.2.4).

One can prove (4.3.1) similarly.

A.5. *Parabolic fixed points with the multiplier* $\neq 1$. We only state the coordinate changes which correspond to A.1. The rest of the arguments in A.2–A.4 are immediately generalized to this case.

First, let f_0 be as in Section 7. There exists a coordinate near 0, in which f_0^q has the form

$$f_0^q(z) = z + z^{q+1} + O(z^{2q+1}).$$

Let us introduce a coordinate w by

$$w = -\frac{1}{qz^q}.$$

Then the corresponding map in this coordinate has the form

$$F_0(w) = w + 1 + O(1/w),$$

where F_0 is multi-valued.

The coordinate w should be understood in terms of the Riemann surface $\mathcal{W}$ as follows. Let $\mathcal{Q}_0^{+,k}$, $\mathcal{Q}_0^{-,k}$ ($k \in \mathbb{Z}/q\mathbb{Z}$) be q copies of $\mathcal{Q}_0^+$, $\mathcal{Q}_0^-$ in A.3. The intersection $\mathcal{Q}_0^+ \cap \mathcal{Q}_0^-$ has two components—the upper and lower sectors. Identify the upper sector of $\mathcal{Q}_0^{+,k}$ with that of $\mathcal{Q}_0^{-,k}$; and the lower sector of $\mathcal{Q}_0^{+,k-1}$ with that of $\mathcal{Q}_0^{-,k}$ ($k \in \mathbb{Z}/q\mathbb{Z}$). The obtained Riemann surface $\mathcal{W}$ is isomorphic to a punctured disc and the map $w \mapsto z = -1/qw^q$ gives a conformal map from $\mathcal{W}$ onto a punctured neighborhood of 0 in $\mathbb{C}$. We consider that the map F_0 sends $\mathcal{Q}_0^{+,k}$ into $\mathcal{Q}_0^{+,k+p}$, and $\mathcal{Q}_0^{-,k} \setminus$ (a neighborhood of the boundary curve) into $\mathcal{Q}_0^{-,k+p}$.

Now let $f \in \mathcal{F}_1$ and $\sigma^{(k)}(f)$ be the periodic points of f as in Section 7. We choose a new coordinate near 0 so that $\sigma^{(k)}(f) = e^{2\pi ik/q}\sigma^{(0)}(f)$, as follows. Put

$$(z - \sigma^{(0)}(f)) \cdots (z - \sigma^{(k-1)}(f)) = z^q - b_{q-1}(f)z^{q-1} - \cdots - b_1(f)z - b_0(f).$$

Then it is easily seen that

$$b_0(f) = -2\pi i\alpha(f)(1+o(1)) \text{ and } b_j(f) = O(\alpha(f)) \ (j = 1, \dots, q-1) \ \text{ as } f \to f_0.$$

Define a new coordinate z' by

$$z' = \frac{z}{\left(1 + \frac{b_1}{b_0}z + \cdots + \frac{b_{q-1}}{b_0}z^{q-1}\right)^{1/q}}.$$

This is a well-defined coordinate near 0, and in this coordinate we have $\sigma^{(k)} = e^{2\pi ik/q}\sigma^{(0)}$.

So we can write

$$[f^q(z)]^q = z^q + z^q(z^q - \sigma(f)^q)u_f(z),$$

where $\sigma(f) = \sigma^{(0)}(f)$ and $u_f(z)$ is a nonzero analytic function defined near 0. Finally, we introduce the coordinate w by

$$\frac{z^q}{z^q - \sigma^q} = e^{2\pi i\alpha w}$$

and the map F_f by

$$F_f(w) = w + \frac{1}{2\pi i\alpha}\log\left(1 - \frac{\sigma^q u_f(z)}{1 + z^q u_f(z)}\right).$$

Here the coordinate w should be interpreted on a suitable Riemann surface which is isomorphic to a neighborhood of 0 in $\mathbb{C}$ with 0 and $\sigma^{(k)}(f)$ removed. Then the z in the definition of F_f can make sense.

For these maps, one can obtain analogous results as in A.2–A.4. The detail is left to the reader.

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