SCHRAMM–LOEWNER EVOLUTION

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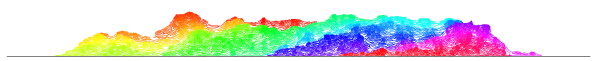

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Preface

The purpose of this book is twofold. On the one hand, I wanted to write a book which is a *concise introduction to* the beautiful topic of *Schramm–Loewner evolution* for anybody, having some mathematical background, interested in it. On the other hand, I wanted to organize some material for a self-contained *textbook for a semester long course* in advanced mathematics.

The history of this book goes back to Fall 2011 — to a lecture course which I gave at University of Helsinki on the subject. The lecture notes of that course formed the seed of this book project. Later, I gave a similar set of lectures during Spring 2016 based on the draft of this book. I added to the book, compared to those lectures, more on regularity and convergence of random curves, a topic presented in Chapter 6. I kept the key parts of the original material including the background in stochastic analysis and complex analysis.

The text intends to be fairly rigorous, but skipping some details; we focus on the core ideas. I have prepared appendices to this text which I have posted on the webpage

`http://www.helsinki.fi/sle-book.`

The reader may choose consult that supplementary material if he/she wishes to learn more on the omitted details.

I want to thank the people I discussed on the material while writing the lecture notes for the courses, most importantly, my colleagues Kalle Kytölä, Miika Nikula and Petri Tuisku. The two last were the teaching assistants of those courses. I also thank the audience of those courses.

Helsinki, *Antti Kemppainen*
June 2017

Contents

Chapter 1
Introduction

In this introductory chapter, we look at iterations of conformal maps, random processes such as random walks and statistical physics and establish some connections.

1.1 Iteration of conformal maps

We assume that the reader has some familiarity with Complex Analysis.[1] Recall that a differentiable function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is a set containing an open neighborhood of a point z_0, is *conformal* at z_0, if the map f preserves angles[2] at z_0. If this holds for all points of U, we call f a *conformal map*. Remember also that a function in a planar domain is conformal if and only if it is holomorphic and one-to-one.

Let $\mathbb{H}$ be the upper half-plane and let $f_k : \mathbb{H} \to \mathbb{H}$ be a sequence of conformal maps where $k \in \mathbb{Z}_{>0}$.[3] Define

$$f^{[\![1,n]\!]}(z) = f_1 \circ f_2 \circ \ldots \circ f_n(z).$$

Suppose that each f_k maps $\mathbb{H}$ onto a set which is the complement (with respect to $\mathbb{H}$) of a bounded set K_k, whose boundary is a curve, and suppose that $|f_k(z)| \to \infty$ as $|z| \to \infty$. Then it turns out that f_k extends continuously to the closure $\overline{\mathbb{H}}$.

Suppose that the set K_k is a line segment $[\xi_k, \zeta_k]$, where $\xi_k \in \mathbb{R}$ is the base point and $\zeta_k \in \mathbb{H}$ is the tip point. By the continuity of f_k to the boundary, we can talk about the point $x_k \in \mathbb{R}$ which is mapped to the tip ζ_k by f_k. Define now $\hat{f}_k(z) =$

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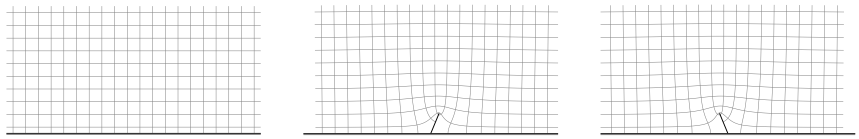
[1] The reader can use, for instance, Rudin's book [7] as a reference. Notice the *supplementary material* (appendices) of this book described in the preface, and also Chapter 3 below.

[2] In the sense that if P_1 and P_2 are smooth curves that form an angle θ at z_0, then also $f \circ P_1$ and $f \circ P_2$ form an angle θ at $f(z_0)$.

[3] Throughout this text we use the notations $\mathbb{Z}_{>0} = \{k \in \mathbb{Z} : k \geq 1\}$, $\mathbb{Z}_{\geq 0} = \{k \in \mathbb{Z} : k \geq 0\}$, $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$, $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ as well as $[\![j,k]\!]$ for the ordered set $j, j+1, j+2, \ldots, k-1, k$, where $j < k$ are integers.



(a) The grid lines in $\mathbb{H}$.

(b) The gridlines transformed by the mapping $z \mapsto \hat{f}_{\mathrm{R}}(z)$.

(c) The gridlines transformed by the mapping $z \mapsto \hat{f}_{\mathrm{L}}(z)$.

Fig. 1.1 Two elementary conformal transformations that are being iterated in the process illustrated in Figure 1.2. Grid lines can be used to illustrate the action of conformal maps.

$f_k(z+x_k)-\xi_k$. Then $\hat{f}_k$ is conformal and it maps $\mathbb{H}$ onto the complement of a line segment, whose base point is 0, it maps ∞ to ∞ and 0 to the tip of the line segment. It turns out (as we will later see) that it is useful to consider conformal maps that for large values of $|z|$ are close to identity, in the sense that they neither expand or shrink the grid as in Figure 1.1 far away from the origin.

If we iterate maps of this form, for instance, $\hat{f}_1 \circ \hat{f}_2$, then the composition will be a map from $\mathbb{H}$ onto the complement of a piecewise smooth curve. The continuity of the curve at the points where the (images of) line segments meet, follows from the fact that 0 is the base point of $\hat{f}_2$ and 0 is mapped to the tip point of $\hat{f}_1$ by $\hat{f}_1$.

Figure 1.2 illustrates the iterates $\hat{f}^{[\![1,n]\!]}$. We have chosen two conformal maps (see Figure 1.1) that correspond to the line segments of the same length forming angles $\alpha\pi$ and $(1-\alpha)\pi$ with the positive real axis, and each f_k is one of the two maps.

The parameter n acts naturally as discrete time of the growth process. If we wish study a continuous time limit of the iterates $\hat{f}^{[\![1,n]\!]}$, we need to take large n and adjust the elementary conformal maps so that the sizes of the line segments are small, but the composed piecewise smooth curve reaches roughly to a constant height. This can be achieved by considering $\hat{\phi}_k(z) = n^{-a}\hat{f}_k(n^a z)$ where $a>0$ is a suitable constant.

Let $F_t^{(n)}$ denote the iterate $\hat{\phi}^{[\![1,\lfloor nt \rfloor]\!]}$ for any $n \in \mathbb{Z}_{>0}$ and $t \in [0,1]$.[4] When n is large, the composed piecewise smooth curve corresponding to $\hat{\phi}^{[\![1,\lfloor nt \rfloor]\!]}$ increases by tiny steps as t is increased. It seems reasonable to expect that the limit $\lim_{n\to\infty} F_t^{(n)}$ exists and defines a continuous-time flow of the points of $\mathbb{H}$.[5] This is indeed the case at least when the sequence f_k are random, symmetrically distributed ($\hat{f}_{\mathrm{R}}$ and $\hat{f}_{\mathrm{L}}$ are equally likely) and independent. The continuous-time versions in the case of random, symmetric and independent sequences are the *Schramm–Loewner evolutions*.

[4] We use a common notion that $\lfloor x \rfloor$ is the largest integer smaller or equal to x.

[5] Such a limit is an example of *scaling limit*. Two typical features of a scaling limit are that there are scaling factor involved, such as n^{-a} and n^a above, which ensure that the limit exists, and that the limiting object will be described by continuous variables (another term is a continuum limit).

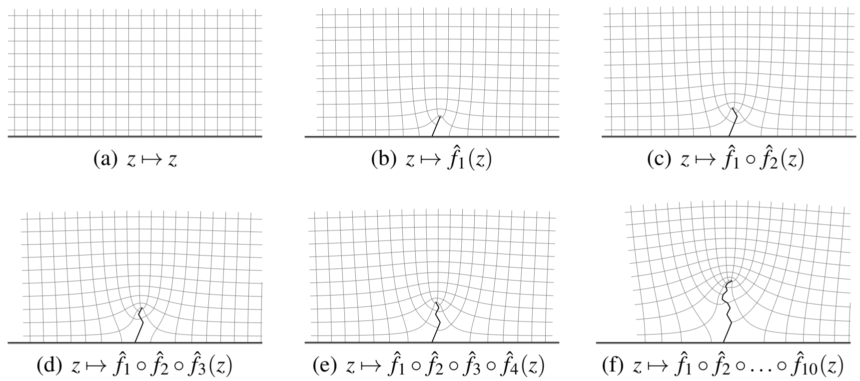


Fig. 1.2 Consider conformal maps from the upper half-plane onto the complements of line segments. We can arrange so that ∞ is mapped to ∞ and that the base point of the line segment is 0 as well as the point which gets mapped to the tip of the segment. The figures here illustrate how iterations of such maps look like.

1.2 On stochastic models and connection to statistical physics

1.2.1 Random walk and Brownian motion

We also assume some familiarity with Probability Theory.[6]

Recall that a *stochastic process* is a collection of random variables indexed by an ordered set which is interpreted as the *time* variable. Let's consider random walks on $\mathbb{Z}$ as an example. We will denote probability measures generally by P. Let X_k, $k \in \mathbb{Z}_{>0}$, be a sequence of random variables which take two possible values ± 1, i.e., $\mathsf{P}[X_k = -1] + \mathsf{P}[X_k = +1] = 1$. Assume that X_k, $k \in \mathbb{Z}_{>0}$, are independent[7] and fix some $x \in \mathbb{Z}$. The formula

$$S_t = x + \sum_{k=1}^{t} X_k$$

defines a stochastic process[8] $(S_t)_{t \in \mathbb{Z}_{\geq 0}}$. If the random variables X_k, $k \in \mathbb{Z}_{>0}$, have symmetric distribution, that is, $\mathsf{P}[X_k = -1] = \mathsf{P}[X_k = +1] = \frac{1}{2}$, then the process is called *symmetric simple random walk* on $\mathbb{Z}$.

Often we wish to derive a continuum limit of the simple random walk or other processes. Such a limit is a scaling limit in the same sense as in the previous section. For that purpose, we choose a constant $a > 0$ and consider the continuous-time process $(n^{-a} S_{\lfloor nt \rfloor})_{t \in \mathbb{R}_{\geq 0}}$. For suitably chosen constant a this process will converge

[6] The reader can use, for instance, Durrett's book [3] as a reference. Notice the *supplementary material* (appendices) of this book described in the preface, and also Chapter 2 below.

[7] Remember that for these given random variables, X_k, $k \in \mathbb{Z}_{>0}$, are independent if for any $n \in \mathbb{Z}_{>0}$ and for any $x_1, x_2, \ldots, x_n \in \{-1, +1\}$, $\mathsf{P}[X_k = x_k \text{ for all } k \in [\![1,n]\!]] = \prod_{k \in [\![1,n]\!]} \mathsf{P}[X_k = x_k]$.

[8] We use the notation $(X_t)_{t \in I}$ where usually $I = \mathbb{Z}_{\geq 0}$ or $I = \mathbb{R}_{\geq 0}$, to denote a stochastic process.

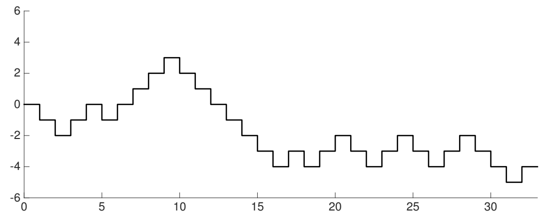


Fig. 1.3 Simple random walk

as $n \to \infty$ to a stochastic process $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ called *Brownian motion*. From the *central limit theorem* (CLT) we know that $a = 1/2$ and that all the finite dimensional distributions (distributions of vectors of type $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$) are Gaussian.[9]

1.2.2 Ising model and other statistical physics models

The study of Schramm–Loewner evolutions is motivated by their applications to statistical physics. Those random curves appear in statistical physics under specific circumstances as *interfaces*, that is, domain walls separating parts of the system which differ in some microscopic property.

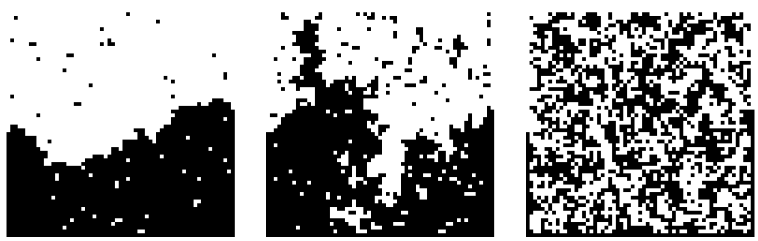


Fig. 1.4 Ising model with Dobrushin boundary conditions for $T < T_c$, $T = T_c$ and $T > T_c$. Here the black pixels are vertices with $\sigma = +1$ and the white pixels are vertices with $\sigma = -1$. An *interface* is a broken line separating white and black regions.

A typical example of a lattice model of statistical physics (i.e., a simplified model defined on a lattice such as $\mathbb{Z}^d$) is the *Ising model*, which models ferromagnetic material. Each site v is occupied by an elementary magnet, *spin*, which takes values $\sigma_v \in \{\pm 1\}$. The Ising model is defined by an energy functional

$$H(\underline{\sigma}) = -\sum \sigma_v \sigma_w .$$

[9] Remember that the result that a sum of independent and identical centered random variables scaled by $n^{-1/2}$ converges to a Gaussian random variable in distribution, is called the central limit theorem.

Here $\underline{\sigma} = (\sigma_v)_{v \in V}$ is the spin configuration of the system and V is a finite subset of the square lattice $\mathbb{Z}^2$ (we focus here on two-dimensional model). The sum in H is over neighboring pairs of sites. The more there are pairs of aligned spins, the more this functional favors the configuration (that is, the configuration has smaller energy) — this can be seen as the source of the ferromagnetic phenomenon.

In the Ising model, we take the configuration $\underline{\sigma} \in \{\pm 1\}^V$ to be random. Its law is given by the Boltzmann distribution corresponding to the energy functional H, i.e., the probability of observing $\underline{\sigma}$ is proportional to $\exp(-\beta H(\underline{\sigma}))$. Here $\beta = 1/T$, the inverse temperature, is a parameter.

The behavior of the systems depends drastically on the temperature T, as the reader can see from Figure 1.4. In the figure we use so called Dobrushin boundary conditions, where we force the spins on the two complementary boundary arcs to be constant -1 on one of them and $+1$ on the other. The *interface* which is the broken line separating the large $+1$-cluster and the large -1-cluster, can be studied when these boundary conditions are used.

The *scaling limit* of the interface is obtained by fixing a shape, say, a square and the Dobrushin boundary conditions on its boundary and then by approximating that shape by finite subsets of a lattice with a lattice mesh parameter. The scaling limit is the limit as the lattice mesh tends to zero.

The phase transition of the model can be explained in terms of interface in the following way. There is a critical temperature T_c such that for $T < T_c$ for large systems looked far away (i.e. in the scaling limit) the interface is close to the minimal energy line with fluctuations of order $\sqrt{N}$, where N is the side length of the box. As T approaches T_c the fluctuations grow and at T_c they are of the size of the system. Therefore $T = T_c$ is the smallest value of the parameter where we expect a non-trivial scaling limit for the interface. The fact that the scaling limit at $T < T_c$ is non-random is a result of [6]. For $T > T_c$, when looked far away, the spins behave more or less independently of the of each other and the interface looks like the interface of $T = \infty$, for which value the spin configuration is truly totally disordered.

1.2.3 Conformal invariance of the scaling limits

Schramm–Loewner evolutions give an efficient tool for verifying conformal invariance in the context of random curves of statistical physics and their scaling limits.

Based on physical arguments, the scaling limit is expected to be scale invariant. In fact, under some hypothesis such as partial rotation invariance of the Hamiltonian ($\pi/4$-rotation invariance of the Ising model on $\mathbb{Z}^2$) and short range of the interactions, it is expected that the scaling limit is even conformally invariant. Conformal invariance could be described to be *local* rotation, scale and translation invariance. Here "local" refers to the fact that the factor that we use in e.g. scale invariance can vary over the domain. Consult, for instance, the introduction of [5] for an introduction to the physical theories of phase transitions. The conformal invariance property of the Ising model should be understood concretely in the following way.

If we start from any two shapes (simply connected domains) and approximate both with sequences of discrete domains then the laws of the interfaces are equal in the scaling limit, in the sense that they are conformal images of each other.

This property is related to the *conformal Markov property* of iterates of conformal maps. Namely consider the conditional law of $\hat{f}^{[\![1,n+m]\!]}$ given that we know $\hat{f}_k$, $k \in [\![1,n]\!]$. That conditional law is just the law of $\hat{f}^{[\![m+1,n+m]\!]}$ transformed by the (known) conformal map $\hat{f}^{[\![1,n]\!]}$. This is an evidence of a connection between statistical physics and the iterates of conformal maps. We call the argument *Schramm's principle*, see [10], the original article by Schramm [8] or Section 5.1.1 below.

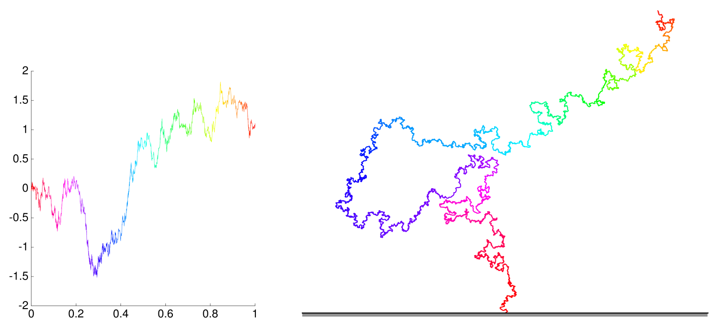


Fig. 1.5 Realizations of a 1D Brownian motion (left) and the corresponding SLE(3) (right) driven by the Brownian motion. SLEs are random curves which are fractal, in the sense, that they contain statistically similar details repeating on different length scales.

1.3 An example: percolation model and Cardy's formula

In this section we will present an example with some details that highlight the main topics of this text and the example is one of the main application of the theory of Schramm–Loewner evolution. The full argument is presented later in the text.

Consider the triangular lattice which is formed by the centers of the regular hexagonal tiling of the plane. Take a finite, simply connected[10] subgraph of the triangular lattice. We call the centers of the hexagons sites.

In the *site percolation model*, each site carries a random variable which takes value *open* or *closed*.[11] In any pictures, we color the corresponding hexagon green

[10] Simply connectedness means that the domain consisting of the hexagons is a simply connected domain (i.e. with no holes) — in other words, if we have a closed path of hexagons in the domain, it cannot disconnect any point in the complement of the domain from infinity.

[11] From the modelling perspective, the open sites represent channels through which a substance, say, water can flow. Therefore if we inject water into the sites of a set A_1, the water will flow to all the sites connected by a path of open sites to A_1. In particular we are interested in connection events that for fixed A_1 and A_2 there exists a connected path from A_1 to A_2 that stays in a set B.

if the site is open and red if the site is closed. The decision, whether a site is open or closed, is made randomly, independently and from the same distribution at each site. This leaves only one parameter in the model, which is the quantity $p := \mathsf{P}[\text{the site } x \text{ is open}] \in [0,1]$ which is independent of x.

Consider first the crossing probability for a fixed shape with varying size. More specifically, take rhombi $R_N = \{x\mathbf{e}_1 + y\mathbf{e}_2 : x,y \in [\![1,N]\!]\}$ where $\mathbf{e}_1 = 1$ and $\mathbf{e}_2 = \exp(i\pi/3)$ are two vectors in the plane that generate the triangular lattice. Denote by $f(p,N)$ the probability of a left-to-right crossing of R_N. Clearly f is monotone in p.[12] As illustrated in Figure 1.6, as N tends to infinity the crossing probability tends to a sharp step function. More accurately $\lim_{N\to\infty} f(p,N)$ equals to 0, $\frac{1}{2}$ and 1, when $p < \frac{1}{2}$, $p = \frac{1}{2}$ and $p > \frac{1}{2}$, respectively. We would arrive to a similar conclusion if we had taken a rhombus with a different aspect ratio. The only difference is that the limit of the crossing probability at $p = \frac{1}{2}$ is not necessarily $\frac{1}{2}$, but it can take some other value in $(0,1)$. The parameter $p = \frac{1}{2}$ is *critical* in the sense that outside criticality limits of crossing probabilities are trivial, either 0 or 1. In fact, the limit when $p = \frac{1}{2}$ depends non-trivially on the aspect ratio of the rhombus.

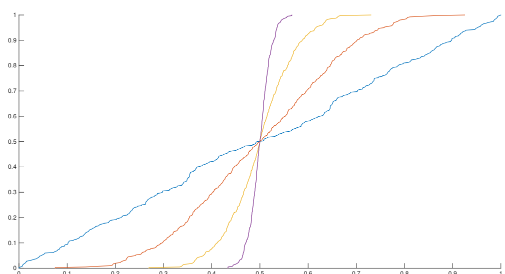


Fig. 1.6 The crossing probabilities of a left-to-right crossing in a rhombus R_N of side length N. The crossing probability is estimated using a computer simulation and plotted as a function of p for different values of N ($N = 1$ blue, $N = 4$ orange, $N = 16$ yellow, $N = 64$ purple). Different values of p are coupled using standard approach that uses uniform random variables. The sample size is 200 for each value of N.

1.3.1 Cardy's formula from SLE(6)

We will describe here how to derive a formula for the crossing probability using a *conformal invariance hypothesis*. Consider for simplicity the crossing probability in a rectangle $[0,aL] \times [0,a]$, where $a > 0$ and $L > 0$, for an open crossing from $\{0\} \times [0,a]$ to $\{aL\} \times [0,a]$. Map the rectangle conformally onto the upper half-plane $\mathbb{H}$ such that $(0,0) \mapsto U_0, (aL,0) \mapsto V_0, (0,a) \mapsto W_0, (aL,a) \mapsto \infty$. The exact form of the mapping doesn't play a role here.

[12] The reader should stop to think this for a moment, though.

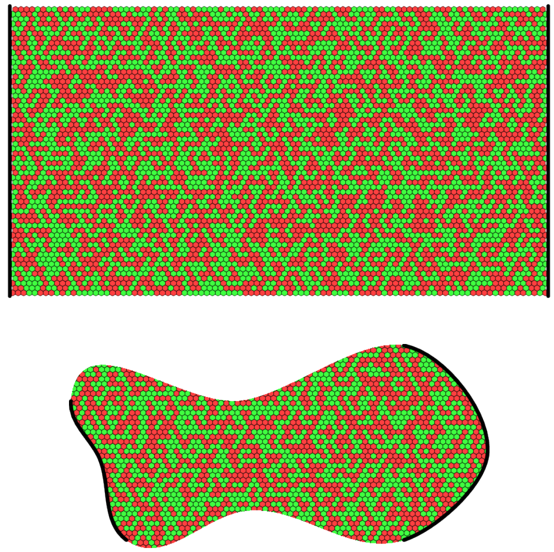


Fig. 1.7 Percolation on two different shapes. Cardy's formula tells that the probability of an open crossing of the quadrilaterals depends only on the conformal modulus in the scaling limit, at criticality, and gives an explicit expression for it.

Next introduce a new layer of hexagons around the rectangle, as in Figure 1.8. Assign boundary conditions such that the hexagons on $([0,aL]\times\{0\})\cup(\{aL\}\times[0,a])$ are closed and on $([0,aL]\times\{a\})\cup(\{0\}\times[0,a])$ open. Then there will be an interface separating the closed cluster and the open cluster that touch the boundary. In Figure 1.8, this is the blue path.

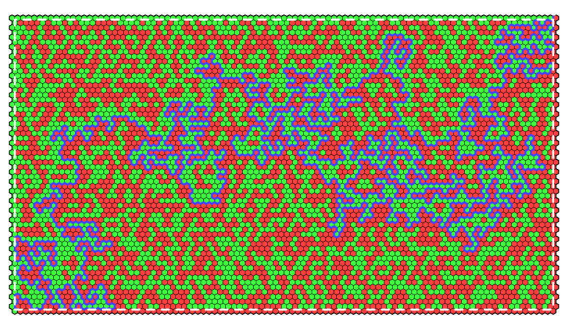


Fig. 1.8 After introducing the extra layer of hexagons for boundary conditions, there will be interface that separates the red and blue clusters that touch the boundary.

We can read the crossing event from the interface. Namely, the left-to-right crossing exists if and only if the interface hits $\{aL\} \times [0,a]$ before $[0,aL] \times \{a\}$.

Let's next consider the probability conditionally on the initial segment of the interface. Suppose that the interface is $\gamma(t)$, $t \in [0,T]$. The conditional probability of an open crossing given the initial segment $\gamma(s)$, $s \in [0,t]$, is a crossing probability but now in the complement of $\gamma[0,t]$ in the rectangle from the union of $\{0\} \times [0,a]$ and the left-hand side of $\gamma[0,t]$ to $\{aL\} \times [0,a]$. It is natural to transform that domain also onto the upper half-plane and take the points $\gamma(t), (aL,0), (0,a), (aL,a)$ to U_t, V_t, W_t, ∞, respectively.

We make an assumption that the scaling limit of the interface is conformally invariant and more specifically, we make a guess that the scaling limit is a process called SLE(6). Under further assumptions it holds that

$$U_t = \sqrt{6}B_t, \qquad \dot{V}_t = \frac{2}{V_t - U_t}, \qquad \dot{W}_t = \frac{2}{W_t - U_t}$$

where $\dot{X}_t = \partial_t X_t$. The first equality is the fact that the process is SLE(6) and the two others represent the Loewner flow of the marked points.

Set $Z_t = (U_t - W_t)/(V_t - W_t)$, which is a quantity called cross-ratio. That is equivalent of mapping $\mathbb{H}$ with marked points U_t, V_t, ∞, W_t onto $\mathbb{H}$ with marked points $z, 1, \infty, 0$. We further map the latter domain using a conformal map of the form $\phi(z) = C\int^z w^{-2/3}(1-w)^{-2/3}\mathrm{d}w$ onto a equilateral triangle PQR. Suppose that $\phi(W) = P$, $\phi(V) = Q$ and ϕ. Then $\zeta_t := \phi(Z_t) \in PQ$.

Based on stochastic calculus we can verify that the process ζ_t is a time change of a Brownian motion on PQ and thus the crossing probability, which can be reformulated as the probability that the process ζ_t hits Q before P, can be calculated. After an argument from stochastics (time-changed Brownian motions are conserved on average) and some algebra we end up to famous *Cardy's formula*

$$\lim_{N\to\infty} f(p_c, N) = \frac{\phi(z) - \phi(0)}{\phi(1) - \phi(0)} = \frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} z^{\frac{1}{3}}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; z\right)$$

where $p_c = \frac{1}{2}$, Γ is the gamma function and ${}_2F_1$ is the hypergeometric function. The original articles on Cardy's formula are [2, 4, 9, 11].

1.4 On reading this book

The next two chapters review background material on Stochastic Calculus and Complex Analysis. The reader familiar with those topics may choose to jump directly to the main chapters, Chapters 4–6. Those chapters build on the prior chapters and are easiest read in the order of presentation. Appendices with additional material are provided in separate documents (see the preface) and cited occasionally here.

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Chapter 2
Introduction to stochastic calculus

In this chapter, we focus on the essential aspects of *stochastic calculus*, a theory of integration with respect to Brownian-motion-type processes and their transformation properties. We take fairly standard approach and the reader can study more from books dedicated to the subject [2, 5, 6, 7, 8].

2.1 Brownian motion

Suppose throughout this text that we are given a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ where Ω is a measurable space, $\mathscr{F}$ a σ-algebra on Ω and P a probability measure on $\mathscr{F}$. For more details, consult any book on probability theory such as Durrett [3], see also Appendix A (which you can find using the instructions of the preface).

A *stochastic process* is a collection of random variables[1] X_t indexed by a variable t which we call time and which belongs to an ordered set I. A notation $(X_t)_{t\in I}$ is used for a stochastic process. Almost always $I = \mathbb{R}_{\geq 0}$ or $I = \mathbb{Z}_{\geq 0}$. Since t is regarded as time, we call the process in those cases *continuous time stochastic process* and *discrete time stochastic process*, respectively. In this text usually $I = \mathbb{R}_{\geq 0}$.

The mapping $t \mapsto X_t(\omega)$ is called the *path* of $(X_t)_{t\in I}$. For continuous time processes the path regularity properties are often essential in their definitions. The most important example of a process whose path is continuous, is the Brownian motion.

Definition 2.1. A stochastic process $(B_t)_{t\geq 0}$ is called a *(standard one-dimensional) Brownian motion* if $B_0 = 0$ and

1. $B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2}, \ldots, B_{t_n} - B_{s_n}$ are independent for any $n \in \mathbb{N}$ and for any $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_n < t_n$.

[1] We use the standard notation $X(\omega)$, where $\omega \in \Omega$ and $X(\omega) \in \mathbb{R}$, for a (real-valued) random variable on $(\Omega, \mathscr{F}, \mathsf{P})$.

2. For any $s,t \geq 0$, $B_{s+t} - B_s$ is normally distributed[2] with mean 0 and variance t.
3. With probability one, $t \mapsto B_t$ is continuous.

Remark 2.1. We say that the process has *independent* and *stationary* increments, if the properties 1. and 2. hold, respectively.

Brownian motion is a Gaussian process meaning that all *finite dimensional distributions* are multivariate Gaussians. From its definition it follows that for any $0 \leq t_1 < t_2 < \ldots < t_n$ and any Borel sets $A_k \subset \mathbb{R}$, $k = 1,2,\ldots,n$,

$$\mathsf{P}[B_{t_k} \in A_k, \forall k \in [\![1,n]\!]] = \int_{A_1 \times A_2 \times \ldots \times A_n} \left[\prod_{k=1}^{n} p(x_{k-1}, x_k, t_k - t_{k-1}) \right] \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

where $p(x,y,s) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(x-y)^2}{2s}\right)$, $t_0 = 0$ and $x_0 = 0$.

The "canonical" probability space for Brownian motion is the space of continuous functions $C(\mathbb{R}_{\geq 0})$ with a certain Borel probability measure P and where the Brownian motion is the coordinate map $B_t(\omega) = \omega_t$. It turns out that there exists a probability space with a Brownian motion[3] and its distribution in $C(\mathbb{R}_{\geq 0})$ defines the 'canonical" Brownian motion. The paths of a Brownian motion are Hölder continuous: for each $\gamma \in (0,1)$ and $T > 0$, there exists a random variable $K > 0$ such that almost surely $|B_t - B_s| \leq K\,|t-s|^\gamma$ for all $s,t \in [0,T]$.[4] The rough appearance of a Brownian path is shown by Figure 1.5.

The following theorem shows that the assumption that the increments are normal is partly redundant in the definition of Brownian motion.

Theorem 2.1. *If $(X_t)_{t \in \mathbb{R}_{\geq 0}}$ is a continuous stochastic process which has independent and stationary increments, then there exists a standard one-dimensional Brownian motion $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ and real numbers $\alpha \geq 0$ and β such that $X_t = \alpha B_t + \beta t$.*

We call $X_t = \alpha B_t + \beta t$ a *Brownian motion with a linear drift*.

Definition 2.2. A *filtration* on $(\Omega, \mathscr{F})$ is a collection $(\mathscr{F}_t)_{t \in \mathbb{R}_{\geq 0}}$ of sub-σ-algebras $\mathscr{F}_t \subset \mathscr{F}$ such that for each $0 \leq s < t$, $\mathscr{F}_s \subset \mathscr{F}_t$.

A stochastic process $(X_t)_{t \in \mathbb{R}_+}$ on $(\Omega, \mathscr{F})$ is *adapted* to the filtration $(\mathscr{F}_t)_{t \in \mathbb{R}_+}$ if for each $t \geq 0$, X_t is $\mathscr{F}_t$-measurable.

---

[2] Remember that X is normally distributed with mean μ and variance σ^2 when $\mathsf{P}[X \in A] = \int_A \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \mathrm{d}x$ for any Borel subset A of $\mathbb{R}$.

[3] The *Brownian bridge construction of the Brownian motion*: Consider a probability space with a countably infinite sequence of independent standard Gaussian random variables.

Suppose that we have constructed $B^{[n]} = (B_{kT2^{-n}})_{k=0,1,2,\ldots,2^n}$. Since the law of $B^{[n+1]} = (B_{kT2^{-n-1}})_{k=0,1,2,\ldots,2^{n+1}}$ is multivariate Gaussian, we can write the conditional law $(B_{kT2^{-n-1}})_{k=1,3,5,\ldots,2^{n+1}-1}$ given $B^{[n]}$ explicitly as multivariate Gaussian which we can construct using the given the sequence of standard Gaussians. Continue this iteration ad infinitum. We leave as an exercise to check that, if $B^{[n]}$ is linearly interpolated to all $t \in [0,T]$, then the sequence $B^{[n]}$ converges uniformly almost surely as $n \to \infty$.

[4] Hölder continuity of a Brownian motion follows as a side product from the Brownian bridge construction of the Brownian motion.

A filtration can be though as refining information on the probability space and $\mathscr{F}_t$ as the information available at time t. For example, the σ-algebras generated by a Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, i.e. $\mathscr{F}_t^B = \sigma(B_s,\, s\in[0,t])$, form a filtration $(\mathscr{F}_t^B)_{t\in\mathbb{R}_+}$.[5]

We will make the following more restrictive definition of Brownian motion.

Definition 2.3. A process $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is called a *(standard one-dimensional) Brownian motion with respect to the filtration* $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$ if it is adapted to $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$, $B_0 = 0$ and

1. $B_t - B_s$ are independent from $\mathscr{F}_s$ for any $0 \leq s < t$,
2. $B_t - B_s$, $0 \leq s < t$, is normally distributed with mean 0 and variance $t-s$
3. With probability one, $t \mapsto B_t$ is continuous.

Remark 2.2. The definition is useful for instance when two Brownian motions $B^{(1)}$ and $B^{(2)}$ are considered on the same probability space. We can weaken $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$ to $(\mathscr{F}_t^B)_{t\in\mathbb{R}_{\geq 0}}$ and therefore Definition 2.3 implies Definition 2.1.

Let $p \geq 1$. Define the *p'th variation* of a process $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ as

$$V_X^{(p)}(t) = \lim_{\text{mesh}(\pi)\to 0} \sum_{k=0}^{m(\pi)-1} |X_{t_{k+1}} - X_{t_k}|^p \tag{2.1}$$

where π is a partitions of $[0,t]$ of the form $\pi = \{0 = t_0 < t_1 < \ldots < t_{m(\pi)} = t\}$ and the limit is in terms of *convergence in probability*[6] as $\text{mesh}(\pi) = \max_k(t_{k+1} - t_k) \to 0$ We call the first variation ($p=1$) as *total variation* and the second variation ($p=2$) as *quadratic variation*.

Proposition 2.1. *The quadratic variation of a Brownian motion exist and* $V_B^{(2)}(t) = t$.

Proof. Let $\varepsilon > 0$ and π be a partition with $\text{mesh}(\pi) < (2t)^{-1}\varepsilon^3$. Let $\Delta_k = (B_{t_{k+1}} - B_{t_k})^2 - (t_{t+1} - t_k)$. Then $\mathsf{E}[\Delta_j\Delta_k] = \delta_{jk}\mathsf{E}\left[\Delta_k^2\right]$ by independence and thus

$$\mathsf{E}\left[\left(\sum_{k=0}^{m(\pi)-1}(B_{t_{k+1}} - B_{t_k})^2 - t\right)^2\right] = \mathsf{E}[(N^2-1)^2]\sum_k (t_{k+1}-t_k)^2 \leq 2\text{mesh}(\pi)t.$$

Here N is normally distributed with mean zero and variance one and we used the scaling property of Brownian motion. Hence by Chebyshev's inequality [3]

$$\mathsf{P}\left[\left|\sum_{k=0}^{m(\pi)-1}(B_{t_{k+1}} - B_{t_k})^2 - t\right| \geq \varepsilon\right] \leq \frac{2\text{mesh}(\pi)t}{\varepsilon^2} < \varepsilon \tag{2.2}$$

and the convergence in probability follows. □

[5] We use the standard notations $\sigma(A,B,\ldots)$ and $\sigma(A_i, i\in I)$ for the σ-algebra generated by the random variables $A,B,\ldots$ and A_i, $i\in I$, respectively.

[6] Convergence in probability means here that that for each $\varepsilon > 0$ there exist $\delta > 0$ such that $\mathsf{P}[|\sum_{k=0}^{m(\pi)-1}|X_{t_{k+1}} - X_{t_k}|^p - V_X^{(p)}(t)| \geq \varepsilon] < \varepsilon$ when $\text{mesh}(\pi) < \delta$.

The above proof and the Borel–Cantelli lemma, see e.g. [3], gives that the total variation of a Brownian motion is almost surely infinite in the sense that if take the limit along the sequence of dyadic partitions $\pi_n = \{t\,k\,2^{-n} \,:\, k = 0,1,2,\ldots,2^n\} = \{t_0 < t_1 < \ldots < t_{2^n}\}$ of $[0,t]$, then

$$\lim_{n\to\infty} \sum_{t_k\in\pi_n, k\le 2^n-1} |B_{t_{k+1}} - B_{t_k}| = \infty \tag{2.3}$$

almost surely. Namely, if we denote $\mathsf{P}[E(\pi)]$ the left-hand side of (2.2), then $\sum_n \mathsf{P}[E(\pi_n)] < \infty$ and hence $\sum_{t_k\in\pi_n, k\le 2^n-1} (B_{t_{k+1}} - B_{t_k})^2 \to t$ almost surely by the Borel–Cantelli lemma. Take any ω for which the convergence occurs. Then (2.3) is implied by the fact that as $n \to \infty$

$$\underbrace{\sum_{t_k\in\pi_n, k\le 2^n-1} (B_{t_{k+1}}(\omega) - B_{t_k}(\omega))^2}_{\to t} \;\le\; \underbrace{\mathrm{mesh}(\pi_n)}_{\to 0} \sum_{t_k\in\pi_n, k\le 2^n-1} |B_{t_{k+1}}(\omega) - B_{t_k}(\omega)|.$$

2.2 Stochastic integration

We define in this section a process X_t which can be interpreted as the integral

$$X_t(\omega) = \int_0^t f(t,\omega)\,\mathrm{d}B_t(\omega).$$

The construction is important because of the following reasons: (i) It is tool for generating new stochastic processes out of Brownian motion. (ii) Coordinate changes such as $f(B_t)$ turn out to have extremely useful representation using the above integral. (iii) In many applications, $\mathrm{d}B_t$ models independent and stationary noise.

The integral doesn't exist as a pathwise Riemann–Stieltjes (or similar) integral even for a continuous f, because the total variation of the Brownian motion is infinite. For instance, for the definition that we use, it holds that $\int_0^t B_s \mathrm{d}B_s \neq (1/2)B_t^2$ and therefore the usual integration by parts formula can't hold.

2.2.1 Stochastic integral as L^2-extension

In this section $(\mathscr{F}_t)_{t\in\mathbb{R}_{\ge 0}}$ is a filtration and $(B_t)_{t\in\mathbb{R}_{\ge 0}}$ is a standard one-dimensional Brownian motion with respect to $(\mathscr{F}_t)_{t\in\mathbb{R}_{\ge 0}}$.

We need to define the correct set of integrands f for the stochastic integral. In this subsection, they'll be the measurable, adapted and square-integrable processes.

Definition 2.4. A stochastic process $(X_t)_{t\in\mathbb{R}_{\ge 0}}$ is *measurable* if the mapping $(t,\omega) \mapsto X_t(\omega)$ is $\mathscr{B}_{\mathbb{R}} \times \mathscr{F}$-measurable.

Definition 2.5. Let $T > 0$. We define $\mathscr{L}^2$ to be the set of measurable, adapted processes f that satisfy

$$\mathsf{E}\left[\int_0^T f(t,\cdot)^2 \mathrm{d}t\right] < \infty \tag{2.4}$$

and we call $f \in \mathscr{L}^2$ *simple* if f can be written in the form

$$f(t,\omega) = \sum_{k=0}^{n-1} X_k(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t) \tag{2.5}$$

where $0 \le t_0 < t_1 < t_2 \ldots < t_n \le T$ and X_k is a $\mathscr{F}_{t_k}$-measurable, square integrable random variable.[7]

Remark 2.3. Notice that $\mathscr{L}^2$ is a closed subspace of $L^2(\mathrm{d}t \times \mathrm{d}\mathsf{P})$.

We would like to define a mapping $f \mapsto I[f]$ which we later denote by

$$I[f](\omega) =: \int_0^T f(t,\omega)\mathrm{d}B_t(\omega).$$

If that notation makes any sense, we need to define $I[\mathbb{1}_{[s,t)}] = B_t - B_s$ for any $0 \le s < t \le T$. Therefore for any f which is of the form (2.5) it is natural to define by linearity that

$$I[f] = \sum_{k=0}^{n-1} X_k(B_{t_{k+1}} - B_{t_k}).$$

It turns out that this definition that works for any simple $f \in \mathscr{L}^2$ has a unique L^2-continuous extension to the whole $\mathscr{L}^2$. Namely, we first observe that the following isometry holds.

Proposition 2.2 (Itô isometry for simple processes). *For any bounded, simple $f \in \mathscr{L}^2$,*

$$\mathsf{E}\left[I[f]^2\right] = \mathsf{E}\left[\int_0^T f(t,\cdot)^2 \mathrm{d}t\right].$$

Proof. Let's calculate both sides explicitly for a bounded, simple $f \in \mathscr{L}^2$ of the form (2.5). Notice that $f^2 = \sum_{k=0}^{n-1} X_k^2 \mathbb{1}_{[t_k,t_{k+1})}$ and hence

$$\mathsf{E}\left[\int_0^T f(t,\cdot)^2 \mathrm{d}t\right] = \sum_{k=0}^{n-1} \mathsf{E}[X_k^2](t_{k+1} - t_k).$$

On the other hand

$$\mathsf{E}[I[f]^2] = \sum_k \mathsf{E}\left[X_k^2 (B_{t_{k+1}} - B_{t_k})^2\right] + 2\sum_{k<l} \mathsf{E}[X_k X_l (B_{t_{k+1}} - B_{t_k})(B_{t_{l+1}} - B_{t_l})].$$

[7] The class $\mathscr{L}^2$ could be instead called $\mathscr{L}^2(T)$ and then we could set $f \in \mathscr{L}^2$ if and only if $f \in \mathscr{L}^2(T)$ for any $T > 0$. For simplicity, we use the notation $\mathscr{L}^2$ for both classes.

The facts that f is adapted, and thus X_k is $\mathscr{F}_{t_k}$-measurable, and that $B_{t_{k+1}} - B_{t_k}$ is independent from $\mathscr{F}_{t_k}$ imply that

$$\mathsf{E}\left[X_k^2\,(B_{t_{k+1}} - B_{t_k})^2\right] = \mathsf{E}[X_k^2]\,\mathsf{E}\left[(B_{t_{k+1}} - B_{t_k})^2\right] = \mathsf{E}[X_k^2]\,(t_{k+1} - t_k)$$
$$\mathsf{E}[X_k X_l (B_{t_{k+1}} - B_{t_k})(B_{t_{l+1}} - B_{t_l})] = \mathsf{E}[X_k X_l (B_{t_{k+1}} - B_{t_k})]\mathsf{E}[B_{t_{l+1}} - B_{t_l}] = 0$$

for $k < l$. The claim follows. □

The simple processes are dense in $\mathscr{L}^2$ by the next result. A sketch of its proof is given in Appendix B.

Proposition 2.3. *For each $f \in \mathscr{L}^2$, there exist a sequence of bounded, simple $f_n \in \mathscr{L}^2$ such that*

$$\mathsf{E}\left[\int_0^T (f(t,\cdot) - f_n(t,\cdot))^2 \mathrm{d}t\right] \to 0,$$

i.e. f_n converges to f in $L^2(\mathrm{d}t \times \mathrm{dP})$.

If $f_n \in \mathscr{L}^2$ is a sequence of simple, bounded processes converging to f, then f_n is a Cauchy sequence in $L^2(\mathrm{d}t \times \mathrm{dP})$ and hence by the isometry property $I[f_n]$ is a Cauchy sequence if $L^2(\mathrm{dP})$ and hence it converges. Therefore we can define $I[f] = \lim_n I[f_n]$. Notice that this limit doesn't depend on the choice of f_n: if f_n and f_n' are two such sequences, then $f_n - f_n'$ goes to zero in $L^2(\mathrm{d}t \times \mathrm{dP})$ and hence by isometry, $\lim_n I(f_n) = \lim_n I(f_n')$ almost surely. This is summarized in the following definition.

Definition 2.6. For any $f \in \mathscr{L}^2$, the *stochastic integral* (or *Itô integral*) is defined to be

$$\int_0^T f\,\mathrm{d}B_t(\omega) := I[f](\omega) := (\lim_n I[f_n)])(\omega) \tag{2.6}$$

where the limit is in $L^2(\mathsf{P})$ and $f_n \in \mathscr{L}^2$ is any sequence of bounded, simple processes converging to f in $L^2(\mathrm{d}t \times \mathrm{dP})$. The integral is defined almost surely.

Proposition 2.4 (Itô isometry for $\mathscr{L}^2$). *For any $f \in \mathscr{L}^2$, $\mathsf{E}[(\int_0^T f\,\mathrm{d}B_t)^2] = \mathsf{E}[\int_0^T f^2 \mathrm{d}t]$.*

Proposition 2.5. *If $f_n \in \mathscr{L}^2$, $f \in \mathscr{L}^2$ and $f_n \to f$ in $L^2(\mathrm{d}t \times \mathrm{dP})$ then $\int_0^T f_n \mathrm{d}B_t \to \int_0^T f \mathrm{d}B_t$ in $L^2(\mathsf{P})$.*

Example 2.1. We'll show that

$$\int_0^t B_s \mathrm{d}B_s = \frac{1}{2}B_t - \frac{1}{2}t.$$

Let π_n be a sequence of partitions of $[0,t]$ such that $\mathrm{mesh}(\pi_n) \to 0$. By the above, the sequence of processes $f_n(s,\omega) = \sum_{t_j \in \pi_n} B_{t_j}(\omega)\mathbb{1}_{[t_j,t_{j+1})}(s)$ is a reasonable choice for a discretization of the integrand. Since

$$\mathsf{E}\left[\int_0^t (B_s - f_n(s,\cdot))^2 \mathrm{d}s\right] = \mathsf{E}\left[\sum_j \int_{t_j}^{t_{j+1}} (B_s - B_{t_j})^2 \mathrm{d}s\right] = \sum_j \frac{1}{2}(t_{j+1} - t_j)^2 \to 0$$

as $n \to \infty$, then by Corollary 2.5, $\int_0^t B_s \mathrm{d}B_s = \lim \int_0^t f_n \mathrm{d}B_s = \lim \sum_j B_{t_j}(B_{t_{j+1}} - B_{t_j})$. Now notice that $B_{t_{j+1}}^2 - B_{t_j}^2 = (B_{t_{j+1}} - B_{t_j})^2 + 2B_{t_j}(B_{t_{j+1}} - B_{t_j})$ and thus

$$\sum_j B_j(B_{t_{j+1}} - B_{t_j}) = \frac{1}{2}B_t^2 - \frac{1}{2}\sum_j (B_{t_{j+1}} - B_{t_j})^2$$

and the second term on the right converges in L^2 to the quadratic variation of Brownian motion which we already showed to be t.

The following proposition states properties of the stochastic integral which hold for simple processes and hence continue to hold for any limit of a sequence of simple processes. We skip further details of the proof.

Proposition 2.6. *Let $f, g \in \mathscr{L}^2$, $a \in \mathbb{R}$ and let $0 \le S < U < T$. Then*

1. $\int_S^T f \mathrm{d}B_t = \int_S^U f \mathrm{d}B_t + \int_U^T f \mathrm{d}B_t$
2. $\int_S^T a f \mathrm{d}B_t = a \int_S^T f \mathrm{d}B_t$
3. $\int_S^T (f+g) \mathrm{d}B_t = \int_S^T f \mathrm{d}B_t + \int_S^T g \mathrm{d}B_t$
4. $\mathsf{E}[\int_S^T f \mathrm{d}B_t] = 0$
5. $\int_S^T f \mathrm{d}B_t$ *is* $\mathscr{F}_T$*-measurable*

2.2.2 Stochastic integral as a process

2.2.2.1 Stochastic integral as a continuous martingale

Based on the results of Section 2.2.1, we try to define a process X_t such that $X_t = \int_0^t f \, \mathrm{d}B_s$ for every t. The problem in defining $X_t = I[f \mathbb{1}_{[0,t]}]$ is that the for each fixed t, X_t is defined in a set of probability one, say, in Ω_t, but it is possible that the probability of the uncountable intersection $\bigcap_t \Omega_t$ is strictly less than 1 or even that $\bigcap_t \Omega_t$ is not an event (a measurable set). Therefore we define X_t in this way in a countable set of t and then extend by continuity of $t \mapsto X_t$ to other values of t as in the following theorem. For the definition of a martingale consult Appendix A.

Theorem 2.2 (Stochastic integral is a continuous L^2-martingale). *For each $f \in \mathscr{L}^2$ there exists a continuous square integrable martingale $(X_t)_{t \in \mathbb{R}_{\ge 0}}$ such that for each t, $X_t = \int_0^t f(s, \cdot) \mathrm{d}B_s$ almost surely.*

Remark 2.4. The process $(X_t)_{t \in \mathbb{R}_{\ge 0}}$ is unique in the sense that if there is another process $(X_t')_{t \in \mathbb{R}_{\ge 0}}$ with the same properties, then almost surely $X_t = X_t'$ for all t.

Proof. Fix some $T > 0$. Take a sequence of simple (and bounded) $f_n \in \mathscr{L}^2$ such that $f_n \to f$ in $L^2(\mathrm{d}t \times \mathrm{dP}, [0,T] \times \Omega)$ and define $X_t^{(n)} = I[f_n \mathbb{1}_{[0,t]}]$ which is well-defined in whole Ω. If $f_n = \sum a_k \mathbb{1}_{[t_k, t_{k+1})}$, then for $t_l \le t < t_{l+1}$ we have an explicit formula

$$X_t^{(n)} = a_l \cdot (B_t - B_{t_l}) + \sum_{k=0}^{l-1} a_k \cdot (B_{t_{k+1}} - B_{t_k}). \tag{2.7}$$

Clearly $t \mapsto X_t$ is continuous. To show that it is a martingale, notice first that it is adapted because all the random variables on the right of (2.7) are $\mathscr{F}_t$-measurable. Next notice that $\mathsf{E}[|X_t^{(n)}|] < \infty$, because it is a finite sum of integrable random variables. Finally, for $0 \le s < t \le T$ we can assume that $s = t_l$ and $t = t_m$ for some l and m (redefine the partition in f_n again if necessary) and then

$$\mathsf{E}[X_t^{(n)}|\mathscr{F}_s] = \mathsf{E}[X_s^{(n)}|\mathscr{F}_s] + \mathsf{E}[\sum_{k=l}^{m-1} a_k \cdot (B_{t_{k+1}} - B_{t_k})|\mathscr{F}_s] = X_s^{(n)}$$

because $X_s^{(n)}$ is $\mathscr{F}_s$-measurable and by the properties of conditional expectation (see Appendix A)

$$\begin{aligned}\mathsf{E}[a_k \cdot (B_{t_{k+1}} - B_{t_k})|\mathscr{F}_s] &= \mathsf{E}[\mathsf{E}[a_k \cdot (B_{t_{k+1}} - B_{t_k})|\mathscr{F}_{t_k}]|\mathscr{F}_s] \\ &= \mathsf{E}[a_k \cdot \mathsf{E}((B_{t_{k+1}} - B_{t_k})|\mathscr{F}_{t_k}]|\mathscr{F}_s] = 0. \end{aligned} \tag{2.8}$$

Since $X_t^{(n)} - X_t^{(m)}$ is a martingale, by Doob's maximal inequality

$$\mathsf{P}\left[\sup_{t\in[0,T]} \left|X_t^{(n)} - X_t^{(m)}\right| \ge \varepsilon\right] \le \frac{1}{\varepsilon^2}\mathsf{E}[|X_T^{(n)} - X_T^{(m)}|^2] = \frac{1}{\varepsilon^2}\|f_n - f_m\|^2_{L^2(\mathrm{d}t\times \mathrm{dP})}$$

for any $\varepsilon > 0$. Choose a subsequence n_k such that $\|f_{n_{k+1}} - f_{n_k}\|^2_{L^2(\mathrm{d}t\times\mathrm{dP})} \le 2^{-3k}$ and use the previous estimate for $\varepsilon = 2^{-k}$ to get $\mathsf{P}[\sup_{t\in[0,T]} |X_t^{(n_{k+1})} - X_t^{(n_k)}| \ge 2^{-k}] \le 2^{-k}$. By the Borel–Cantelli lemma, there exist random variable N which is almost surely finite and for $k \ge N(\omega)$,

$$\sup_{t\in[0,T]} \left|X_t^{(n_{k+1})} - X_t^{(n_k)}\right| < 2^{-k}.$$

Hence the sequence of the continuous processes $(X_t^{(n_k)})$ converges almost surely uniformly to a continuous process (X_t). Since for any fixed t, $\lim X_t^{(n_k)}$ in $L^2(\mathsf{P})$ is $\int_0^t f \mathrm{d}B_s$ then

$$X_t = \int_0^t f\,\mathrm{d}B_s$$

almost surely. This also shows that (X_t) is adapted and square integrable.

Finally the martingale property of $(X_t^{(n)})$, for any $0 \le s < t \le T$

$$X_s^{(n)} = \mathsf{E}[X_t^{(n)}|\mathscr{F}_s].$$

Since the random variables $X_s^{(n)}$ and $X_s^{(n)}$ converge in $L^2(\mathsf{P})$ to X_s and X_t, respectively, then by the properties of conditional expectation (see Appendix A)

$$X_s = \mathsf{E}[X_t|\mathscr{F}_s].$$

for any $0 \le s < t \le T$. For the whole $\mathbb{R}_+$, the claim follows from the above by taking a countable sequence $T \nearrow \infty$ and using the uniqueness. □

Remark 2.5. The property that we used in (2.8) could be reformulated in the following way: if $(M_t)_{t\in\mathbb{R}_+}$ is a martingale and if $0 \le s \le t \le u$ and Y is a $\mathscr{F}_t$-measurable bounded random variable, then $\mathsf{E}[Y(M_u - M_t)\,|\,\mathscr{F}_s] = 0$. We say that martingale increments $M_u - M_t$ are orthogonal to $\mathscr{F}_t$.

Definition 2.7. For any $f \in \mathscr{L}^2$, the *stochastic integral* (or Itô integral) is redefined to be a continuous version of $\int_0^t f \mathrm{d}B_s$, which exists by the previous Theorem.

Remark 2.6. The processes $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ are versions of each other if $\mathsf{P}[X_t = Y_t] = 1$ for each t.

Definition 2.8. For any process $X_t = \int_0^t f \mathrm{d}B_s$, define the *quadratic variation process* as

$$\langle X \rangle_t(\omega) = \int_0^t f(s,\omega)^2 \mathrm{d}t.$$

The process $\langle X \rangle$ is the quadratic variation in the sense of the equation (2.1). We postpone the statement of that result. The following result gives a second interpretation of the quadratic variation process.

Theorem 2.3. *Let $f \in \mathscr{L}^2$, $X_t = \int_0^t f \mathrm{d}B_s$ and $\langle X \rangle_t$ as above. Then $X_t^2 - \langle X \rangle_t$ is a martingale.*

Proof. We leave as an exercise to check this for bounded, simple $f \in \mathscr{L}^2$. In the general case take a sequence of bounded, simple $f_n \in \mathscr{L}^2$ and define $X_t^{(n)} = \int_0^t f_n \mathrm{d}B_s$. The claim follows easily from the $L^1(\mathsf{P})$ convergence of $(X_t^{(n)})^2 - \langle X^{(n)} \rangle_t$ which implies the $L^1(\mathsf{P})$ convergence of $\mathsf{E}[(X_t^{(n)})^2 - \langle X^{(n)} \rangle_t \,|\, \mathscr{F}_s]$ by the properties of conditional expectation (see Appendix A). □

Next we define a stopping time which can be taught as the time when "some event occurs" so that for each time instant, the question whether this event has already occurred or not before or at that time is a "measurable question".

Definition 2.9. A random variable $\tau : \Omega \to [0,\infty]$ is called a *stopping time* with respect to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ if for all $t \ge 0, \{\omega : \tau(\omega) \le t\} \in \mathscr{F}_t$.

An example of a stopping time is $\tau_A = \inf\{t \in \mathbb{R}_{\ge 0} : B_t \in A\}$ where A is a closed or open set in $\mathbb{R}$.

One way to describe the following result is that by that proposition, the pathwise interpretation of the Itô integral makes sense: if two integrands have the same paths up to a stopping time, then the integrals also agree up to that stopping time. For the proof see Appendix B.

Proposition 2.7. *If τ is a stopping time and $f \in \mathscr{L}^2$ and $g \in \mathscr{L}^2$ processes such that $f(t,\omega) = g(t,\omega)$ for any (t,ω) such that $t \le \tau(\omega)$, then for almost all ω*

$$\int_0^t f \,\mathrm{d}B_s(\omega) = \int_0^t g \,\mathrm{d}B_s(\omega)$$

for all $t \le \tau(\omega)$.

2.2.2.2 Localization and a general class of integrands

At this point, we have the Itô integral defined for any measurable, adapted process f such that $\mathsf{E}[\int_0^T f^2 \mathrm{d}t] < \infty$ for any $T \in (0,\infty)$. However, we would like to have a larger class of processes that includes at least all the continuous processes, such as $f(t,\omega) = \exp(B_t(\omega)^3)$ which is an example of a process that doesn't belong to $\mathscr{L}^2$.

Definition 2.10. The class $\mathscr{L}^2_{\mathrm{loc}}$ is defined to be the set of measurable, adapted process f such that almost surely

$$\int_0^T f(t,\cdot)^2 \,\mathrm{d}t < \infty$$

for any $T \in (0,\infty)$.

Fix some $f \in \mathscr{L}^2_{\mathrm{loc}}$. Define a stopping time

$$\tau_n(\omega) = \inf\left\{ t \in \mathbb{R}_+ \,:\, \int_0^t f(s,\omega)^2 \mathrm{d}s \geq n \right\}.$$

It follows from $f \in \mathscr{L}^2_{\mathrm{loc}}$, that $\tau_n \nearrow \infty$ almost surely as $n \to \infty$.

Let $f_n(t,\omega) = f(t,\omega)\mathbb{1}_{t \leq \tau_n(\omega)}$. Then $f_n \in \mathscr{L}^2$ and we can define the Itô integral $X_t^{(n)} = \int_0^t f_n \mathrm{d}B_s$. Since $f_n(t,\omega) = f_m(t,\omega)$ for all (t,ω) such that $t \leq (\tau_n \wedge \tau_m)(\omega)$ and since $\tau_n \wedge \tau_m$ is a stopping time[8], by Proposition 2.7 for almost all ω,

$$X_t^{(n)}(\omega) = X_t^{(m)}(\omega)$$

for $t \leq (\tau_n \wedge \tau_m)(\omega)$.

For each fixed ω, this is a strong mode of convergence: there is a finite $n_0(\omega)$ such that $X_t^{(n)}(\omega) = X_t^{(m)}(\omega)$ for any $n,m \geq n_0(\omega)$. Define now a process $(X_t)_{t\in\mathbb{R}_+}$ on the event $\{\tau_n \nearrow \infty\}$

$$X_t(\omega) = X_t^{(n)}(\omega)$$

where $n \in \mathbb{N}$ is any number satisfying $\tau_n(\omega) \geq t$. The complement of the event $\{\tau_n \nearrow \infty\}$ has zero probability and there we can define $X_t = 0$ identically, say.

Definition 2.11. The Itô integral of $f \in \mathscr{L}^2_{\mathrm{loc}}$ is defined as

$$\int_0^t f \mathrm{d}B_s(\omega) = X_t(\omega) = X_t^n(\omega)$$

where $n \in \mathbb{N}$ is any number satisfying $\tau_n(\omega) \geq t$ and $X_t^n(\omega)$ is as above.

For any continuous process $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ and for any stopping time τ, define a stopped process $(X_t^\tau)_{t\in\mathbb{R}_{\geq 0}}$ by $X_t^\tau = X_{t\wedge\tau}$. The continuity of $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ guarantees that X_t^τ is measurable.

---

[8] The minimum of two stopping times is a stopping time.

Definition 2.12. A continuous process $(M_t)_{t\in\mathbb{R}_{\geq 0}}$ adapted to $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$ is called *local martingale* if there exist a sequence of stopping times $0 \leq \tau_1 \leq \tau_2 \leq \ldots$ such that $\mathsf{P}(\tau_k \nearrow \infty) = 1$ and for each k, M^{τ_k} is a martingale. It is a *local square integrable martingale*, if each $(M_t^{\tau_k})_{t\in\mathbb{R}_{\geq 0}}$ is a square integrable martingale.

Remark 2.7. The use of stopping times of similar to τ_n, in Definitions 2.11 and 2.12, is called *localization* of the processes.

The next theorem lists the properties of Itô integral in its most general form. The theorem follows from the properties of the Itô integral for $\mathscr{L}^2$-integrands and from the construction of the integral using localization. See Appendix B for the proof of the last statement.

Theorem 2.4. *For any $f \in \mathscr{L}^2_{\mathrm{loc}}$, the processes $X_t = \int_0^t f\,\mathrm{d}B_s$ and $X_t^2 - \langle X\rangle_t$ are continuous local martingale. Furthermore, $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ has finite quadratic variation and almost surely for any t, $V_X^{(2)}(t) = \langle X\rangle_t$.*

2.3 Itô's formula

2.3.1 Itô's formula for a Brownian motion

Itô's formula is a result of central importance in stochastic calculus. We present first its version for a Brownian motion. By Itô's formula, functions of Brownian motion can be written as sum of a stochastic integral and an integral with respect to dt.

Theorem 2.5 (Itô's formula for a Brownian motion). *Let $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\dot{F}, F', F''$ exist and are continuous, where*

$$\dot{F}(t,x) = \frac{\partial F}{\partial t}(t,x), \quad F'(t,x) = \frac{\partial F}{\partial x}(t,x) \quad \text{and } F''(t,x) = \frac{\partial^2 F}{\partial x^2}(t,x).$$

Then almost surely

$$F(t,B_t) = F(0,B_0) + \int_0^t \dot{F}(s,B_s)\mathrm{d}s + \int_0^t F'(s,B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t F''(s,B_s)\mathrm{d}s \tag{2.9}$$

for any $t \in \mathbb{R}_+$. For the previous equation we will use the shorthand notation

$$\mathrm{d}F(t,B_t) = \dot{F}(t,B_t)\mathrm{d}t + F'(t,B_t)\mathrm{d}B_t + \frac{1}{2}F''(t,B_t)\mathrm{d}t.$$

Proof (A sketch). The proof is based on the Taylor expansion of $F(t,x)$ in both of its variables. Take a partition π of $[0,t]$ and write a telescoping sum

$$F(t,B_t) - F(0,B_0) = \sum_{k=0}^{m(\pi)-1} (F(t_{k+1}, B_{t_{k+1}}) - F(t_k, B_{t_k})).$$

By the mean value theorem

$$F(t_{k+1},B_{t_{k+1}})-F(t_k,B_{t_k})=[F(t_{k+1},B_{t_{k+1}})-F(t_k,B_{t_{k+1}})]+[F(t_k,B_{t_{k+1}})-F(t_k,B_{t_k})]$$
$$=\underbrace{[F(t_{k+1},B_{t_{k+1}})-F(t_k,B_{t_{k+1}})]}_{=:a_k}+\underbrace{F'(t_k,B_{t_k})(B_{t_{k+1}}-B_{t_k})}_{=:b_k}+\frac{1}{2}\underbrace{F''(t_k,\eta_k)(B_{t_{k+1}}-B_{t_k})^2}_{=:c_k}$$

where η_k is a $\mathscr{F}_{t_{k+1}}$-measurable random variable that lies on the interval between B_{t_k} and $B_{t_{k+1}}$. Take a sequence of partitions π_n such that $\mathrm{mesh}(\pi_n)\to 0$ as $n\to\infty$. The claim is that the sums $\sum a_k$, $\sum b_k$ and $\sum c_k$ will converge to each of the three integrals in (2.9), respectively. The convergence will be almost sure along suitable subsequences of π_n. For the rest of the proof see Appendix B. □

Example 2.2. Let $F(x)=x^2/2$ and let $(B_t)_{t\in\mathbb{R}_+}$ be a one-dimensional Brownian motion with $B_0=0$, then $\frac{1}{2}B_t^2=\int_0^t B_s\mathrm{d}B_s+\frac{1}{2}\int_0^t \mathrm{d}s$ by Theorem 2.5 and hence after rearranging the terms $\int_0^t B_s\mathrm{d}B_s=\frac{1}{2}B_t^2-\frac{1}{2}t$ which is in agreement with the result we obtained by directly applying the definition of Itô integral in Example 2.1.

Remark 2.8. The proof of Theorem 2.5 uses (i) continuity of $t\mapsto B_t$ and the facts that (ii) B_t and (iii) B_t^2-t are martingales. It is possible to use the same proof to derive the formula $\mathsf{E}[f(X_t)|\mathscr{F}_s]=f(X_s)+\frac{1}{2}\int_s^t \mathsf{E}[f(X_u)|\mathscr{F}_s]\mathrm{d}u$ for any $s<t$ and any adapted process X_t satisfying the properties (i)–(iii). If this formula is applied to the function $f(x)=\exp(\mathrm{i}\theta x)$, an argument using the characteristic function very similar to the proof of Theorem 2.8 shows the next results.

Theorem 2.6 (Lévy's characterization of Brownian motion). *Let $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ be a continuous local martingale with $X_0=0$. If $(X_t^2-t)_{t\in\mathbb{R}_{\geq 0}}$ is a local martingale, then $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ is a standard Brownian motion.*

2.3.2 Itô's formula for semimartingales

Henceforth, we'll write the time parameter of the integrands explicitly. Let's first study two stochastic integrals with respect to a common Brownian motion

$$X_t=\int_0^t f_s\mathrm{d}B_s,\quad Y_t=\int_0^t g_s\mathrm{d}B_s$$

where $f,g\in\mathscr{L}^2_{\mathrm{loc}}$. Their *(quadratic) covariation process* is defined as

$$\langle X,Y\rangle_t=\int_0^t f_s g_s\mathrm{d}s.$$

Then we notice that it satisfies the relation $4\langle X,Y\rangle_t=\langle X+Y\rangle_t-\langle X-Y\rangle_t$. A similar relation can be written for the product X_tY_t and a sum of the form $\sum_k(X_{t_{k+1}}-X_{t_k})(Y_{t_{k+1}}-Y_{t_k})$. Consequently, $X_tY_t-\langle X,Y\rangle_t$ is a local martingale and along partitions of $[0,t]$

$$\lim_{\mathrm{mesh}(\pi)\to 0} \sum_{k=0}^{m(\pi)-1} (X_{t_{k+1}} - X_{t_k})(Y_{t_{k+1}} - Y_{t_k}) = \langle X, Y\rangle_t \tag{2.10}$$

in probability.

Let's then consider the case of two stochastic integrals with respect to independent Brownian motions. If $(B^{(1)}, B^{(2)})$ is a standard two-dimensional Brownian motion and

$$X_t = \int_0^t f_s \, \mathrm{d}B_s^{(1)}, \quad Y_t = \int_0^t g_s \, \mathrm{d}B_s^{(2)}$$

where $f, g \in \mathscr{L}^2_{\mathrm{loc}}$, then $X_t Y_t$ is a local martingale. The covariation process is $\langle X, Y\rangle_t = 0$ and it satisfies (2.10) together with $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ and $(Y_t)_{t\in\mathbb{R}_{\geq 0}}$. These statements can be verified in the same manner as Theorem 2.4.

In the most general case, let $(B_t^{(1)}, B_t^{(2)}, \ldots, B_t^{(m)})$ be a standard m-dimensional Brownian motion. Let

$$X_t = X_0 + \int_0^t f_s \mathrm{d}s + \sum_{k=1}^m \int_0^t g_s^{(k)} \, \mathrm{d}B_s^{(k)} \tag{2.11}$$

$$Y_t = Y_0 + \int_0^t \hat{f}_s \mathrm{d}s + \sum_{k=1}^m \int_0^t \hat{g}_s^{(k)} \, \mathrm{d}B_s^{(k)}$$

where X_0 and Y_0 are $\mathscr{F}_0$-measurable random variables, $g^{(k)}, \hat{g}^{(k)} \in \mathscr{L}^2_{\mathrm{loc}}$ and $f, \hat{f}$ are measurable, adapted to $(F_t)_{t\in\mathbb{R}_+}$ and satisfy

$$\mathsf{P}\left[\int_0^t |f_s| \mathrm{d}s < \infty \text{ for all } t \in \mathbb{R}_+\right] = 1.$$

Then since integrals $\int_0^t f_s \mathrm{d}s$ have (locally) finite total variation, by the above it is natural to define

$$\langle X\rangle_t = \sum_{k=1}^m \int_0^t \left(g_s^{(k)}\right)^2 \mathrm{d}s, \quad \langle Y\rangle_t = \sum_{k=1}^m \int_0^t \left(\hat{g}_s^{(k)}\right)^2 \mathrm{d}s, \quad \langle X, Y\rangle_t = \sum_{k=1}^m \int_0^t g_s^{(k)} \hat{g}_s^{(k)} \, \mathrm{d}s$$

which are the quadratic variation and covariation processes also in the sense of (2.1) and (2.10).

Definition 2.13. We call a process of the form (2.11) a *semimartingale* and use a shorthand notation $\mathrm{d}X_t = f_t \mathrm{d}t + \sum_{k=1}^m g_t^{(k)} \, \mathrm{d}B_t^{(k)}$.

Remark 2.9. This is a slight abuse of standard terminology. More generally, semimartingale is any process that is sum of an adapted finite variation process and a local martingale.

Next we present a version of Itô's formula for semimartingales. An interesting viewpoint to this result is that the class of semimartingales is closed under forming new processes $F(X_t^{(1)}, \ldots, X_t^{(n)})$ from semimartingales $(X_t^{(k)})_{t\in\mathbb{R}_{\geq 0}}$, $k = 1, 2, \ldots, n$.

Theorem 2.7 (Itô's formula for semimartingales). *Let $1 \le l \le n$. Let $X_t^{(j)}$ be semimartingales*

$$\mathrm{d}X_t^{(j)} = f_t^{(j)}\,\mathrm{d}t + \sum_{k=1}^{m} g_t^{(j,k)}\,\mathrm{d}B_t^{(k)}$$

for $1 \le j \le n$ where $f^{(j)}$ and $g^{(j,k)}$ are as above. Assume that $g^{(j,k)} = 0$ identically for $j > l$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous function such that $\partial_{x_i} F$ exists and is continuous for all $1 \le i \le n$ and that $\partial_{x_i x_j} F$ exists and is continuous for all $1 \le i, j \le l$.

Then $Y_t = F(X_t^{(1)}, \ldots, X_t^{(n)})$ is a semimartingale and almost surely

$$\mathrm{d}Y_t = \sum_{j=1}^{n} \left\{ \partial_{x_j} F(X_t^{(1)}, \ldots, X_t^{(n)}) f_t^{(j)}\,\mathrm{d}t + \sum_{k=1}^{m} \partial_{x_j} F(X_t^{(1)}, \ldots, X_t^{(n)}) g_t^{(j,k)}\,\mathrm{d}B_t^{(k)} \right\}$$
$$+ \frac{1}{2} \sum_{i,j=1}^{l} \sum_{k=1}^{m} \partial_{x_i, x_j} F(X_t^{(1)}, \ldots, X_t^{(n)})\, g_t^{(i,k)} g_t^{(j,k)}\,\mathrm{d}t$$

for all $t \in \mathbb{R}_+$. A compact way to write the expression is $\mathrm{d}Y_t = \sum_{j=1}^{n} (\partial_j F)\,\mathrm{d}X_t^{(j)} + \frac{1}{2}\sum_{i,j=1}^{l} (\partial_{ij} F)\,\mathrm{d}\left\langle X^{(i)}, X^{(j)} \right\rangle_t$.

Remark 2.10. Note that the theorem includes the case when F depends explicitly on time: let $l < n$ and take $X_t^{(n)} = t$. Theorem 2.5 is a special case of Theorem 2.7.

Remark 2.11 (Rules of stochastic calculus). Let $Y_t = F(X_t^{(1)}, \ldots, X_t^{(n)})$. The reader can memorize Itô's formula for Y_t by writing formally $Z_{t+\mathrm{d}t} = Z_t + \mathrm{d}Z_t$ for any semimartingale Z_t, taking the Taylor expansion of F at $(X_t^{(1)}, \ldots, X_t^{(n)})$ and using

$$\mathrm{d}t^2 = 0, \qquad \mathrm{d}t\,\mathrm{d}B_t^{(i)} = 0, \qquad \mathrm{d}B_t^{(i)}\,\mathrm{d}B_t^{(j)} = \delta_{ij}\mathrm{d}t.$$

Example 2.3 (Norm of a Brownian motion). Let $(B_t^{(1)}, \ldots, B_t^{(m)})$ be m-dimensional standard Brownian motion, $m \ge 2$, started away from the origin and let $F(x_1, \ldots, x_m) = (\sum_{k=1}^{m} x_k^2)^{1/2}$. By Itô's formula, $Y_t = F(B_t^{(1)}, \ldots, B_t^{(n)})$ satisfies

$$\mathrm{d}Y_t = \sum_k \frac{B_t^{(k)}\mathrm{d}B_t^{(k)}}{Y_t} + \frac{m-1}{2Y_t}\mathrm{d}t.$$

2.4 Further topics in stochastic calculus

2.4.1 When is a semimartingale a local martingale?

The following result is based on the observation that $\int_0^t f_s \mathrm{d}s$ has finite total variation, where as for every continuous martingale it is infinite. The proof is given in Appendix B. The result will be extremely useful in conjunction with Itô's formula.

Lemma 2.1. *Let* $\mathrm{d}X_t = \sum g_t^{(k)} \mathrm{d}B_t^{(k)} + f_t \mathrm{d}t$ *be a semimartingale. Then it is a local martingale if and only if almost surely* $f_t = 0$ *for almost all* t.

Example 2.4. Let $F : \mathbb{R} \to \mathbb{R}$ be smooth, $\lambda \in \mathbb{R} \setminus \{0\}$ and suppose that $F(B_t)\, e^{\lambda t}$ is a martingale. Then $\mathrm{d}(F(B_t)\, e^{\lambda t}) = F'(B_t)\, e^{\lambda t} \mathrm{d}B_t + (\lambda F(B_t) + \frac{1}{2} F''(B_t))\, e^{\lambda t} \mathrm{d}t$ by Itô's formula. By Lemma 2.1, it holds that $\lambda F(B_t) + \frac{1}{2} F''(B_t) = 0$ for all t. This is possible only if F satisfies $\lambda F(x) + \frac{1}{2} F''(x) = 0$ for all x. Thus

$$
\begin{aligned}
F(x) &= C_1 \exp\left(\sqrt{-2\lambda t}\right) + C_2 \exp\left(-\sqrt{-2\lambda t}\right), && \text{when } \lambda < 0, \text{ and} \\
F(x) &= C_1 \sin(\sqrt{2\lambda} t) + C_2 \cos(\sqrt{2\lambda} t), && \text{when } \lambda > 0.
\end{aligned}
$$

Here $C_1, C_2 \in \mathbb{R}$ are constants.

2.4.2 Time changes of local martingales and semimartingales

As usual, let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space with a filtration $(\mathscr{F}_t)_{t \in \mathbb{R}_{\geq 0}}$ and let $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ be a standard one-dimensional Brownian motion with respect to $(\mathscr{F}_t)_{t \in \mathbb{R}_{\geq 0}}$. Let's start this section by making the following definition.

Definition 2.14. If τ is a stopping time with respect to $(\mathscr{F}_t)_{t \in \mathbb{R}_{\geq 0}}$, define the *stopping time* σ*-algebra* as $\mathscr{F}_\tau = \{A \in \mathscr{F} : A \cap \{\tau \leq t\} \in \mathscr{F}_t \text{ for all } t \in \mathbb{R}_{\geq 0}\}$.

Remark 2.12. If $s \in \mathbb{R}_{\geq 0}$ is a constant and $\tau = s$ almost surely, then it's easy to check that $\mathscr{F}_\tau = \mathscr{F}_s$. So the notation $\mathscr{F}_\tau$ and the concept of stopping time σ-algebra is consistent with the earlier definitions.

In the same way, as $\mathscr{F}_t$ can be thought as the information available at time t, a stopping time σ-algebra $\mathscr{F}_\tau$ can be thought as the information available at a random time τ. The main reason to introduce the stopping time σ-algebra is time changes and analysis of martingales under time changes. See Appendix A for the optional stopping theorem which a result that that extends the martingale property from non-random time instances to stopping times.

The following theorem is an application of Itô's formula. It is a special case of more general result that *any continuous local martingale is a time-change of a Brownian motion*. The proof of the general result would follow the same lines if we had established the theory of the stochastic integral with respect to local martingales and we had corresponding Itô's formula available.

Theorem 2.8. *Let* $(X_t)_{t \in \mathbb{R}_{\geq 0}}$ *be a local martingale defined by*

$$X_t = \sum_{k=1}^{m} \int_0^t g_s^{(k)}\, \mathrm{d}B_s^{(k)}$$

where $g_t^{(k)} \in \mathscr{L}^2_{\mathrm{loc}}$. *Let* $(\sigma_r)_{r \in \mathbb{R}_{\geq 0}}$ *be the set of stopping times*

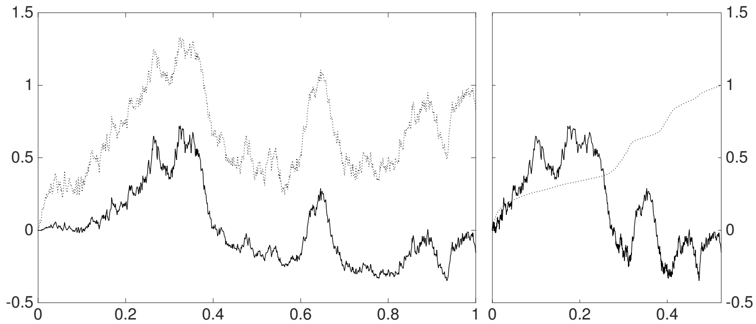


Fig. 2.1 On the left, an instance of Brownian motion $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is plotted with dots and the corresponding instance of $(X_t)_{t\in\mathbb{R}_{\geq 0}} = ((1/2)\,(B_t^2 - t))_{t\in\mathbb{R}_{\geq 0}}$ with a solid line. On the right, the time change of $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ plotted with a solid line and the change of time, which is the inverse of the map $t \mapsto \langle X\rangle_t$, is plotted with dots.

$$\sigma_r = \inf\{t \geq 0 \,:\, \langle X\rangle_t \geq r\}$$

where $\langle X\rangle_t = \sum_{k=1}^m \int_0^t (g_s^{(k)})^2\,\mathrm{d}s$ *is the quadratic variation process as before. Assume that almost surely* $\langle X\rangle_t \to \infty$ *as* $t \to \infty$*. Then the process* $(Y_t)_{t\in\mathbb{R}_{\geq 0}}$ *defined by* $Y_t = X_{\sigma_t}$ *is a standard one-dimensional Brownian motion with respect to the filtration* $(\mathscr{F}_{\sigma_t})_{t\in\mathbb{R}_{\geq 0}}$.

Proof. Since $\langle X\rangle_t \to \infty$ as $t \to \infty$, each σ_r is almost surely finite. By the continuity of the mapping $t \mapsto \langle X\rangle_t$, we have that $\langle X\rangle_{\sigma_r} = r$.

Let $M_t = \exp\left(\mathrm{i}\theta X_t + \theta^2\langle X\rangle_t/2\right)$. By Itô's formula $(M_t)_{t\in\mathbb{R}_{\geq 0}}$ is a continuous local martingale, see also Example 2.4. Note that $(M_t)_{t\in\mathbb{R}_{\geq 0}}$ is a complex valued process, but this causes no problems: we can apply Itô's formula separately for its real and imaginary parts. The statement that it is a local martingale means that both its real and imaginary parts are local martingales. Since $M_t^{\sigma_r} = M_{t\wedge\sigma_r}$ is bounded, $(M_t^{\sigma_r})_{t\in\mathbb{R}_{\geq 0}}$ is a martingale. Namely, if τ_n is the localizing sequence of $(M_t)_{t\in\mathbb{R}_{\geq 0}}$, then $(M_t^{\sigma_r\wedge\tau_n})_{t\in\mathbb{R}_{\geq 0}}$ is a martingale. Hence by boundedness of $(M_t^{\sigma_r})_{t\in\mathbb{R}_{\geq 0}}$ and by the fact that $\tau_n \nearrow \infty$ almost surely as $n \to \infty$,

$$\mathsf{E}[\underbrace{M_t^{\sigma_r\wedge\tau_n}}_{\to M_t^{\sigma_r} \text{ in } L^1}|\mathscr{F}_s] = \underbrace{M_s^{\sigma_r\wedge\tau_n}}_{\to M_s^{\sigma_r} \text{ in } L^1,\text{ as } n\to\infty}$$

and therefore by properties of conditional expected value, see Appendix A,

$$\mathsf{E}[M_t^{\sigma_r}|\mathscr{F}_s] = M_s^{\sigma_r}.$$

Thus $(M_t^{\sigma_r})_{t\in\mathbb{R}_{\geq 0}}$ is a continuous bounded martingale.

Next we apply the optional stopping theorem for stopping times $\sigma_s \leq \sigma_r$, where $0 \leq s \leq r$, to show that

$$\mathsf{E}[M_{\sigma_r}|\mathscr{F}_{\sigma_s}] = M_{\sigma_s}.$$

This implies that for any $0 \le s \le r$ and for any $\theta \in \mathbb{R}$,

$$\mathsf{E}\left[\exp\left(\mathrm{i}\theta(X_{\sigma_r} - X_{\sigma_s})\right) | \mathscr{F}_{\sigma_s}\right] = \exp\left(-\frac{\theta^2}{2}(r-s)\right).$$

The right-hand side of this equation is the characteristic function of a normal random variable with mean 0 and variance $r-s$. The left-hand side is a conditional version of characteristic function of $X_{\sigma_r} - X_{\sigma_s}$. That characteristic function is now constant as a $\mathscr{F}_{\sigma_s}$-measurable random variable. Therefore the fact that the characteristic function determines the distribution uniquely shows that $X_{\sigma_r} - X_{\sigma_s}$ is independent from $\mathscr{F}_{\sigma_s}$ and that $X_{\sigma_r} - X_{\sigma_s}$ is normally distributed with mean 0 and variance $r-s$.[9] □

Example 2.5. Let us continue the setup of Example 2.3. Let $(W_t)_{t\in\mathbb{R}_{\ge 0}}$ be a process defined by W_0 and $\mathrm{d}W_t = \sum_{k=1}^{n}(B_t^{(k)}/Y_t)\mathrm{d}B_t^{(k)}$, where $Y_t = F(B_t^{(1)}, \ldots, B_t^{(n)})$. Then

$$\langle W\rangle_t = \sum_{k=1}^{n}\int_0^t \frac{(B_s^{(k)})^2}{Y_s^2}\mathrm{d}s = t.$$

By Theorem 2.8, $(W_t)_{t\in\mathbb{R}_{\ge 0}}$ is a $(\mathscr{F}_t)_{t\in\mathbb{R}_{\ge 0}}$-Brownian motion.

The next result gives a general form of a *time change for semimartingales*. The proof is left as an exercise.

Proposition 2.8. *Let $a_t(\omega)$ be a continuous, positive, adapted process. Define a random time change by setting:*

$$S(t,\omega) = \int_0^t a_r(\omega)^2\,\mathrm{d}r, \qquad \sigma(s,\omega) = \inf\{t \in \mathbb{R}_{\ge 0} : S(t,\omega) \ge s\}$$

Let $(\tilde{B}_s)_{s\in\mathbb{R}_{\ge 0}}$ be the process defined by

$$\tilde{B}_s(\omega) = \int_0^{\sigma(s)} a_r\,\mathrm{d}B_r(\omega).$$

Then $(\tilde{B}_s)_{s\in\mathbb{R}_{\ge 0}}$ is a standard one-dimensional Brownian motion with respect to $(\mathscr{F}_{\sigma(s)})_{s\in\mathbb{R}_{\ge 0}}$, and for any continuous, adapted process $v_t(\omega)$ the following time-change formula holds

$$\int_0^s v_{\sigma(q)}\,\mathrm{d}\tilde{B}_q = \int_0^{\sigma(s)} v_r a_r\,\mathrm{d}B_r.$$

[9] If $\mathsf{E}[\exp(\mathrm{i}\theta_1 X)|\mathscr{G}] = \psi(\theta_1)$ is a constant as a function of $\omega \in \Omega$, then for any $\mathscr{G}$-measurable random variable Z, it holds that $\phi_{(X,Z)}(\theta_1,\theta_2) = \mathsf{E}[\exp(\mathrm{i}\theta_1 X + \mathrm{i}\theta_2 Z)] = \mathsf{E}[\exp(\mathrm{i}\theta_2 Z)\,\mathsf{E}[\exp(\mathrm{i}\theta_1 X)|\mathscr{G}]] = \psi(\theta_1)\phi_Z(\theta_2)$ by properties of conditional expected value. Here $\phi_{\underline{\theta}} = \mathsf{E}[\exp(\mathrm{i}\underline{\theta}\cdot\underline{Y})]$, $\underline{\theta} \in \mathbb{R}^n$ is the characteristic function of a $\mathbb{R}^n$-valued random variable $\underline{Y}$. Thus by the uniqueness theorem of the characteristic function, it follows that X and Z are independent and that the law of X is the unique probability measure on $\mathbb{R}$ such that $\phi_X = \psi$. Since this holds in particular for any random variable $Z = \mathbb{1}_E$, $E \in \mathscr{G}$, it follows that X is independent from $\mathscr{G}$.

Moreover if X_t is a semimartingale $\mathrm{d}X_t = u_t \mathrm{d}t + v_t \mathrm{d}B_t$ then the process $(\tilde{X}_s)_{s\in\mathbb{R}_{\geq 0}}$ defined by $\tilde{X}_s = X_{\sigma(s)}$ is a semimartingale with respect to $(\mathscr{F}_{\sigma(s)})_{s\in\mathbb{R}_{\geq 0}}$ and $(\tilde{B}_s)_{s\in\mathbb{R}_{\geq 0}}$ and satisfies

$$\mathrm{d}\tilde{X}_s = \frac{u_{\sigma(s)}}{a^2_{\sigma(s)}}\mathrm{d}s + \frac{v_{\sigma(s)}}{a_{\sigma(s)}}\mathrm{d}\tilde{B}_s.$$

2.4.3 Stochastic differential equations

We present here the rudiments of stochastic differential equations for single variable.

Let $(X_t)_{t\in[0,T]}$ be an $\mathbb{R}$-valued continuous stochastic process and let $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ be a standard one-dimensional Brownian motion. We say that X_t satisfies the *stochastic differential equation* (SDE)

$$\mathrm{d}X_t = F(t,X_t)\mathrm{d}t + G(t,X_t)\mathrm{d}B_t \tag{2.12}$$

with initial value $X_0 = Z$ if for each $t \in [0,T]$

$$X_t = Z + \int_0^t F(s,X_s)\,\mathrm{d}s + \int_0^t G(s,X_s)\,\mathrm{d}B_s.$$

If the process can be constructed in a given probability space with a given filtration and Brownian motion, then $(X_t)_{t\in[0,T]}$ is called a strong solution of the SDE.

Theorem 2.9. *Let $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ be one-dimensional Brownian motion and let $F : [0,T]\times\mathbb{R}\to\mathbb{R}$ and $G : [0,T]\times\mathbb{R}\to\mathbb{R}$ be measurable maps. Let Z be $\mathbb{R}$-valued square integrable random variable which is independent from $\sigma(B_t, t\in\mathbb{R}_{\geq 0})$. Suppose that*

$$|F(t,x)| + |G(t,x)| \leq C(1+|x|)$$
$$|F(t,x) - F(t,y)| + |G(t,x) - G(t,y)| \leq D\,|x-y|.$$

Then there exist a unique continuous solution $(X_t)_{t\in[0,T]}$ to the stochastic differential equation (2.12) with initial value $X_0 = Z$ with the property that X_t is adapted to the filtration $\mathscr{F}_t^{(B,Z)}$ generated by Z and B_s, $s\in[0,t]$. Furthermore $\mathsf{E}\left[\int_0^T |X_t|^2\mathrm{d}t\right] < \infty$.

The proof of the theorem is very similar to the proofs of existence and uniqueness of solutions of ordinary differential equations and is based on Picard–Lindelöf iteration. We leave the details to the reader.

Example 2.6. Let us continue Examples 2.3 and 2.5. The solution of the SDE

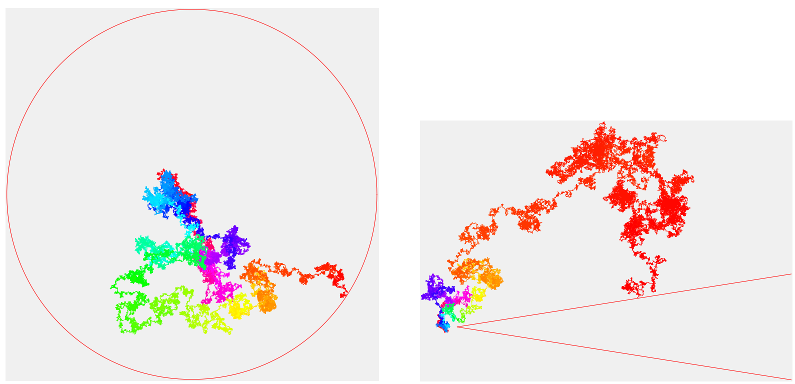
$$\mathrm{d}X_t = \mathrm{d}B_t + \frac{\delta-1}{2X_t}\mathrm{d}t,$$

$X_0 = x$ is called a *Bessel process* of dimension $\delta\in\mathbb{R}$ sent from x. In Examples 2.3 and 2.5 we saw that the norm of n-dimensional Brownian motion is a n-dimensional

Bessel process. We can use Theorem 2.9 with Proposition 2.7 to show that the solution exists and is unique for all $\delta \in \mathbb{R}$ up to the time $\tau = \inf\{t \in \mathbb{R}_{\geq 0} : \inf_{s\in[0,t)} X_s = 0\}$ which is the hitting time of 0. Using other methods, we could define it beyond the hitting of 0 for the parameter values $\delta > 0$.

Remark 2.13. In the time-homogeneous case, $F(t,x) = F(x)$ and $G(t,x) = G(x)$, these solutions X_t are called *diffusions*. From another viewpoint, diffusion defines a family of processes with one element for each starting point $x \in \mathbb{R}$.

2.5 Conformal invariance of two-dimensional Brownian motion



(a) A sample of complex Brownian motion stopped at its exit time from the disc.

(b) Conformal image of the same sample under a map taking the disc onto a sector.

Fig. 2.2 Illustration of conformal invariance of Brownian motion. The colors indicate time. We notice that the appearances of the paths are similar in both pictures except that the time is changed by a local factor when we move from the first picture to the second one.

As usual complex number z is represented in terms of its real and imaginary parts as $z = x + \mathrm{i}\,y$, similarly complex valued function of a complex variable is divided into its real and imaginary parts as $f(z) = u(z) + \mathrm{i}\,v(z)$. Let U be an open set in the complex plane $\mathbb{C}$ and let $z_0 \in U$. The starting point of complex analysis is that the following statements about a function $f : U \to \mathbb{C}$ are equivalent:

- The function f is *holomorphic* near z_0 in the sense that the complex derivative $f'(z) = \lim_{h\to 0} \frac{1}{h}(f(z+h) - f(z))$ exists and is continuous in a neighborhood of z_0. This is equivalent to that the statement that f has continuous partial derivatives $\partial_x f$, $\partial_y f$ and satisfies $\overline{\partial} f(z) = 0$ in a neighborhood of z_0.[10] The complex derivative f' satisfies $f'(z) = \partial f(z) = \partial_x f(z) = -\mathrm{i}\partial_y f(z)$.

[10] Define as usual the following partial differential operators $\partial = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and $\overline{\partial} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$.

- The real and imaginary parts of f satisfies *Cauchy–Riemann equations* near z_0: $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$.
- The function f is *(complex) analytic* at z_0: $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ which converges absolutely when $|z-z_0| \leq r$ for some $r > 0$.

Remember that u and v are *harmonic*: $\Delta u = 0 = \Delta v$.[11]

We conclude the introduction to stochastic calculus by showing that the complex Brownian motion is conformally invariant (up to a time-change). This justifies more or less all the time that we invested on the technical steps in this chapter and also works as motivation for the treatise of conformally invariant scaling limit in later chapters. Define a complex Brownian motion send from $z_0 \in \mathbb{C}$ as

$$B_t = B_t^{\mathbb{C}} = z_0 + B_t^{(1)} + \mathrm{i}\, B_t^{(2)}.$$

Theorem 2.10. *Let $U \subset \mathbb{C}$ be a domain (non-empty connected open set) and let $f : U \to \mathbb{C}$ be analytic. Let $z_0 \in U$ and let B_t be a complex Brownian motion send from $z_0 \in \mathbb{C}$. Let $\tau = \inf\{t \geq 0 : B_t \notin U\}$. Let $Z_t = f(B_{\sigma(t)})$ for $0 \leq t < S(\tau)$ where $\sigma(t) = S^{-1}(t)$ and $S(t) = \int_0^t |f'(B_s)|^2 \mathrm{d}s$ for $0 \leq 0 < \tau$. Then Z_t is a complex Brownian motion send from $f(z_0)$ and stopped at $S(\tau)$.*

Proof. As above write $f = u + \mathrm{i}\, v$. Define $X_t = u(B_t)$ and $Y_t = v(B_t)$. Since u and v are harmonic and satisfy the Cauchy–Riemann equations, by Itô's formula

$$\mathrm{d}X_t = u_1(B_t)\mathrm{d}B_t^{(1)} + u_2(B_t)\mathrm{d}B_t^{(2)},$$
$$\mathrm{d}Y_t = -u_2(B_t)\mathrm{d}B_t^{(1)} + u_1(B_t)\mathrm{d}B_t^{(2)},$$

where $u_1 = \partial_x u$ and $u_2 = \partial_y u$ are the partial derivatives of u. The dt-terms vanished by $\Delta u = 0 = \Delta v$. Therefore $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ are local martingales.

By a direct calculation $\langle X, Y\rangle_t = 0$ and

$$\langle X\rangle_t = \langle Y\rangle_t = \int_0^t u_1(B_s)^2 + u_2(B_s)^2 \,\mathrm{d}s = \int_0^t |f'(B_s)|^2 \mathrm{d}s.$$

Here we used the fact that $f'(z) = u_1(z) - \mathrm{i}\, u_2(z)$. A modification of the proof of Theorem 2.8 shows that the process $\exp(\mathrm{i}\theta_1 X_t + \theta_1^2 \langle X\rangle_t/2) \exp(\mathrm{i}\theta_2 Y_t + \theta_2^2 \langle Y\rangle_t/2)$ is a local martingale for any $\theta_1, \theta_2 \in \mathbb{R}$, and consequently, $(X_{\sigma_t})_{t\in\mathbb{R}_+}$ and $(Y_{\sigma_t})_{t\in\mathbb{R}_+}$ are independent Brownian motions. □

2.6 Weak convergence of probability measures

The concept of a scaling limit (such as the scaling limit of the Ising model in Chapter 1) involves sequence objects defined on different probability spaces or even if

[11] By the Cauchy–Riemann equations, $u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0$ and similarly for v.

they could be embedded to the same probability space, there is no hope to find an embedding easily under which the convergence would be almost sure. Therefore it is natural to consider so called weak convergence of probability measures. Typically we consider curves to be elements of the space of continuous functions $C(\mathbb{R}_{\geq 0},\mathbb{C})$ and thus we are interested in probability measures on that space.

Let $\mathscr{X}$ be a metric space (such as the space $C(\mathbb{R}_{\geq 0},\mathbb{C})$ of continuous complex-valued functions on $\mathbb{R}_{\geq 0}$). Then every Borel probability measure on $\mathscr{X}$ is regular and any measure is determined by the expected values of continuous bounded functions on $\mathscr{X}$, see [1]. Let $(\mathsf{Q}_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of Borel probability measures on $\mathscr{X}$. The link to our setup is that we consider our random objects as $\mathscr{X}$-valued random variables and their laws are given by Q_n, $n \in \mathbb{Z}_{>0}$. The following definition is a very practical approach to convergence of probability measures.

Definition 2.15. We say that $(\mathsf{Q}_n)_{n\in\mathbb{Z}_{>0}}$ *converges weakly* to a Borel probability measure $\hat{\mathsf{Q}}$ on $\mathscr{X}$ if for every continuous bounded $f : \mathscr{X} \to \mathbb{R}$ it holds that $\mathsf{Q}_n(f) \to \hat{\mathsf{Q}}(f)$ as n tends to ∞.[12,13]

The fact that makes the previous definition useful is that there is a practical way to verify the precompactness of the sequence $(\mathsf{Q}_n)_{n\in\mathbb{Z}_{>0}}$.

Definition 2.16. A sequence $(\mathsf{Q}_n)_{n\in\mathbb{Z}_{>0}}$ is said to be *tight* if for each $\varepsilon > 0$ there exists a compact set $K \subset \mathscr{X}$ such that $\mathsf{Q}_n[K] > 1-\varepsilon$ for all $n \in \mathbb{Z}_{>0}$.

Theorem 2.11 (Prohorov). *If $(\mathsf{Q}_n)_{n\in\mathbb{Z}_{>0}}$ is tight, then it is precompact, that is, every subsequence of $(\mathsf{Q}_n)_{n\in\mathbb{Z}_{>0}}$ contains a subsequence that converges weakly. Conversely, if $\mathscr{X}$ is separable and complete and if $(\mathsf{Q}_n)_{n\in\mathbb{Z}_{>0}}$ is precompact, then $(\mathsf{Q}_n)_{n\in\mathbb{Z}_{>0}}$ is tight.*

As a conclusion, a convergent sequence needs to be tight, and establishing tightness is often the first step in showing the weak convergence of probability measures.

For more technical purposes, let us still introduce the following definition.

Definition 2.17. A real-valued random variable X is *tight* with respect the sequence $(\mathsf{Q}_n)_{n\in\mathbb{Z}_{>0}}$ if for each $\varepsilon > 0$ there exists a compact set $M > 0$ such that $\mathsf{Q}_n[|X| \leq M] > 1-\varepsilon$ for all $n \in \mathbb{Z}_{>0}$.

Let's still study tightness in space of continuous function $C(\mathbb{R}_{\geq 0},\mathbb{C})$. A standard metric in $C(\mathbb{R}_{\geq 0},\mathbb{C})$ is given by

$$\mathrm{d}(f,g) = \sum_{k=1}^{\infty} 2^{-k} \left(1 \wedge \sup\{|f(t)-g(t)| : t \in [0,k]\}\right)$$

[12] The notation $\mathsf{Q}(f)$ denotes the expected value of f with respect to the measure Q.

[13] There are many equivalent definitions of the weak convergence — by a result named the Portmanteau theorem [1]. The above definition is equivalent to any of the following statements (i) $\limsup \mathsf{Q}_n(C) \leq \hat{\mathsf{Q}}(C)$ for all closed sets C of the space $\mathscr{X}$, (ii) $\liminf \mathsf{Q}_n(U) \geq \hat{\mathsf{Q}}(U)$ for all open sets U of the space $\mathscr{X}$, (iii) $\lim \mathsf{Q}_n(A) = \hat{\mathsf{Q}}(A)$ for all continuity sets A of the measure $\hat{\mathsf{Q}}$. A set A is a continuity set if $\hat{\mathsf{Q}}(\partial A) = 0$.

and it is easy to see that the associated topology is given by uniform convergence on compact subsets of $\mathbb{R}_{\geq 0}$. Define the modulus of continuity of f on $[0,k]$ as

$$\omega_{f,k}(\varepsilon) = \sup\{|f(t)-f(s)| : t,s \in [0,k] \text{ s.t. } |t-s| \leq \varepsilon\}.$$

The next result, based on the Arzelà–Ascoli theorem, characterizes the precompact sets in $C(\mathbb{R}_{\geq 0},\mathbb{C})$.

Proposition 2.9. *A set $\mathscr{A} \subset C(\mathbb{R}_{\geq 0},\mathbb{C})$ is precompact if and only if*

$$\sup_{f\in\mathscr{A}} |f(0)| < \infty \tag{2.13}$$

and

$$\lim_{\varepsilon\to 0} \sup_{f\in\mathscr{A}} \omega_{f,k}(\varepsilon) = 0 \tag{2.14}$$

for all $k \in \mathbb{Z}_{>0}$.

Proof. Since $C(\mathbb{R}_{\geq 0},\mathbb{C})$ is a metric space, $\mathscr{A}$ is precompact if and only if it is sequentially precompact.[14] Any sequence $f_n \in \mathscr{A}$ contains a subsequence converging uniformly on $[0,k]$ if and only if (2.13) and (2.14) hold by the Arzelà–Ascoli theorem, see [1], Theorem 7.2. A diagonal argument shows that $f_n \in \mathscr{A}$ contains a subsequence converging with respect to the metric d if and only if (2.13) and (2.14) hold for all $k \in \mathbb{Z}_{>0}$. □

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[14] A set is sequentially precompact if every sequence in the set contains a converging subsequence.

Chapter 3
Introduction to conformal mappings

In this chapter we present briefly some result of complex analysis which are useful for our theory.

3.1 Harmonic functions

We assume that reader is familiar with complex analysis on the level of Rudin's book [7]. This chapter is supplemented by Appendix C.

3.1.1 Mean value property and Poisson kernel

A *domain* is a non-empty, open and connected set. For a set A, $\overline{A}$ usually denotes its closure, whereas the meaning of A^* depends on the context, but it is often related to the complex conjugation or a similar symmetry. Some domains that we will consider are the *unit disc* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the *exterior of the unit disc* $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}$ and the *upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Let U be a domain in the complex plane. A twice continuously differentiable function $u : U \to \mathbb{R}$ is *harmonic* if $\Delta u = 0$. A harmonic function $u : U \to \mathbb{R}$ has mean-value property in the sense that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \, \mathrm{d}\theta \tag{3.1}$$

for any $z \in U$ and $r > 0$ such that $\overline{B(0,r)} \subset U$. Conversely, if $u : U \to \mathbb{R}$ is continuous function satisfying (3.1) for every $z \in U$ and for every $0 < r < r_0(z)$ (where $r_0(z) > 0$ can be less than the inradius at z_0) then u is smooth and harmonic. See, for instance, [5] pp. 210, 218-220.

When the mean value property is applied together with a Möbius transformation,[1] the mean value property can be written for any point in the disc (not just for the center) as an integral over the boundary of the disc. Namely, if $u : \overline{B(0,R)} \to \mathbb{R}$ is continuous function that is harmonic in $B(0,R)$, then

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z|^2}{|z - Re^{i\theta}|^2} u(Re^{i\theta}) \, \mathrm{d}\theta \tag{3.2}$$

where the quantity $P_{B(0,R)}(z,\theta) = (R^2 - |z|^2)/(|z - Re^{i\theta}|^2)$ is called the *Poisson kernel* in $B(0,R)$. This quantity extends to discs of the form $B(z_0,R)$ in an obvious way by translation.

Similarly in $\mathbb{H}$, if $u : \overline{\mathbb{H}} \to \mathbb{R}$ is continuous and bounded and if u is harmonic in $\mathbb{H}$ then u is given in terms of an integral of the *Poisson kernel* in $\mathbb{H}$ as

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} z}{|z - \xi|^2} u(\xi) \, \mathrm{d}\xi \tag{3.3}$$

for any $z \in \mathbb{H}$.

The *harmonic conjugate* of u is any harmonic function v such that $f = u + \mathrm{i}v$ is holomorphic. The function v is unique up to an additive constant and it exists at least locally in each ball contained in the domain. This can be seen from the Poisson kernel which can be written as $P_{B(0,R)}(z,\theta) = \operatorname{Re}[(Re^{i\theta} - z)/(Re^{i\theta} + z)]$. Therefore if we take the imaginary part of the complex valued kernel $(Re^{i\theta} - z)/(Re^{i\theta} + z)$, then the corresponding integral gives the harmonic conjugate in the disc. This can be summarized by an explicit formula for f in $B(0,R)$ given u

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{i\theta} + z}{Re^{i\theta} - z} u(Re^{i\theta}) \, \mathrm{d}\theta + iC$$

where $C \in \mathbb{R}$ is a constant. Globally in a non-simply connected domain, v might be multivalued.

3.1.2 Schwarz reflection principle

Another consequence of the mean value property is the *Schwarz reflection principle*: if $f = u + \mathrm{i}v$ is holomorphic in $D_+ = B(0,r) \cap \mathbb{H}$ and if $\lim_{z \to x} v(z) = 0$ as $z \in D_+$ tends to any $x \in (-r,r)$, then f has a unique holomorphic extension to $B(0,r)$. Namely, $v(\bar{z}) = -v(z)$ for any $z \in D_-$ defines a continuous extension of v to $B(0,r)$ and this extension satisfies the mean value property in $B(0,r)$. Hence v is smooth and harmonic in $B(0,r)$ and it has a harmonic conjugate which is unique if we require that $f = u + \mathrm{i}v$ is in D_+. Hence f extends holomorphically to $B(0,r)$ and satisfies

$$f(\bar{z}) = \overline{f(z)}. \tag{3.4}$$

[1] Harmonicity is preserved by any holomorphic change of coordinates.

More generally, if $U \subset \mathbb{H}$ is a domain and $J \subset \mathbb{R} \cap \partial U$ is non-empty set such that each point $x \in J$ satisfies the condition that $B(x,r) \cap \mathbb{H} \subset U$ for some $r > 0$ and if $f : U \to \mathbb{C}$ is holomorphic function such that $\lim \operatorname{Im} f(z) = 0$ as z tends to J, then there exists a unique holomorphic extension of f to $U \cup J \cup U^*$ and the extension satisfies (3.4). Here U^* is the reflection of U with respect to the real axis.

3.1.3 Harmonicity and complex Brownian motion

Under suitable conditions on the domain U and on the function $h : \overline{U} \to \mathbb{R}$ harmonic in U and its boundary values $\phi = h|_{\partial U}$, the function h can be represented using the complex Brownian motion as $h(z) = \mathsf{E}^z[\phi(B_\tau)]$ where τ is the exit time of $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ from U and E^z is the expected value with respect to the law of the complex Brownian motion $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ sent from z. The following result is a result of this type.

Lemma 3.1. *Let U be a domain and $h : \overline{U} \to \mathbb{R}$ be a bounded continuous function such that h is harmonic in U. Let P^z be the law of a complex Brownian motion $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ started from $z \in U$ and E^z be the corresponding expected value. Assume that $\tau = \inf\{t \in \mathbb{R}_{\geq 0} : B_t \notin U\}$ is almost surely finite. Then $h(B_{t \wedge \tau})$ is a bounded continuous martingale and $h(z) = \mathsf{E}^z[h(B_\tau)]$.*

Proof. The fact that $M_t = h(B^{(z)}_{t \wedge \tau_z})$ is a local martingale follows from Itô's formula similarly as in the proof of the conformal invariance of Brownian motion. Since h is bounded, M_t is a bounded continuous martingale and we can use optional stopping to show that $M_0 = \mathsf{E}[M_{\tau_z}]$. □

3.2 Conformal maps

Definition 3.1. A map $f : U \to \mathbb{C}$ is a *conformal map* if and only if it is holomorphic and injective. A *univalent function* is the same as a conformal map. When a map $f : U \to U'$ is *conformal and onto*, i.e., f is conformal and $f(U) = U'$, we state explicitly the fact that the map is onto.

If f is conformal, locally near z_0, we have the absolutely convergent expansion

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \dots$$

It is necessary that $f'(z_0) \neq 0$ based on the expansion, otherwise f wouldn't be injective near z_0. Thus if we ignore the small correction of order $|z - z_0|^2$, locally the map f translates z_0 to $f(z_0)$, rotates around that point by multiplying by the complex number (of unit modulus) $f'(z_0)/|f'(z_0)|$ and scales by the factor $|f'(z_0)|$, that is, f is *locally a combination of translation, rotation and scaling*.

If $f : U \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$, then it is continuously invertible near z_0 and the inverse is holomorphic, [5] p. 165. Therefore the inverse of a conformal map is conformal. However, the fact that $f' \neq 0$ everywhere is not sufficient for f to be injective globally. For example, consider the map $z \mapsto z^2$ in the domain $\mathbb{C} \setminus \{0\}$. Its derivative is non-zero everywhere, but it is not injective because $z^2 = (-z)^2$.

Example 3.1. The most elementary examples of conformal maps are the *Möbius maps* which are the linear fractional transformations and can be interpreted as the family of conformal self-maps of the *Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Recall that the conformal self-maps of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ are of the form[2] $z \mapsto \lambda \frac{z-a}{z-b}$, where $a, b \in \mathbb{R}$ with $a \neq b$ and $\lambda \operatorname{sgn}(a-b) > 0$, and the conformal maps from the upper-half plane $\mathbb{H}$ onto the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ are of the form $z \mapsto \nu \frac{z-w}{z-\bar{w}}$, where $\nu \in \mathbb{T} = \partial \mathbb{D}$ and $w \in \mathbb{H}$.

The Riemann mapping theorem establishes existence of conformal maps between simply connected domains. A domain $U \subset \mathbb{C}$ is *simply connected* if its complement $\hat{\mathbb{C}} \setminus U$ in the Riemann sphere is connected. For example $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$ is simply connected because the parts $\operatorname{Im} z \leq 0$ and $\operatorname{Im} z \geq 1$ can be connected through infinity. An equivalent definition of simply connectedness is that each closed loop in U is null-homotopic, that is, each loop can be continuously shrunk to a trivial loop. See [7] for more details and for the proof of the next theorem.

Theorem 3.1 (Riemann mapping theorem). *Suppose $U \subset \mathbb{C}$ is a simply connected domain other than $\mathbb{C}$ and $w \in U$. Then there exist a unique conformal map f from U onto $\mathbb{D}$ such that $f(w) = 0$ and $f(w) > 0$.*

Remark 3.1. All the other conformal maps from U onto $\mathbb{D}$ are obtained by composing f with a Möbius self-map of the disc.

Remark 3.2. If $U \subset \hat{\mathbb{C}}$ is a simply connected domain and $w \notin U$, then the image $\tilde{U}$ of U under $z \mapsto 1/(z-w)$ is bounded. Therefore $\tilde{U}$ is a subset of $\mathbb{C}$. Consequently, by the Riemann mapping theorem, if $U_1, U_2 \subset \hat{\mathbb{C}}$ are simply connected domains and $\hat{\mathbb{C}} \setminus U_k$ contains at least two distinct points for $k = 1, 2$, then there exists a conformal map from U_1 onto U_2 and we say that U_1 and U_2 are conformally equivalent.

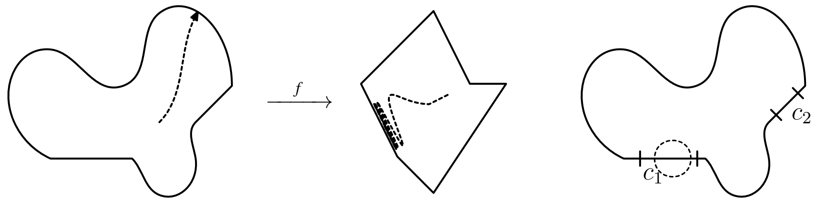
3.2.1 Continuity up to the boundary

In this section, we follow Ahlfors [1] and Pommerenke [6], see also [2].

Let's first see what type of continuity up to the boundary follows from the fact that ϕ is a homeomorphism, that is, a continuous map with a continuous inverse.

For that purpose, we define what we mean when we say that a sequence or a curve tends to the boundary domain. Let U be a non-empty open set, $z_n \in U$ a sequence and $\gamma : [0,1) \to U$ a curve. Remember that a *curve* in a topological space X is a continuous map from an interval of $\mathbb{R}$ into X. We say that (z_n) or $\gamma(t)$ *tends*

[2] Or the special cases of this form as a or b tends to infinity so that λa or λb^{-1} remains finite, respectively.



(a) If $f : U \to U'$ is a homeomorphism and $\gamma(t)$ is a curve that tends to the boundary, then the image $f(\gamma(t))$ tends to the boundary. However, it is not always true that $f(\gamma(t))$ extends continuously to its end point, not even when γ extends continuously to its end point.

(b) Schwarz reflection principle can be applied in those boundary arcs that are straight line segments and away from other parts of the boundary.

Fig. 3.1 By Theorems 3.2 and 3.3 conformal map maps boundary to boundary and extends continuously and injectively to a piece of boundary which is a straight line segment, an arc of a circle or an analytic curve.

to the boundary if (z_n) or $\gamma(t)$ will stay eventually away from any point in U, more formally, for each $z \in U$ there exist $\varepsilon(z) > 0$ and $n_0(z) \in \mathbb{N}$ such that $|z - z_n| \geq \varepsilon(z)$ for $n \geq n_0(z)$ or there exist $\varepsilon(z) > 0$ and $0 \leq t_0(z) < 1$ such that $|z - \gamma(t)| \geq \varepsilon(z)$ for $t_0(z) \leq t < 1$.

The discs $B(z, \varepsilon(z))$ form an open covering of U and for any compact $K \subset U$ there is a finite subcover. Hence we see that z_n or $\gamma(t)$ will stay eventually away from any compact $K \subset U$ in the sense that there exist $n_0(K) \in \mathbb{N}$ and $0 \leq t_0(K) < 1$ such that $z_n \notin K$ for $n \geq n_0(K)$ and $\gamma(t) \notin K$ for $t_0(K) \leq t < 1$. After noticing this the following theorem is almost trivial.

Theorem 3.2. *Let U and U' be non-empty open subsets of $\mathbb{C}$ and let $f : U \to U'$ be a homeomorphism. If (z_n) or $\gamma(t)$ tends to the boundary of U, then $(f(z_n))$ or $f(\gamma(t))$ tends to the boundary of U'.*

Proof. Let $K \subset U'$ be compact. Then by continuity of f^{-1}, the set $f^{-1}(K)$ is compact and there is $n_0 \in \mathbb{N}$ and $0 \leq t_0 < 1$ such that $z_n \notin f^{-1}(K)$ for $n \geq n_0$ and $\gamma(t) \notin f^{-1}(K)$ for $t_0 \leq t < 1$. Therefore $f(z_n) \notin K$ for $n \geq n_0$ and $f(\gamma(t)) \notin K$ for $t_0 \leq t < 1$. The claim follows by taking K to be a closed ball. □

Next we state and prove a theorem based on the Schwarz reflection principle that gives the continuity of f to the boundary arcs which are straight line segments.

Suppose that the boundary of U contains an open straight line segment c. By applying rotation and translation, we can assume that c is the interval $a < x < b$ on the real line. Suppose also that every point on c has an open neighborhood in $\mathbb{C}$ whose intersection with the whole boundary ∂U is the same as with the arc c. By this assumption each point in c is now a center of a disc whose diameter is a subset of c, and which c divides into two half-discs which are either completely inside or outside of U. Notice that at least one of the half-discs is inside U. Since c is connected, the property, whether one or two half-discs are inside U, is the same in each point. Therefore we can name these cases as *one-sided free arc* and *two-sided free arc*. See Figure 3.1(b) where c_1 and c_2 are one-sided free arcs.

Theorem 3.3 (Schwarz reflection principle for conformal maps). *Let U be a domain with one-sided free arc c. Then any conformal onto map $f : U \to \mathbb{D}$ can be extended to a holomorphic and injective map on $U \cup c$. The image of c is an arc c' on the unit circle $\partial\mathbb{D}$. Furthermore, if we apply the same extension to two or more one-sided free arcs, then the resulting extension is holomorphic and injective.*

Proof. Let c be one-sided free arc and $x \in c$ and D a half-disc neighborhood of x which is contained in U. We can assume that the point $f^{-1}(0)$ is not in D by choosing smaller D if necessary. Then $\log f(z)$ has single valued branch in D and its real part tends to 0 as $z \in D$ tends to c, because by the previous theorem $|f(z)|$ goes to 1. Therefore by the Schwarz reflection principle (3.4), $\log f(z)$ has holomorphic extension to $D \cup c \cup D*$ where D^* is the reflection of D with respect to $\mathbb{R}$. Therefore $f(z)$ can be extended holomorphically to a disc around z. The extensions in overlapping disc must coincide and therefore f has holomorphic extension to c and $|f(z)| = 1$ when $z \in c$. Call the neighborhood of c which lies outside U as U_-. Then f is now defined on $U \cup c \cup U_-$.

Clearly the extension is one-to-one if we manage to prove that $f(x) \neq f(x')$ for any $x, x' \in c$, $x \neq x'$ after all in $|f| < 1$ in U, $|f| = 1$ on c and $|f| > 1$ in U_- and in addition in U_-, f is by construction one-to-one. Assume that for some $x, x' \in c$, $x \neq x'$, $f(x) = f(x')$. We can assume that $f(x) = 1$.

Notice that $f'(x) \neq 0$ and $f'(x') \neq 0$. Otherwise $f(z) = c_0 + c_n(z-x)^n + \ldots$ around x, say, where $n \geq 2$ and $c_n \neq 0$. The interval $(1-\varepsilon, 1]$ would have n fold preimage under f and those paths would meet at angles $2\pi/n$ at x or x'. Since $n \geq 2$, at least one of them would intersect with D^* which leads to a contradiction. Thus $f'(x) \neq 0$ and $f'(x') \neq 0$ and f is locally holomorphically invertible near x and x'. A similar argument shows that any neighborhoods of x and x' are intersected by $f^{-1}(\{1-\varepsilon\})$ for small $\varepsilon > 0$. This leads to a contradiction with the injectivity of f in U. The last claim follows from the same argument. ◻

Remark 3.3. The previous theorem has a modification for c which is an arc of a circle or more generally for c which is an image of line segment under a holomorphic map (c is called an analytic arc).

A compact set $A \subset \mathbb{C}$ is said to be *locally connected* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any two points $a, b \in A$ with $|a-b| < \delta$, there exist a closed connected set B with $a, b \in B \subset A$ and $\operatorname{diam} B < \varepsilon$. For non-bounded closed $A \subset \hat{\mathbb{C}}$, we could adjust this definition and the next theorem by defining metric on the Riemann sphere $\hat{\mathbb{C}}$ that makes $\hat{\mathbb{C}}$ a compact space. For the proof, see Appendix C.

Theorem 3.4. *Let $U \subset \mathbb{C}$ be a bounded domain. A conformal onto map $f : \mathbb{D} \to U$ extends continuously to $\mathbb{D} \cup \partial\mathbb{D}$ if and only if ∂U is locally connected.*

If $f : \mathbb{D} \to U$ is conformal and extends continuously to the boundary, then ∂U is a closed curve that can be parametrized as $\theta \mapsto f(e^{i\theta})$. On the other hand, any closed curve is locally connected. Hence f extends continuously to the boundary if and only if the boundary is a curve. Clearly the extension is injective if and only if $\theta \mapsto f(e^{i\theta})$ is a simple curve. Hence Theorem 3.4 implies that f extends to a continuous and injective map from $\overline{\mathbb{D}}$ onto $\overline{U}$ if and only if U is a Jordan domain.[3] In fact, the inverse map is continuous to the boundary in that case. Thus any conformal map between two Jordan domains extends to a homeomorphism between their closures.

3.2.2 *Schwarz–Christoffel mappings*

Conformal mappings that map the unit disc or the upper half-plane onto the interior of a polygon are useful, because they have fairly explicit formulas. If a point p is mapped to a vertex of the polygon with interior angle $\alpha\pi$, the map looks locally like $(z-p)^\alpha$. The following theorem gives the precise statement.

Theorem 3.5 (Schwarz-Christoffel mapping). *Let U be the interior of a polygon γ with vertices $w_1, w_2, \dots, w_n$ and interior angles $\alpha_1\pi, \alpha_2\pi, \dots, \alpha_n\pi$. Then any conformal and onto map $f : \mathbb{H} \to U$ with $f(\infty) = w_n$ is of the form*

$$f(z) = C_1 + C_2 \int^z \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} \mathrm{d}\zeta \tag{3.5}$$

where C_1 and C_2 are constants and $w_k = f(z_k)$, $k = 1, 2, \dots, n-1$.

Example 3.2. Consider the case $n = 3$ and $\alpha_k = 1/3$ for all $k = 1, 2, 3$. Then any conformal map from $\mathbb{H}$ onto an equilateral triangle T such that $0, 1, \infty$ are mapped to the vertices of T, is of the form $f(z) = C_1 + C_2 \int^z \zeta^{-\frac{2}{3}} (\zeta - 1)^{-\frac{2}{3}} \mathrm{d}\zeta$.

Example 3.3. Consider the case $n = 4$ and $\alpha_k = 1/2$ for all $k = 1, 2, 3, 4$. Then any conformal map from $\mathbb{H}$ onto a rectangle R such that $0, x, 1, \infty$ are mapped to the vertices R, is of the form $f(z) = C_1 + C_2 \int^z \zeta^{-\frac{1}{2}} (\zeta - x)^{-\frac{1}{2}} (\zeta - 1)^{-\frac{1}{2}} \mathrm{d}\zeta$. The value of $x \in (0, 1)$ is treated as a parameter and it in one-to-one correspondence with the aspect ratio of R. In this example and the previous example, the constants C_1, C_2 determine the position, orientation and size of T or R.

3.3 From Area theorem to distortion

In this section we present some classical result on two classes of functions:

Definition 3.2. The *class S* consists of all holomorphic and univalent functions in $\mathbb{D}$ such that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad |z| < 1. \tag{3.6}$$

The *class Σ* consists of all holomorphic and univalent functions in $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}$ such that

[3] A curve is Jordan if it is simple closed curve. A domain is Jordan if it's boundary is Jordan curve.

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots, \quad |z| > 1. \tag{3.7}$$

Notice that if $f \in S$, then

$$g(z) = 1/f(z^{-1}) = z - a_2 + (a_2^2 - a_3)z^{-1} + \dots \tag{3.8}$$

belongs to Σ and $g(z) \neq 0$ for all $z \in \mathbb{D}^*$. Conversely if g belongs to Σ and $g(z) \neq 0$ for all $z \in \mathbb{D}^*$, then $f(z) = 1/g(z^{-1}) = z - b_0 z^2 + (b_0^2 - b_1)z^3 + \dots$ belongs to S.

The area of a bounded domain U, whose boundary is a smooth curve, can be computed as $\mathrm{Area}(U) = \frac{1}{2}\int_{\partial U} x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{1}{2\mathrm{i}}\int_{\partial U} \overline{w}\,\mathrm{d}w$. This is a consequence of Green's theorem, a special case of Stokes' theorem for two dimensions. If $g \in \Sigma$ and we apply formula for the area inside $\theta \mapsto g(re^{i\theta})$, $r > 1$, we get the formula

$$\mathrm{Area}\left(\mathbb{C} \setminus g\left(\mathbb{D}^*\right)\right) = \pi \left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right).$$

The reader can verify the details or see [3], p. 29, for the proof. The next theorem follows immediately from the area formula.

Theorem 3.6 (Area theorem). *For any $g \in \Sigma$, $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.*

If $f \in S$ and the coefficients are as in (3.6), then there exists odd function[4] $h \in S$ such that $h(z) = \sqrt{f(z^2)}$ and $h(z) = z + \frac{1}{2}a_2 z^3 + \dots$ The function h can be constructed as follows: The function $\phi(z) = \log(f(z)/z)$ has single-valued branch in $\mathbb{D}$, because $f(z)/z$ is holomorphic and doesn't have zeros in $\mathbb{D}$. Choose the branch for instance so that $\phi(0) = 0$. Write $f(z) = z \exp \phi(z)$. Then $h(z) := z \exp(\phi(z^2)/2)$ is in S and satisfies the required properties. Therefore (3.8) and the Area theorem imply that for any $f \in S$

$$|a_2| \leq 2. \tag{3.9}$$

The result (3.9) is called Bieberbach's theorem and it is a special case of the following famous and hard theorem.

Theorem 3.7 (Bieberbach conjecture – de Branges theorem). *For any $f \in S$, $|a_n| \leq n$, $n = 2, 3, \dots$*

Remark 3.4 (A historical remark). In 1923, Charles Loewner was studying the Bieberbach conjecture in the paper where he introduced the Loewner equation. He was studying conformal maps from the unit-disc, and therefore he introduced the Loewner equation in $\mathbb{D}$ (for maps $f_t : \mathbb{D} \to D_t$) where it is written as

$$\partial_t f_t(z) = f_t'(z)\, z \frac{z + e^{\mathrm{i}U_t}}{z - e^{\mathrm{i}U_t}}$$

for a conformal map f_t from $\mathbb{D}$ onto a simply connected domain $D_t \subset \mathbb{D}$, $0 \in D_t$, normalized by the expansion $f_t(z) = e^{-t}z + \dots$ near 0. The Loewner equation holds,

[4] The function h is odd if $h(-z) = -h(z)$.

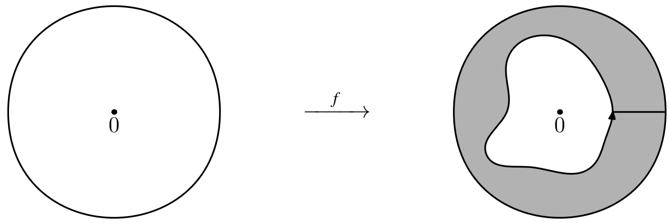


Fig. 3.2 A map f from $\mathbb{D}$ into $\mathbb{D}$ can be studied by the Loewner equation in $\mathbb{D}$ by defining a curve that first goes from $\partial\mathbb{D}$ to $\partial f(\mathbb{D})$ and then follows the boundary of the image domain $\partial f(\mathbb{D})$.

for instance, when $D_t = \mathbb{D} \setminus \gamma((0,t])$ where $\gamma : [0,T] \to \mathbb{C}$ is a simple curve with $\gamma(0) \in \partial\mathbb{D}$ and $\gamma((0,T]) \subset \mathbb{D}$. The function $t \mapsto U_t$ is real and continuous.

Let $0 \in \hat{D} \subset \mathbb{D}$ be a simply connected domain. By approximation we can always assume that the boundary of $\hat{D}$ is a simple curve. By considering a curve $\gamma(t)$, $t \in [0,T]$, as in Figure 3.2 which first follows a curve from $\partial\mathbb{D}$ to $\partial\hat{D}$ (a line segment, say) and then follows $\partial\hat{D}$ in counterclockwise direction, say, we can use the Loewner equation to study the conformal map ϕ from $\mathbb{D}$ onto $\hat{D}$ satisfying $\phi(0) = 0$, $\phi'(0) > 0$, because $\phi = f_T$. Using this approach Charles Loewner was able to show that for any $f \in S$ (which has an expansion of the form (3.6)) $|a_3| \le 3$ which is another special case of the Bieberbach–de Branges theorem.

One of the consequences Bieberbach's theorem (3.9) is the following. Let's use $\mathrm{dist}(x,A)$ to denote the Euclidean distance from a point x to a set A.

Theorem 3.8 (Koebe $1/4$ theorem). *Let $f \in S$ and $U = f(\mathbb{D})$ then*

$$\frac{1}{4} \le \mathrm{dist}(0, \partial U) \le 1$$

Proof. Let $f \in S$ and $w \notin f(\mathbb{D})$. Suppose that the expansion of f is given by (3.6). Then $w \neq 0$ and the map $g(z) = \frac{wf(z)}{w-f(z)} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \dots$ is holomorphic and univalent in $\mathbb{D}$. The details are left to the reader. Thus by Bieberbach's theorem (3.9), $1/|w| - |a_2| \le |1/w - a_2| \le 2$ and consequently, $1/|w| \le 4$, which gives the lower bound.

Let $d = \mathrm{dist}(0, \partial U)$. Define a conformal map h from $\mathbb{D}$ into $\mathbb{D}$ by $h(z) = f^{-1}(dz)$. Now $h'(0) = d/f'(0) = d$ and by the Schwarz lemma $|h'(0)| \le 1$. □

To apply Bieberbach's theorem (3.9) to a less restricted class of functions, define for any f univalent in $\mathbb{D}$ and for any $w \in \mathbb{D}$ a function

$$h(z) = \frac{f\left(\frac{z+w}{1+\overline{w}z}\right) - f(w)}{(1-|w|^2)\, f'(w)} = z + \left(\frac{1}{2}(1-|w|^2)\frac{f''(w)}{f'(w)} - \overline{w}\right) z^2 + \dots$$

We leave as an exercise to verify the expansion. Since $h \in S$, this expansion and (3.9) imply the following result.

Proposition 3.1. *If f is a conformal map on $\mathbb{D}$ and $z \in \mathbb{D}$ then*

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leq 4.$$

The previous result can be integrated (see [3], p. 32) to give Koebe's theorem, a result which, for example, tells how f distorts circles $\theta \mapsto re^{\mathrm{i}\theta}$. The first of the inequalities tells that $\theta \mapsto f(re^{\mathrm{i}\theta})$ lies between two particular circles centered at $f(0)$ and the second inequality tells that the length of this curve is bounded from below and from above by certain constants.

Theorem 3.9 (Koebe distortion theorem). *If f is a conformal map on $\mathbb{D}$ and $z \in \mathbb{D}$ then*

$$|f'(0)|\frac{|z|}{(1+|z|)^2} \leq |f(z)-f(0)| \leq |f'(0)|\frac{|z|}{(1-|z|)^2}$$
$$|f'(0)|\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq |f'(0)|\frac{1+|z|}{(1-|z|)^3}.$$

3.4 Conformally invariant tools

Finally, we have collected some definitions and results on the harmonic measure and the extremal length to this final section of this chapter.

3.4.1 Harmonic measure

The representation of harmonic functions using a Brownian motion was presented in Section 3.1.3. It is thus meaningful to study more the exit distribution of a Brownian motion from a domain.

Let U is a simply connected domain in $\mathbb{C}$ with a non-empty, locally connected boundary. Let $\phi : U \to \mathbb{D}$ is a conformal and onto map.

Definition 3.3. Let $z \in U$ and $E \subset \partial U$ a Borel set. Then *harmonic measure* of E relative to U seen from z is defined as

$$\mathrm{HM}(z,E,U) := \mathrm{HM}(w,\phi^{-1}(E),\mathbb{D}) := \frac{1}{2\pi}\int_{\phi^{-1}(E)} \frac{1-|z|^2}{|z-e^{i\theta}|^2}\,\mathrm{d}\theta$$

Remark 3.5. The function $z \mapsto \mathrm{HM}(z,E,U)$ is harmonic in U and tends to one as z tends to an interior (with respect to ∂U) point of E and to zero as z tends to an interior point of $\partial U \setminus E$. This observation leads to generalizations of the concept of harmonic measure to non-simply connected domains. For general domains,

$\mathrm{HM}(z,E,U)$ can be defined as supremum over $h(z)$ where h is harmonic with continuous boundary values $f = h|_{\partial U}$ such that $f \leq 1$ on E and $f = 0$ on $\Omega \setminus E$.

Remark 3.6. If the boundary is non-simple and we wish to separate the "left-hand and right-hand sides" of boundary points, we can use the formula $\mathrm{HM}(w, \phi^{-1}(E), \mathbb{D})$ to do so. In this approach, the points in $\partial\mathbb{D}$ parametrize $\partial\Omega$ and a generalized boundary point (also called prime end [2, 6]) is an equivalence class of convergent sequences z_n tending to the boundary of Ω such that $\phi(z_n)$ converges. Two sequences z_n and w_n are equivalent if $\phi(z_n)$ and $\phi(w_n)$ tend to the same limit point in $\partial\mathbb{D}$.

Lemma 3.2 (Weak Beurling estimate). *There exist constant $\alpha > 0$ and $C > 0$ such that the following holds: Let $D = \mathbb{D} \setminus \gamma[0,1)$ where $\gamma : [0,1) \to \mathbb{D}$ be a simple curve with $\gamma(0) = 0$ and $\lim_{t \nearrow 1} |\gamma(t)| = 1$. Let P^z be the law of complex Brownian motion $(B_t)_{t \in \mathbb{R}_+}$ sent from $z \in D$ and let τ be its exit time from D. Then for any $z \in D$*

$$\mathsf{P}^z[|B_\tau| = 1] \leq C|z|^\alpha$$

Remark 3.7. The result is called "weak" since the proof doesn't give the optimal exponent $\alpha = \frac{1}{2}$. See for instance [4] for the strong estimate.

Proof. Consider a complex brownian motion sent from $w \in \mathbb{C}$ with $|w| = 2$ and let $\sigma = \inf\{t \in \mathbb{R}_+ : |B_t| = 1 \text{ or } 4\}$. By rotational invariance of the complex Brownian motion $q := \mathsf{P}^w[B([0,\tau])$ contains a loop around $0]$ is independent of w. Now $q > 0$ follows from a more general fact that the probability that d-dimensional Brownian motion follows any given continuous path segment with a given precision is positive.

Let $\rho = |z|$ and define $r_k = \rho\, 2^k$, for any $k \in \mathbb{Z}$. If $|B_\tau| = 1$ then the Brownian motion $B_t, 0 \leq t \leq \tau$, will hit the circles of radii r_k, $k \in [\![0, n_0(\rho)]\!]$ centered at 0 where $n_0(\rho)$ is the largest integer n such that $\rho\, 2^n \leq 1$. Denote the hitting times of those circles by T_k, $k \in [\![0, n_0(\rho)]\!]$. If for some $k \in [\![0, n_0(\rho) - 1]\!]$, B_t, $t \geq T_k$, makes a loop around 0 before hitting the circles of radii r_{k-1} or r_{k+1}, then the Brownian motion hits ∂D and $|B_\tau| < 1$. Apply the strong Markov property of Brownian motion for T_k, $k \in [\![0, n_0(\rho) - 1]\!]$, to show that $\mathsf{P}^z[|B_\tau| = 1] \leq (1-q)^{n_0(\rho)}$. Then note that $n_0(\rho) > (\log(1/\rho))/(\log 2) - 1$ and hence the claim holds for $C = 1/(1-q)$ and $\alpha = (\log 1/(1-q))/(\log 2)$. □

3.4.2 Extremal length

For a domain U, a rectifiable curve $\gamma : [0,1] \to U$ and a non-negative function $\rho : U \to \mathbb{R}_{\geq 0}$, define *$\rho$-area* of U as $\int_U \rho(z)^2\, \mathrm{d}^2 z$ and *ρ-length* of γ as $\int_\gamma \rho(z)\, |\mathrm{d}z|$.[5]

---

[5] Remember that a curve is rectifiable, if its arc length is finite, and also that $\int \mathrm{d}^2 z$ denotes the integral with respect to the Lebesgue area measure and $\int |\mathrm{d}z|$ is the integral w.r.t. the arc length measure.

Definition 3.4. For a collection of paths Γ, the *extremal length* of Γ is defined as

$$\mathrm{EL}(\Gamma) = \sup_{\rho} \frac{\inf\{(\int_{\gamma} \rho(z)\,|\mathrm{d}z|)^2 \,:\, \gamma \in \Gamma\}}{\int_U \rho(z)^2 \,\mathrm{d}^2 z} \tag{3.10}$$

where the supremum is over all non-negative ρ such that the fraction is well-defined (i.e., the numerator and the denominator are not equal, when at least one of them is equal to 0 or ∞).

Let us list some properties of extremal length with references to their proofs.

- For a rectangle $U = (x, x+a) \times (y, y+b)$ and for a path family Γ with sufficiently regular curves γ that connect the two vertical sides of U in U, it holds that $\mathrm{EL}(\Gamma) = \frac{a}{b}$, see [4], Example IV.1.1.
- For a annulus $U = A(z_0, r, R)$ and for a path family Γ with sufficiently regular curves γ that connect the two components of ∂U in U, it holds that $\mathrm{EL}(\Gamma) = \frac{1}{2\pi} \log \frac{R}{r}$, see [4], Example IV.1.2.
- If U and Γ are as above and $\phi : U \to \mathbb{C}$ is a conformal map, then $\mathrm{EL}(\phi(\Gamma)) = \mathrm{EL}(\Gamma)$, see [4], Section IV.1, that is, the extremal length is conformal invariant.
- There are many natural monotonicity properties of the extremal length, see [4], Section IV.3.
- A very important class of Γ are the following. Let U be a simply connected domain that for simplicity is assumed to be a Jordan domain. Take four distinct boundary points ζ_k, $k = 1,2,3,4$, in the counterclockwise order and denote the boundary arcs by $\zeta_k \zeta_{k+1}$ We call $(U, \zeta_1, \zeta_2, \zeta_3, \zeta_4)$ a *topological quadrilateral*. Then Γ is the path family with sufficiently regular curves γ that connect the $\zeta_1\zeta_2$ and $\zeta_3\zeta_4$ in U and Γ^* is the path family with sufficiently regular curves γ that connect the $\zeta_2\zeta_3$ and $\zeta_4\zeta_1$ in U. Then by conformal invariance, we can map these curve families to a rectangle and we find easily that $\mathrm{EL}(\Gamma) = 1/\mathrm{EL}(\Gamma^*)$. This is called reciprocity of the extremal length of topological quadrilaterals. See also the equation (1.7) in [4].

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Chapter 4
Loewner equation

This chapter develops in detail the connection of growing hulls and the Loewner differential equation satisfied by families of conformal maps.

4.1 Conformal maps of the upper half-plane

4.1.1 Hydrodynamically normalized maps and half-plane capacity

We start by defining a basic object of our study. We'll consider conformal maps from subsets of the upper half-plane onto the upper half-plane. By the Riemann mapping theorem that subset needs to be simply connected, which leads to the following definition.

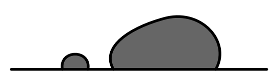


Fig. 4.1 A hull K is the shaded area in the picture. The hull K is a compact subset of the closed upper half-plane $\overline{\mathbb{H}}$ such that its complement $\mathbb{H} \setminus K$ is simply connected.

Definition 4.1. A set $K \subset \overline{\mathbb{H}}$ is called a *hull* if K is compact and $\mathbb{H} \setminus K$ is simply connected.

Remark 4.1. An alternative definition of a hull as a subset of the open upper half-plane can be made in the following way: $K \subset \mathbb{H}$ is a hull if K is bounded, relatively closed in $\mathbb{H}$ and $\mathbb{H} \setminus K$ is simply connected. The two definitions are practically the same. To move from the former to the latter, one needs just to take the intersection of the set with the open upper half-plane and to move to the other direction, one needs to take the closure of the set.

Lemma 4.1. *For any hull K, there exists a unique conformal and onto map $g_K : \mathbb{H}\setminus K \to \mathbb{H}$ such that*

$$\lim_{z\to\infty}(g_K(z)-z)=0 \tag{4.1}$$

where the limit holds along any sequence $z_n \in \mathbb{H}$ such that $|z_n|\to\infty$. Such g_K is said to have hydrodynamic normalization. *Near ∞, g_K has the expansion*

$$g_K(z)=z+a_1z^{-1}+a_2z^{-2}+\dots$$

where the coefficients a_k, $k\in\mathbb{N}$, are real.

Proof. If $\tilde{g}:\mathbb{H}\setminus K\to\mathbb{D}$ is a conformal onto map, then $\tilde{g}(\infty)\in\partial\mathbb{D}$ is well-defined since there is a holomorphic extension of $z\mapsto\tilde{g}(-1/z)$ to a neighborhood of 0 by Theorem 3.3. Hence we can compose $\tilde{g}$ with a Möbius map from $\mathbb{D}$ onto $\mathbb{H}$ mapping $\tilde{g}(\infty)$ to ∞ and get this way a conformal map from $\mathbb{H}\setminus K$ onto $\mathbb{H}$ mapping ∞ to ∞. By this observation and by the Riemann mapping theorem, there are conformal onto maps from $H=\mathbb{H}\setminus K$ onto $\mathbb{H}$ which map ∞ to ∞. Pick one of them and call it $\hat{g}$. Let $H'=\{-1/z : z\in H\}$ and

$$f(z)=-1/\hat{g}(-1/z). \tag{4.2}$$

By Theorem 3.3, f extends holomorphically and injectively to a neighborhood of 0. Let $\varepsilon>0$ be such that $B(0,\varepsilon)\cap\mathbb{H}\subset H'$. Then f maps $(-\varepsilon,\varepsilon)$ into $\mathbb{R}$. Moreover, if $f=u+iv$, then $f'(0)=\partial_x u(0)=\partial_y v(0)>0$ because f maps $B(0,\varepsilon)\cap\mathbb{H}$ into $\mathbb{H}$. Hence

$$f(z)=b_1z+b_2z^2+\dots$$

near 0 where the coefficients satisfy $b_1>0$ and $b_j\in\mathbb{R}$. For $\hat{g}$ this implies that for large $|z|$

$$\hat{g}(z)=\hat{a}_{-1}z+\hat{a}_0+\hat{a}_1z^{-1}+\hat{a}_2z^{-2}+\dots \tag{4.3}$$

where the coefficients satisfy $\hat{a}_{-1}>0$ and $\hat{a}_j\in\mathbb{R}$. Now we notice that $\hat{g}$ satisfies (4.1), if and only if $\hat{a}_{-1}=1$ and $\hat{a}_0=0$.

By the remark after the Riemann mapping theorem (Theorem 3.1), if $g:H\mapsto\mathbb{H}$ is a conformal onto map taking ∞ to ∞, then all the other such maps can be written as $\phi\circ g$ where ϕ is a Möbius self-map of $\mathbb{H}$ fixing ∞. The Möbius self-maps of $\mathbb{H}$ that fix ∞ are of the form $z\mapsto\alpha z+\beta$ where $\alpha>0$ and $\beta\in\mathbb{R}$. Hence for given $\hat{g}$ there is a unique choice for ϕ such that $g_K=\phi\circ\hat{g}$ has the expansion

$$g_K(z)=z+a_1\,z^{-1}+a_2\,z^{-2}+\dots$$

for $z\in\mathbb{H}\setminus B(0,R)$. □

Lemma 4.2. *The coefficient a_1 is non-negative and $a_1=0$ only if g_K is the identity map.*

Proof. Define a harmonic function h in $\mathbb{H}\setminus K$ by $h(z)=\operatorname{Im}(z-g_K(z))$. Then the boundary values of h are non-negative: it is zero on $\mathbb{R}$ away from K and on $\partial K\cap\mathbb{H}$ it is positive. Also $h(z)\to 0$ as $|z|\to\infty$. Hence by the minimum principle, h is non-negative in $\mathbb{H}\setminus K$. In fact, h is strictly positive unless $h=0$ identically and g_K is an identity map. Now

$$\lim_{y\nearrow\infty} y\,h(\mathrm{i}\,y) = \lim_{y\nearrow\infty} y\,\mathrm{Im}\left(-\frac{a_1}{\mathrm{i}\,y} + \mathscr{O}\left(|y|^{-2}\right)\right) = a_1$$

which shows that $a_1 \geq 0$. The strict positivity follows when we notice that

$$a_1 = \frac{2R}{\pi}\int_0^{\pi} h(R\,e^{i\theta})\sin\theta\,\mathrm{d}\theta. \tag{4.4}$$

That formula follows from the previous formula and from the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \{z\in\mathbb{H} : |z|>R\} \\ u(x) = 0 & \text{for } x\in\mathbb{R}, |x|\geq R \\ u(Re^{i\theta}) = \phi(\theta) & \text{for } \theta\in(0,\pi) \end{cases}$$

in terms of a Poisson kernel. The proof of the formula (4.4) is left as an exercise. □

Definition 4.2. If K is a hull and g_K satisfies the hydrodynamic normalization, then the coefficient of z^{-1} in the expansion of g_K is denoted by $a_1(K)$. We call $a_1(K)$ as the *half-plane capacity* (h-capacity) of K.

The half-plane capacity satisfies the following properties:

- Scaling rule: $a_1(\lambda K) = \lambda^2 a_1(K)$ because

$$g_{\lambda K}(z) = \lambda g_K(\lambda^{-1}z) = z + \lambda^2 a_1(K) z^{-1} + \dots$$

- Summation rule: $a_1(K\cup L) = a_1(K) + a_1(g_K(L))$. Let $L' = g_K(L)$. Then

$$g_{K\cup L}(z) = g_{L'}\circ g_K(z) = z + (a_1(K) + a_1(L'))z^{-1} + \dots$$

- Translation invariance: $a_1(K+x) = a_1(K)$

$$g_{K+x}(z) = x + g_K(z-x) = z + a_1(K)z^{-1} + \dots$$

From the summation rule and from Lemma 4.2 it follows that if $J\subset K$ are hulls then $a_1(J)\leq a_1(K)$ and $a_1(J) = a_1(K)$ only if $\mathbb{H}\cap(K\setminus J) = \emptyset$. We say that *half-plane capacity is increasing*. These properties make the half-plane capacity very natural measure for the size of the hull K (as seen from the point ∞ in the domain $\mathbb{H}$).

Example 4.1 (Half-disc). When $K = \overline{\mathbb{H}\cap B(x_0,R)}$, the corresponding map is

$$g_K(z) = z + \frac{R^2}{z-x_0} = z + \frac{R^2}{z} + \frac{R^2 x_0}{z^2} + \dots$$

We can verify that this formula defines a map from $\mathbb{H}\setminus K$ to $\mathbb{H}$ a direct computation: namely $g_K(x)\in\mathbb{R}$ when $x\in\mathbb{R}$, $|x-x_0|\geq R$, and for $\theta\in(0,\pi)$, $g_K(x_0+Re^{i\theta}) = x_0 + 2R\cos\theta\in\mathbb{R}$. The half-plane capacity of the half-disc of radius R is $a_1(K) = R^2$.

Example 4.2 (Vertical line segment). When $K = [x_0, x_0 + \mathrm{i}\,h] = \{x_0 + \mathrm{i}\,y : y \in [0,h]\}$

$$g_K(z) = x_0 + \sqrt{(z-x_0)^2 + h^2} = x_0 + z\sqrt{1 - \frac{2x_0}{z} + \frac{x_0^2 + h^2}{z^2}}$$
$$= x_0 + z\left(1 - \frac{x_0}{z} + \frac{x_0^2 + h^2}{2z^2} - \frac{1}{8}\frac{4x_0^2}{z^2} + \dots\right) = z + \frac{h^2}{2z} + \dots$$

where we used the expansion $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$ Thus we find that the half-plane capacity of the vertical line segment of length h is $a_1(K) = h^2/2$.

The following result on the inverse maps f_K of g_K can be shown, for instance, by imitating the proof of Lemma 4.1. The last claim follows from composing $g_K \circ f_K$ which is a conformal self-map in the upper half-plane which has the expansion $z + (a_1(K) + b_1(K))z^{-1} + \mathscr{O}(|z|^{-2}) = z + \mathscr{O}(|z|^{-1})$ as $z \to \infty$. Consequently, $g_K \circ f_K$ is the identity map and $a_1(K) + b_1(K) = 0$.

Lemma 4.3. *For any hull K, there exists a unique conformal and onto map $f_K : \mathbb{H} \to \mathbb{H} \setminus K$ such that*

$$\lim_{z\to\infty} (f_K(z) - z) = 0$$

where the limit holds along any sequence $z_n \in \mathbb{H}$ such that $|z_n| \to \infty$. Such f_K is also said to have hydrodynamic normalization. *Near ∞, f_K has the expansion*

$$f_K(z) = z + b_1 z^{-1} + b_2 z^{-2} + \dots$$

where the coefficients b_k, $k \in \mathbb{N}$, are real. Furthermore, $f_K = g_K^{-1}$ and $b_1(K) = -a_1(K)$.

We conclude this section by showing that the *half-plane capacity is a continuous function of the hull*. For a hull K and $\varepsilon > 0$, let K^ε be the ε*-thickening* of K, that is, K^ε is the smallest hull containing the set $\mathbb{H} \cap \bigcup_{z\in K} \overline{B(z,\varepsilon)}$.

Lemma 4.4. *There are constants $C(R) > 0$ and $\alpha > 0$ such that the following holds: If $K \subset K^\varepsilon \subset B(z_0, R)$ for some $z_0 \in \mathbb{R}$, then*

$$a_1(K) \le a_1(K^\varepsilon) \le a_1(K) + C(R)\varepsilon^\alpha$$

Proof. The inequality on the left follows from the summation rule and positivity of the half-plane capacity.

To show the other inequality consider the harmonic functions $h_K(z) = \mathrm{Im}(z - g_K(z))$ and $h_{K^\varepsilon}(z) = \mathrm{Im}(z - g_{K^\varepsilon}(z))$. Notice that they are both non-negative and bounded by R, and also that they are continuous in $\overline{\mathbb{H} \setminus K}$ and $\overline{\mathbb{H} \setminus K^\varepsilon}$, respectively.

Let $z \in \mathbb{H} \cap \partial K^\varepsilon$. Then $\mathrm{dist}(z,K) = \varepsilon$. Let P^z be the law of a complex Brownian motion send from z and let τ be the hitting time of $\mathbb{R} \cup K$. Then by Lemma 3.1 $h_K(z) = \mathsf{E}^z[\mathrm{Im}\, B_\tau]$ and by definition $h_{K^\varepsilon}(z) = \mathrm{Im}\, z$. Write

$$\begin{aligned}|h_K(z) - h_{K^\varepsilon}(z)| &\le \mathsf{E}^z[|\operatorname{Im} B_\tau - \operatorname{Im} z|] \\ &= \mathsf{E}^z[|\operatorname{Im} B_\tau - \operatorname{Im} z|\,;\, \sigma < \tau] + \mathsf{E}^z[|\operatorname{Im} B_\tau - \operatorname{Im} z|\,;\, \sigma = \tau] \qquad (4.5)\end{aligned}$$

where is σ be the exit time from $(\mathbb{H} \setminus K) \cap B(z, \sqrt{\varepsilon})$.[1] The first term on the right of (4.5) is at most $R\,\mathsf{P}^z[\sigma < \tau]$ and hence by Lemma 3.2, there are constants $\tilde{\alpha} > 0$ and $\tilde{C} > 0$ such that the first term is bounded by $\tilde{C}R(\varepsilon/\sqrt{\varepsilon})^{\tilde{\alpha}} = \tilde{C}R\varepsilon^{\tilde{\alpha}/2}$. The second term is at most $\sqrt{\varepsilon}$.

Now since for some constants $C(R) > 0$ and α, $|h_K(z) - h_{K^\varepsilon}(z)| \le C(R)\varepsilon^\alpha$ on the boundary of $\mathbb{H} \setminus K^\varepsilon$ and $h_K - h_{K^\varepsilon}$ is a bounded harmonic function on $\mathbb{H} \setminus K^\varepsilon$, the maximum principle gives that $|h_K(z) - h_{K^\varepsilon}(z)| \le C(R)\varepsilon^\alpha$ on $\mathbb{H} \setminus K^\varepsilon$. Therefore the formula (4.4) can be applied to show that $|a_1(K) - a_1(K^\varepsilon)| \le C(R)\varepsilon^\alpha$. □

4.1.2 Growing families of hulls

Let I be an interval of the form $[0,\infty), [0,T]$ or $[0,T)$ where $T \in (0,\infty)$. Let $\gamma : I \to \overline{\mathbb{H}}$ be curve such that $\gamma(0) \in \mathbb{R}$. We can define a family of hulls $(K_t)_{t \in I}$ associated to $\gamma(t)$, $t \in I$, in the following way:

- If γ is simple (a curve is simple if and only if it is injective) and $\gamma(t) \subset \mathbb{H}$, $t > 0$, then define $K_t = \gamma([0,t])$ for any $t \in I$.
- If γ is not simple let H_t be the unbounded connected component of $\mathbb{H} \setminus \gamma([0,t])$ and let $K_t = \overline{\mathbb{H} \setminus H_t}$.

If γ is simple both of the above definition would give the same hulls $(K_t)_{t \in I}$.

More generally, let $(K_t)_{t \in I}$ be a family of hulls parametrized by a real variable $t \in I$ where I is as above. The family of hulls associated to a curve is a good example of such family. If the family $(K_t)_{t \in I}$ is *growing* in the sense that $K_s \subset K_t$ for $s \le t$ and if *the growth is continuous* in the sense that for any $\varepsilon > 0$ and for any $S \in (0,\infty)$ such that $[0,S] \subset I$ there exist $\delta > 0$ such that $K_{t+\delta} \subset K_t^\varepsilon$ for any $0 \le t \le S - \delta$, then by Lemmas 4.2 and 4.4, the function $\phi : t \mapsto a_1(K_t)$ is continuous and non-decreasing. If we assume that $K_0 \subset \mathbb{R}$ and that $\mathbb{H} \cap (K_t \setminus K_s) \neq \emptyset$ for any $0 \le s < t \le T$, then $\phi(0) = 0$ and by the summation rule and by the positivity of the half-plane capacity $\phi(t) > \phi(s)$ for any $0 \le s < t \le T$. Hence we can reparameterize the family of hulls by setting $\tilde{K}_t = K_{\phi^{-1}(2t)}$. In this parametrization $a_1(\tilde{K}_t) = \phi(\phi^{-1}(2t)) = 2t$. As a summary, continuously growing families of hulls can be parametrized by capacity.

Definition 4.3. A family of hulls $(K_t)_{t \in [0,T)}$ is said to be *parametrized with the half-plane capacity* if $a_1(K_t) = 2t$. A curve $\gamma : [0,T) \to \overline{\mathbb{H}}$ is said to be *parametrized with the half-plane capacity* if the associated hulls are parametrized with the half-plane capacity.

For a given family of hulls $(K_t)_{t \in I}$ it is convenient to set

[1] Remember the notation $\mathsf{E}[X;A] = \int_A X \, d\mathsf{P}$.

$$g_t = g_{K_t}.$$

If $(K_t)_{t\in[0,T)}$ is parametrized by the capacity then

$$g_t(z) = z + \frac{2t}{z} + \dots$$

From now on we assume (almost without exceptions) that g_t is a conformal map with this form of an expansion near ∞. It is useful to call the parameter t time.

Remark 4.2. The factor 2 is because of historical reasons: using that normalization the Loewner equation in $\mathbb{H}$ will be better compatible with the Loewner equation in $\mathbb{D}$. The choice, that the half-plane capacity is linear in t, is consistent with the summation rule of the half-plane capacity.

4.2 Loewner chains

4.2.1 Loewner equation holds for simple curves

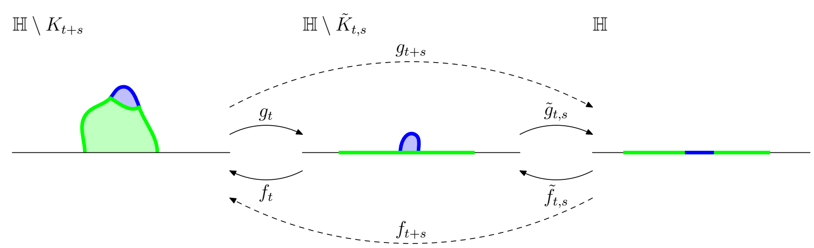


Fig. 4.2 Composition of hydrodynamical conformal maps uses the uniqueness of such maps. Therefore for instance $g_{t+s}(z) = \tilde{g}_{t,s} \circ g_t(z)$ for $z \in \mathbb{H} \setminus K_{t+s}$ and $f_{t+s}(z) = f_t \circ \tilde{f}_{t,s}(z)$ for $z \in \mathbb{H}$.

Let $(K_t)_{t\in[0,T]}$ be a growing family of hulls parametrized with the half-plane capacity. The Loewner differential equation describes infinitesimal changes in the conformal maps g_t as t increases. In its simplest version, the growth need to be *local* and consequently the Loewner equation contains one driving term. We will first look at this statement somewhat heuristically and then formulate and prove the result that the conformal maps g_t associated to a simple curve γ satisfy the Loewner equation.

We can write g_{t+s} as a composition of conformal maps in the following way, see also Figure 4.2. Let for t, s such that $t, t+s \in [0,T]$,

$$\tilde{K}_{t,s} = \overline{g_t(K_{t+s} \setminus K_t)}, \qquad \tilde{g}_{t,s} = g_{\tilde{K}_{t,s}}.$$

Notice that $a_1(\tilde{K}_{t,s}) = 2s$ by additivity of the half-plane capacity and that $g_{t+s} = \tilde{g}_{t,s} \circ g_t$ by uniqueness of the hydrodynamically normalized conformal maps.

Next we observe that we can apply the Poisson kernel of the upper half-plane to the inverses of intermediate conformal maps $\tilde{g}_{t,s}$. For any f_K, the function $h(z) = \mathrm{Im}(f_K(z) - z)$ is harmonic in $\mathbb{H}$ and bounded and non-negative in $\overline{\mathbb{H}}$. Consequently, by taking (3.3) and its harmonic conjugate, it follows that

$$f_K(z) - z = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\, f_K(\xi)}{z-\xi}\, \mathrm{d}\xi. \tag{4.6}$$

Define $\tilde{f}_{t,s} = \tilde{g}_{t,s}^{-1}$. Then it holds that $\tilde{f}_{t,s} = g_t \circ f_{t+s}$. Let $t \geq 0$ and $\delta > 0$. We write

$$g_{t+\delta}(z) - g_t(z) = g_{t+\delta}(z) - \tilde{f}_{t,\delta} \circ g_{t+\delta}(z). \tag{4.7}$$

We say that the growth is local, when the support $\{\xi \in \mathbb{R} : \mathrm{Im}\, \tilde{f}_{t,\delta}(\xi) > 0\}$, tends to a point, which we denote by $W(t)$, as $\delta \to 0$ (we will give more precise definition in the next subsection). Then it follows (4.6) and (4.7) under suitable conditions that g_t satisfies the *Loewner differential equation in the upper half-plane* $\mathbb{H}$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}.$$

Here the number 2 in the numerator follows from the choice of the parametrization so that $a_1(\tilde{K}_{t,\delta}) = 2\delta$. We interpret that K_t is growing *locally at the point* $P(t) = g_t^{-1}(W(t))$ if g_t^{-1} extends continuously to $W(t)$.

The following theorem makes the above argument more formal.

Theorem 4.1. *Let $T > 0$ and let $\gamma : [0,T] \to \mathbb{C}$ be a simple curve such that $\gamma(0) \in \mathbb{R}$ and $\gamma(\,(0,T]\,) \subset \mathbb{H}$. Suppose that γ is parametrized by the capacity. Then*

$$W(t) = \lim_{z \to \gamma(t)} g_t(z) \tag{4.8}$$

exists for any $t \in [0,T]$ and $t \mapsto W(t)$ is continuous. Here the limit is along any sequence $z_n \in \mathbb{H} \setminus \gamma(0,t]$ converging to $\gamma(t)$. Moreover the hydrodynamically normalized conformal maps $(g_t)_{t\in[0,T]}$ related to γ satisfy the Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)} \tag{4.9}$$

with the initial value $g_0(z) = z$.

Before the proof of this theorem we present three auxiliary results.

Lemma 4.5. *Let K be a hull and $H = \mathbb{H} \setminus K$. If $K \subset B(x_0, r)$, then g_K maps $H \cap B(x_0, 2r)$ into $B(x_0, 3r)$ and $\sup_{z\in H} |g_K(z) - z| \leq 5r$.*

Proof. We can assume that $x_0 = 0$. Otherwise consider the map $g_{K-x_0}(z) = g_K(z + x_0) - x_0$.

Let $\tilde{g}$ be the holomorphic extension of $r^{-1}g_K(rz)$ to $\mathbb{D}^*$. Then $\tilde{g} \in \Sigma$ and by the Area theorem $\sum_{n=1}^{\infty} n|a_n(K)|^2 r^{-2(n+1)} \leq 1$ and therefore $|a_n(K)| \leq r^{n+1}$. Hence

$$|g_K(z) - z| \leq \sum_{n=1}^{\infty} |a_n(K)|\, |z|^{-n} \leq r \sum_{n=1}^{\infty} (r/|z|)^n = \frac{r^2}{|z| - r} \leq r$$

for $|z| \geq 2r$. And therefore $g_K(\mathbb{H} \cap B(0,2r)) \subset B(0,3r)$.

If $z \in H \cap B(0,2r)$, then $|g_K(z) - z| \leq |g_K(z)| + |z| < 5r$. □

Using the next lemma we can control the length distortion under conformal maps. This lemma could be used in the proof of the general result Theorem 3.4 about the continuity of conformal maps to the boundary. The same principle of proof when systematized gives estimates for the extremal length (Definition 3.4).

Lemma 4.6. *Let ϕ be a conformal map from open set $U \subset \mathbb{C}$ into $B(0,R)$. Let $z_0 \in \mathbb{C}$ and let $C(r) = U \cap \{z : |z - z_0| = r\}$ for any $r > 0$. Then*

$$\inf_{\rho < r < \sqrt{\rho}} \{\text{Length}\,(\phi(C(r)))\} \leq \frac{2\pi R}{\sqrt{\log 1/\rho}}. \tag{4.10}$$

Proof. Let $l(r) = \text{Length}\,(\phi(C(r)))$. By the Cauchy–Schwarz inequality

$$\begin{aligned} l(r)^2 = \left(\int_{C(r)} |\phi'(z)|\, |\mathrm{d}z| \right)^2 &\leq \int_{C(r)} |\mathrm{d}z| \int_{C(r)} |\phi'(z)|^2\, |\mathrm{d}z| \\ &\leq 2\pi r \int_{z_0 + re^{i\theta} \in U} \left| \phi'\left(z_0 + re^{i\theta}\right) \right|^2 r\, \mathrm{d}\theta. \end{aligned}$$

Divide this by r and then integrate over r to find that

$$\int_0^\infty l(r)^2\, r^{-1}\, \mathrm{d}r \leq 2\pi \int_U |\phi'(z)|^2\, \mathrm{d}x\mathrm{d}y = 2\pi \text{Area}(\phi(U))$$

which implies that

$$\frac{1}{2} \log \frac{1}{\rho} \left(\inf_{\rho < r < \sqrt{\rho}} l(r)^2 \right) \leq \int_\rho^{\sqrt{\rho}} l(r)^2\, r^{-1}\, \mathrm{d}r \leq 2\pi^2 R^2.$$

The claim follows by taking square root. □

The next lemma is the application of the Poisson kernel to establish Theorem 4.1.

Lemma 4.7. *There exist an absolute constant $C > 0$ such that the following holds: If $K \subset B(x_0, r) \cap \mathbb{H}$ and $z \in \mathbb{H}$, $|z - x_0| \geq Cr$, then*

$$\left| f_K(z) - z + \frac{a_1(K)}{z - x_0} \right| \leq \frac{Cra_1(K)}{|z - x_0|^2}$$

where $f_K = g_K^{-1}$

Proof. We can assume $x_0 = 0$. We can also assume that the boundary of K is a continuous. If not, then take a sequence K_n each having a continuous boundary and such that $f_{K_n} \to f_K$ uniformly in compact subsets of $\mathbb{H}$.

By Lemma 4.3, f_K has near ∞ the expansion

$$f_K(z) = z - a_1 z^{-1} + \dots$$

Let $h(z) = \operatorname{Im}(f_K(z) - z)$. Then h is a bounded continuous function in $\overline{\mathbb{H}}$ and harmonic in $\mathbb{H}$. Hence we can write h using the Poisson kernel of $\mathbb{H}$ as

$$h(z) = \operatorname{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - z} h(\xi)\, \mathrm{d}\xi.$$

We can use this formula to derive the harmonic conjugate of h. Notice that $h = \operatorname{Im} f_K$ on $\mathbb{R}$ and therefore

$$f_K(z) = z + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - z} \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi. \tag{4.11}$$

The additive constant in the harmonic conjugate of h was fixed by the expansion near ∞.

Clearly $\operatorname{Im} f_K(\xi)$ is zero outside a bounded interval I which is defined as the smallest interval containing $\{\xi \in \mathbb{R} : f_K(\xi) \in \mathbb{H} \cap \partial K\}$. From this it follows that

$$f_K(z) = z + \frac{1}{\pi} \int_I \frac{1}{\xi - z} \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi = z - \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_I \xi^{n-1} \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi \right) z^{-n}$$

for large enough $|z|$. Hence $a_1 = \frac{1}{\pi} \int_I \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi$ and

$$\begin{aligned} \left| f_K(z) - z + \frac{a_1(K)}{z} \right| &= \left| \frac{1}{\pi} \int_I \left(\frac{1}{\xi - z} + \frac{1}{z} \right) \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi \right| \\ &\leq a_1(K) \sup \left\{ \left| \frac{1}{x - z} + \frac{1}{z} \right| : x \in I \right\} \end{aligned}$$

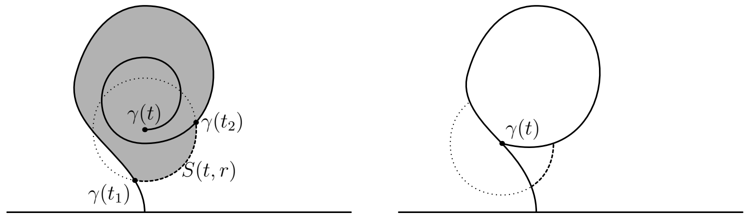
By Lemma 4.5, $I \subset (-3r, 3r)$ and hence

$$\left| \frac{x}{(x - z)z} \right| \leq \frac{6r}{|z|^2}$$

for any $|z| \geq 6r$ and $x \in I$. □

Proof (Proof of Theorem 4.1). As usual, denote $H_t = \mathbb{H} \setminus \gamma(0,t]$. Since $\gamma[0,T]$ is bounded, we can define $R = \sup_{t \in [0,T]} |\gamma(t)| < \infty$.

For each $t \in [0,T]$ and $r > 0$, let $S(t,r)$ be the outermost of all the connected components of $H_t \cap \partial B(\gamma(t), r)$ which separate $\gamma(t)$ from ∞ in H_t. See Figure 4.3. Since by Lemma 4.5, g_t maps $H_t \cap B(0,2R)$ into $B(0,3R)$, we can apply Lemma 4.6 to g_t and show that the diameter of $g_t(S(t,r))$ is at most $6\pi R / \sqrt{\log(1/r)}$. Since the



(a) The arc $S(t,r)$ of the circle of radius r centered at $\gamma(t)$ separates the tip $\gamma(t)$ from ∞ in H_t and it is outermost of all such arcs (in the sense that it separates all other such arcs from ∞ in H_t). When $\gamma(t) \in \mathbb{H}$ and r is small, the end points of $S(t,r)$ lie on the curve.

(b) When the curve is not simple, the conformal map doesn't extend continuously to the tip $\gamma(t)$ when that point is a double point visible from more than one side of the curve. The correct solution for this problem is to define the corresponding generalized boundary point (called *prime end*) as a nested sequence of arcs of circles.

Fig. 4.3 Continuity of the conformal map g_t at the tip point $\gamma(t)$ follows from the fact that an arc $S(t,r)$ of a small circle is mapped to a set of small diameter.

curves $g_t(S(t,r))$, $r > 0$ are disjoint and nested (in the sense that for any $0 < r_1 < r_2$, $g_t(S(t,r_2))$ separates $g_t(S(t,r_1))$ from ∞ in $\mathbb{H}$) and their diameters go to zero as $r \searrow 0$, there exist $W(t) \in \mathbb{R}$ such that

$$\{W(t)\} = \bigcap_{r>0} \overline{V(t,r)}$$

where $V(t,r)$ is the bounded component of $\mathbb{H} \setminus g_t(S(t,r))$.

Since γ is simple, $g_t(H_t \cap B(\gamma(t),r')) \subset V(r,t)$ for small enough $r' > 0$. Namely, when $r < \operatorname{Im}\gamma(t)$, the end points of $S(t,r)$ are points $\gamma(t_1), \gamma(t_2)$, for some $0 < t_1 \leq t_2 < t$. Since the distance from $\gamma(t)$ to $\gamma([t_1,t_2]) \cup S(t,r)$ is positive, then for small enough $r' > 0$, $S(t,r)$ separates $H_t \cap B(\gamma(t),r')$ from ∞ in H_t. See Figure 4.3(a). Therefore $g_t(H_t \cap B(\gamma(t),r')) \subset V(r,t)$ and

$$\{W(t)\} = \bigcap_{r'>0} \overline{g_t(H_t \cap B(\gamma(t),r'))}.$$

Hence $g_t(\gamma(t)) = \lim_{z \to \gamma(t)} g_t(z)$ is well-defined and the first claim follows.

Now for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $t \in [0, T-\delta]$ we have that $g_t(\gamma(t,t+\delta]) \subset V(t,\varepsilon)$. Denote the conformal map associated to the hull $g_t(\gamma([t,t+\delta]))$ by $\tilde{g}_{t,\delta}$ and let $\tilde{\gamma}(\delta) = g_t(\gamma(t+\delta))$. Since $\operatorname{diam} V(t,\varepsilon) \leq r_0(\varepsilon) = 6\pi R/\sqrt{\log(1/\varepsilon)}$, $\tilde{\gamma}(\delta) \in \mathbb{H} \cap B(W(t), r_0(\varepsilon))$ and therefore by Lemma 4.5

$$|W(t+\delta) - W(t)| = |\tilde{g}_{t,\delta}(\tilde{\gamma}(\delta)) - W(t)| \leq 3 r_0(\varepsilon). \tag{4.12}$$

Therefore $t \mapsto W(t)$ is continuous.

Let $C > 0$ be as in Lemma 4.7. Let $t \in [0,T)$, $z \in H_t$ and choose $\varepsilon > 0$ so small that $(C+5)r_0(\varepsilon) < \operatorname{Im} g_t(z)$. If $0 < \delta \leq T-t$ is such that $g_t(\gamma(t,t+\delta]) \subset V(t,\varepsilon)$ then

$$|g_{t+\delta}(z) - W(t)| \geq |g_t(z) - W(t)| - |\tilde{g}_{t,\delta} \circ g_t(z) - g_t(z)| \geq C r_0(\varepsilon).$$

Use Lemma 4.7 for the map $f_K = \tilde{g}_{t,\delta}^{-1}$ at point $g_{t+\delta}(z)$ with $r = r_0(\varepsilon)$ and $x_0 = W(t)$ to find that

$$\left| g_{t+\delta}(z) - g_t(z) - \frac{2\delta}{g_{t+\delta}(z) - W(t)} \right| \leq \frac{2\delta C r_0(\varepsilon)}{|g_{t+\delta}(z) - W(t)|^2}.$$

Since we can take $r_0(\varepsilon) \searrow 0$ as $\delta \searrow 0$, the derivative from the right exists and satisfies

$$\partial_{t+} g_t(z) = \lim_{\delta \searrow 0} \frac{g_{t+\delta}(z) - g_t(z)}{\delta} = \frac{2}{g_t(z) - W(t)}$$

Since the right-hand side is continuous in t, actually, $\partial_t g_t(z)$ exists and we have shown that (4.9) holds. □

Example 4.3. Let $\delta(t) = 2\sqrt{t}$ and let $g_t(z) = \sqrt{z^2 + \delta(t)^2}$. Then $\partial_t g_t(z) = \frac{2}{g_t(z)}$. Thus the driving term of the straight vertical line $t \mapsto \mathrm{i}\delta(t)$, $t \geq 0$, is $W_t = 0$ for all t.

4.2.1.1 The Loewner equation of the inverse function $f_t = g_t^{-1}$

It is customary to denote the inverse of g_t by $f_t = g_t^{-1}$. By differentiating the expression $f_t(g_t(w)) = w$ on both sides, we arrive to the identity $(\partial_t f_t)(g_t(w)) + f_t'(g_t(w))\,\partial_t g_t(w) = 0$. Replacing $g_t(w)$ by z gives the *Loewner equation of f_t*

$$\partial_t f_t(z) = -\frac{2 f_t'(z)}{z - W_t}. \tag{4.13}$$

4.2.2 Solving Loewner equation with a continuous driving term

In this section, we study the solution of Loewner equation with a continuous driving term and show that there is a growing family of hulls parametrized with the half-plane capacity. In fact, we will show that there is one-to-one correspondence between *locally growing hulls* and the solutions of the Loewner equation with continuous driving terms.

Let $t \mapsto W_t$ be a given real valued function on $[0,T]$. We will investigate whether there is a family of conformal maps $(g_t)_{t\in[0,T]}$ that satisfy the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z. \tag{4.14}$$

4.2.2.1 The solution of Loewner equation is a normalized conformal map

First fix $z \in \overline{\mathbb{H}}$. Then (4.14) is just an ordinary differential equation (ODE)

$$\dot{z}_t = \frac{2}{z_t - W_t}, \qquad z_0 = z \tag{4.15}$$

in the parameter t.

Lemma 4.8. *For $z \in \overline{\mathbb{H}} \setminus \{W_0\}$, the solution of (4.15) is unique and exists for all $t \in [0,T] \cap [0,\tau(z))$ where $\tau(z) = \inf\{t \geq 0 : \liminf_{s \nearrow t} |z_s - W_s| = 0\}$. Furthermore, for fixed t, the map $z \mapsto z_t$ is continuous at any point $z \in \overline{\mathbb{H}} \setminus \{W_0\}$ such that $t < \tau(z)$.*

Proof. The mapping $\zeta \mapsto \frac{2}{\zeta - W_t}$ is continuous in t and Lipschitz continuous in ζ in the set of points $\{(t,\zeta) \in [0,T] \times \overline{\mathbb{H}} : |\zeta - W_t| \geq \varepsilon\}$ where $\varepsilon > 0$. Thus by the theory of ODEs, there exists a unique solution to (4.15) and the solution at given time is a continuous function of the initial condition. □

Now set $g_t(z) := z_t$ for $t \in [0,T] \cap [0,\tau(z))$ and $z \in \overline{\mathbb{H}} \setminus \{W_0\}$. We claim that this defines a conformal map. Define

$$H_t = \{z \in \mathbb{H} : \tau(z) > t\}, \qquad K_t = \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}.$$

Then H_t is open by continuity of $z \mapsto g_t(z)$ and similarly K_t is closed.

Proposition 4.1. *The function g_t restricted to H_t is a conformal map onto $\mathbb{H}$. The set K_t is a hull.*

Proof. Let $z, z' \in \mathbb{H}$ and $D_t(z,z') = g_t(z) - g_t(z')$ for any $t \in [0,T] \cap [0,\tau(z) \wedge \tau(z'))$. It satisfies the differential equation

$$\partial_t D_t(z,z') = -D_t(z,z') \frac{2}{(g_t(z) - W_t)(g_t(z') - W_t)}$$

which can be integrated as

$$D_t(z,z') = (z - z') \exp\left(-\int_0^t \frac{2\mathrm{d}s}{(g_s(z) - W_s)(g_s(z') - W_s)} \right).$$

This show that g_t is one-to-one. Furthermore, the complex derivative $g_t'(z)$ exists and equals to

$$g_t'(z) = \lim_{z' \to z} \frac{D_t(z,z')}{z - z'} = \exp\left(-\int_0^t \frac{2\mathrm{d}s}{(g_s(z) - W_s)^2} \right).$$

This shows that g_t is holomorphic. Thus $g_t : H_t \to \mathbb{C}$ is a conformal map.

We will show that $g_t(H_t) = \mathbb{H}$. Note first, that

$$\partial_t \operatorname{Im} g_t(z) = -2 \frac{\operatorname{Im} g_t(z)}{|g_t(z) - W_t|^2}$$

and hence $\operatorname{Im} g_t(z)$ is strictly decreasing and positive, since the previous formula can be integrated as

$$\operatorname{Im} g_t(z) = (\operatorname{Im} z) \exp\left(-\int_0^t \frac{2\mathrm{d}s}{|g_s(z) - W_s|^2} \right)$$

which holds for any $t \in [0,T] \cap [0,\tau(z))$. Therefore $g_t(H_t) \subset \mathbb{H}$. Next fix $t \in (0,T]$ and let $w \in \mathbb{H}$. Define $h_s(w)$ as the solution of the *backward Loewner equation*

$$\partial_s h_s(w) = -\frac{2}{h_s(w) - W_{t-s}}, \qquad h_0(w) = w. \tag{4.16}$$

Then $h_s(w)$, $0 \le s \le t$, is well-defined and lies in the upper half-plane, because $\operatorname{Im} h_s(w)$ is strictly increasing. Let $z = h_t(w)$. Then then $g_s(z) = h_{t-s}(w)$ because $s \mapsto h_{t-s}(w)$ solves the (forward) Loewner equation with the initial condition $h_t(w) = z$. In particular, it holds that $g_t(z) = w$ and we have shown that $g_t(H_t) = \mathbb{H}$.

Next we will show that K_t is a hull. Observe first that since g_t is conformal and $g_t(H_t) = \mathbb{H}$, $\mathbb{H} \setminus K_t = H_t$ is simply connected. As we stated above K_t is closed. To show that K_t is bounded let $M = \sup_{t\in[0,T]} |W_t|$. The first observation is that for any $z \in \overline{\mathbb{H}}$ with $\operatorname{Re} z > M$, $\operatorname{Re} g_s(z)$ is strictly increasing since

$$\partial_s \operatorname{Re} g_s(z) = 2\frac{\operatorname{Re}(g_s(z)) - W_s}{|g_s(z) - W_s|^2} > 0$$

when $\operatorname{Re} g_s(z) > M$. Similarly for any $z \in \overline{\mathbb{H}}$ with $\operatorname{Re} z < -M$, $\operatorname{Re} g_s(z)$ is strictly decreasing. The second observations is that for any $z \in \mathbb{H}$

$$\partial_s \operatorname{Im} g_s(z) = -2\frac{\operatorname{Im}(g_s(z))}{|g_s(z) - W_s|^2} \ge -\frac{2}{\operatorname{Im}(g_s(z))}$$

and hence $(\operatorname{Im} g_t(z))^2 \ge (\operatorname{Im} z)^2 - 4t > 0$, when $\operatorname{Im} z > 2\sqrt{t}$. Therefore

$$\left\{ z \in \mathbb{H} : |\operatorname{Re} z| > M \text{ or } \operatorname{Im} z > 2\sqrt{T} \right\} \subset H_t \tag{4.17}$$

and

$$K_t \subset \left\{ z \in \overline{\mathbb{H}} : |\operatorname{Re} z| \le M \text{ and } \operatorname{Im} z \le 2\sqrt{T} \right\}. \tag{4.18}$$

Now we have established that g_t is a conformal map from $H_t = \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ and that K_t is a hull. Notice then that by using $(\operatorname{Re} g_t(z))^2 + M^2 \ge \max\{(\operatorname{Re} g_t(z))^2, M^2\} \ge (\operatorname{Re} z)^2$ and $(\operatorname{Im} g_t(z))^2 \ge (\operatorname{Im} z)^2 - 4t$, it follows that

$$g_t(z) = g_0(z) + \int_0^t \frac{2\mathrm{d}s}{g_s(z) - W_s} = z + \mathscr{O}(|z|^{-1})$$

as $z \to \infty$. We can apply Lemma 4.1 to show that g_t has the expansion $g_t(z) = z + \sum_{k=1}^{\infty} a_k(t) z^{-k}$ which converges uniformly for $|z| > R$ where $R > 0$ satisfies $K_t \subset \overline{B(0,R) \cap \mathbb{H}}$. Hence $\partial_t g_t(z) = \frac{2}{z} + \dots$ and consequently, $a_1(t) = 2t$. □

4.2.2.2 Local growth and Loewner chains

The following theorem generalizes Theorem 4.1 and will give a sufficient and necessary condition to the fact that g_t has a continuous driving term. This condition is called *local growth*.

Theorem 4.2. *Let $(K_t)_{t\in[0,T]}$ be a growing family of hulls and g_t be the associated conformal maps. Then the following statements are equivalent:*

- *For all $t \in [0,T]$, $a_1(K_t) = 2t$ and for any $\varepsilon > 0$ there is $\delta > 0$ such that for each $t \in [0,T-\delta]$, there exists a bounded connected set $C \subset \mathbb{H} \setminus K_t$ with $\operatorname{diam}(C) < \varepsilon$ such that C separates $K_{t+\delta} \setminus K_t$ from infinity in $\mathbb{H} \setminus K_t$.*
- *There is a continuous $W(t)$, $t \in [0,T]$ such that g_t is the solution of* (4.14).

Definition 4.4. A *Loewner chain* is the solution g_t of the Loewner equation with a continuous driving term.

Remark 4.3. By the previous theorem, any one of the quantities $W(t), K_t, g_t$ could be taken as the most fundamental object. The concept of a Loewner chain covers all those related quantities.

Example 4.4 (Some examples and counterexamples). See Figure 4.4 for some examples related to the theorem.

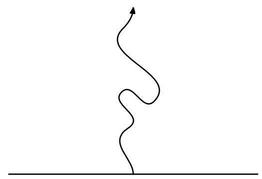
The curves in Figures 4.4(a) and 4.4(b) grow locally and hence they generate a Loewner chain. The former is a simple curve and already covered by Theorem 4.1. While the latter one isn't simple, it is non-self-crossing and thus satisfies the local growth conditions. We don't give the definition of non-self-crossing here formally, but roughly speaking, if there is a double point on the curve, the curve exits to the same side as it entered from to the double point.

Figure 4.4(c) indicates that the set of the locally growing hull collections is a strictly bigger class than the locally growing curves. In that example as the capacity time approaches t_0, the growing hull winds infinitely many times around a disc, and afterwards it unwinds infinitely many times. Hence it can't be continuous at t_0. For more information, see [5].

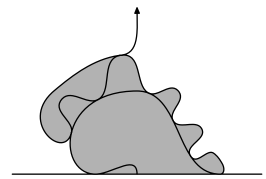
Finally Figure 4.4(d) shows a curve that violates the local growth. A self-crossing leads to a jump in the Loewner driving term. Although we omit discussing that kind of processes, considering Loewner driving terms with jumps is also very fruitful.

Proof (Proof of Theorem 4.2). The fact that the first statement implies the second one is a straightforward generalization of the proof of Theorem 4.1. Namely if $R > 0$ is such that $K_T \subset B(0,R)$ and $t, \varepsilon, \delta, C$ are as in the statement of the theorem, then $\operatorname{diam}(g_t(C)) \leq r_0(\varepsilon) = 6\pi R/\sqrt{\log 1/\varepsilon}$, because $C \subset \overline{B(z_0,\varepsilon)}$ for some $z_0 \in \mathbb{C}$ and by Lemmas 4.5 and 4.6 there is a circle of radius $\rho \in (\varepsilon, \sqrt{\varepsilon})$ which is mapped by g_t to a curve which has length less than $r_0(\varepsilon)$. Since $g_t(C)$ separates $\tilde{K}_{t,\delta} = \overline{g_t(K_{t+\delta} \setminus K_t)}$ from ∞ in $\mathbb{H}$, also the diameter of $\tilde{K}_{t,\delta}$ is less than $r_0(\varepsilon)$.

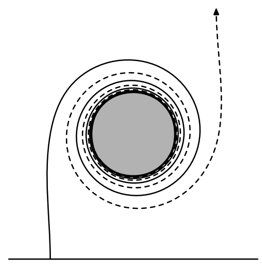
The intersection $\bigcap_{s>0} \tilde{K}_{t,s}$ is non-empty because the sets $\tilde{K}_{t,s}$ are compact and any finite intersection is non-empty. Since the diameter of $\bigcap_{s>0} \tilde{K}_{t,s}$ is less than $r_0(\varepsilon')$



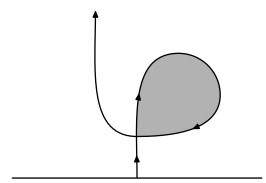
(a) A simple curve in $\mathbb{H}$ starting from the boundary



(b) A non-simple curve which doesn't have self-crossings and the increments $\gamma((t,t+s])$ are always visible to ∞ in $\mathbb{H}\setminus\gamma((0,t])$.



(c) A growing hull, which makes infinite number of turns around a disc



(d) A curve with a self-crossing

Fig. 4.4 Some examples and counterexamples based on Theorem 4.2: The growing hulls of (a)-(c) satisfy the "local growth" condition but (d) doesn't satisfy the condition. However it is possible to use the Loewner equation for (d), but then the driving term would have a discontinuity at the time of the self-crossing.

for any $\varepsilon' > 0$, there exists $W(t) \in \mathbb{R}$ such that $\{W(t)\} = \bigcap_{s>0} \tilde{K}_{t,s}$. Now $\tilde{K}_{t,\delta} \subset B(W(t), r_0(\varepsilon))$ and therefore as in (4.12), the function $t \mapsto W(t)$ is continuous. The Loewner equation now holds by the same argument as in the end of the proof of Theorem 4.1. We have shown that the first statement implies the second one.

To prove that the second statement implies the first one, define for any $\delta > 0$, the oscillation of W by

$$O(W,\delta) = \sup\{|W(t) - W(s)| \,:\, s,t \in [0,T], |s-t| \leq \delta\}.$$

By continuity of W, $O(W,\delta) \searrow 0$ as $\delta \searrow 0$. Let $r_1(\delta) = \left((2\sqrt{\delta})^2 + O(W,\delta)^2\right)^{1/2}$. By the inclusion (4.17), $\tilde{K}_{t,\delta} = \overline{g_t(K_{t+\delta}\setminus K_t)} \subset B(W(t), r_1(\delta))$. By Lemmas 4.5 and 4.6 there exists an arc of a circle of radius $r \in (r_1(\delta), \sqrt{r_1(\delta)})$

$$S = \mathbb{H} \cap \partial B(W(t), r)$$

such that the length of $C = g_t^{-1}(S)$ is less than $cR/\sqrt{\log(1/r_1(\delta))}$, where $R > 0$ is such that $K_T \subset B(0,R)$ and $c > 0$ is some universal constant. Since S separates

$\tilde{K}_{t,t+\delta}$ from ∞ in $\mathbb{H}$, C separates $K_{t+\delta} \setminus K_t$ from ∞ in H_t. Hence we have existence of the separating set C with a uniformly small diameter. The claim now follows. □

4.2.2.3 Reverse Loewner equation

Let's introduce the *reverse Loewner equation*

$$\partial_t h_t(z) = -\frac{2}{h_t(z) - V_t}, \qquad h_0(z) = z. \tag{4.19}$$

The next result was shown in the proof of Proposition 4.1.

Lemma 4.9. *Let $h_t(z)$ be the solution of* (4.19) *where $(V_t)_{t\in\mathbb{R}_{\geq 0}}$ is continuous. Then the solution is well-defined for all $t \in \mathbb{R}_{\geq 0}$. More over $t \mapsto \operatorname{Im} h_t(z)$ is strictly increasing.*

The following lemma gives the significance of the reverse flow.

Lemma 4.10. *Let $(W_t)_{t\in[0,T]}$ be continuous and $\tilde{V}_t = W_{T-t}$, $t \in [0,T]$. Let $\tilde{h}_t(z)$ be the reverse Loewner flow with the driving term $(\tilde{V}_t)_{t\in[0,T]}$ and the $f_t(z)$ be the inverse Loewner flow with the driving term $(W_t)_{t\in[0,T]}$. Then the functions $z \mapsto f_T(z)$ are $z \mapsto \tilde{h}_T(z)$ equal.*

Proof. We will show that $g_T \circ \tilde{h}_T(z) = z$ for all $z \in \mathbb{H}$.

Fix $z \in \mathbb{H}$ and let $\zeta_t = \tilde{h}_{T-t}(z)$ for all $t \in [0,T]$. Then $\zeta_0 = \tilde{h}_T(z)$ and

$$\dot{\zeta}_t = \frac{2}{\tilde{h}_{T-t}(z) - \tilde{V}_{T-t}} = \frac{2}{\zeta_t - W_t}.$$

Hence $\zeta_t = g_t(\zeta_0)$ for all $t \in [0,T]$. In particular, $z = \zeta_T = g_T(\tilde{h}_T(z))$. □

4.3 Loewner equations in $\mathbb{D}$ and $\mathbb{S}_\pi$

In this subsection, we develop the Loewner theory similar to the theory presented in Section 4.2, for the unit disc $\mathbb{D}$ and for the *strip* $\mathbb{S}_\pi = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$. The reader can skip this section and return here when reaching Section 5.4 where this theory is used.

4.3.1 Loewner equation in $\mathbb{D}$

Let $K \subset \overline{\mathbb{D}}$ be a closed set such that the complement $\mathbb{D} \setminus K$ is simply connected and contains 0. We call it a *d-hull* (or $\mathbb{D}$-hull). For any d-hull K, there exists a unique conformal and onto map $g_K : \mathbb{D} \setminus K \to \mathbb{D}$ satisfying $g_K(0) = 0$ and $g_K'(0) > 0$ by the Riemann mapping theorem. The quantity $\mathrm{cap}_{\mathbb{D}}(K) = \log g_K'(0)$ is called the *d-capacity* (or $\mathbb{D}$-capacity) of K. The function g_K can be expanded around $z = 0$ as

$$g_K(z) = e^{\mathrm{cap}_{\mathbb{D}}(K)}\, z + \sum_{k=2}^{\infty} c_k z^k$$

where $c_k \in \mathbb{C}$ are some K dependent coefficients. In composition of conformal maps of this form, the d-capacity $\mathrm{cap}_{\mathbb{D}}(K)$ is additive in a way similar to the h-capacity.

Suppose, for simplicity, that the boundary of K is locally connected and thus the conformal maps we consider extend continuously to the boundary. Let $f_K = g_K^{-1}$. Define a harmonic function h on $\mathbb{D}$ by $h(z) = -\mathrm{Re}\log(f_K(z)/z) = -\log|f_K(z)/z|$. Notice that the holomorphic function inside log is non-zero for all $z \in \mathbb{D}$ and thus the function h is well-defined and harmonic in $\mathbb{D}$. The boundary values of h are non-negative and in fact zero outside of the set $I = g_K(K \cap \mathbb{D})$. Since h is continuous in $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$, we can use the Poisson kernel of $\mathbb{D}$ to write it as

$$h(z) = \frac{1}{2\pi} \int_I h(w)\, \mathrm{Re}\, \frac{w+z}{w-z}\, |\mathrm{d}w|. \tag{4.20}$$

The formula evaluated at $z = 0$ is the mean value property of h and it implies that

$$\mathrm{cap}_{\mathbb{D}}(K) = -\log f_K'(0) = h(0) = \frac{1}{2\pi} \int_I h(w)\, |\mathrm{d}w|.$$

In particular, the d-capacity $\mathrm{cap}_{\mathbb{D}}(K)$ is strictly positive for any non-empty K. The identity (4.20) can be complemented with its harmonic conjugate to arrive to[2]

$$f(z) = z \exp\left(-\frac{1}{2\pi} \int_I h(w) \frac{w+z}{w-z}\, |\mathrm{d}w| \right). \tag{4.21}$$

Suppose that we have a continuously growing chain of d-hulls $(K_t)_{t\in\mathbb{R}_{\geq 0}}$. Call the parameter t time. Due to the additivity of the d-capacity it is natural to reparameterize so that $\log g_t'(0)$ is linear in time. Thus we reparameterize so that $g_t'(0) = e^t$. If the support of the increment of $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ at time t is at $W_t \in \partial\mathbb{D}$, then

$$\begin{aligned} \partial_t g_t(z) &= \lim_{\delta \searrow 0} \frac{g_t(z) - g_{t-\delta}(z)}{\delta} = \lim_{\delta \searrow 0} \frac{g_t(z) - \tilde{f}_{t,\delta} \circ g_t(z)}{\delta} \\ &= g_t(z) \frac{W_t + g_t(z)}{W_t - g_t(z)}. \end{aligned} \tag{4.22}$$

The right sided difference quotient has the same limit, for example, by the continuity of the right-hand side of (4.22). The equation is called *Loewner equation in* $\mathbb{D}$.

[2] The multiplicative constant in (4.21) is fixed by evaluating both sides of (4.20) and its conjugate at 0.

The family of hulls $(K_t)_{t\in[0,T)}$ corresponding to a solution to Loewner equation in $\mathbb{D}$ with a continuous driving term will satisfy a local growth condition analogous to the one given in Theorem 4.2. We call any of such a family a *d-Loewner chain*.

4.3.2 Loewner equation in the strip $\mathbb{S}_\pi$

Next we consider the strip $\mathbb{S}_\pi = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$ in the same manner. A compact set $K \subset \overline{\mathbb{S}_\pi}$ whose complement in $\mathbb{S}_\pi$ is simply connected is called *s-hull*. For any s-hull K there is a unique conformal and onto map $g_K : \mathbb{S}_\pi \setminus K \to \mathbb{S}_\pi$, such that $\lim_{z\to\pm\infty}(g_K(z) - z) = \pm\mathrm{cap}_{\mathbb{S}_\pi}(K)$, where the constant $\mathrm{cap}_{\mathbb{S}_\pi}(K)$ is called *s-capacity*. The s-capacity is additive in composition of normalized conformal maps.

Suppose that we have a curve γ in $\mathbb{S}_\pi$ that generates a family of s-hulls $(K_t)_{t\in[0,T)}$, in an analogous sense as we have learned earlier. Since $t \mapsto \mathrm{cap}_{\mathbb{S}_\pi}(K_t)$ is non-negative and strictly increasing, it is possible to reparameterize the curve so that $\mathrm{cap}_{\mathbb{S}_\pi}(K_t) = t$. With this *s*-capacity parametrization, the Loewner maps $g_t = g_{K_t}$ satisfy the *Loewner equation in* $\mathbb{S}_\pi$

$$\partial_t g_t(z) = \coth\frac{g_t(z) - W_t}{2}, \qquad g_0(z) = z. \tag{4.23}$$

This equation generalizes to a general family of hulls $(K_t)_{t\in[0,T)}$. Namely, being a solution to the Loewner equation is equivalent to local growth of the family of hulls. We leave as an exercise to verify that the s-capacity is positive and that the Loewner equation holds under suitable assumptions.

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Chapter 5
Schramm–Loewner evolution

We define the Schramm–Loewner evolutions in this chapter and study their basic properties.

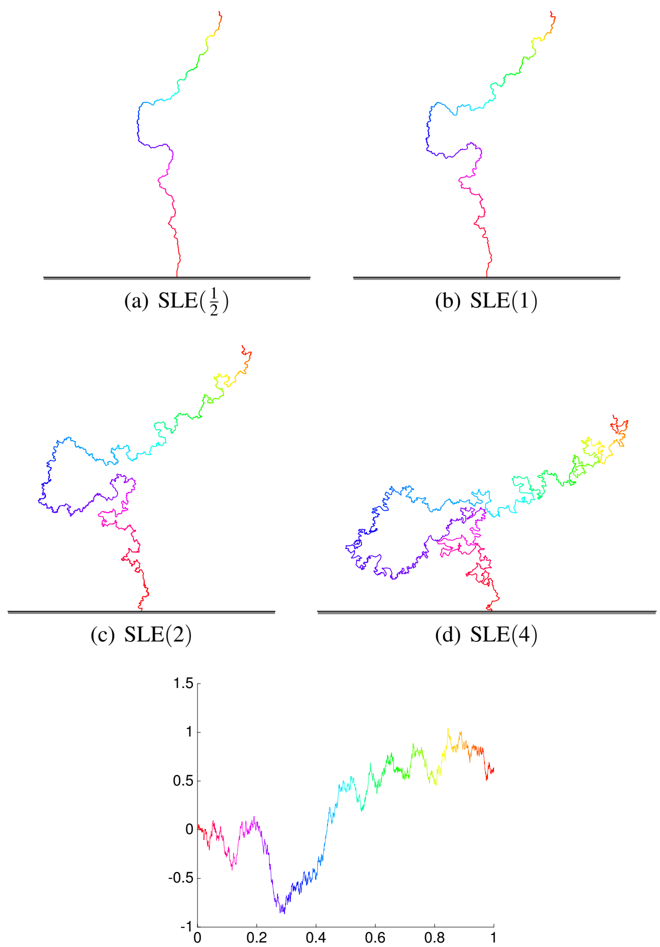
5.1 Schramm–Loewner evolution and its elementary properties

The *Schramm–Loewner evolutions*, which first were called *stochastic Loewner evolutions* (for instance, in Schramm's original article [11]), were invented by Oded Schramm[1]. His groundbreaking innovation in his paper released in 1999 was that random curves can be described using the Loewner equation with a random driving term. This enabled him to define Schramm–Loewner evolutions, which in general are Loewner chains driven by stochastic processes which satisfy a Markov type property and which locally resemble Brownian motions.

The motivation of studying SLE comes from predictions of theoretical physics. Given a lattice model of statistical physics with a temperature-like parameter, for instance, the Ising model, it is believed based on so called Renormalization group analysis that (under some circumstances) there is a critical value of this parameter so that which separates two regimes: in the large system limit, inside each of the regimes the system looks macroscopically the same for all parameter values. We say that the system renormalizes either to the zero temperature system or the infinite temperature system. In between the regimes there is a *critical parameter*. The critical point doesn't renormalize to either of the above mentioned fixed points in the large system limit, but it will be third fixed point.

By the renormalization phenomena (i.e. larger systems can be seen as slightly smaller systems with renormalized parameters through coarse-graining) one expects

[1] Oded Schramm (1961–2008) was an Israeli-American mathematician, who was a highly influential researcher in the fields of complex analysis and probability theory and is best known for inventing SLE and deriving many of its properties (with his co-authors) as well as many other insightful results around random processes related to statistical physics. He died tragically in a climbing accident.



(a) SLE($\frac{1}{2}$)

(b) SLE(1)

(c) SLE(2)

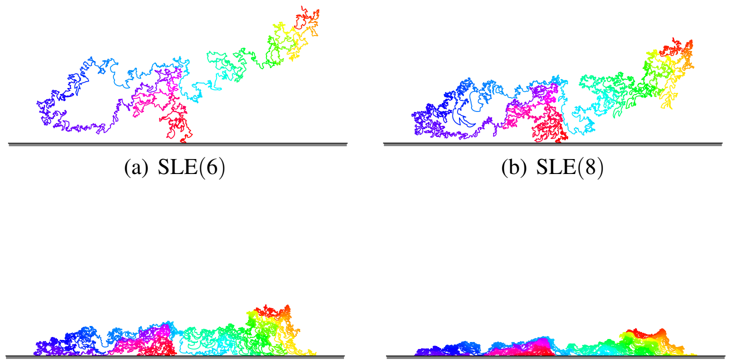
(d) SLE(4)

(e) The standard Brownian motion sample driving the SLE's.

Fig. 5.1 Instances of SLE(κ), for $\kappa = \frac{1}{2}, 1, 2, 4$, which is a stochastic Loewner chain driven by continuous driving process (i.e., stochastic driving term), which is a standard Brownian motion (drawn in Figure (e)) multiplied by a factor $\sqrt{\kappa}$.

that the fixed points are *scale invariant*. The predictions based on the Renormalization group analysis also yield that the critical systems should be *conformally invariant* which is a stronger symmetry than scale invariance. In physics, the continuum theory with conformal symmetry is called *Conformal field theory* (CFT).

Based on so called *Schramm's principle*, which we will go through in Section 5.1.1, a random curve with a Markovian property and conformal symmetry is a Schramm–Loewner evolution (SLE), that is, it is a stochastic Loewner chain with a Brownian motion as its (stochastic) driving term.



(a) SLE(6) (b) SLE(8)

(c) SLE(16) (d) SLE(32)

Fig. 5.2 Instances of SLE(κ), for $\kappa = 6, 8, 16, 32$, driven by the instance of a standard Brownian motion drawn in Figure 5.1(e) multiplied by a factor $\sqrt{\kappa}$.

Scale invariant random curves can be either smooth or fractal, but if the large system limit of a random curve is probabilistically non-trivial, it has to be fractal. If there are fluctuations on some scale, similar fluctuations are seen in all other scales, too, by scale invariance. Fractality of SLE's is well seen in Figures 5.1 and 5.2.

5.1.1 Schramm's principle

Schramm's principle is a calculation that characterizes a family of random Loewner chains that have connection to statistical physics. We present this principle on a heuristical level, but with some additional definitions and assumptions this could be made a theorem that states the following.

Schramm's principle. *Schramm–Loewner evolutions are the only random curves satisfying conformal invariance and the domain Markov property.*

We expect those two properties to be satisfied by scaling limits of random interfaces of statistical physics models at criticality as discussed above in this chapter.

Assume that we are given a collection of probability measures $(\mu^{(U,a,b)})$ indexed by the set of all triplets (U,a,b) where U is any simply connected domain and $a \neq b$ are any two boundary points of U. Assume that $\mu^{(U,a,b)}$ is the law of a random curve $\gamma : [0,\infty) \to \mathbb{C}$ (the parametrization is arbitrary) such that $\gamma([0,\infty)) \subset \overline{U}$ and $\gamma(0) = a$, $\gamma(\infty) = b$. We assume that the family $(\mu^{(U,a,b)})$ satisfies the following properties:

1. Let ϕ_* denote the *pushforward* defined by $\phi_* \mathsf{P} = \mathsf{P} \circ \phi^{-1}$. The family $(\mu^{(U,a,b)})$ satisfies **conformal invariance** (CI): for all (U,a,b)

$$\phi_* \mu^{(U,a,b)} = \mu^{(\phi(U),\phi(a),\phi(b))}$$

2. Let $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$ be the filtration generated by $(\gamma(t))_{t\in\mathbb{R}_{\geq 0}}$. The family $(\mu^{(U,a,b)})$ satisfies **domain Markov property** (DMP): for all (U,a,b), for every (random[2]) $t\in\mathbb{R}_{\geq 0}$ and for any measurable set B in the space of curves (in what ever way that space is defined. . .)
$$\mu^{(U,a,b)}(\gamma|_{[t,\infty)}\in B\,|\,\mathscr{F}_t)=\mu^{(U\setminus\gamma([0,t]),\gamma(t),b)}(\gamma\in B).$$
3. Assume that we can describe the curve γ by the Loewner equation in the sense that there is a $\mu^{(\mathbb{H},0,\infty)}$-almost sure event on which γ satisfies Theorem 4.2.

In Schramm's principle, we'll investigate the consequences of these assumptions.

The first observation is that we need to describe only one of the measures in the family. Then CI fixes the rest of them. Let us choose to work with $\mu^{(\mathbb{H},0,\infty)}$. By Theorem 4.2 for each realization of γ there is a driving term $(W_t(\gamma))_{t\in\mathbb{R}_{\geq 0}}$ such that the corresponding conformal maps g_t satisfy the Loewner equation. Here we also make a reparameterization with the half-plane capacity.[3] Let's call the stochastic driving term $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ as *driving process* of the random curve γ.

Fix some $t\in\mathbb{R}_{\geq 0}$. Define $\hat{\gamma}(s)=g_t(\gamma(t+s))-W_t$ for all $s\in\mathbb{R}_{\geq 0}$. By CI and the DMP, $\hat{\gamma}$ is distributed as γ and independent of the realization of $\gamma|_{[0,t]}$. The conformal map associated to the hull $\hat{\gamma}([0,s])$ is
$$\hat{g}_s(z)=\tilde{g}_{t,s}(z+W_t)-W_t=g_{t+s}\circ g_t^{-1}(z+W_t)-W_t.$$
Now by differentiating this with respect to s
$$\begin{aligned}\partial_s\hat{g}_s(z)&=(\partial_s g_{t+s})(g_t^{-1}(z+W_t))\\&=\frac{2}{g_{t+s}(g_t^{-1}(z+W_t))-W_{t+s}}=\frac{2}{\hat{g}_s(z)-(W_{t+s}-W_t)}\end{aligned}$$
Hence the driving process of $\hat{\gamma}$ is $\hat{W}_s=W_{t+s}-W_t$. Since $\hat{\gamma}$ is distributed as γ and is independent of $\mathscr{F}_t$, which is the σ-algebra generated by $\gamma(s)$, $s\in[0,t]$, $(\hat{W}_s)_{s\in\mathbb{R}_{\geq 0}}$ is independent of $\mathscr{F}_t$ and it is distributed as $(W_s)_{s\in\mathbb{R}_{\geq 0}}$. Hence the continuous stochastic process $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ has *independent and stationary increments*. Theorem 2.1 shows that $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ is a Brownian motion with drift. Thus the driving process of a random curve γ distributed according to $\mu^{(\mathbb{H},0,\infty)}$ is
$$W_t=\sqrt{\kappa}B_t+\alpha t$$
for some $\kappa\geq 0$ and $\alpha\in\mathbb{R}$. We will next show that $\alpha=0$.

Notice that by CI, $\mu^{(\mathbb{H},0,\infty)}$ is invariant under any scaling $z\mapsto\lambda z$, $\lambda>0$, that is, $\gamma^{(\lambda)}$ defined by $\gamma^{(\lambda)}(t)=\lambda\gamma(t/\lambda^2)$ is distributed as γ. Defined this way $\gamma^{(\lambda)}$ is parametrized by capacity as we can verify by using the scaling property of the half-

[2] Technically here we should restrict to so called stopping times.

[3] By an argument which we leave as an exercise, when CI and DMP are satisfied, the half-plane capacity of the hull $\gamma[0,t]$ will tend to infinity as t tends to infinity. Therefore the reparameterized curve will be parametrized by the set $\mathbb{R}_{\geq 0}$.

plane capacity. By a calculation similar to the one above, it follows that the driving process of $\gamma^{(\lambda)}$ is $W_t^{(\lambda)} := \lambda W_{t/\lambda^2}$. Since $(W_t^{(\lambda)})_{t\in\mathbb{R}_{\geq 0}}$ is distributed as $(W_t)_{t\in\mathbb{R}_{\geq 0}}$, the driving process satisfies the Brownian scaling and hence $\alpha = 0$ and

$$W_t = \sqrt{\kappa} B_t.$$

As a conclusion, we have shown that the families $(\mu^{(U,a,b)})$ satisfying CI and the DMP are those where $\mu^{(\mathbb{H},0,\infty)}$ is the law of a random curve whose Loewner driving process is equal to a constant multiple of a one-dimensional Brownian motion.

5.1.2 Definition of SLE as a stochastic Loewner chain

We wish to define a random Loewner chain. We start by a short comment on the measurability of such a construction.

It is possible to show that for any compact $J \subset \mathbb{H}$, there exists a constant C such that the following holds. If $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ and $(\tilde{K}_t)_{t\in\mathbb{R}_{\geq 0}}$ are two Loewner chains such that K_T and $\tilde{K}_T$ are subsets of $\overline{\mathbb{H}} \setminus J$, then $\|g_T - \tilde{g}_T\|_{\infty,J} \leq C\|W - \tilde{W}\|_{\infty,[0,T]}$. For the proof, see Lemma 6.2 below.

Thus the mapping from the continuous functions $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ to the corresponding Loewner chains $(g_t)_{t\in\mathbb{R}_{\geq 0}}$ (the solution of (4.14)) is continuous if we use the following topologies in these spaces. The topology of the driving functions is given by the locally uniform convergence, that is, a sequence converges if it converges uniformly on every compact subinterval of $\mathbb{R}_{\geq 0}$. The topology of the Loewner chains is given by a form of Carathéodory convergence. More specifically, a sequence of Loewner chains[4] $(g_n(t,\cdot), K_n(t))_{t\in\mathbb{R}_{\geq 0}}$ converges to $(g(t,\cdot), K(t))_{t\in\mathbb{R}_{\geq 0}}$, if for any $T > 0$ and any compact $J \subset \mathbb{H} \setminus K_T$, the sequence of functions $(t,z) \mapsto g_n(t,z)$ converges uniformly to $(t,z) \mapsto g(t,z)$ on $[0,T] \times J$.

In particular, the map of the previous paragraph is measurable and hence if we have a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ with a continuous stochastic process $(W_t)_{t\in\mathbb{R}_{\geq 0}}$, we can define a Loewner chain valued random variable $(g_t)_{t\in\mathbb{R}_{\geq 0}}$ corresponding to the stochastic driving term $(W_t)_{t\in\mathbb{R}_{\geq 0}}$. We call them a *driving process* and a *stochastic Loewner chain*.

Next we define SLE as a stochastic Loewner chain. This is slightly unsatisfactory since ultimately we aim to define it as a random curve — the point of view which we took earlier in this chapter. We fix that problem in Theorem 5.2.

Definition 5.1 (Chordal SLE in $\mathbb{H}$). Let $\kappa \geq 0$. A *chordal Schramm–Loewner evolution* SLE(κ) is a stochastic Loewner chain with a driving process $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ equal to a Brownian motion with variance parameter κ, that is, $W_t = \sqrt{\kappa} B_t$ where $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion.

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[4] We use here a variant of the notation so that $W_t, K_t, g_t(z)$ etc. are replaced by $W(t), K(t), g(t,z)$.

Remark 5.1. We call this kind of SLEs *chordal* because we *expect* that they will be random curves that connect two boundary points, namely, 0 and ∞. A *radial* SLE would be a random curve connecting a boundary point to an interior point.

Example 5.1 (SLE(0) is trivial). If W_t is identically zero, then $g_t(z) = \sqrt{z^2+4t}$ and the Loewner chain is equal to the vertical line segment $t \mapsto \mathrm{i}2\sqrt{t}$ as we saw in Example 4.3. To exclude this trivial example, we *make the assumption that* $\kappa > 0$.

5.1.2.1 Elementary properties of SLE

The next theorem captures the elementary consequences of the definition of SLE. For the review of stopping times, Markov properties etc. consult Chapter 2 and references therein.

Theorem 5.1. *Let* $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ *be SLE*(κ)*,* $\kappa > 0$*, and* $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ *the corresponding driving process which is a Brownian motion with respect to a filtration* $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$*. SLE*$(\kappa)$ *satisfies the following properties.*

1. Scale invariance*: For any* $\lambda > 0$*,* $(\lambda K_{t/\lambda^2})_{t\in\mathbb{R}_{\geq 0}} \overset{d}{=} (K_t)_{t\in\mathbb{R}_{\geq 0}}$.[5]
2. Conformal Markov property*: For any* $s \in \mathbb{R}_{\geq 0}$*, the family of hulls*

$$(\hat{K}_{s,t})_{t\in\mathbb{R}_{\geq 0}} = \big(\overline{g_s(K_{s+t}\setminus K_s) - W_s}\big)_{t\in\mathbb{R}_{\geq 0}}$$

is independent of $\mathscr{F}_s$ *and* $(\hat{K}_{s,t})_{t\in\mathbb{R}_{\geq 0}} \overset{d}{=} (K_t)_{t\in\mathbb{R}_{\geq 0}}$.
3. Strong conformal Markov property*: For any almost surely finite stopping time* τ *with respect to* $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$*, the family of hulls*

$$(\hat{K}_{\tau,t})_{t\in\mathbb{R}_{\geq 0}} = \big(\overline{g_\tau(K_{\tau+t}\setminus K_\tau) - W_\tau}\big)_{t\in\mathbb{R}_{\geq 0}}$$

is independent of $\mathscr{F}_\tau$ *and* $(\hat{K}_{\tau,t})_{t\in\mathbb{R}_{\geq 0}} \overset{d}{=} (K_t)_{t\in\mathbb{R}_{\geq 0}}$.

Proof. In the cases 1–3, the hulls and the corresponding conformal maps are

$$\begin{array}{rrr} \lambda K_{t/\lambda^2}, & \overline{g_s(K_{s+t}\setminus K_s) - W_s}, & \overline{g_\tau(K_{\tau+t}\setminus K_\tau) - W_\tau} \\ \lambda g_{t/\lambda^2}(z/\lambda), & \hat{g}_{s,t}(z), & \hat{g}_{\tau,t}(z), \end{array}$$

respectively, where $\hat{g}_{s,t}(z) = g_{s+t}\circ g_s^{-1}(z+W_s) - W_s$. By differentiating these functions with respect to t, we find that the Loewner chains satisfy the Loewner equation with the driving processes $\lambda W_{t/\lambda^2}$, $W_{s+t} - W_s$ and $W_{\tau+t} - W_\tau$, respectively. The claims now follow from the scaling property, the Markov property and the strong Markov property of Brownian motion. □

We leave as an exercise to verify that SLE is also *symmetric under the reflection* with respect to the y-axis, that is, $(m(K_t))_{t\in\mathbb{R}_{\geq 0}} \overset{\mathrm{d}}{=} (K_t)_{t\in\mathbb{R}_{\geq 0}}$, where $m(z) = -\bar{z}$.

[5] Remember that $\overset{\mathrm{d}}{=}$ denotes equal in distribution.

5.1.2.2 SLE in a general simply-connected domain

At the moment, we have defined only $\mu^{(\mathbb{H},0,\infty)}$. Guided by the Schramm's principle, we now extend the definition to include $\mu^{(U,a,b)}$ for a general simply connected domain with two distinguished boundary points. It is natural and consistent with the conformal Markov property to use the conformal invariance requirement for doing this and define SLE(κ) in other domains by the conformal image of a SLE(κ) in $\mathbb{H}$. This definition relies on the fact that SLE(κ) in $\mathbb{H}$ started from $W_0 = 0$ is scale invariant, see Theorem 5.1.

Definition 5.2 (Chordal SLE in a general simply connected domain). Let $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ be a chordal SLE(κ) and let U be a simply connected domain and a and b two boundary points of U with $a \neq b$. We define *(chordal) SLE(κ) in a domain U going from a to b* to be the image of $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ under any conformal onto map $\phi : \mathbb{H} \to U$ with $\phi(0) = a$ and $\phi(\infty) = b$.

Remark 5.2. The definition is unique only up to a linear time change, because all the conformal onto maps from $\mathbb{H}$ to U with the above properties are of the form $z \mapsto \phi(\lambda z)$ where $\lambda > 0$ is a constant. By the scaling property of SLE, the choice of this conformal map only affects the time parametrization of the hulls in U. Naturally we make here the exception that SLE(κ) in $\mathbb{H}$ from x to ∞ is always parametrized with the half-plane capacity.

Remark 5.3. If the boundary of U is not locally connected and ϕ doesn't extend continuously to the boundary, a and b have to be understood as "generalized boundary points", more specifically as prime ends (see references in Remark 3.6).

5.2 Advanced properties of SLE

In this section, we review some properties of SLE. The proofs of most these facts will be given in the later sections or chapters.

5.2.1 SLE is generated by a curve

At this point SLE(κ) is a stochastic Loewner chain. It turns out that it can be defined as a random curve in the sense of the theorem below.

Definition 5.3. A growing family of hulls $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ is *generated by a curve* γ if $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ for all $t \in \mathbb{R}_{\geq 0}$.

For any Loewner chain g_t, $t \in \mathbb{R}_{\geq 0}$, we try to define the generating curve γ as $\gamma(t) = \lim_{\varepsilon\searrow 0} f_t(W(t) + \mathrm{i}\varepsilon)$ where $f_t = g_t^{-1}$. The function γ, if it exists, is called the *trace* of the Loewner chain.

Theorem 5.2. *For each κ, the trace γ exists and is a random curve such that the hulls $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ of SLE(κ) are generated by γ almost surely.*[6]

We will *make the assumption that the previous result holds for SLE(κ)*. We will use that assumption without mentioning it. The result will be proven in Section 6.2 using estimates established for $f_t'(z+\mathrm{i}W_t)$ in this chapter.

5.2.2 Phases of SLE

The next theorem summarizes the important facts on the random curve γ. We will prove those statements at least partly in this section and we are going to do it in several stages.

Theorem 5.3. *Let the random curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ be SLE(κ) (in the sense of Theorem 5.2). Then*

- *For all $0 < \kappa \leq 4$, γ is simple and $\gamma(0,\infty)\cap\mathbb{R} = \emptyset$.*
- *For all $4 < \kappa < 8$, γ is not simple, in fact, it is not simple on any interval:*

$$\text{for any } 0 \leq t_1 < t_2 \text{ there exists } t_1 < s_1 < s_2 < t_2 \text{ such that } \gamma(s_1) = \gamma(s_2). \quad (5.1)$$

  However, γ is not space-filling: for any $z \in \mathbb{H}$, $\operatorname{dist}(z,\gamma[0,\infty)) > 0$ or equivalently $z \notin \gamma[0,\infty)$ almost surely.
- *For all $\kappa \geq 8$, γ is not simple, it satisfies (5.1), but γ is space-filling: $z \in \gamma[0,\infty)$ almost surely.*

Moreover, γ is transient in the sense that $|\gamma(t)| \to \infty$ as $t \to \infty$.

Remark 5.4. A formulation of transience is that under the law $\mu^{(U,a,b)}$ (of Schramm's principle) the random curve γ tends to b as the time tends to its terminal value.

5.2.3 Dimension of SLE

Remember that the *Hausdorff dimension* $d = \dim_{\mathscr{H}}(\Gamma)$ of a point set $\Gamma \subset \mathbb{C}$ is such that the s-dimensional Hausdorff measure[7] $\mathscr{H}^s[\Gamma]$ is infinite for all $s < d$ and zero for all $s > d$, or more definitely, $\dim_{\mathscr{H}}(\Gamma) = \inf\{s \geq 0 : \mathscr{H}^s[\Gamma] < \infty\}$.

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[6] We will prove the theorem only for $\kappa \neq 8$. The case $\kappa = 8$ is a consequence of the results of [9].

[7] The s-dimensional Hausdorff measure $\mathscr{H}^s$ is defined by

$$\mathscr{H}^s[\Gamma] = \lim_{\delta\to 0} \inf\left\{ \sum_{k=1}^{\infty} (\operatorname{diam}(V_k))^s : \Gamma \subset \bigcup_{k=1}^{\infty} V_k \text{ and } \operatorname{diam}(V_k) < \delta \right\}$$

where the infimum is over all countable covers V_k, $k = 1,2,\ldots,n$, of Γ satisfying $\operatorname{diam}(V_k) < \delta$ for all k.

The following result shows that SLE(κ)'s are random fractals, which we anticipated based on the statistical scale invariance. It also gives a very nice interpretation for the parameter κ.

Theorem 5.4. *For $\kappa > 0$, let $\Gamma = \gamma[0,\infty)$, where γ is the trace of SLE(κ). Then* $\dim_{\mathscr{H}}(\Gamma) = 2 \wedge (1 + \frac{\kappa}{8})$.

5.3 Proofs for some of the advanced properties

5.3.1 SLE and Bessel processes

We start the investigation to prove Theorem 5.3 by continuing our review of topics in stochastic analysis.

5.3.1.1 SLE as a complex Bessel process

Fix $z \in \overline{\mathbb{H}}$ with $z \neq 0$ for a moment. Let g_t be a chordal SLE(κ) with a driving process $W_t = -\sqrt{\kappa} B_t$, where the minus sign is for convenience. Define the processes

$$\hat{Z}_t = g_t(z) - W_t, \qquad Z_t = \hat{Z}_t / \sqrt{\kappa}.$$

By the Loewner equation, these processes have the Itô differentials[8]

$$\mathrm{d}\hat{Z}_t = \frac{2}{\hat{Z}_t}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t, \qquad \mathrm{d}Z_t = \frac{2/\kappa}{Z_t}\mathrm{d}t + \mathrm{d}B_t,$$

respectively. Therefore $(Z_t)_{t \in [0,\tau(z))}$, where $\tau(z)$ is as in the section 4.2.2, could be called as a $\delta(\kappa)$*-dimensional complex Bessel process sent from* $z/\sqrt{\kappa}$ where

$$\delta(\kappa) = 1 + \frac{4}{\kappa} \in (1, \infty).$$

The standard use of the parameter δ in the context of Bessel processes is presented below in the stochastic differential equation (5.2).

5.3.1.2 Some properties of Bessel processes

In the next proposition we list some properties of (real-valued) Bessel processes.

[8] It is convenient to use complex valued Itô processes. It is understood that an equality of the form $\mathrm{d}Z(t) = \xi(t)\mathrm{d}t + \sum_{k=1}^{n} \zeta_k(t)\mathrm{d}B_k(t)$, where $(B_k(t))_{t \in \mathbb{R}_{\geq 0}}$ are standard one-dimensional Brownian motions, means that the real and imaginary part of both of the sides are equal when we consider $\mathrm{d}t$ and $\mathrm{d}B_k(t)$ to be real.

Proposition 5.1. *Let $\delta \in \mathbb{R}$ and let $(X_t)_{t\in[0,T)}$ be a δ-*dimensional Bessel process sent from $x > 0$*, that is, $(X_t)_{t\in[0,T)}$ is the unique solution*[9] *of*

$$\mathrm{d}X_t = \frac{\delta - 1}{2X_t}\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x \tag{5.2}$$

and $T \in (0,+\infty]$ is the maximal time such that the solution exists and is positive for any $t \in [0,T)$. Then

1. $\mathsf{P}[T < \infty] = 1$ *if and only if* $\delta < 2$,
2. $\mathsf{P}[T = \infty] = 1$ *if and only if* $\delta \geq 2$,
3. $\mathsf{P}[\inf_{0\leq t<T} X_t > 0] = 1$ *if and only if* $\delta > 2$,
4. $\mathsf{P}[\lim_{t\to T} X_t = 0] = 1$ *when* $\delta < 2$.

Remark 5.5. As we saw in Example 2.3, the Euclidean norm of a d-dimensional Brownian motion is a d-dimensional Bessel process.

By this proposition when $\delta \geq 2$, the Brownian motion of dimension δ won't hit the origin. In the case $\delta = 2$, the Brownian motion will eventually get arbitrarily close to zero, but doesn't hit it in finite time.

Proof. These claim can be proven using the fact that for δ-dimensional Bessel process $(X_t)_{t\in[0,T)}$, the process $M_t = X_t^{2-\delta}$ when $\delta \neq 2$ or $M_t = \log X_t$ when $\delta = 2$ is a local martingale for $t < T$. We leave as an exercise to apply Itô's formula to M_t to verify the claim.

Notice also that the Bessel processes are scale invariant so that they satisfy the Brownian scaling $\lambda X_{t/\lambda^2} \stackrel{\mathrm{d}}{=} X_t$ for all $\lambda > 0$.

For $t < T$, notice that $X_t - B_t = x + ((\delta-1)/2)\int_0^t X_s^{-1}\mathrm{d}s$. Since $X_s > 0$ for all $s \in [0,T)$ it follows that

$$\begin{cases} X_t \geq B_t & \text{when } \delta \geq 1 \\ X_t \leq B_t & \text{when } \delta \leq 1. \end{cases} \tag{5.3}$$

For $a \in (0,x)$, $b \in (x,\infty)$, $c \in (0,x) \cup (x,\infty)$, define $\tau_c = \inf\{[0,T) : X_t = c\}$ and using it $\tau_{a,b} = \tau_a \wedge \tau_b$. Define also $\tau_0 = \lim_{a\to 0} \tau_a$ and $\tau_{0,b} = \lim_{a\to 0} \tau_{a,b}$ which exist since they are decreasing functions of a. By the comparison inequalities (5.3), we can show that for any $a \in [0,x)$ and $b \in (x,\infty)$, the stopping time $\tau_{a,b}$ is finite almost surely. For example, when $\delta \geq 1$, $\tau_{a,b}$ is bounded from above by the almost surely finite time that the Brownian motion hits b. The same argument works for $\delta < 1$.

Since $(M_{t\wedge\tau_{a,b}})_{t\in\mathbb{R}_{\geq 0}}$ is a bounded martingale and $\tau_{a,b}$ is an almost surely finite stopping time, by the optional stopping theorem (see Appendix A and references therein)

$$f(x) = f(a)\mathsf{P}[X_{\tau_{a,b}} = a] + f(b)\mathsf{P}[X_{\tau_{a,b}} = b] \tag{5.4}$$

[9] The existence and uniqueness of the solution follows from Theorem 2.9, for instance, using the following trick. For any $n \in \mathbb{N}$, replace the drift term by a smooth continuation of the function that maps $x \mapsto (\delta-1)/(2x)$, $x > 1/n$ and $x \mapsto 0$, $x < 0$. The drift term and the approximating drift term are identical on the interval $[1/n,+\infty)$ and thus the solutions agree until the process exits the interval.

where $f(x) = x^{2-\delta}$ when $\delta \neq 2$ and $f(x) = -\log(x)$ when $\delta = 2$. Writing (5.4) in the form

$$\mathsf{P}[\tau_a = \tau_{a,b}] = \frac{f(x) - f(b)}{f(a) - f(b)}$$

allows us to make the following conclusions

- When $\delta < 2$, $\lim_{a \to 0} f(a) = 0$. Thus $\mathsf{P}[\tau_0 = \tau_{0,b}] = 1 - f(x)/f(b) > 0$. Notice that $\tau_0 = T$. Thus for some $t > 0$, it holds that $\mathsf{P}[T \leq t] > 0$. By scale invariance of the Bessel process, $p = \mathsf{P}[T = \infty]$ is independent of x. It follows that $p = \mathsf{P}[T = \infty | \mathscr{F}_t]$ on the event $T > t$. Taking expected value from both sides it follows that $p = p\mathsf{P}[T > t]$. Since $\mathsf{P}[T \leq t] > 0$ it follows that $p = 0$.
- When $\delta \geq 2$, $\lim_{a \to 0} f(a) = \infty$. Thus $\mathsf{P}[\tau_0 = \tau_{0,b}] = 0$. It is then possible to argue that $\tau_{0,b} = \tau_b$ tends to ∞ as $b \to \infty$, since there is no blow-up by the solution in finite time by the uniqueness and existence theorem of SDEs. This shows that $T = \tau_0 = \infty$ almost surely.
- When $\delta > 2$, since $\lim_{b \to \infty} \tau_b = \infty$ almost surely and $\lim_{b \to \infty} f(b) = 0$, it holds that $\mathsf{P}[\tau_{2^{-n}} < \infty] = f(x)/f(2^{-n}) = x^{2-\delta}\, 2^{-(\delta-2)n}$ for any $n \in \mathbb{Z}_{>0}$ such that $2^{-n} < x$, which is summable over n. Thus by the Borel–Cantelli lemma, $\inf_{t \in \mathbb{R}_{\geq 0}} X_t > 0$ almost surely.
- When $\delta < 2$, then choose $r_n > 1$ such that $\mathsf{P}[\tau_{r_n x} = \tau_{0, r_n x}] = 2^{-n}$. Then by scale invariance and the Markov property of the Bessel process,

  $$\mathsf{P}[X_t \geq n^{-1} \text{ for some } t > \tau_{n^{-1} r_n^{-1}}] = 2^{-n}$$

  for all $n \in \mathbb{Z}_{>0}$ such that $n^{-1} r_n^{-1} < x$. Consequently $\limsup_{t \to T} X_t = 0$.

All the claims follow. □

5.3.2 *Phase transition from simple to non-simple curve at* $\kappa = 4$

We will show in this subsection that γ is simple for $\kappa \in (0, 4]$ and non-simple for $\kappa \in (4, +\infty)$.

Remember that $z \in K_t$ if and only if $\tau(z) \leq t$, where $\tau(z)$ is as in Section 4.2.2. Notice also that for $0 < x_1 < x_2$ or for $x_2 < x_1 < 0$, it holds that $W_t < g_t(x_1) < g_2(x_2)$ or $g_2(x_2) < g_t(x_1) < W_t$, respectively, for all $0 < t < \tau(x_1) \wedge \tau(x_2)$ and consequently, it holds that $\tau(x_1) \leq \tau(x_2)$.

Now we use the fact that the process $X_t = (g_t(x) - W_t)/\sqrt{\kappa}$, $x \in \mathbb{R} \setminus \{0\}$, is a Bessel process of dimension $\delta = 1 + \frac{4}{\kappa}$. Proposition 5.1 applied to X_t shows that $\mathsf{P}[\tau(x) < \infty]$ is equal to 1 when $\kappa > 4$ and 0 when $\kappa \in (0, 4)$. This together with the above monotonicity property of $\tau(x)$ implies the following result easily.

Proposition 5.2. *For* $0 < \kappa \leq 4$, $(\bigcup_{t \in \mathbb{R}_{\geq 0}} K_t) \cap \mathbb{R} = \{0\}$ *and for* $\kappa > 4$, $\mathbb{R} \subset \bigcup_{t \in \mathbb{R}_{\geq 0}} K_t$ *almost surely. Equivalently, almost surely* $\tau(x) = \infty$ *for all* $x \in \mathbb{R} \setminus \{0\}$*, when* $\kappa \in (0, 4]$*, and* $\tau(x) < \infty$ *for all* $x \in \mathbb{R}$*, when* $\kappa > 4$

Based on this result, let's first show that SLE(κ), $0 < \kappa \leq 4$, is simple, based on this result. Let $s > 0$ and let x_- and x_+ be the two images of 0 under the map $g_s - W_s$. By the previous proposition and by the conformal Markov property, $\hat{\gamma}(t) = g_s(\gamma(s+t)) - W_s$, $t \in \mathbb{R}_{\geq 0}$, intersect the real axis only at 0. In particular it doesn't intersect $[x_-, 0) \cup (0, x_+]$. Since $f_s = g_s^{-1}$ is continuous to the boundary by Theorem 3.4 and the assumed Theorem 5.2, this implies that

$$\gamma[0,s] \cap \gamma[s,\infty) = \{\gamma(s)\} \tag{5.5}$$

almost surely. In fact this holds almost surely for all s (we can show it first for all rational s and then by continuity to all s). If $t_1 < t_2$ are such that $\gamma(t_1) = \gamma(t_2)$, then pick $t_1 < s < t_2$ such that $\gamma(s) \neq \gamma(t_1)$. Then $\gamma(t_1) = \gamma(t_2)$ contradicts with (5.5). Thus γ is simple.

Let's then show that SLE(κ), $\kappa > 4$, is not simple. Let $0 \leq s_1 < u < s_2$. Let $\hat{x}_- \leq 0 \leq \hat{x}_+$ be such that the image of $\gamma[s_1, u]$ under $g_u - W_u$ is $[\hat{x}_-, \hat{x}_+]$. Since $\tau(1)$ is finite almost surely, for fixed $t > 0$, by scaling $\mathsf{P}[\tau(x) \leq t] \to 1$ as $x > 0$ tends to 0. Therefore

$$\mathsf{P}[\gamma[0,t] \cap (0,x] \neq \emptyset] = 1$$

for all $t > 0$ and $x > 0$. Hence we can find $u < t_2 < s_2$ such that $g_u(\gamma(t_2)) - W_u \in [\hat{x}_-, 0) \cup (0, \hat{x}_+]$. And hence there exists $s_1 \leq t_1 < u$ such that $\gamma(t_1) = \gamma(t_2)$ and we have shown the property (5.1).

As conclusion, we have shown the following result.

Proposition 5.3. *When $\kappa \in (0,4]$, γ is simple almost surely and when $\kappa > 4$, γ is non-simple almost surely in the sense of Theorem 5.3.*

5.3.3 Transience for $\kappa \in (0,4]$

We will show in this subsection that a chordal SLE(κ) curve γ is transient when $\kappa \in (0,4]$, in the sense that $|\gamma(t)| \to \infty$ as $t \to \infty$. In particular, $\overline{\gamma[0,\infty)} = \gamma[0,\infty)$ when γ is transient.

Proposition 5.4. *When $0 < \kappa \leq 4$, $\mathsf{P}[\mathrm{dist}(\overline{\gamma[0,\infty)}, [x,x']) > 0] = 1$ for any $0 < x < x'$ or $x < x' < 0$.*

Proof. Suppose first that $\kappa \in (0,4)$. By symmetry and the scale invariance of SLE(κ), it is enough to show that $\mathsf{P}(\mathrm{dist}(\overline{\gamma[0,\infty)}, [1,x]) > 0) = 1$ for all $x > 1$. Let $0 < \delta < 1/4$ and define $\sigma_\delta = \inf\{t \in \mathbb{R}_{\geq 0} : \mathrm{dist}(\gamma(t), [1,x]) \leq \delta\}$.

Let's consider the event $\sigma_\delta < \infty$. Let R_{σ_δ} be the union of the right-hand side of $\gamma[0,\sigma_\delta]$ and $[0,1/2]$ and $h(z)$ be the bounded harmonic function on $H_{\sigma_\delta} = \mathbb{H} \setminus \gamma[0,\sigma_\delta]$ that has boundary value 1 on R_{σ_δ} and 0 elsewhere. See Figure 5.3. Then $h(z)$ can be written as the harmonic measure $\mathrm{HM}(z, R_{\sigma_\delta}, H_{\sigma_\delta})$ as in Definition 3.3. By the conformal invariance of harmonic measure, after applying the conformal map g_{σ_δ} we see that for $y \in \mathbb{R}_{>0}$,

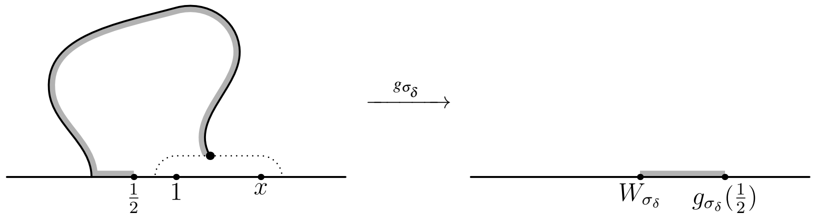

Fig. 5.3 The harmonic function h in the proof of Proposition 5.4 has boundary value 1 in the shaded boundary arc and 0 elsewhere on the boundary. The dotted curve is the set of points at distance δ from the interval $[1,x]$.

$$h(\mathrm{i}y) = \frac{1}{\pi y}(g_{\sigma_\delta}(1/2) - W_{\sigma_\delta}) + \mathscr{O}\left(\frac{1}{y^2}\right)$$

as $y \to \infty$. This can be derived, for instance, by further mapping conformally to the unit disc and sending $g_{\sigma_\delta}(\mathrm{i}y)$ to 0. We leave the details to the reader.

On the other hand, we can write the harmonic measure $h(\mathrm{i}y)$ as the probability that a Brownian motion sent from $\mathrm{i}y$ exits H_{σ_δ} through R_{σ_δ}. On this event the Brownian motion has to intersect the vertical line connecting the interval $[1,x]$ to $\gamma(\sigma_\delta)$. Let $x_0 = \mathrm{Re}\,\gamma(\sigma_\delta)$. Then the probability that a complex Brownian motion sent from $\mathrm{i}y$ will hit the right-hand side of the segment $[x_0, x_0 + \mathrm{i}\delta]$ before hitting the left-hand side or the real axis is equal to $\frac{\delta}{\pi y} + \mathscr{O}(\frac{1}{y^2})$ as $y \to \infty$. This can be derived similarly as above.

Using the latter harmonic measure as an upper bound for the former one, multiplying by y and taking the limit $y \to \infty$, we conclude that $g_{\sigma_\delta}(1/2) - W_{\sigma_\delta} \le \delta$. Because the infimum of a Bessel process is positive (Proposition 5.1), there exists a positive random variable δ_0 such that $\sigma_\delta = \infty$ for all $\delta \in (0, \delta_0)$. The claim follows for $\kappa \in (0,4)$.

For $\kappa = 4$, it holds that the infimum of a Bessel process with corresponding dimension ($\delta = 2$) is zero. The above argument cannot therefore be used for $\kappa = 4$. One can give an argument similar to that given in Section 5.3.4, see also [10]. □

Proposition 5.5. *For* $0 < \kappa \le 4$, $|\gamma(t)| \to \infty$ *as* $t \to \infty$.

Proof. First of all $|\gamma(t_k)| \to \infty$ along some sequence $t_k \to \infty$, because otherwise γ would be bounded and hence had bounded half-plane capacity.

Let T_1 be the hitting time of $\partial B(0,1)$ by γ and let $x_- < 0 < x+$ be the two images of 0 under the map $g_{T_1} - W_{T_1}$. Then by Proposition 5.4 almost surely distance from the SLE(κ) curve $\hat{\gamma}(t) = g_{T_1}(\gamma(T_1 + t)) - W_{T_1}$, $t \in \mathbb{R}_{\ge 0}$, to $[-n, -1/n] \cup [1/n, n]$ is positive for all $n \in \mathbb{N}$. Hence it will stay at a positive distance from x_- and x_+ and consequently, there exists a random variable $r > 0$ such that $|\gamma(t)| \ge r$ for all $t \ge T_1$. Using this property, scaling and the Borel–Cantelli lemma, we can construct a sequence of random variables $0 < R_1 < R_2 < \ldots$ such that $R_k \to \infty$ almost surely and γ doesn't enter to $B(0, R_{k-1})$ after hitting $\partial B(0, R_k)$. □

5.3.4 Phase transition of distance from a point to γ at $\kappa = 8$

In this subsection, we analyze the distance from z to $\overline{\gamma[0,\tau(z))}$ or $\overline{\gamma[0,\infty)}$.

5.3.4.1 The time evolution of the conformal radius and $\arg Z_t$

We first need a conformally covariant[10] version of the distance to the boundary. For any simply connected domain $U \subset \mathbb{C}$ (with $U \neq \mathbb{C}$) and for any $z_0 \in U$, let ψ be the unique conformal map from U onto $\mathbb{D}$ such that $\psi(z_0) = 0$ and $\psi'(z_0) > 0$. Then the *conformal radius of U from* z_0 is defined as

$$\rho(z_0, U) = \psi'(z_0)^{-1}. \tag{5.6}$$

The conformal radius is proportional to the inradius as shown by the next result.

Lemma 5.1. *For any simply connected domain $U \neq \mathbb{C}$ and any $z_0 \in U$, it holds that $(1/4)\rho(z_0,U) \leq \operatorname{dist}(z_0, \partial U) \leq \rho(z_0,U)$.*

Proof. Let $\phi : \mathbb{D} \to (\psi'(z_0)U)$ be defined by $\phi(z) = \psi'(z_0)\,(\psi^{-1}(z) - z_0)$. Then ϕ is a conformal map, $\phi(0) = 0$ and $\phi'(0) = 1$. Thus $(1/4) \leq \psi'(z_0)\operatorname{dist}(z_0, \partial U) \leq 1$ by Theorem 3.8. □

We leave as an exercise to verify that the conformal radius of $H_t = \mathbb{H} \setminus K_t$ from z_0 is equal to

$$\rho(z_0, H_t) = \frac{2Y_t}{|g_t'(z_0)|}$$

when $t < \tau(z_0)$. The proportionality to the inradius, Lemma 5.1, implies that

$$\frac{1}{2}\operatorname{dist}\left(z_0, \mathbb{R} \cup \overline{\gamma[0,\infty)}\right) \leq \lim_{t \nearrow \tau(z_0)} \frac{Y_t}{|g_t'(z_0)|} \leq 2\operatorname{dist}\left(z_0, \mathbb{R} \cup \overline{\gamma[0,\infty)}\right).$$

For any fixed $z_0 \in \overline{\mathbb{H}}$, let $Z_t = g_t(z_0) - W_t$, $t \in [0, \tau(z_0))$ and let X_t and Y_t be the real and imaginary parts of Z_t, respectively, and as usual let $W_t = -\sqrt{\kappa}B_t$. Then from the Loewner equation it follows that

$$\mathrm{d}X_t = \frac{2X_t}{X_t^2 + Y_t^2}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t \tag{5.7}$$

$$\partial_t Y_t = -\frac{2Y_t}{X_t^2 + Y_t^2} \tag{5.8}$$

$$\partial_t \log|g_t'(z)| = -2\frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2}. \tag{5.9}$$

[10] Conformally covariant here means that the transformation rule under conformal maps is simple.

The first two equations follow from taking the real and imaginary parts on both sides and the equation (5.9) by taking derivative of the Loewner equation with respect to z. Write using (5.8) and (5.9) that $\partial_t \log(Y_t/|g_t'(z)|) = -4\,Y_t^2\,(X_t^2+Y_t^2)^{-2}$ and define

$$S(t) = 4\int_0^t \frac{Y_s^2 \mathrm{d}s}{(X_s^2+Y_s^2)^2} = 4\int_0^t \frac{(\sin\arg Z_t)^2}{X_s^2+Y_s^2}\mathrm{d}s. \tag{5.10}$$

Then it follow that

$$\rho(z_0, H_t) = \frac{2Y_t}{|g_t'(z_0)|} = 2y_0 \exp(-S(t)) \tag{5.11}$$

where $y_0 = \operatorname{Im} z_0$.

Since $z \mapsto \log z$ is holomorphic, using Itô's formula for the real and imaginary parts of $\log Z_t$ gives

$$\mathrm{d}\log Z_t = (2-\kappa/2)\frac{\mathrm{d}t}{Z_t^2} + \sqrt{\kappa}\frac{\mathrm{d}B_t}{Z_t}$$

and therefore by taking real and imaginary parts we find that

$$\mathrm{d}\log|Z_t| = (2-\kappa/2)\frac{X_t^2-Y_t^2}{(X_t^2+Y_t^2)^2}\mathrm{d}t + \sqrt{\kappa}\frac{X_t}{X_t^2+Y_t^2}\mathrm{d}B_t \tag{5.12}$$

$$\mathrm{d}\arg Z_t = -(2-\kappa/2)\frac{2X_tY_t}{(X_t^2+Y_t^2)^2}\mathrm{d}t - \sqrt{\kappa}\frac{Y_t}{X_t^2+Y_t^2}\mathrm{d}B_t. \tag{5.13}$$

Let now $\theta_t = \arg Z_t$. Then we can rewrite the previous equation as

$$\mathrm{d}\theta_t = \frac{1}{2}(\kappa-4)\sin(2\theta_t)\frac{\mathrm{d}t}{X_t^2+Y_t^2} - \sqrt{\kappa}\sin(\theta_t)\frac{\mathrm{d}B_t}{\sqrt{X_t^2+Y_t^2}}.$$

5.3.4.2 A time change

Define a time change using the definitions (5.10) and $\sigma(s) = S^{-1}(s)$ in Proposition 2.8. We use the definition $\hat{\theta}_s = 2\theta_{\sigma(s)}$ since it corresponds to the coordinate change from $\mathbb{H}$ to $\mathbb{D}$. After the time-changed quantities obey the time evolution

$$\mathrm{d}\hat{\theta}_s = \frac{\kappa-4}{2}\cot\left(\frac{\hat{\theta}_s}{2}\right)\mathrm{d}s + \sqrt{\kappa}\mathrm{d}\hat{B}_s \tag{5.14}$$

$$\rho(z_0, H_{\sigma(s)}) = 2y_0 e^{-s}. \tag{5.15}$$

The solution to this stochastic differential equation exists and is unique until the exist time from the interval $(0, 2\pi)$

$$\hat{\tau} = \sup\{s \in \mathbb{R}_{\geq 0} : \hat{\theta}_u \in (0, 2\pi) \text{ for all } u \in [0, s]\} \tag{5.16}$$

which is a stopping time. By comparing to Bessel processes, we know that if $\hat{\tau}$ is finite, then $\lim_{t\to\hat{\tau}} \hat{\theta}_t$ exists and belongs to $\{0, 2\pi\}$. Since $\theta_t \in (0,\pi)$ for all $t < \tau(z_0)$, this implies that $S(t) < \hat{\tau}$ for all $t < \tau(z_0)$ and

$$S(\tau(z_0)) := \lim_{t\to\tau(z_0)} S(t) \leq \hat{\tau}. \tag{5.17}$$

We will later show that equality will indeed hold in this inequality for SLE(κ).

Furthermore, we can compare $\hat{\theta}_s$ to Bessel processes when $\hat{\theta}_s \approx 0$ or $\hat{\theta}_s \approx 2\pi$. Namely, then $\cot(\hat{\theta}_s/2)$ is close to $2/\hat{\theta}_s$ or $2/(2\pi - \hat{\theta}_s)$, respectively. The corresponding Bessel process has dimension δ such that $\delta = (3\kappa - 8)/\kappa$. This implies that $\hat{\tau}$ is almost surely finite for $\kappa < 8$ and almost surely infinite for $\kappa \geq 8$. The next result follows from this observation, the equation (5.11) and the equation (5.17).

Proposition 5.6. *When* $\kappa < 8$*, for any* $z \in \mathbb{H}$*, almost surely* $\mathrm{dist}(z, \overline{\gamma[0,\tau(z))}) > 0$.

5.3.5 Phase transition of $\tau(z)$ *at* $\kappa = 4$

We will investigate in this subsection whether $\tau(z)$ is finite or infinite, that is, whether or not z belongs to $\bigcup_{t\in\mathbb{R}_{\geq 0}} K_t$.

Proposition 5.7. *When* $\kappa > 4$*, for any* $z \in \mathbb{H}$*,* $\tau(z) < \infty$ *almost surely.*

Proof. Let $Z_t = g_t(z) - W_t$, $z \in \mathbb{H}$. Since $\arg Z_t \in (0,\pi)$ for all $t < \tau(z)$, Z_t can exit the set $B_R = \{z \in \mathbb{H} : |z| < R\}$ only through $\{0\} \cup \{z \in \mathbb{H} : |z| = R\}$.

Let σ_R be the exit time of $(Z_t)_{t\in\mathbb{R}_{\geq 0}}$ from B_r. Then always $\sigma_R \leq \tau(z)$. We claim that for all κ, $\sigma_R < \infty$ almost surely. To see this let $X_t = \mathrm{Re}\, Z_t$ and write

$$\mathrm{d}X_t = \sqrt{\kappa} B_t + \frac{2X_t \mathrm{d}t}{|Z_t|^2}. \tag{5.18}$$

For each $k \in \mathbb{Z}_{\geq 0}$ and $R > 0$, let $E_{k,R}$ be the event that $\min_{t\in[k,k+1]} B_t \leq B_{k+1} - R$ or $\max_{t\in[k,k+1]} B_t \geq B_{k+1} + R$. On the event $E_{k,R}$, denote by $\eta_{k,R}$ the maximal s such that $|B_s - B_{k+1}| = R$. It is fairly easy to see, for instance, by the fact that the time reversal of Brownian motion is a Brownian motion, that

$$\begin{aligned}
&\mathsf{P}\left[B_{\eta_{k,R}} - B_{k+1} = m_1 R,\ m_2 X_{\eta_{k,R}} \geq 0 \,\middle|\, E_{k,R}\right] \\
=&\mathsf{P}\left[B_{\eta_{k,R}} - B_{k+1} = m_1 R \,\middle|\, E_{k,R}\right] \mathsf{P}\left[m_2 X_{\eta_{k,R}} \geq 0 \,\middle|\, E_{k,R}\right] \\
=&\frac{1}{2}\,\mathsf{P}\left[m_2 X_{\eta_{k,R}} \geq 0 \,\middle|\, E_{k,R}\right]
\end{aligned} \tag{5.19}$$

for all $m_1, m_2 = \pm 1$. Therefore

$$\mathsf{P}\left[\begin{array}{c} B_{\eta_{k,R}} - B_{k+1} = -R,\ X_{\eta_{k,R}} \geq 0 \\ \text{or } B_{\eta_{k,R}} - B_{k+1} = +R,\ X_{\eta_{k,R}} \leq 0 \end{array} \,\middle|\, E_{k,R}\right] = \frac{1}{2}. \tag{5.20}$$

Also it is easy (by the same argument) to see that $\mathsf{P}[E_{k,R}] = \mathsf{P}[E_{1,R}] > 0$ for all k and R. Notice that the events $E_{k,R}$, $k \in \mathbb{Z}_{\geq 0}$, are independent and $\sum_{k\in\mathbb{Z}_{\geq 0}} \mathsf{P}[E_{k,R}] = \infty$, and thus by the second Borel-Cantelli lemma, see [2], $\mathsf{P}[\bigcap_{n=0}^{\infty}\bigcup_{k=n}^{\infty} E_{k,R}] = 1$.

We claim that the conditional probability of $\sigma_R \leq k+1$ given $E_{k,R}$ is at least $1/2$. If $\tau(z) \leq k+1$, then always $\sigma_R \leq k+1$. Suppose therefore that $\tau(z) > k+1$. If either one of the events on the left of (5.20) (separated by or) occurs, then by (5.18), $|X_{k+1}| \geq |X_{k+1} - X_{\eta_{k,R}}| \geq |B_{k+1} - B_{\eta_{k,R}}| = R$ and it follows that $\sigma_R - \leq k+1$. Therefore we have shown that $\sigma < \infty$ almost surely.

Next notice that $Z_t^{1-4/\kappa}$ is a local martingale, in the sense that its real and imaginary parts are local martingales.[11] We leave as an exercise to apply Itô's formula to verify this. Let $\alpha = 1 - 4/\kappa$ and $\nu = e^{\mathrm{i}\pi(1-\alpha)/2}$. Then $h(z) = \mathrm{Im}(\nu z^{\alpha})$ is a positive function on $\overline{\mathbb{H}} \setminus \{0\}$ and $h(Z_t)$ is a real-valued local martingale. It is straightforward to check that there exists a constant $c \in (0,1)$ such that $c|z|^{\alpha} \leq h(z) \leq |z|^{\alpha}$ for all $z \in \overline{\mathbb{H}}$.

If we apply the optional stopping theorem to the bounded martingale $h(Z_{t\wedge\sigma_R})$ and the stopping time σ_R, which is almost surely finite, then

$$h(z) = \mathsf{E}[h(Z_{\sigma}) \,|\, \sigma_R < \tau(z)]\; \mathsf{P}[\sigma_R < \tau(z)] \tag{5.21}$$

Therefore

$$c\left(\frac{|z|}{R}\right)^{\alpha} \leq \mathsf{P}[\sigma_R < \tau(z)] \leq c^{-1}\left(\frac{|z|}{R}\right)^{\alpha} \tag{5.22}$$

Thus $\mathsf{P}[\tau(z) < \infty] \geq 1 - \mathsf{P}[\sigma_R < \tau(z)] \geq 1 - c^{-1}\left(\frac{|z|}{R}\right)^{\alpha}$ and since $R > |z|$ is arbitrary, it follows that $\mathsf{P}[\tau(z) < \infty] = 1$. ☐

Proposition 5.8. *When $\kappa \in (0,4]$, for any $z \in \mathbb{H}$, $\tau(z) = \infty$ almost surely .*

Proof. The claim follows from Proposition 5.6, since $K_t = \gamma[0,t]$. ☐

5.3.6 One-point function of SLE(κ)

5.3.6.1 The behavior of θ_t as t tends to $\tau(z)$

Lemma 5.2 **(**$\lim\arg Z_t$ **for simple, transient** γ**).** *Let $z \in \mathbb{H}$ and γ be a simple curve in $\mathbb{H}$ such that $\gamma(0) \in \mathbb{H}$, $\gamma(0,\infty) \subset \mathbb{H}$, $z \notin \gamma(0,\infty)$ and $\lim_{t\to\infty}|\gamma(t)| = \infty$. Let $(g_t)_{t\in\mathbb{R}_{\geq 0}}$ be the Loewner chain corresponding to γ with the driving term $(W_t)_{t\in\mathbb{R}_{\geq 0}}$. Then $\lim_{t\to\infty}\arg(g_t(z) - W_t) \in \{0, 2\pi\}$.*

Proof. Let $r = |z|$ and $R > r$. By the assumptions, there exists $s \in \mathbb{R}_{\geq 0}$ such that $|\gamma(t)| \geq R$ for all $t \geq s$. By symmetry, we can suppose that z is to the right of $\gamma[0,s]$ in the sense that z can be connected by a path in $(\mathbb{H} \setminus \gamma[0,s]) \cap B(0,r)$ to the "right

[11] Here and below z^{α} is defined as $e^{\alpha \log z}$ where the branch of log is such that $\mathrm{Im}\log z \in [0,\pi]$ for $z \in \overline{\mathbb{H}}$.

side" of $\gamma[0,s]$. Notice that then z is to the right of $\gamma[0,t]$ in the same sense for all $t \geq s$.

Denote by L_t the union of $\mathbb{R}_{<0}$ and the "left side" of $\gamma[0,t]$. We can write using the harmonic measure $\arg(g_t(z)-W_t) = \pi\,\mathrm{HM}(z,L_t,\mathbb{H}\setminus\gamma(0,t])$ (recall Definition 3.3 and remarks after it). By the weak Beurling estimate of the harmonic measure, $0 \leq \arg(g_t(z)-W_t) \leq C(r/R)^{\alpha}$ with some universal constants $C>0$ and $\alpha>0$. The claim follows by taking R to ∞.

The proof of the statement, that $\arg(g_t(z)-W_t)$ tends to π, when z is to the left of $\gamma[0,s]$, can be done completely symmetrically. □

Lemma 5.3 **(**$\lim \arg Z_t$ **when point is swallowed and not hit).** *Let* $z \in \mathbb{H}$ *and* $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ *be a Loewner chain generated by a curve* γ*. Suppose that* $\tau(z)<\infty$ *and* $\mathrm{dist}(z,\gamma[0,\tau(z)])>0$*. Then* $\lim_{t\to\tau(z)} \arg(g_t(z)-W_t) \in \{0,2\pi\}$*.*

Proof. The proof is very similar to the proof of Lemma 5.2. When z is to the right of $\gamma[0,\tau(z)]$, then the harmonic measure of the union of $\mathbb{R}_{<0}$ and the left-hand side of $\gamma[0,t]$ tends to zero as t tends to $\tau(z)$. Similarly when z is to the left of $\gamma[0,\tau(z)]$. □

If we combine Lemmas 5.2 and 5.3 with Propositions 5.3, 5.5, 5.6 and 5.7, we see that the inequality (5.17) is actually equality for SLE(κ). Namely, we can deduce in the following way.

- When $\kappa \in (0,4]$, γ is simple, transient and avoids the point z almost surely. Thus by Lemma 5.2, $\lim_{t\to\infty}\theta_t \in \{0,\pi\}$ and consequently $S(\infty)=\hat{\tau}<\infty$.
- When $\kappa \in (4,8)$, similarly using Lemma 5.3 it follows that $\lim_{t\to\tau(z)}\theta_t \in \{0,\pi\}$ and consequently $S(\tau(z))=\hat{\tau}<\infty$.
- When $\kappa \in [8,\infty)$, $\tau(z)<\infty$ and $\hat{\tau}=\infty$. If it would happen that $S(\tau(z))<\infty$, then by Lemma 5.3, $\lim_{t\to\tau(z)}\theta_t \in \{0,\pi\}$ and therefore $\hat{\tau}\leq S(\tau(z))<\infty$, which would lead to a contradiction. Consequently, $\hat{\tau}=S(\tau(z))=\infty$.

We have shown the next result. The notion $(U)_{z_0}$ is used for the connected component of z_0 in U.

Proposition 5.9. *For all* κ*, it holds that* $\rho(z_0,(\mathbb{H}\setminus\gamma(0,\infty))_{z_0}) = 2y_0 e^{-\hat{\tau}}$ *where* $\hat{\tau}$ *is as in* (5.16)*.*

5.3.6.2 One-point function

Let's continue the calculation of Section 5.3.4 using Proposition 5.9. Write $z_0 = r\exp(\mathrm{i}\hat{\theta}_0/2)$, $r>0$. Define a function, which doesn't depend on $r>0$,

$$F(\hat{\theta}_0,u) = \mathsf{P}\left[\rho(z_0,(\mathbb{H}\setminus\gamma(0,\infty))_{z_0}) \leq 2y_0 e^{-u}\right] = \mathsf{P}[\hat{\tau}\geq u].$$

Its conditional version given $\hat{\mathscr{F}}_s = \mathscr{F}_{\sigma(s)}$, can be written in the form

$$\mathsf{P}\left[\hat{\tau}\geq u \,\middle|\, \hat{\mathscr{F}}_s\right] = F(\hat{\theta}_s, u-s) \tag{5.23}$$

by the conformal Markov property of SLE(κ).

The left-hand side of (5.23) is by construction a martingale and therefore F satisfies, provided F is smooth

$$\frac{\partial F}{\partial t} = LF \tag{5.24}$$

by Itô's formula, where L is the second order differential operator

$$L = -\frac{\kappa}{2}\frac{\partial^2}{\partial x^2} - \frac{\kappa-4}{2}\cot\frac{x}{2}\frac{\partial}{\partial x}.$$

The function F satisfies the boundary conditions

$$F(x,0) = 1, \quad 0 < x < 2\pi \qquad \text{and} \qquad F(0,u) = 0 = F(2\pi,u), \quad u > 0. \tag{5.25}$$

In a suitable function space L is a self-adjoint operator. Moreover, there exists a eigenbasis $(f_k)_{k\in\mathbb{N}}$ of L such that $Lf_k = \lambda_k f_k$, $0 < \lambda_1 < \lambda_2 < \ldots$ and for each k, f_k has $k-1$ zeros on the interval $(0,2\pi)$. It is straightforward to check that

$$f(x) = \sin\left(\frac{x}{2}\right)^{\beta}$$

satisfies $Lf = \lambda f$ if and only if $\lambda = 1 - \frac{\kappa}{8}$, $\beta = \frac{8}{\kappa} - 1$. Since this eigenfunction is positive and thus doesn't have any zeros, it must be the eigenfunction with the smallest eigenvalue. Consequently, by an argument that we will skip, since the boundary values (5.25) are non-negative, it is possible to prove the following version of the maximum principle: there exists a constant $C > 0$ such that

$$C^{-1} f(x) e^{-\lambda u} \leq F(x,u) \leq C f(x) e^{-\lambda u}$$

for all $x \in [0,2\pi]$ and $u \geq 1$.

This shows that for all $z_0 \in \mathbb{H}$ and $r \in (0, \operatorname{Im} z_0)$, it holds that

$$\mathsf{P}\left[\overline{\gamma[0,\infty)} \cap \overline{B(z_0,r)} \neq \emptyset\right] \asymp \left(\frac{r}{\operatorname{Im} z_0}\right)^{\lambda} \sin\left(\arg z_0\right)^{\beta} \tag{5.26}$$

where $A \asymp B$ means that there exist positive constants c_1 and c_2 such that $c_1 B \leq A \leq c_2 B$. We call the right-hand side of (5.26) the *one-point function of SLE*(κ).

5.3.6.3 An upper bound for the dimension of SLE

For a non-empty bounded Borel set $K \subset \mathbb{C}$, let N_ε be the number of sets of the form $[(j-1)\varepsilon, j\varepsilon] \times [(k-1)\varepsilon, k\varepsilon]$, where $(j,k) \in \mathbb{Z}^2$, that intersect the set K. We define the *upper and lower box-counting dimensions* (or Minkowski dimensions) as

$$\dim_{\overline{\mathrm{M}}}(K) = \limsup_{\varepsilon \searrow 0} \frac{\log N_\varepsilon}{\log\frac{1}{\varepsilon}}, \qquad \dim_{\underline{\mathrm{M}}}(K) = \liminf_{\varepsilon \searrow 0} \frac{\log N_\varepsilon}{\log\frac{1}{\varepsilon}},$$

respectively. If the upper and lower limits are equal, then $\lim_{\varepsilon\searrow 0}(\log N_\varepsilon)/(\log \frac{1}{\varepsilon})$ exists and is equal to the *box-counting dimension* $\dim_{\mathrm{M}}(K) = \dim_{\overline{\mathrm{M}}}(K) = \dim_{\underline{\mathrm{M}}}(K)$ of K. It always holds that the (upper and lower) box-counting dimension is not less than the Hausdorff dimension of K. Hence any upper bound for the box-counting dimension is an upper bound for the Hausdorff dimension.

Consider now SLE(κ), $0 < \kappa < 8$, curve γ. By (5.26), for some $C > 0$ and $\lambda > 0$, the left-hand side of (5.26) is bounded from above by $C\,(\frac{r}{\operatorname{Im} z_0})^\lambda$ for all $z_0 \in \mathbb{H}$ and $r > 0$. If $\gamma[0,\infty)$ intersects the box $R_{j,k} := [(j-1)\,2^{-n}, j\,2^{-n}] \times [(k-1)\,2^{-n}, k\,2^{-n}]$, then $\operatorname{dist}\left(\left(j-\frac{1}{2}\right)2^{-n} + i\left(k-\frac{1}{2}\right)2^{-n}, \gamma[0,\infty)\right) \le 2^{-n-1/2}$. Hence

$$\begin{aligned} \mathsf{E} N_{2^{-n}} &= \sum_{\substack{-2^n < j \le 2^n \\ 0 < k \le 2^n}} \mathsf{P}\left[\gamma[0,\infty) \cap R_{j,k} \neq \emptyset\right] \le C\, 2^{-\lambda/2} \sum_{\substack{-2^n < j \le 2^n \\ 0 < k \le 2^n}} (k-1/2)^{-\lambda} \\ &\le C' 2^{(2-\lambda)n} \end{aligned}$$

where C' is a constant that depends only on C and λ.

By Chebyshev inequality, for each $\delta > 0$

$$\mathsf{P}\left[N_{2^{-n}} \ge 2^{(2-\lambda+\delta)n}\right] \le C' 2^{-\delta n}.$$

Since these probabilities are summable over n, by Borel–Cantelli lemma there exist a random variable $n_0(\delta)$ such that $N_{2^{-n}} < 2^{(2-\lambda+\delta)n}$ for $n > n_0(\delta)$. Thus $\limsup_{n\to\infty}(\log N_{2^{-n}})/(n \log 2) \le 2 - \lambda + \delta$. Since $\delta > 0$ is arbitrary and $\varepsilon \mapsto N_\varepsilon$ is non-decreasing, it follows that $\limsup_{\varepsilon\searrow 0}(\log N_\varepsilon)/(\log \frac{1}{\varepsilon}) \le 2 - \lambda$ and therefore the upper box-counting dimension is at most $2 - \lambda$. Thus we have shown

$$\dim_{\overline{\mathrm{M}}}(([-1,1] \times [0,1]) \cap \gamma[0,\infty)) \le 1 + \frac{\kappa}{8}. \tag{5.27}$$

which implies that $\dim_{\mathscr{H}}(\gamma[0,\infty)) \le 1 + \frac{\kappa}{8}$. Remember that almost surely $\gamma[0,\infty) = \overline{\mathbb{H}}$ and thus $\dim_{\mathscr{H}}(\gamma[0,\infty)) = 2$, when $\kappa \ge 8$.

5.4 Variants of SLE

In this section, we will go through some variants of SLE. Besides the chordal SLE(κ), the most important variants of SLE are the radial SLE(κ), the dipolar SLE$[\kappa,\alpha]$ and (chordal) SLE(κ,ρ). We will also remind about the Loewner equations for the inverse maps $f_t = g_t^{-1}$ and the time-reversed maps $h_t = g_{T-t}$, because they are needed in later sections.

5.4.1 *Radial SLE*(κ)

Definition 5.4. *Radial SLE*(κ) *in* $(\mathbb{D},1,0)$ is a stochastic d-Loewner chain $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ driven by the process $W_t = \exp(\mathrm{i}\sqrt{\kappa}B_t)$ where $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion.

Remark 5.6. As we commented in Remark 4.2, the $\mathbb{H}$-capacity parametrization is consistent with the d-capacity parametrization in the sense that choral and radial SLE(κ)'s look the same locally. We leave this argument as an exercise.

Definition 5.5. Let (U,a,w) be a triplet where U is a simply connected domain, a its boundary point and w its interior point. *Radial SLE*(κ) *in* (U,a,w) is defined as the conformal image of radial SLE in the domain $(\mathbb{D},1,0)$ under the (unique) conformal map that takes $(\mathbb{D},1,0)$ to (U,a,w).

Remark 5.7. For fixed $\kappa > 0$, the family of laws of radial SLE(κ) in (U,a,w), where (U,a,w) runs over all simply connected domains $U \neq \mathbb{C}$ as well as all $a \in \partial U$ and $w \in U$, satisfies a version of *Schramm's principle* for a domain with a marked boundary point and a marked interior point. Namely, in this version Schramm's principle, the family $(\mu^{(U,a,w)})$ of laws of random curves satisfies

- $\mu^{(U,a,w)}$ is supported on curves $\gamma(t)$ starting at a and tending to w as $t \to \infty$. The curve γ is well-described by the Loewner equation of $\mathbb{D}$.
- **Conformal invariance** (CI): $\phi_*\mu^{(U,a,w)} = \mu^{(\phi(U),\phi(a),\phi(w))}$.
- **Domain Markov property** (DMP): for any measurable set B in the space of curves, $\mu^{(U,a,w)}[\gamma|_{[t,\infty)} \in B \mid \mathscr{F}_t] = \mu^{(U\setminus\gamma([0,t]),\gamma(t),w)}[\gamma \in B]$.

Unlike in the chordal case, from the point of view of Schramm's principle, here it would be reasonable to include a linear drift to the Brownian motion and extend the definition of radial SLE to two parameter family. It would be interpreted as chirality of the random curve as the non-zero drift would give the curve tendency to swirl around the marked interior point to the direction specified by the sign of the drift.

5.4.2 *Dipolar SLE*$[\kappa,\alpha]$

Definition 5.6. *Dipolar SLE*$[\kappa,\alpha]$ *in* $(\mathbb{S}_\pi,0,+\infty,-\infty)$ is a stochastic s-Loewner chain $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ driven by the process $W_t = \sqrt{\kappa}B_t + \alpha t$ where $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion.

Definition 5.7. Let (U,a,b,c) be a quadruplet where U is a simply connected domain and a,b,c its distinct boundary points in counterclockwise order. *Dipolar SLE*$[\kappa,\alpha]$ *in* (U,a,b,c) is defined as the conformal image of dipolar SLE$[\kappa,\alpha]$ of the domain $(\mathbb{S}_\pi,0,+\infty,-\infty)$ under the (unique) conformal map that takes the quadruplet $(\mathbb{S}_\pi,0,+\infty,-\infty)$ to (U,a,b,c).

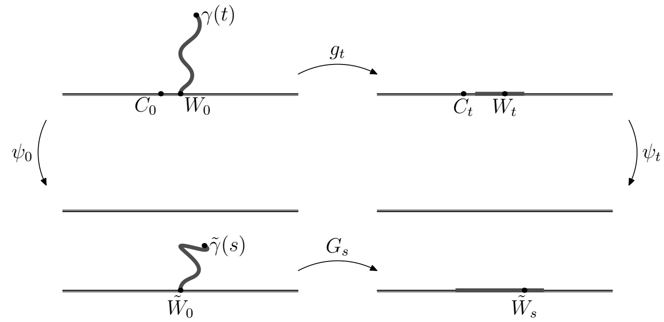

Fig. 5.4 Coordinate transform of a SLE-type process from $\mathbb{H}$ to $\mathbb{S}_\pi$

Remark 5.8. For fixed $\kappa > 0$ and $\alpha \in \mathbb{R}$, the family of laws of dipolar SLE$[\kappa, \alpha]$ in (U,a,b,c), where (U,a,b,c) runs over all simply connected domains $U \neq \mathbb{C}$ as well as all $a,b,c \in \partial U$, satisfies a version of *Schramm's principle* for a domain with distinct marked boundary points. Namely, in this version Schramm's principle, the family $(\mu^{(U,a,b,c)})$ of laws of random curves satisfies

- $\mu^{(U,a,w)}$ is supported on curves $\gamma(t)$ starting at a and tending to the boundary arc between b and c as $t \to \infty$. The curve γ is well-described by the Loewner equation of $\mathbb{S}_\pi$.
- **Conformal invariance** (CI): $\phi_* \mu^{(U,a,b,c)} = \mu^{(\phi(U),\phi(a),\phi(b),\phi(c))}$.
- **Domain Markov property** (DMP): for any measurable set B in the space of curves, $\mu^{(U,a,b,c)}[\,\gamma|_{[t,\infty)} \in B \mid \mathscr{F}_t\,] = \mu^{(U\setminus\gamma([0,t]),\gamma(t),b,c)}[\gamma \in B]$.

5.4.3 Coordinate changes and SLE(κ,ρ)

5.4.3.1 Coordinate transform of SLE(κ) from $\mathbb{H}$ to $\mathbb{S}_\pi$

Let's consider SLE(κ) on $\mathbb{H}$ and a conformal transformation from $\mathbb{H}$ onto the strip $\mathbb{S}_\pi = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$ Let $c < 0$. The unique conformal map ψ_0 from with $\psi_0(0) = 0$, $\psi_0(c) = -\infty$ and $\psi_0(\infty) = +\infty$ is given by $\psi_0(z) = \log(z-c) - \log|c|$. The point c evolves under the SLE(κ) flow as $C_t = g_t(c)$. Let $\psi_t(z) = \log(z - C_t) + \delta(t)$ where $\delta(t)$ is a constant such that the map

$$\hat{g}_t = \psi_t \circ g_t \circ \psi_0^{-1} \tag{5.28}$$

satisfies the normalization introduced in Section 4.3.2. After a calculation we find that $-\frac{1}{2}\log g_t'(c) - \log|c|$. Consequently, the s-capacity $S(t)$ of the s-hull $\psi_0(K_t)$ can be written as

$$S(t) = -\frac{1}{2}\log g_t'(c) = \int_0^t \frac{\mathrm{d}u}{(W_t - C_t)^2},$$

and the driving term transforms to

$$\hat{W}_t = \log(W_t - C_t) + S(t) - \log|c| \in \mathbb{R} \subset \partial\mathbb{S}_\pi.$$

Define a time-change $\sigma = S^{-1}$ and set $\tilde{g}_s = \hat{g}_{\sigma(s)}$ and $\tilde{W}_s = \hat{W}_{\sigma(s)}$. A straightforward calculation shows that $\tilde{g}_s$ satisfies the Loewner equation (4.23) of the strip $\mathbb{S}_\pi$. If the driving term in the upper half-plane is a Brownian motion then the driving term of the strip is a Brownian motion with a drift. See Table 5.1 for more details.

	$\mathbb{H}$	$\mathbb{S}_\pi$
Normalization	$g_t(z) = z + \frac{2t}{z} + \ldots,\ z \to \infty$	$\tilde{g}_s(z) = \begin{cases} z - s + o(1), & z \to -\infty \\ z + s + o(1), & z \to +\infty \end{cases}$
Loewner equation	$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$	$\partial_s \tilde{g}_s(z) = \coth \frac{\tilde{g}_s(z) - \tilde{W}_s}{2}$
Driving term	$W_t \in \mathbb{R}$	$\tilde{W}_s \in \mathbb{R}$
Chordal SLE(κ)	$W_t = \sqrt{\kappa} B_t$	$\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha_0(\kappa) s$
SLE(κ,ρ) and Dipolar SLE[κ,α]	$\begin{cases} \mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \dfrac{\rho}{W_t - C_t}\mathrm{d}t \\ \mathrm{d}C_t = \dfrac{2}{C_t - W_t}\mathrm{d}t \end{cases}$	$\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha s$
Relations between parameters	$\alpha = \rho + 3 - \frac{\kappa}{2}, \qquad \alpha_0(\kappa) = 3 - \frac{\kappa}{2}$	

Table 5.1 A comparison between SLE in $\mathbb{H}$ and in $\mathbb{S}_\pi$. For clarity we use separate notations for quantities in $\mathbb{H}$ and in $\mathbb{S}_\pi$.

5.4.3.2 SLE(κ,ρ) and coordinate transform of dipolar SLE$[\kappa,\alpha]$

Definition 5.8. Let $\kappa \geq 0$ and $\rho \in \mathbb{R}$. Let $w_0, c_0 \in \mathbb{R}$ with $w_0 \neq c_0$ and let $(W_t, C_t)_{t\in[0,\tau(c_0))}$ be the solution to the system of stochastic differential equations

$$\begin{cases} \mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \dfrac{\rho}{W_t - C_t}\mathrm{d}t \\ \mathrm{d}C_t = \dfrac{2}{C_t - W_t}\mathrm{d}t \end{cases}, \qquad \begin{cases} W_0 = w_0 \\ C_0 = c_0 \end{cases} \tag{5.29}$$

which exists for $t \in [0, \tau(c_0))$ where $\tau(c_0) = \sup\{t \in \mathbb{R}_{\geq 0} : \inf_{s\in[0,t]} |W_t - C_t| > 0\}$. Then the Loewner chain $(g_t, K_t)_{t\in[0,\tau(c_0))}$ with the driving process $(W_t)_{t\in[0,\tau(c_0))}$ is called *SLE*(κ,ρ).

Remark 5.9. The chordal SLE(κ) is a special case SLE$(\kappa, 0)$ of this definition.

Remark 5.10. It is possible to construct SLE(κ,ρ) using a Bessel process. This construction is especially useful when we want to consider the process beyond $\tau(c_0)$, which we don't do in this text, but we'll give this construction here. Let $w_0, c_0 \in \mathbb{R}$

with $w_0 \neq c_0$ and let $\eta = \operatorname{sgn}(w_0 - c_0)$. Let D_t be the Bessel process (with an unusual time-parametrization)

$$\mathrm{d}D_t = \frac{\rho+2}{D_t}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}\tilde{B}_t, \qquad D_0 = |w_0 - c_0|.$$

Define

$$C_t = c_0 - 2\eta \int_0^t \frac{\mathrm{d}u}{D_u}, \qquad W_t = C_t + \eta D_t.$$

Then they satisfy (5.29) with $B_t = \eta \tilde{B}_t$.

Definition 5.9. Let (U,a,b,c) be a quadruplet where U is a simply connected domain and a,b,c its distinct boundary points in counterclockwise order. *SLE*(κ,ρ) *in* (U,a,b,c) is defined as the conformal image of SLE(κ,ρ) of the domain $(\mathbb{H},0,\infty,-1)$ under the (unique) conformal map that takes $(\mathbb{H},0,\infty,-1)$ to (U,a,b,c).

The above coordinate change calculation shows the following result, which is also in the last row of Table 5.1.

Lemma 5.4. *When $\alpha = \rho + 3 - \frac{\kappa}{2}$, the dipolar SLE$[\kappa,\alpha]$ and SLE(κ,ρ) in the domain (U,a,b,c) considered up to time of disconnection are equal (up to a time change) in distribution.*

5.4.4 Special parameters values of SLE(κ,ρ)

5.4.4.1 SLE$(\kappa,(\kappa-6)/2)$ is symmetric

Denote the reflection with respect to the y-axis by

$$m(z) = -\bar{z}. \tag{5.30}$$

Then m is an antiholomorphic map from $\mathbb{C}$ onto itself. Since the process $\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s$ is invariant under $\tilde{W}_s \mapsto -\tilde{W}_s$, SLE$(\kappa,(\kappa-6)/2)$ on $\mathbb{S}_\pi$ is invariant under m and for fixed $\kappa > 0$, it is the unique SLE(κ,ρ) process with this property. We say that SLE$(\kappa,(\kappa-6)/2)$ on $\mathbb{S}_\pi$ is *symmetric*.

Suppose that we know that some discrete random curve arising from statistical physics converges to SLE(κ) as the mesh goes to zero. For example, suppose we know that the interface of Ising model with boundary conditions changing at two marked points (boundary conditions are $+$ spins on one arc and $-$ spins on the other arc) converges to SLE(3). Can we conclude something about the scaling limit for other boundary conditions? If we consider the Ising model with three marked points $a,b,c \in \partial U$ (in counterclockwise order), instead, and boundary conditions are set to be $-$ on the arc ab, $+$ on the arc ca and free on the arc bc, then by Schramm's principle we expect that the scaling limit of the interface starting from the point a should be SLE$(3,\rho)$ process. And since the law of that interface is invariant under flipping all the spins $\sigma \to -\sigma$, the scaling limit should be symmetric on $\mathbb{S}_\pi$ and hence it should be SLE$(3,-3/2)$.

5.4.4.2 SLE(6) satisfies locality

Consider following map

$$\psi = m \circ \phi^{-1} \circ m \circ \phi \tag{5.31}$$

where m is as in (5.30). Under those maps SLE(κ) is transformed as

$$(\mathbb{H}, \kappa = 6, \rho = 0) \xrightarrow{\phi} (\mathbb{S}_\pi, \kappa = 6, \alpha = 0) \xrightarrow{m} (\mathbb{S}_\pi, \kappa = 6, \alpha = 0) \\ \xrightarrow{\phi^{-1}} (\mathbb{H}, \kappa = 6, \rho = 0) \xrightarrow{m} (\mathbb{H}, \kappa = 6, \rho = 0)\,.$$

On the other hand ψ is a holomorphic and bijective self map of $\mathbb{H}$ with $\psi(0) = 0$, $\psi(\infty) = \infty$ and $\psi(c) = |c|$. Hence $\psi(z) = \frac{|c|z}{z-c}$. Therefore SLE(6) has the following *locality* property: the image of SLE(6) under any conformal self-map of $\mathbb{H}$ is again (a time-change of) SLE(6). If $\psi : \mathbb{H} \to \mathbb{H}$ is this Möbius map, then we consider the first process until it disconnects $\psi^{-1}(\infty)$ from ∞ and the second one until it disconnects $\psi(\infty)$ from ∞. Actually SLE(6) has even stronger locality property because SLE(6) sent from 0 is invariant up to a time-change under any conformal transformation defined in a neighborhood of 0 such that it maps a neighborhood of 0 in $\mathbb{R}$ into $\mathbb{R}$.

5.4.4.3 SLE($\kappa, \kappa-6$) is target independent

For other values of κ, the argument of the section 5.4.4.2 gives that if $(K_t)_{t\in[0,\tau(c))}$ is a chordal SLE(κ) stopped at the time $\tau(c)$ then $(\psi(K_t))_{t\in[0,\tau(c))}$ is a time-change of the SLE($\kappa, \kappa-6$) process stopped at the time when the process disconnects $|c|$ from ∞. Namely, under the map ψ of the form (5.31) the processes are transformed in the following way:

$$(\mathbb{H}, \kappa, \rho = 0) \xrightarrow{\phi} (\mathbb{S}_\pi, \kappa, \alpha = 3 - \kappa/2) \xrightarrow{m} (\mathbb{S}_\pi, \kappa, \alpha = \kappa/2 - 3) \\ \xrightarrow{\phi^{-1}} (\mathbb{H}, \kappa, \rho = \kappa - 6) \xrightarrow{m} (\mathbb{H}, \kappa, \rho = \kappa - 6)\,.$$

5.5 Moments of the derivative of the Loewner map of SLE(κ)

We will present in this section the auxiliary results needed for the proof of Theorem 5.2.

Continue the setup of Lemma 4.10 and set $h_t(z) = \tilde{h}_t(z + W_T) - W_T$. Then by a straightforward calculation, h_t satisfies the reverse Loewner equation with a driving term $V_t = W_{T-t} - W_t$. This observation leads to the following "symmetry" of the chordal SLE(κ).

Lemma 5.5. *Let $h_t(z)$ be the solution of* (4.19) *for $V_t = \sqrt{\kappa} B_t$ and let $f_t(z)$ be the solution of* (4.13) *for $W_t = \sqrt{\kappa} B_t$. Then for any $t \in \mathbb{R}_{\geq 0}$, the functions $z \mapsto f_t(z + W_t) - W_t$ and $z \mapsto h_t(z)$ have the same distribution. In particular, $f_t'(z+W_t)$ has the same distribution as $h_t'(z)$.*

Remark 5.11. This result holds only for a single time instant. It is not true that the joint law of $(f_t(z+W_t) - W_t)_{t \in \mathbb{R}_{\geq 0}}$, is the same as the joint law of $(h_t(z))_{t \in \mathbb{R}_{\geq 0}}$.

It is useful to define

$$\tilde{f}_t(z) = f_t(z + W_t). \tag{5.32}$$

The goal of this section is to have good bounds for $|\tilde{f}_t'(iy)|$, $t \in [0,1], y \in (0,1]$. The proof of the theorem will be given in Section 6.2 below. We follow here [8, 10].

Let's deal with the forward and reverse Schramm–Loewner evolution at the same time by fixing $\nu = \pm 1$ and letting $h_t(z)$ be the solution of the equation

$$\partial_t h_t(z) = \nu \frac{2}{h_t(z) - W_t}, \qquad h_0(z) = z$$

where $W_t = -\sqrt{\kappa} B_t$. For fixed $z_0 = x_0 + iy_0 \in \mathbb{H}$, let $Z_t = h_t(z_0) - W_t$ and let X_t and Y_t be the real and imaginary parts of Z_t, respectively. Let's list some useful formulas

$$\begin{gathered}
\mathrm{d}X_t = 2\nu \frac{X_t}{X_t^2 + Y_t^2} \mathrm{d}t + \sqrt{\kappa} \mathrm{d}B_t, \qquad \partial_t Y_t = -2\nu \frac{Y_t}{X_t^2 + Y_t^2}, \\
\partial_t |h_t'(z_0)| = -2\nu |h_t'(z_0)| \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2}, \qquad \partial_t \frac{|h_t'(z_0)|}{Y_t} = 4\nu \frac{|h_t'(z_0)|}{Y_t} \frac{Y_t^2}{(X_t^2 + Y_t^2)^2}, \\
\mathrm{d} \arg Z_t = (\kappa - 4\nu) \frac{X_t Y_t}{(X_t^2 + Y_t^2)^2} \mathrm{d}t - \sqrt{\kappa} \frac{Y_t}{X_t^2 + Y_t^2} \mathrm{d}B_t, \\
\mathrm{d} \log |Z_t| = -\frac{1}{2} (\kappa - 4\nu) \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2} \mathrm{d}t + \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2} \mathrm{d}B_t, \\
\mathrm{d} \sin \arg Z_t = (\sin \arg Z_t) \left[\frac{(\kappa - 4\nu) X_t^2 - \frac{\kappa}{2} Y_t^2}{(X_t^2 + Y_t^2)^2} \mathrm{d}t - \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2} \mathrm{d}B_t \right].
\end{gathered}$$

They are mostly familiar from Section 5.3 and we leave as an exercise to verify these formulas using Itô's formula. Now we fix $\nu = -1$. Then all the processes above are well-defined for all $t \in \mathbb{R}_{\geq 0}$.

Let $p, q, r \in \mathbb{R}$ and define $M_t = |h_t'(z_0)|^p \, Y_t^q \, (\sin \arg Z_t)^{-2r}$. By Itô's formula,

$$\begin{aligned}
\mathrm{d}M_t = M_t \Bigg(& 2p \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2} + 2q \frac{Y_t}{X_t^2 + Y_t^2} - 2r \frac{(\kappa + 4) X_t^2 - \frac{\kappa}{2} Y_t^2}{(X_t^2 + Y_t^2)^2} \\
& + r(2r+1) \frac{\kappa X_t^2}{(X_t^2 + Y_t^2)^2} \Bigg) \mathrm{d}t - 2r\sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2} M_t \mathrm{d}B_t
\end{aligned} \tag{5.33}$$

Therefore M_t is a local martingale if and only if $q = p - \frac{\kappa}{2} r$, and $r^2 - \left(1 + \frac{4}{\kappa}\right) r + \frac{2}{\kappa} p = 0$. In that case (5.33) simplifies to

$$\mathrm{d}M_t = -2r\sqrt{\kappa}\,\frac{X_t}{X_t^2 + Y_t^2}\,M_t\,\mathrm{d}B_t.$$

Next we define a time change that simplifies the above formula. Let

$$S(t) = \int_0^t \frac{\mathrm{d}u}{X_u^2 + Y_u^2}, \qquad \sigma(s) = S^{-1}(s) \tag{5.34}$$

and $\hat{\mathscr{F}}_s = \mathscr{F}_{\sigma(s)}$ and use Proposition 2.8. Then

$$\hat{B}_s = \int_0^{\sigma(s)} \frac{\mathrm{d}B_u}{\sqrt{X_u^2 + Y_u^2}}$$

is a standard one-dimensional Brownian motion with respect to the filtration $(\hat{\mathscr{F}}_s)_{s \in \mathbb{R}_{\geq 0}}$. Denote the time-changed processes by

$$\hat{Z}_s = Z_{\sigma(s)}, \qquad \hat{X}_s = X_{\sigma(s)}, \qquad \hat{Y}_s = Y_{\sigma(s)}, \qquad \hat{h}_s(z_0) = h_{\sigma(s)}(z_0).$$

Notice that the equations

$$\partial_s \hat{Y}_s = 2\hat{Y}_s, \qquad \partial_s \frac{|\hat{h}_s'(z_0)|}{\hat{Y}_s} = -4\frac{|\hat{h}_s'(z_0)|}{\hat{Y}_s}(\sin\arg\hat{Z}_s)^2$$

hold and therefore

$$\hat{Y}_s = y_0 e^{2s} \tag{5.35}$$

$$|\hat{h}_s'(z_0)| = \exp\left(2s - 4\int_0^s (\sin\arg\hat{Z}_u)^2 \mathrm{d}u\right). \tag{5.36}$$

Hence (5.35) shows that the time change (5.34) is such that $\hat{Y}_s$ is deterministically exponentially increasing. The equation (5.36) implies that

$$e^{-2s'} \leq \frac{|\hat{h}_{s+s'}'(z_0)|}{|\hat{h}_s'(z_0)|} \leq e^{2s'}. \tag{5.37}$$

Observe also that $Y_t \leq \sqrt{y_0^2 + 4t}$ This shows that $y_0 e^{2s} \leq \sqrt{y_0^2 + 4\sigma(s)}$ and hence $\sigma(s) \to \infty$ as $s \to \infty$.

Under this time-change, the local martingale $\hat{M}_s = M_{\sigma(s)}$ satisfies

$$\mathrm{d}\hat{M}_s = -2r\sqrt{\kappa}\,(\cos\arg\hat{Z}_s)\,\hat{M}_s\,\mathrm{d}B_s.$$

It is not hard to show that $(\hat{M}_s)_{s \in \mathbb{R}_{\geq 0}}$ is a martingale.

Lemma 5.6. *Let N_0 be a constant and let $(N_t)_{t\in\mathbb{R}_{\geq 0}}$ be a local martingale with*

$$N_t = N_0 + \int_0^t A_s N_s \mathrm{d}B_s.$$

If for every $t>0$ there is a constant $c(t)$ such that $|A_s| \leq c(t)$ for all $s \in [0,t]$, then N_t is a martingale.

Proof. Let $M_t = N_t - N_0$. Then $M_t = \int_0^t (A_s M_s + A_s N_0)\mathrm{d}B_s$. Let $n \in \mathbb{N}$ and define $T = \inf\{t \in \mathbb{R}_{\geq 0} : \langle M \rangle_t = n\}$. Then $M_{t\wedge T}$ is an Itô integral with a $\mathscr{L}^2$ integrand. Define $f(t) = \mathsf{E}[M_{t\wedge T}^2]$. By the Itô isometry, $f(t) = \mathsf{E}[\int_0^t (A_s M_s + A_s N_0)^2 \mathbb{1}_{s\leq T} \mathrm{d}s]$. Therefore for any $t' \in [0,t]$

$$f(t') \leq 2c(t)^2 N_0^2 t' + 2c(t)^2 \int_0^{t'} f(s)\,\mathrm{d}s \leq \tilde{c}(t)\,t' + \tilde{c}(t) \int_0^{t'} f(s)\,\mathrm{d}s \tag{5.38}$$

where $\tilde{c}(t) = 2c(t)^2 \max\{1, N_0^2\}$. This implies that $f(t') < \exp(2\tilde{c}(t)t')$ because no $t' \in [0,t]$ can be the smallest s such that $f(s) \geq \exp(2\tilde{c}(t)s)$ by (5.38). Therefore

$$\mathsf{E}[\langle N \rangle_{t\wedge T}] = \mathsf{E}[\langle M \rangle_{t\wedge T}] = f(t) < \exp(2\tilde{c}(t)t)$$

Taking $n \to \infty$ we get by the monotone convergence theorem that $\mathsf{E}[\langle N \rangle_t] \leq \exp(2\tilde{c}(t)t)$. This shows that the integrand $A_t N_t$ is in $\mathscr{L}^2$ and hence by the construction of the Itô integral, N_t is a martingale. □

The next theorem is the main result of this section.

Theorem 5.5. *Let $(p,r) \in \mathbb{R}^2$ be a solution of the equation $r^2 - \left(1+\frac{4}{\kappa}\right) r + \frac{2}{\kappa} p = 0$. Then*

$$\hat{M}_s = |\hat{h}_s'(z_0)|^p \, \hat{Y}_s^{p-\frac{\kappa}{2}r} \, (\sin\arg \hat{Z}_s)^{-2r}$$

is a martingale and

$$\mathsf{E}\left[|\hat{h}_s'(z_0)|^p \,(\sin\arg\hat{Z}_s)^{-2r}\right] = e^{-2s\left(p-\frac{\kappa}{2}r\right)} \left(\frac{y_0}{|z_0|}\right)^{-2r}.$$

Furthermore, if $r \geq 0$ and $p \geq 0$, then

$$\mathsf{P}\left[|\hat{h}_s'(z_0)| \geq \lambda\right] \leq \lambda^{-p} e^{-2s\left(p-\frac{\kappa}{2}r\right)} \left(\frac{y_0}{|z_0|}\right)^{-2r}.$$

Proof. We have already shown the first claim. For the second one notice that $\hat{M}_s = y_0^{p-\frac{\kappa}{2}r} \, e^{2s\left(p-\frac{\kappa}{2}r\right)} \, |\hat{h}_s'(z_0)|^p \,(\sin\arg\hat{Z}_s)^{-2r}$. If $r \geq 0$, then $(\sin\arg\hat{Z}_s)^{-2r} \geq 1$ and the last claim follows from the Chebyshev inequality. □

Corollary 5.1. *Let $\tilde{f}_t$ be defined as in (5.32) for SLE(κ). For every $0 \leq r \leq 1+4/\kappa$, there is a constant $c = c(\kappa, r) < \infty$ such that for all $0 \leq t \leq 1$, $0 < y_0 \leq 1$, $e^6 \leq \lambda \leq y_0^{-1}$,*

$$\mathsf{P}\left[|\tilde{f}_t'(z_0)| \geq \lambda\right] \leq c\lambda^{-p}\left(\frac{y_0}{|z_0|}\right)^{-2r}\delta(y_0,\lambda). \tag{5.39}$$

Here $p = \frac{\kappa}{2}\left(\left(1+\frac{4}{\kappa}\right)r - r^2\right) \geq 0$ *and*

$$\delta(y_0,\lambda) = \begin{cases} \lambda^{-p+\frac{\kappa}{2}r} & \text{when } p - \frac{\kappa}{2}r > 0 \\ 1 + \log\frac{1}{\lambda y_0} & \text{when } p - \frac{\kappa}{2}r = 0 \\ y_0^{p-\frac{\kappa}{2}r} & \text{when } p - \frac{\kappa}{2}r < 0 \end{cases}.$$

Proof. Since $\tilde{f}_t'$ and h_t' have the same distribution, it is enough to show (5.39) when $\tilde{f}_t'$ is replaced by h_t'. Notice first that $Y_t \leq \sqrt{y_0^2+4t} \leq \sqrt{5}$. Therefore

$$\mathsf{P}\left[|h_t'(z_0)| \geq \lambda\right] \leq \mathsf{P}\left[\sup_{0\leq s\leq T}|\hat{h}_s'(z_0)| \geq \lambda\right]$$

where $T = (\log(\sqrt{5}/y_0))/2$. Next notice that by (5.37), $|\hat{h}_{s+s'}'(z_0)| \leq e^{2s'}|\hat{h}_s'(z_0)|$ and therefore

$$\mathsf{P}\left[\sup_{0\leq s\leq T}|\hat{h}_s'(z_0)| \geq \lambda\right] \leq \sum_{j=0}^{\lfloor T\rfloor}\mathsf{P}\left[|\hat{h}_j'(z_0)| \geq e^{-2}\lambda\right]$$

Also by (5.37), $|\hat{h}_s'(z_0)| \leq e^{2s}$ and therefore

$$\begin{aligned}\mathsf{P}\left[\sup_{0\leq s\leq T}|\hat{h}_s'(z_0)| \geq \lambda\right] &\leq \sum_{j=\lceil\log(\lambda)/2-1\rceil}^{\lfloor T\rfloor}\mathsf{P}\left[|\hat{h}_j'(z_0)| \geq e^{-2}\lambda\right] \\ &\leq e^{2p}\lambda^{-p}\left(\frac{y_0}{|z_0|}\right)^{-2r}\sum_{j=\lceil\log(\lambda)/2-1\rceil}^{\lfloor T\rfloor}e^{-2j\left(p-\frac{\kappa}{2}r\right)} \leq c\,\lambda^{-p}\left(\frac{y_0}{|z_0|}\right)^{-2r}\delta(y_0,\lambda).\end{aligned}$$

Here we use that $\sum_{k=n}^{m}\beta^k \leq \beta^n/(1-\beta)$ when $0<\beta<1$ and similar bounds for $\beta = 1$ and $\beta > 1$. □

Next we apply the previous result and optimize over the parameters for fixed κ. Let's parametrize p in terms of r as $p(r) = \frac{\kappa}{2}\left(\left(1+\frac{4}{\kappa}\right)r - r^2\right)$ and study the quantity

$$\alpha(r) = 2p(r) - \frac{\kappa}{2}r = \kappa r\left(\left(\frac{1}{2}+\frac{4}{\kappa}\right) - r\right).$$

Notice that $\alpha(r)$ is maximized by $r_0 = 1/4 + 2/\kappa$ and $\alpha(r_0) = \kappa(\frac{1}{4}+\frac{2}{\kappa})^2 = \frac{\kappa}{16} + 1 + \frac{4}{\kappa} \geq 2$ and $\alpha(r_0) = 2$ only if $\kappa = 8$.

Let $\kappa \neq 8$ and set $p_0 = p(r_0)$. Then $p_0 > \kappa r_0/2$ if $\kappa < 8$ and $p_0 < \kappa r_0/2$ if $\kappa > 8$. Let $\theta \in (0, 1 - \frac{2}{2p_0 - \kappa r_0/2})$. Let $t \in [0,1]$ and $n \in \mathbb{N}$. By the estimate (5.39) for $r = r_0$ and $p = p_0$, we have that for large enough n

$$\begin{aligned}\mathsf{P}\left[\left|\tilde{f}_t'(i2^{-n})\right| \geq 2^{n(1-\theta)}\right] &\leq c\,2^{-p_0(1-\theta)n}\,\delta\left(2^{-n},2^{n(1-\theta)}\right)\\ &= c\,2^{-p_0(1-\theta)n}\times\begin{cases}2^{-(1-\theta)(p_0-\frac{\kappa}{2}r_0)n} & \text{when } \kappa<8\\ 2^{-(p_0-\frac{\kappa}{2}r_0)n} & \text{when } \kappa>8\end{cases}\\ &\leq c\,2^{-(1-\theta)(2p_0-\frac{\kappa}{2}r_0)n} = c2^{-(2+\varepsilon)n}\end{aligned}$$

for some $\varepsilon > 0$. Let $\mathscr{D}_{2n} = \{k\,2^{-2n} : k \in [\![0,2^{2n}]\!]\}$ which is the dyadic partitioning of $[0,1]$ into intervals of length 2^{-2n}. Then

$$\sum_{n\in\mathbb{N}}\sum_{t\in\mathscr{D}_{2n}}\mathsf{P}\left[\left|\tilde{f}_t'(i2^{-n})\right| \geq 2^{n(1-\theta)}\right] < \infty$$

and hence the Borel–Cantelli lemma implies the following result.

Proposition 5.10. *Let $\tilde{f}_t$ be defined as in* (5.32) *for SLE(κ). For each $\kappa \neq 8$ there exists $\theta_0(\kappa) > 0$ such that the following holds: For any $\theta \in (0,\theta_0(\kappa))$, there exists a random variable C such that $C < \infty$ almost surely and*

$$\left|\tilde{f}_t'(i2^{-n})\right| \leq C\,2^{n(1-\theta)} \tag{5.40}$$

for any $t \in \mathscr{D}_{2n}$ and for any $n \in \mathbb{N}$.

Remark 5.12. By the above, we can choose $\theta_0(\kappa) = (\frac{\kappa}{16}+\frac{4}{\kappa}-1)/(\frac{\kappa}{16}+\frac{4}{\kappa}+1)$.

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Chapter 6
Regularity and convergence of random curves

In this chapter, we study regularity properties curves in the capacity parametrization and their convergence with respect to the uniform norm on compact time intervals. Sometimes the term *strong convergence* is used for the convergence under the uniform norm and either given or unspecified parametrization.

When we consider random curves, we are interested, in probability theoretic sense, in so called *weak convergence of probability measures* or equivalently *convergence of random curves in distribution* with respect to the above topology. Before that subject, we aim to prove Theorem 5.2 in Section 6.2. Section 6.1 can be seen as an introduction to Sections 6.2 and 6.3.

6.1 Continuity properties of the Loewner chains

In this section we will switch to the following notation for Loewner chains.

$$W(t), \quad \gamma(t), \quad g(t,z), \text{ etc.} \quad \text{and} \quad W_n(t), \quad \gamma_n(t), \quad g_n(t,z), \text{ etc.} \tag{6.1}$$

This allows us to denote, for instance, a sequence of driving terms by $(W_n(t))_{t\in[0,T_n)}$. Let us also use the notation

$$F(t,y) = f(t, W(t) + \mathrm{i}y). \tag{6.2}$$

The path $y \mapsto F(t,y)$, $y > 0$ is the shortest path in a conformal sense between the point ∞ and the "tip" of K_t, that is, the points $\lim_{y\to\infty} F(t,y)$ and $\lim_{y\to 0} F(t,y)$.

The variable t takes values in $[0,T)$ for $W(t), \gamma(t), g(t,z)$, etc. and in $[0,T_n)$ for $W_n(t), \gamma_n(t), g_n(t,z)$, etc. The variables T and T_n can be finite or infinite. Since we often consider uniform convergence in compact sets of the time variable we often restrict to $t \in [0,T']$ and consider any T' which is finite and less than T or T_n.

We study in this section the dependencies of modes of convergence for Loewner chains, Loewner curves and Loewner driving terms. The most important motivation for this is to clarify the relations of different aspects of Loewner chains. In a later section, that insight is used when studying an example case of a random curve converging to SLE(κ).

6.1.1 Carathéodory kernel convergence

Let us first define what is meant by the convergence of sequences of Loewner chains or simply connected domains.

For a given point z_0 and a sequence of simply connected domains $(\Omega_n)_{n\in\mathbb{Z}_{>0}}$ such that $z_0 \in \Omega_n \neq \mathbb{C}$, define conformal and onto maps $\phi_n : \mathbb{D} \to \Omega_n$ such that $\phi_n(0) = z_0$ and $\phi_n'(0) > 0$.

Definition 6.1. We say that $(\Omega_n)_{n\in\mathbb{Z}_{>0}}$ *converges in the Carathéodory sense* if the sequence $(\phi_n)_{n\in\mathbb{Z}_{>0}}$ converges uniformly on compact subsets of $\mathbb{D}$.

Remark 6.1. Either $\lim_n \phi_n = \text{const.} = w_0$ or $\lim_n \phi_n$ is a conformal map.

Definition 6.2. We say that $((K_n(t))_{t\in[0,T']})_{n\in\mathbb{Z}_{>0}}$ *converges in the Carathéodory sense* if for each compact $G \subset \mathbb{H}$, the sequence $(f_n)_{n\in\mathbb{Z}_{>0}}$ converges uniformly on $[0,T'] \times G$.

The Carathéodory convergence is convenient since it depends on conformal maps, which we can estimate in many ways. The next theorem makes the definition more concrete by giving an equivalent geometric description. See [10] for the proof.

Theorem 6.1 (Carathéodory kernel theorem). *The locally uniform convergence of* $(\phi_n)_{n\in\mathbb{Z}_{>0}}$ *is equivalent to the* kernel convergence *of* $U_n \to U$ *as* $n \to \infty$ *with respect to* w_0 *in the sense that*

- *either* $U = \{w_0\}$ *or* U *is a domain* $\neq \mathbb{C}$ *with* $w_0 \in U$ *such that every* $w \in U$ *has a neighborhood that lies in* U_n *for large enough* n*,*
- *for each* $w \in \partial U$ *there exists a sequence* $w_n \in \partial U_n$ *such that* $w_n \to w$ *as* $n \to \infty$*.*

6.1.2 Continuity properties of the mappings $W \mapsto f$ and $W \mapsto g$

We first present auxiliary lemmas.

Lemma 6.1. *For each* $\delta > 0$ *and* $T > 0$ *there exists a constant* $C(T,\delta)$ *such that the following holds. Let* $h_k(t,z)$*,* $k = 1,2$ *be the solutions of* (4.19) *with the continuous driving terms* $(W_k(t))_{t\in[0,T]}$*,* $k = 1,2$*, respectively. Then they satisfy*

$$|h_1(T,z_1) - h_2(T,z_2)| \le C(T,\delta)(\|W_1 - W_2\|_{\infty,[0,T]} + |z_1 - z_2|) \tag{6.3}$$

for any z_1, z_2 *such that* $\operatorname{Im} z_k > \delta > 0$*.*

Proof. Fix $\delta > 0$, $T > 0$ and $z_k \in \mathbb{H}$, $k = 1,2$, such that $\mathrm{Im}\, z_k > \delta$, $k = 1,2$. Let $h_k(t,z)$, $k = 1,2$ be the solutions of (4.19) with the continuous driving terms $(W_k(t))_{t\in[0,T]}$, which we also consider to be fixed. Write $\psi(t) = h_1(t,z_1) - h_2(t,z_2)$. Then

$$\partial_t \psi(t) = \zeta(t)\,(\psi(t) - D(t))$$

where $\zeta(t) = 2/((h_1(t,z_1) - W_1(t))(h_2(t,z_2) - W_2(t)))$ and $D(t) = W_1(t) - W_2(t)$.

We can write $\partial_t \left(e^{-\int_0^t \zeta(s)\,\mathrm{d}s}\,\psi(t)\right) = -\zeta(t)e^{-\int_0^t \zeta(s)\,\mathrm{d}s} D(t)$ using an integrating factor. Hence

$$\psi(t) = e^{\int_0^t \zeta(s)\,\mathrm{d}s}\,\psi(0) - \int_0^t \zeta(u)e^{\int_u^t \zeta(s)\,\mathrm{d}s} D(u)\mathrm{d}u.$$

We find using $|e^{\int_0^t \zeta(s)\,\mathrm{d}s}| \le e^{\int_0^t |\zeta(s)|\,\mathrm{d}s}$ that

$$\begin{aligned}\left|\int_0^t \zeta(u)e^{\int_u^t \zeta(s)\mathrm{d}s} D(u)\mathrm{d}u\right| &\le \|D\|_{\infty,[0,T]} \int_0^t |\zeta(u)|e^{\int_0^u |\zeta(s)|\mathrm{d}s}\mathrm{d}u \\ &= \|D\|_{\infty,[0,T]} \left(e^{\int_0^t |\zeta(s)|\,\mathrm{d}s} - 1\right).\end{aligned}$$

By the Cauchy–Schwarz inequality

$$\int_0^t |\zeta(s)|\,\mathrm{d}s \le \sqrt{I_1 I_2}$$

where $I_k = \int_0^t \frac{2\mathrm{d}s}{|h_k(t,z_k) - W_k(t)|^2}$. On the other hand

$$\partial_t \log \mathrm{Im}\, h_k(t,z_k) = \frac{2\mathrm{d}s}{|h_k(t,z_k) - W_k(t)|^2}$$

and therefore

$$I_k = \log \frac{\mathrm{Im}\, h_k(t,z_k)}{y} \le \log \frac{\sqrt{y^2 + 4t}}{y}.$$

Here we used the upper bound $\partial_t \,\mathrm{Im}\, h_k(t,z_k) \le 2(\mathrm{Im}\, h_k(t,z_k))^{-1}$ to derive an upper bound for $\mathrm{Im}\, h_k(t,z_k)$.

Thus easily from the bounds above, we establish the bound

$$|\psi(t)| \le \frac{\sqrt{\delta^2 + 4t}}{\delta}\,|\psi(0)| + \left(\frac{\sqrt{\delta^2 + 4t}}{\delta} - 1\right) \|D\|_{\infty,[0,T]}.$$

This gives the claim. □

A similar result for the forward Loewner equation is the following.

Lemma 6.2. *For each $\delta > 0$ and $T > 0$ there exists a constant $C(T,\delta)$ such that the following holds. Let $g_k(t,z)$, $k = 1,2$ be the solutions of the Loewner equation* (4.14) *with the continuous driving terms $(W_k(t))_{t\in[0,T]}$, $k = 1,2$, respectively. Then they* *satisfy*

$$|g_1(T,z_1) - g_2(T,z_2)| \le C(T,\delta)(\|W_1 - W_2\|_{\infty,[0,T]} + |z_1 - z_2|) \tag{6.4}$$

for any z_1, z_2 *such that* $\operatorname{Im} g_k(T,z_k) > \delta > 0$.

Proof. The proof is similar to the proof of Lemma 6.1. The only difference is that we replace $\psi(t)$ by $\psi(t) = g_1(t,z_1) - g_2(t,z_1)$ and $\zeta(t)$ by $\zeta(t) = -2/((g_1(t,z_1) - W_1(t))(g_2(t,z_2) - W_2(t)))$. Then I_k is given as and bounded by $I_k = \int_0^t 2|g_k(s,z_k) - W_k(s)|^{-2} \mathrm{d}s \le \log \frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(t,z_k)} \le \log \frac{\operatorname{Im} z_k}{\max\{\delta, \sqrt{((\operatorname{Im} z_k)^2 - 4t)^+}\}}$ where $a^+ = \max\{a, 0\}$. □

The following results establish continuous dependency of solutions of Loewner equations on the driving term (i.e. continuity of the mappings $W \mapsto f$ and $W \mapsto g$).

Proposition 6.1. *The mapping $W \mapsto f$ is continuous with respect to the convergence in the Carathéodory sense. More specifically, for any compact $G \subset \mathbb{H}$, there exists a constant $C > 0$ such that if f_k, $k = 1,2$, are two (inverse) Loewner chains, then*

$$\|f_1 - f_2\|_{\infty,[0,T]\times G} \le C\|W_1 - W_2\|_{\infty,[0,T]}. \tag{6.5}$$

Proof. The claim follows directly from Lemma 6.1 and from the fact that h_k and f_k are related a time reversal, see Lemma 4.10. □

A similar result for the (direct) Loewner maps is the following.

Proposition 6.2. *Let K_0 be a hull and $G \subset \mathbb{H} \setminus K_0$ be a compact set. Then there exists a constant $C > 0$ such that if if g_k, $k = 1,2$, are two Loewner chains such that $K_k(T) \subset K_0$ for $k = 1,2$, then*

$$\|g_1 - g_2\|_{\infty,[0,T]\times G} \le C\|W_1 - W_2\|_{\infty,[0,T]}. \tag{6.6}$$

Proof. The claim follows directly from Lemma 6.2. □

6.1.3 Continuity properties of the mapping $\gamma \mapsto g$

The mapping from $(\gamma(t))_{t\in[0,T']}$ to $g(T',\cdot)$ is continuous by Theorem 6.1 with respect to the convergence in the Carathéodory sense. It is easy to extend this to continuity of the mapping $(\gamma(t))_{t\in[0,T']}$ to $(g(t,\cdot))_{t\in[0,T']}$ the convergence in the Carathéodory sense. For example, this can be using an compactness argument in the following way.

Make a counter assumption that there are γ, γ_n, $t_n \in [0,T']$ and $z_n \in G$, where G is a compact set as in Definition 6.2, such that γ_n converges to γ uniformly on $[0,T']$ but $\liminf_n |g_n(t_n,z_n) - g(t_n,z_n)| =: \varepsilon > 0$. Using compactness we can suppose that t_n and z_n converge to $t \in [0,T']$ and z, respectively, as $n \to \infty$ and then again by Theorem 6.1 $g_n(t_n,\cdot)$ converges to $g(t,\cdot)$ and $g(t_n,\cdot)$ converges to $g(t,\cdot)$ as $n \to \infty$.

Therefore $\lim_n |g_n(t_n, z_n) - g(t_n, z_n)| = 0$ (along a subsequence), which leads to a contradiction.

Proposition 6.3. *The mapping $\gamma \mapsto g$ is continuous with respect to the convergence in the Carathéodory sense.*[1]

The next corollary follows when this proposition is combined with the continuous dependency of the solution of the Loewner equation on the driving term.

Corollary 6.1. *Suppose that γ_n is a sequence of curves and W_n are their driving terms. If γ_n tends to γ and W_n tends to W uniformly on $[0,T]$ as n tends to ∞, then γ is driven by W, that is, the Loewner chain g of γ satisfies the Loewner equation with the driving term W.*

6.1.4 Continuity properties of the mapping $\gamma \mapsto W$

Below we will study the continuity properties of the mapping $(\gamma(t))_{t\in[0,T)} \mapsto (W_t)_{t\in[0,T)}$.

Theorem 6.2. *Let $X \subset C([0,T),\mathbb{C})$ be the set of curves in the upper half-plane that generate a Loewner chain. The mapping $(\gamma(t))_{t\in[0,T)} \mapsto (W_t)_{t\in[0,T)}$ from X to $C([0,T),\mathbb{R})$ is continuous. However, it is not uniformly continuous.*

Proof. Both the domain and the range of the map $(\gamma(t))_{t\in[0,T)} \mapsto (W_t)_{t\in[0,T)}$ are metrizable topological spaces. Hence it is sufficient to consider $T' \in (0,T)$ and a sequence γ_n that converges to γ in the uniform norm on $[0,T']$ and to establish that the corresponding driving terms W_n of γ_n converge to a limit W in the uniform norm on $[0,T']$ and that γ is driven by W.

Since γ_n tends to γ, the family γ_n is equicontinuous on $[0,T']$. By the same argument as in the proof of Theorem 4.2 (specifically the proof that the first statement implies the second one) the family W_n is equicontinuous on $[0,T']$. It is also uniformly bounded and we can apply Arzelà–Ascoli theorem to find a converging subsequence W_{n_i}. Denote its limit by W. By taking a sequence T'_n that increases to T, we can suppose that W_{n_i} converges to W uniformly on any $[0,T'] \subset [0,T)$. By Corollary 6.1, γ is driven by W.

To prove the first claim, we need to show that W_n converges to W. Assume the contrary and suppose that there is a subsequence of W_n that stays at a positive distance away from W in the uniform norm on $[0,T']$, for some $T' \in (0,T)$. By the same argument as above we can find a subsequence of that sequence such that it converges to some W_1. By Corollary 6.1, γ is driven by W_1 which leads to a contradiction since $W_1 \neq W$ by the assumption we made. Thus we have shown the first claim.

[1] It is possible to give a more quantitative estimates for this convergence using the harmonic measure.

To prove the second claim let γ_n be the broken line 0, i, $\mathrm{i}+1/n$, $(1/2)\mathrm{i}+1/n$ parametrized with the half-plane capacity and let $\tilde{\gamma}_n(t) = -\overline{\gamma_n(t)}$, i.e., the reflection with respect to the y-axis. Let the corresponding driving terms be W_n and $\tilde{W}_n$. Let then by a simple argument using, say, harmonic measure there exists $c > 0$ and T' such that the capacity of γ_n and $\tilde{\gamma}_n$ is greater than T' and $W_n(T') > c$ and $\tilde{W}_n(T') < -c$. Hence $\|\gamma_n - \tilde{\gamma}_n\|_{\infty,[0,T']} = 2/n$ and $\|W_n - \tilde{W}_n\|_{\infty,[0,T']} > 2c$. This shows that the map $(\gamma(t))_{t\in[0,T)} \mapsto (W_t)_{t\in[0,T)}$ is not uniformly continuous. □

Remark 6.2. To get uniform continuity, we need to improve the topology of curves, say, by keeping track of the harmonic measures of the left-hand and right-hand sides of $\mathbb{R}\cup\gamma[0,t]$. We don't try to formulate the topology here, but we wanted to mention that it can be done in principle.

6.1.5 Continuity properties of the mapping $W \mapsto \gamma$

Lemma 6.3. *The mapping* $(W_t)_{t\in[0,T)} \mapsto (\gamma(t))_{t\in[0,T)}$ *is not continuous.*

For the proof see Figure 6.1 (See also [8], Example 4.49 where the example is originally from.) where an example of a sequence of curves is given such that their driving terms converge uniformly to a constant[2], while the sequence of curves doesn't have any subsequence that would converge. A similar example of an obstruction to the convergence of γ_n is related to the tip being hidden from infinity during a non-trivial interval. Consider a path γ_n of Figure 6.2. Then it is quite standard using, say, harmonic measure, to notice that W_n converges to some W while γ_n doesn't converge uniformly in the capacity parametrization.

Based on the observation of the previous lemma, we need to consider more restrictive class of curves. Denote by F the function

$$F(t,y) = f(t, W_t + y\mathrm{i}). \tag{6.7}$$

For any $T' \in \mathbb{R}_{>0}$, $\delta_0 > 0$ and any function $\lambda : (0,\delta_0] \to \mathbb{R}_{\geq 0}$ such that $\lim_{y\to 0}\lambda(y) = 0$, define

$$\mathscr{E}_{\lambda,T',\delta_0} = \left\{ W \in C([0,T]) : \begin{array}{c} W \text{ drives a curve } \gamma \text{ and} \\ |F(t,y) - \gamma(t)| \leq \lambda(y) \text{ for all } (t,y) \in [0,T'] \times (0,\delta_0] \end{array} \right\}.$$

By Theorem 6.4 below, the uniform convergence of the mappings $t \mapsto F_t(y)$ as y tends to zero is sufficient and necessary for the Loewner chain to be driven by a curve.

Lemma 6.4. *For each* T', δ *there exists* $C(T',\delta)$ *such that the following holds. If* $W_1, W_2 \in \mathscr{E}_{\lambda,T',\delta_0}$ *for some* $\lambda : [0,\delta_0] \to \mathbb{R}_{\geq 0}$*, then*

[2] Remember that constant driving term corresponds to the trivial Loewner chain, which is a straight vertical line segment.

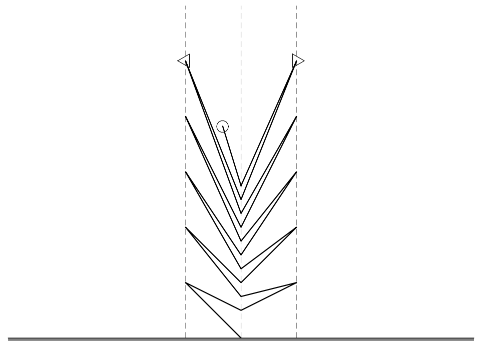


Fig. 6.1 The proof of Lemma 6.3 by a picture: consider a sequence of curves γ_n which all are broken lines. The corners of γ_n lie on three lines $\operatorname{Re} z = -\frac{1}{n}, 0, \frac{1}{n}$ which are the three dashed lines in the picture. On $\operatorname{Re} z = \pm\frac{1}{n}$, the corners are at height $\operatorname{Im} z = \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots$ and on $\operatorname{Re} z = 0$, they are at height $\operatorname{Im} z = 0, \frac{1}{2n}, \frac{3}{4n}, \frac{1}{n}, \frac{5}{4n}, \frac{3}{2n} \ldots$ These points are connected by line segments in the order indicated in the picture ($\operatorname{Re} z = 0, -\frac{1}{n}, 0, \frac{1}{n}, 0, -\frac{1}{n}, \ldots$ and the height is ordered on each line). The tip of the curve at time t is indicated by the circle in the picture. The value of $W_n(t)$ is between the images of the triangles z_{left}, z_{right} under $g(t,\cdot)$. Concentrate on the rightmost triangle z_{right}. Let L be the broken line $-\frac{1}{n}, z_{\text{left}}, z_{\text{right}}$. The value of $g(t, z_{\text{right}}) = g_{K_t}(z_{\text{right}})$ increases if we add to K_t the path L and all the points disconnected from infinity as a consequence. In a similar manner, removing "matter" from the right-hand side of L increases $g_K(z_{\text{right}})$. Therefore W_t is at most $g_L(z_{\text{right}})$, which can be estimated using the explicit Loewner map of the vertical slit to be at most $C \max\{t^{1/4} n^{-1/2}, n^{-1}\}$ where $C > 0$ is some constant. By symmetry we have shown that $|W_n(t)| \leq C \max\{t^{1/4} n^{-1/2}, n^{-1}\}$. We leave as an exercise to verify the fact that γ_n doesn't contain any subsequence that would converge uniformly on any $[0,T]$, $T > 0$

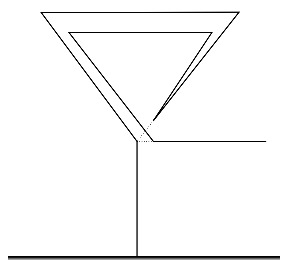


Fig. 6.2 The solid horizontal line in the bottom of the picture is the real axis and the solid broken line is γ_n. Suppose that the vertices are $z_0, z_1, \ldots, z_8$ and that the other vertices are fixed except that z_4 and z_7 converge towards z_1 as $n \to \infty$ (along the dashed lines).

$$\|\gamma_1 - \gamma_2\|_{\infty,[0,T']} \leq C(T',\delta)\|W_1 - W_2\|_{\infty,[0,T']} + 2\lambda(\delta). \tag{6.8}$$

Proof. Write using the triangle inequality that

$$|\gamma_1(t) - \gamma_2(t)| \leq |F_1(t,\delta) - F_2(t,\delta)| + |F_1(t,\delta) - \gamma_1(t)| + |F_2(t,\delta) - \gamma_2(t)|. \tag{6.9}$$

By the assumption, when $t \in [0,T']$, the second and third terms on the right-hand side are bounded by $\lambda(\delta)$. Notice that $F_k(t,\delta) = f_k(t, z_k(t))$ where $z_k(t) = W_k(t) +$

iδ. Hence by Lemma 6.1 there exists an explicit constant $C(T',\delta)$ such that the first term on the right-hand side of the above inequality is bounded by $C(T',\delta)\|W_1 - W_2\|_{\infty,[0,T']}$. □

Proposition 6.4. *Let $\mathscr{E} \subset C([0,T))$ be a collection of driving terms of curves such that for each $T \in (0,T')$ there exists δ_0 and λ as above such that any $W \in \mathscr{E}$ when restricted to the time interval $[0,T')$ belongs to $\mathscr{E}_{\lambda,T',\delta_0}$. Then the mapping $(W(t))_{t\in[0,T)} \mapsto (\gamma(t))_{t\in[0,T)}$ is uniformly continuous in $\mathscr{E}$ with respect to the topology of local uniform convergence for $(W(t))_{t\in[0,T)}$ and $(\gamma(t))_{t\in[0,T)}$.*

Proof. Let $T' \in (0,T)$ and let λ and δ_0 be such that any $W \in \mathscr{E}$ belongs to $\mathscr{E}_{\lambda,T',\delta_0}$.

Let $\varepsilon > 0$. Take $\delta_1 \in (0,\delta_0]$ such that $4\lambda(\delta_1) \leq \varepsilon$. Then take $\delta = \varepsilon/(2C(T',\delta_1))$ where $C(T',\delta_1)$ is as in Lemma 6.4. Now by Lemma 6.4, $\|\gamma_1 - \gamma_2\|_{\infty,[0,T']} \leq \varepsilon$ for any $W_1, W_2 \in \mathscr{E}$ such that $\|W_1 - W_2\|_{\infty,[0,T']} \leq \delta$. □

6.1.5.1 The modulus of continuity for γ driven by $W \in \mathscr{E}_{\lambda,T',\delta_0}$

Using the extending the idea of Lemma 6.4, it is possible to prove the following result on the modulus of continuity of the curve driving a Loewner chain. The proof is written in Appendix D.

Theorem 6.3. *For any T', δ_0, λ as above and any function $\psi : (0,1] \to (0,\infty)$ such that $\lim_{\delta\to 0}\psi(\delta) = 0$, there exists ϕ such that $\lim_{\delta\to 0}\phi(\delta) = 0$ and the following holds. If $W \in \mathscr{E}_{\lambda,T',\delta_0}$ and $|W(t) - W(s)| \leq \psi(|t-s|)$, then γ satisfies*

$$|\gamma(t) - \gamma(s)| \leq \phi(|t-s|). \tag{6.10}$$

6.2 Continuity of SLE(κ)

6.2.1 Existence of the trace for Loewner chains

The following lemma shows that a Loewner chain is tame at certain time instances, in the sense that $\lim_{y\to 0} F_t(y)$ exists.

Lemma 6.5. *Let $z_0 \in \overline{\mathbb{H}} \setminus \{W_0\}$, $0 < r < |z_0 - W_0|$ and $B = B(z_0,r) \cap \mathbb{H}$. Suppose $t > 0$ is such that $K_t \cap \overline{B}$ is non-empty and $K_s \cap \overline{B}$ is empty for each $s \in [0,t)$. Suppose also that $\overline{B} \setminus K_t$ is non-empty, that is, $\overline{B}$ is hit, but not swallowed by K_t. Then there exists $z_1 \in \partial B$ such that $K_t \cap \overline{B} = \{z_1\}$ and moreover, $z_1 = \lim_{y\to 0} F_t(y)$.*

Proof. Let z_0, r, B and t be as stated above. Then by the assumptions $(\partial K_t) \cap \overline{B} \neq \emptyset$. Let $z_1 \in (\partial K_t) \cap \overline{B}$. By the Carathéodory kernel convergence theorem there exists $w_n \in \partial K_{s_n}$ such that s_n increases to t and w_n tends to z_1 as $n \to \infty$. Since $w_n \in \mathbb{C} \setminus \overline{B}$ for all n, it follows that $z_1 \in \partial B$. Hence $(K_t \cap \overline{B}) \subset \partial B$.

Since the line segment from z_1 to z_0 lies in H_t except the endpoint z_1, the end point z_1 is accessible,[3] see [10] Section 2, Exercise 5, and thus there exists $x \in \mathbb{R}$ such that $z_1 = \lim_{y\to 0} f_t(x+\mathrm{i}y)$. From the facts that $\sup\{|z - W_{t-\delta}| : z \in g_{t-\delta}(K_t \setminus K_{t-\delta})\} = o(1)$ as $\delta \to 0$, see Theorem 4.2, and that $s \mapsto W_s$ is continuous, it follows that $x = W_s$. □

Let $H \subset \mathbb{H}$. Denote by $\partial_+ H$ the set points, which have the property that every neighborhood (in $\mathbb{C}$) of the point intersects both H and $\mathbb{H} \setminus H$. In other words, $\partial_+ H$ is the boundary of H in $\mathbb{H}$.

Remember that $(K_t)_{t\in[0,T)}$ is generated by a curve $\gamma : [0,T) \to \mathbb{C}$, if H_t is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$. The next result is a basic tool to verify that a Loewner chain is generated by a curve.

Theorem 6.4. *If $t \mapsto F_t(y)$ converges to some γ uniformly on compact subsets of $[0,T)$ as $y > 0$ tends to 0, then γ is a continuous curve and $(K_t)_{t\in[0,T)}$ is generated by γ. Furthermore for each $t \in [0,T)$, the map $z \mapsto f_t(z)$ extends continuously to $\overline{\mathbb{H}}$.*

Proof. For each fixed $y > 0$, the map $t \mapsto F_t(y)$ is continuous by Lemma 6.1. Hence the uniform convergence of $F_t(y) \to \gamma(t)$ as $y \to 0$ on compact subsets of $[0,T)$ implies that $\gamma : [0,T) \to \mathbb{C}$ is continuous.

It remains to show that for each $t \in [0,T)$, H_t is the unbounded component of $\mathbb{H} \setminus (\gamma[0,t])$. Since $\gamma(s) \in \partial_+ H_s$, it follows that $\gamma[0,t] \subset \bigcup_{s\in[0,t]} \partial_+ H_s$. Hence it is sufficient to show that $\partial_+ H_t \subset \gamma[0,t]$.

Let $z_0 \in \partial_+ H_t$. If $z_0 = W_0$, then clearly $z_0 \in \gamma[0,t]$. Suppose then that $z_0 \neq W_0$ and take any $\varepsilon \in (0, |z_0 - W_0|)$. Let

$$t_\varepsilon = \inf\left\{ s \in \mathbb{R}_{\geq 0} : K_s \cap \overline{B(z_0,\varepsilon)} \neq \emptyset \right\}.$$

Then $0 < t_\varepsilon < t$. Since $z_0 \in \partial_+ H_t$, the set $\overline{\mathbb{H} \cap B(z_0,\varepsilon)} \setminus K_{t_\varepsilon}$ is non-empty. By Lemma 6.5, $|z_0 - \gamma(t_\varepsilon)| = \varepsilon$. Therefore $z_0 = \lim_{\varepsilon\to 0} \gamma(t_\varepsilon)$ and consequently $z_0 \in \overline{\gamma[0,t)} = \gamma[0,t]$. Thus $\partial_+ H_t \subset \gamma[0,t]$ as was claimed. The set H_t is therefore the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ and since ∂H_t is locally connected f_t extends continuously to $\overline{\mathbb{H}}$ as claimed. □

Let's formulate the following corollary to stress the importance of the result.

Corollary 6.2. *A Loewner chain $(K_t)_{t\in[0,T)}$ is generated by a curve if and only if for each $T' \in [0,T)$, there exists a function $\lambda : (0,1] \to \mathbb{R}_{\geq 0}$ such that $\lim_{y\to 0} \lambda(y) = 0$ and*

$$|F_t(y_1) - F_t(y_2)| \leq \lambda(y) \tag{6.11}$$

for all $y \in (0,1]$, $y_1, y_2 \in (0,y]$ and $t \in [0,T']$.

[3] A boundary point is accessible, if there is a Jordan arc in the domain ending at that point. If we apply a conformal map from the domain onto $\mathbb{D}$, say, then the image of that arc is continuous up to the boundary. Consequently, the accessible point is always a limit of a image of a Jordan arc in $\mathbb{D}$ under a conformal map from $\mathbb{D}$ onto the domain. The radial limit of the conformal map at the same boundary point of $\mathbb{D}$ follows from Corollary 2.17 of [10].

6.2.2 Auxiliary results on conformal maps

The next result is a version of Koebe distortion theorem in $\mathbb{H}$. The proof, which is straightforward, is given in Appendix D.

Lemma 6.6 (Koebe distortion in $\mathbb{H}$). *There exists a constant C such that for any $y > 0$, $s \in [\frac{1}{2}, 2]$, $x \in \mathbb{R}$ and any conformal map $f : \mathbb{H} \to \mathbb{C}$,*

$$C^{-1}|f'(\mathrm{i}y)| \le |f'(\mathrm{i}sy)| \le C|f'(\mathrm{i}y)| \tag{6.12}$$

$$C^{-1}(1+x^2)^{-3}|f'(\mathrm{i}y)| \le |f'(y(x+\mathrm{i}))| \le C(1+x^2)^3|f'(\mathrm{i}y)|. \tag{6.13}$$

The next result is based on the Loewner equation and thus the proof is given here.

Lemma 6.7. *There exists a constant C such that for any solution f_t of the Loewner equation for the inverse Loewner map and for any $x+\mathrm{i}y \in \mathbb{H}$, $t \in \mathbb{R}_{\ge 0}$ and $s \in [0, y^2]$*

$$C^{-1}\,|f_t'(x+\mathrm{i}y)| \le |f_{t+s}'(x+\mathrm{i}y)| \le C\,|f_t'(x+\mathrm{i}y)| \tag{6.14}$$

$$|f_{t+s}(x+\mathrm{i}y) - f_t(x+\mathrm{i}y)| \le C\,y\,|f_t'(x+\mathrm{i}y)|. \tag{6.15}$$

Proof. By differentiating the Loewner equation and using the triangle inequality and the inequality $|x+\mathrm{i}y - W_t| \ge y$, it follows that

$$|\partial_t f_t'(x+\mathrm{i}y)| \le \frac{2|f_t''(x+\mathrm{i}y)|}{y} + \frac{2|f_t'(x+\mathrm{i}y)|}{y^2}.$$

To estimate $|f_t''(z)|$, for fixed $z = x+\mathrm{i}y \in \mathbb{H}$, let $\phi(\zeta) = x + \mathrm{i}y\frac{1-\zeta}{1+\zeta}$. Then ϕ is a Möbius map from $\mathbb{D}$ onto $\mathbb{H}$ and it has expansion $\phi(\zeta) = x + \mathrm{i}y(1 + 2\sum_{n=1}^{\infty}(-1)^n\zeta^n)$. Thus $\phi(0) = z$, $\phi'(0) = -2\mathrm{i}y$ and $\phi''(0) = 4\mathrm{i}y$.

The function $(f_t \circ \phi(\zeta) - f_t(z))/(f_t'(z)\phi'(0))$ has expansion

$$\zeta + \frac{f_t''(z)(\phi'(0))^2 + f_t'(z)\phi''(0)}{2f_t'(z)\phi'(0)}\zeta^2 + \dots$$

around $\zeta = 0$. Using (3.9), it follows that $|f_t''(z)||\phi'(0)|^2 \le |f_t'(z)|(|\phi''(0)| + 4|\phi'(0)|)$ and thus $|f_t''(z)| \le 6\,|f_t'(z)|\,y^{-1}$. Combining this with the above estimate gives $|\partial_t f_t'(x+\mathrm{i}y)| \le 14|f_t'(x+\mathrm{i}y)|\,y^{-2}$. Thus $|\partial_t \log f_t'(x+\mathrm{i}y)| \le 14\,y^{-2}$ and hence

$$-\frac{14}{y^2} \le \partial_t \log|f_t'(x+\mathrm{i}y)| \le \frac{14}{y^2}$$

where we used the inequality $-|z| \le \operatorname{Re} z \le |z|$.

By integrating this inequality with respect to t, we get the first claim easily. The second claim is derived from the first one by plugging it in to the Loewner equation, which is then integrated with respect to t. This gives an upper bound which is proportional to $|f_t'(x+\mathrm{i}y)|\frac{s}{y} \le |f_t'(x+\mathrm{i}y)|y$. □

6.2.3 Proof of Theorem 5.2

Definition 6.3. An increasing, continuous function $\psi : [0,\infty) \to (0,\infty)$ is said to be a *subpower function* if $\lim_{x\to\infty} \frac{\log\psi(x)}{\log x} = 0$ or equivalently if $\lim_{x\to\infty} x^{-\mu}\psi(x) = 0$ for all $\mu > 0$.

Remark 6.3. One way to write this is $\psi(x) = e^{o(\log x)}$. If ψ_1 and ψ_2 are subpower functions also $\psi_1\psi_2$, $\psi_1+\psi_2$ and $\psi(x) = \psi_1(x^p)$, $p>0$, are subpower functions.

Proof (Proof of Theorem 5.2). By Theorem 6.4, it is enough to prove that the functions $t \mapsto f_t(W_t + \mathrm{i}y)$ converges uniformly as y tends to 0.

Our goal is to prove this based on the following bounds: As we saw above for each $\kappa \neq 8$, there exist a constant $\theta > 0$ and a random variable C which is almost surely finite such that

$$|\tilde{f}_t'(i2^{-n})| \le C\,2^{n(1-\theta)} \tag{6.16}$$

for all $t \in \mathscr{D}_{2n}$ and for any $n \in \mathbb{N}$. Remember also that since $(W_t)_{t\in\mathbb{R}_{\ge 0}}$ is a Brownian motion, there is an almost surely finite random variable $\tilde{C}$ such that

$$|W_{t+s} - W_t| \le \tilde{C}\sqrt{s\log(1/s)} \tag{6.17}$$

for any $t,s \in [0,1]$. Fix a realization of the driving process and the Loewner chain such that the bounds (6.16) and (6.17) hold for some finite C and $\tilde{C}$.

Let $t \in [0,1], y \in (0,1)$. Take $n \in \mathbb{N}$ and $t_0 \in \mathscr{D}_{2n}$ such that

$$2^{-n} \le y < 2^{-n+1}, \qquad t_0 \le t < t_0 + 2^{-2n},$$

that is, $n = \lceil \log_2(1/y) \rceil$ and $t_0 = \lfloor t2^{2n} \rfloor 2^{-2n}$. By (6.16), (6.17) and Lemmas 6.6 and 6.7, it follows that

$$\begin{aligned} |\tilde{f}_t'(iy)| &= |f_t'(W_t + iy)| \le c|f_{t_0}'(W_t + iy)| \\ &\le c|f_{t_0}'(W_t + iy_0)| \le c\left(1 + \frac{|W_t - W_{t_0}|^2}{y_0^2}\right)^3 |f_{t_0}'(W_{t_0} + iy)| \\ &\le cn^r 2^{n(1-\theta)} \le y^{\theta-1}\psi(1/y) \end{aligned} \tag{6.18}$$

for some subpower function ψ. Here c is a generic constant that might change from line to line .

Let's integrate the bound (6.18). For any $0 < y_1 < y_2 \le y < 1$, by the triangle inequality and a change of integration variable

$$|\tilde{f}_t(\mathrm{i}y_2) - \tilde{f}_t(\mathrm{i}y_1)| \le \int_{y_1}^{y_2} |\tilde{f}_t'(\mathrm{i}u)|\,\mathrm{d}u \le \int_0^y u^{\theta-1}\psi(1/u)\,\mathrm{d}u = y^\theta\tilde{\psi}(1/y)$$

where $\tilde{\psi}(x) = \int_0^1 u^{\theta-1}\psi(x/u)\,\mathrm{d}u$. It is not difficult to check that $\tilde{\psi}$ is a subpower function. Hence $\gamma(t) = \lim_{y\searrow 0}\tilde{f}_t(\mathrm{i}y)$ exists and satisfies

$$|\gamma(t) - \tilde{f}_t(\mathrm{i}y)| \leq y^{\theta} \tilde{\psi}(1/y). \tag{6.19}$$

Now by Theorem 6.4, γ continuous and generates $(K_t)_{t \in \mathbb{R}_{\geq 0}}$. □

It turns out that the following more quantitative result holds. The proof is given in Appendix D.

Proposition 6.5. *For each $\kappa \neq 8$, there exist a constant $\alpha_0 > 0$ such that $t \mapsto \gamma(t)$ is Hölder continuous for any exponent $\alpha < \alpha_0$*

Remark 6.4. We can choose $\alpha_0 = \theta_0/2$ where θ_0 is as in Remark 5.12.

6.3 Convergence of interfaces in the site percolation model

In this section, we apply the ideas of Section 6.1 to the convergence of random curves to SLEs in a specific example case of the site percolation model on triangular lattice. Most of the proofs are based on estimates established for this discrete model.

6.3.1 Definition of the site percolation model

A *graph* G is a pair $G = (V, E)$ where V is a finite set and $E \subset \{\{v, w\} \subset V : v \neq w\}$. An element $v \in V$ is called a *vertex* or a *site* and an element $e \in E$ is called an *edge* or a *link*. Two vertices connected by an edge are said to be neighbors on the graph, and the number of neighbors is called the degree of the vertex. All the graphs we are going to consider are *planar*, that is, they come together with an embedding to the plane so that the vertices are distinct points and edges are represented by simple paths which connect the vertices and which are pair-wise disjoint.

The complement (as a planar point set) of the embedded graph consist of a finite number of components, that we can interpret as polygons whose edges and vertices are edges and vertices of the graph. (The face of) Such a polygon is said to be a *face* of the graph. The centers (or otherwise chosen points, one for each face) of the faces form a finite set V^*. The set E^* is defined to be edges that connect two centers v, w of faces if and only if the faces are adjacent, i.e., the distinct faces share at least one edge. The graph $G^* = (V^*, E^*)$ is called the *dual graph* of G. We can arrange so that there is one-to-one correspondence between edges E and dual edges E^* by saying that $e \in E$ and $f \in E^*$ are dual to each other if and only they cross.[4] In this case we define $e^* := f$.

Let us extend the notion of graphs so that we allow V to be an infinite set. An approach is to take $G = (V, E)$ to be a pair (together with an embedding to the plane)

[4] A careful reader can notice that E^* defined in the latter way, which is more general, doesn't necessarily define a (simple) graph, but a multigraph where a pair of vertices can be linked by several edges and where the endpoints of an edge don't need to be distinct vertices.

such that the restriction of V to any bounded region is finite and all vertices have finite degree. An important class of infinite graphs are the *lattices*. We define a lattice to be an infinite graph such that its faces are translates of each other and they cover the plane, or more generally, its faces tile the plane and the tiling is formed from translates of a finite pattern of polygons. Examples are the square $\mathbb{Z}^2$, triangular $\mathbb{L}_{\text{tri}}$ and hexagonal $\mathbb{L}_{\text{hex}}$ lattices which are formed from regular squares, triangles and hexagons, respectively.

Let $G = (V,E)$ be a finite of infinite graph. We always think that G is either a lattice or a subgraph of lattice. Let $p \in [0,1]$ be a parameter and consider a family of random variables, one for each site of G, taking values in the set $\{\text{open},\text{closed}\}$. We say that a site $v \in V$ is open or closed depending on the value that random variable takes. To define the law precisely, it is assumed that

$$\mathsf{P}[v \text{ is open}] = p, \qquad \mathsf{P}[v \text{ is closed}] = 1-p \tag{6.20}$$

for each $v \in V$ and that the random variables are independent. A probability space with the percolation model can be constructed as a product space $\{\text{open},\text{closed}\}^V$ with a product measure of laws of Bernoulli random variables.

The model is called the *site percolation* on G. The closed sites model a random media and the open sites form cavities through which a fluid can flow. Fundamental questions in the percolation model are therefore connectivity properties of the subgraph of open sites. For small values of p, we expect that it is rare to see large clusters of open sites and that for large p, in an infinite graph, almost surely there exists an open infinite cluster.

If we want to stress the dependency of the model from the parameter, we denote the probability measure as P_p.

6.3.1.1 The percolation exploration process

We wish to define a process that explores a part of a percolation configuration. This process will be a simple path on the hexagonal lattice that keeps open sites of the triangular lattice on its left, say, and closed sites on its right.

Let us first define a type of domain on which we will define the exploration. For later purposes, let's introduce a lattice mesh $\delta > 0$. Suppose that Ω_δ is a simply connected domain such that $\partial\Omega_\delta \neq \emptyset$ is a path on the lattice $\delta\mathbb{L}_{\text{hex}}$. Let $\tilde{V}$ be the set of sites on $\delta\mathbb{L}_{\text{tri}}$ that lie inside Ω_δ and let V_1 be the set of those $v \in \tilde{V}$ such that the hexagon corresponding to v has at least one common edge with $\partial\Omega_\delta$. Let $V = \tilde{V} \setminus V_1$. The set V is now the one where we are going to put the percolation configuration. So if we want that V is a given shape, say, a rhombic domain $V = \delta R(v, \lceil a\delta^{-1}\rceil, \lceil b\delta^{-1}\rceil)$, then for Ω_δ we need to add a layer of hexagons around V.

Suppose now that we have defined a percolation configuration on V. The set V_1 is connected and we can interpret it as a unique non-self-crossing closed path π (with counterclockwise orientation) on the triangular lattice. Let a and b be two distinct points on the hexagonal lattice such that at a and b, exactly one of the three edges of $\mathbb{L}_{\text{hex}}$ belongs to $\partial\Omega_\delta$ or that property holds and the property that Ω_δ is a simply connected domain remains valid if a suitable edge is added to a and b or one of them. Then at a and b, of the three neighboring hexagons, two have centers in V_1 (and correspond to the above mentioned edge) and third in either V_1 or V. We say that a and b are points on the boundary of V.

To define the exploration process, pick for both a and b one of the neighboring edges that cross π and denote the two halves of π as ab and ba.[5] The *exploration process* is the unique simple path from a to b on the hexagonal lattice such that all the hexagons on its right are closed and all on its left are open in the way that ab is considered to be closed and ba open when defining the process.

6.3.1.2 Percolation interface and its scaling limit

Definition 6.4. For a domain Ω and its distinct boundary points a and b and an approximating sequence $(\Omega_\delta, a_\delta, b_\delta)_\delta$, where $\delta > 0$ runs over a set of points accumulating to 0, we define $\mu_\delta^{(\Omega,a,b)}$ to be the probability law of the percolation interface γ in Ω_δ starting at a_δ and ending at b_δ. The *scaling limit* of the percolation interface is $\mu^{(\Omega,a,b)} = \lim_{\delta\to 0} \mu_\delta^{(\Omega,a,b)}$ where the mode of convergence is specified soon.

Remark 6.5. Let us collect the approximating sequence $(\Omega_\delta, a_\delta, b_\delta)_\delta$ to a family $\mathscr{D}$ of triplets (U,a,b) and let's assume that the parameter δ is implicitly given (it is the length of any edge (line segment) in $\partial\Omega_\delta$). Let's also collect $\mu_\delta^{(\Omega,a,b)}$ to a family $\mathscr{M}$ of probability measures.

In what follows, we *mostly don't explicitly refer $\mathscr{D}$ or $\mathscr{M}$* before Theorem 6.8. In practice, the statements hold for any collection of domain triplets (Ω,a,b) and they don't need to form an approximating sequence. Hence we drop the notation for the lattice mesh $\delta > 0$ and think that any $(U,a,b) \in \mathscr{D}$ is a discrete domain. We also use notation $\mu \in \mathscr{M}$ without superscripts or even just the standard notation for probability P.

Remark 6.6. We will use the topology of a uniform convergence of continuous functions for curves parametrized with the capacity. The mode of convergence of (generalized, curve-valued) random variables is specified in the next section.

6.3.2 *A probability bound on crossings by multiple open percolation paths*

In this section, we aim to establish bounds on multiple crossings of open paths in percolation.

[5] It is natural to select the edge that belongs to $\partial\Omega_\delta$ if that exists. If it doesn't exist, we can always add such an edge to $\partial\Omega_\delta$ without disturbing any of the required properties.

6.3.2.1 Increasing events and the FKG inequality

There is a natural ordering among percolation configurations, namely, a configuration $(\omega_x)_{x\in V}$ is greater that a configuration $(\omega'_x)_{x\in V}$ if and only if for each $x \in V$, $\omega_x \geq \omega'_x$. This order relation is denoted by $\succ$. A random variable that respects the order $\succ$ is said to be *increasing*, that is, for an increasing random variable $X : \Omega \to \mathbb{R}$, it holds that $X((\omega_x)_{x\in V}) \geq X((\omega'_x)_{x\in V})$ for any configurations $(\omega_x)_{x\in V} \succ (\omega'_x)_{x\in V}$. An event said to be *increasing* if and only if its indicator function is an increasing random variable.

The following result shows that increasing events are positively correlated. One way to formulate this is that if A, B are increasing events in a percolation model, then $\mathsf{P}[A\,|\,B] \geq \mathsf{P}[A]$. For the proof of the theorem, see for instance [3] or [4].

Theorem 6.5 (Fortuin-Kasteleyn-Ginibre (FKG) inequality). *For increasing, non-negative random variables X, Y in a percolation model, it holds that $\mathsf{E}[XY] \geq \mathsf{E}[X]\,\mathsf{E}[Y]$. In particular, this holds for indicator functions of increasing events.*

6.3.2.2 Critical point and RSW estimates

There are many (equivalent) ways to characterize the *critical parameter* $p = p_c$ of the site percolation model on a given lattice. We will choose the following definition which is suitable for our needs.

Let $e_1 = 1$ and $e_2 = e^{\mathrm{i}\pi/3}$, which are unit vectors that generate the triangular lattice $\mathbb{L}_{\mathrm{tri}}$. Consider rhombi

$$R(v,a,b) = \{v + s\,e_1 + t\,e_2 \,:\, s \in [\![0,a]\!], t \in [\![0,b]\!]\}, \tag{6.21}$$

where $v \in V(\mathbb{L}_{\mathrm{tri}})$ is the lower left corner and $a, b \in \mathbb{Z}_{>0}$ are the side lengths of the rhombus. In any percolation configuration on $R(v,a,b)$, either there is an open path from left to right in $R(v,a,b)$ or a closed path from top to bottom. We set $p_c = 1/2$, since satisfies the following crossing property. At $p = p_c$, by symmetry

$$\mathsf{P}[\text{left to right crossing in } R(v,a,a)] = \frac{1}{2}. \tag{6.22}$$

Notice that the equality (6.22) holds for all a. In particular, we see that the crossing probability remains bounded away from 0 and 1 as a tends to ∞. For subcritical p, that is $p < p_c$, the crossing probability would tend to zero and for supercritical p, that is $p > p_c$, it would tend to one.[6]

Define also a triangle $T(v,a) = \{v + s\,e_1 + t\,e_2 \,:\, s,t \in \mathbb{Z}_{\geq 0}, s+t \leq a\}$ and a trapezoid $\tilde{R}(v,a,b) = R(v,a,b) \cup T(v + a\,e_1, b)$. Let's call the bottom of $T(v + a\,e_1, b)$ the bottom-right side of the trapezoid $\tilde{R}(v,a,b)$. Denote by

---

[6] For the proof see [3]. The fact is based on so called sharp threshold phenomenon by which for all p the crossing probability is either close to zero or close to one or the its derivative with respect to p is large.

$$\mathscr{S}_{\text{L-R}}(R) \qquad \text{or} \qquad \mathscr{S}_{\text{L-R}}(\tilde{R})$$

the *event of left–right crossing of a rhombus R* or *trapezoid $\tilde{R}$* and by

$$\mathscr{S}_{\text{L-BR}}(\tilde{R})$$

the *event of a crossing from the left side to the bottom-right of the trapezoid $\tilde{R}$*.

The *Russo–Seymour–Welsh estimates* (RSW) allow us to extend the equation (6.22) as inequalities for other rhombi $R(v,a,b)$, $a \neq b$, and other shapes.

Lemma 6.8. *For $a,b \in \mathbb{Z}_{>0}$ such that $b \leq a$, the following inequalities hold*

- *When $p \geq p_c$, $\mathsf{P}_p[\mathscr{S}_{\text{L-BR}}(\tilde{R}(v,a,b))] \geq \frac{1}{2}\mathsf{P}_p[\mathscr{S}_{\text{L-BR}}(R(v,a,b))]$*
- *For all p, $\mathsf{P}_p[\mathscr{S}_{\text{L-R}}(\tilde{R}(v,a,b))] \geq \mathsf{P}_p[\mathscr{S}_{\text{L-BR}}(\tilde{R}(v,a,b))]^2$*
- *When $p \geq p_c$, $\mathsf{P}_p[\mathscr{S}_{\text{L-R}}(R(v,2a,b))] \geq \frac{1}{2}\mathsf{P}_p[\mathscr{S}_{\text{L-R}}(\tilde{R}(v,a,b))]^2$*

Consequently, $\mathsf{P}_p[\mathscr{S}_{\text{L-R}}(R(v,2a,b))] \geq \frac{1}{32}\mathsf{P}_p[\mathscr{S}_{\text{L-R}}(R(v,a,b))]^4$ when $p \geq p_c$.

Proof. Notice first that the probabilities of the crossing events we consider are independent of v and their values remain invariant under lattice rotations.

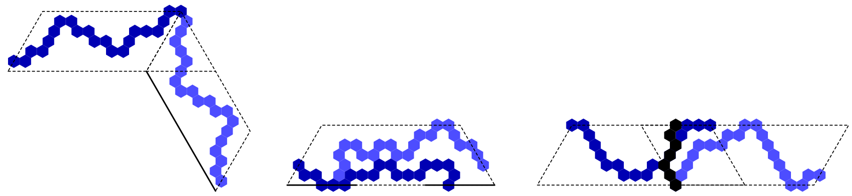
The argument for the first claim is similar to one given in [14]. It is the main observation of the present proof. Namely, if the event $\mathscr{S}_{\text{L-R}}(R(v,a,b))$ occurs, we can find the topmost simple open path π in the percolation configuration crossing the rhombus $R(v,a,b)$. It is illustrated as the dark blue path in Figure 6.3(a). It turns out that such path can be found so that we have to "reveal" only the site above and on the path, for details see [14]. Consequently, conditionally on π, the configuration restricted to the sites below π has still the percolation distribution. Consider the mirror image R' of $R(v,a,b)$ with respect to the right side of $R(v,a,b)$ and the mirror image π' of π. See Figure 6.3(a). In the kite shaped domain formed by the union of $R(v,a,b)$ and R' on the sites below the path concatenated from π and π', by symmetry, the probability of an open path from sites next to π to the lower left side of R' (the solid line in Figure 6.3(a)) is at least $\frac{1}{2}$. The occurrence of this event together with existence of π implies that $\mathscr{S}_{\text{L-BR}}(\tilde{R}(v,a,b))$ occurs. Summing over the paths π gives the claim.

The second claim follows from the FKG inequality, when we consider the events $\mathscr{S}_{\text{L-BR}}(\tilde{R}(v,a,b))$ and the similar crossing event from bottom left to right side of $\tilde{R}(v,a,b)$. See also Figure 6.3(b).

The third claim follows also from the FKG inequality. Namely, if we put two trapezoid so that they partially overlap as in Figure 6.3(b) and form an equilateral rhombus in the middle, then by FKG inequality the joint occurrence of open crossings from left to right in both trapezoids and from bottom to top in the rhombus has probability at least the product of the probabilities of the three events. Thus the claim follows.

The last claim follows by combining all the other inequalities. □

The previous bounds will give us easily the following bounds.



(a) Arrowhead-shaped domain obtained from two rhombi symmetric to each other and the corresponding mirrored paths. By symmetry, the probability of a crossing from dark blue path to the solid black line within the domain is at least $\frac{1}{2}$ for $p \geq p_c$.

(b) Two open crossings in the trapezoid from a side to a bottom corner implies the left to right crossing event.

(c) Two congruent trapezoids are superimposed so that the intersection is a equilateral rhombus. The open crossings of these three shapes implies the open crossing of the resulting long rhombus.

Fig. 6.3 The proof of Lemma 6.8 is based on ideas presented in these figures.

Corollary 6.3 (Crossing probability of long rhombi or rectangles). *For any* $\rho \geq 1$, *there exists* $\varepsilon \in (0,1)$ *such that for every* $n \in \mathbb{Z}_{>0}$

$$\varepsilon \leq \mathsf{P}_{p_c}[\mathscr{S}_{\text{L-R}}(R(v, \lceil \rho n \rceil, n))] \leq 1 - \varepsilon.$$

Similar bound holds for rectangles.

Proof. The upper bound follows from the fact that $\mathsf{P}_{p_c}[\mathscr{S}_{\text{L-R}}(R(v, \lceil \rho n \rceil, n))]$ is bounded from above by $\mathsf{P}_{p_c}[\mathscr{S}_{\text{L-R}}(R(v,n,n))] = 1/2$.

The lower bound follow from $\mathsf{P}_{p_c}[\mathscr{S}_{\text{L-R}}(R(v, 2^k n, n))] \geq 32^{-k} \mathsf{P}_{p_c}[\mathscr{S}_{\text{L-R}}(R(v,n,n))]^{4k}$ and $\mathsf{P}_{p_c}[\mathscr{S}_{\text{L-R}}(R(v,n,n))] = 1/2$. □

Together with the FKG inequality this implies the following bound. Consider an annulus $A = A(z_0, r, R)$. We say that a path π *crosses*[7] A if π intersects both of the connected components of $\mathbb{C} \setminus A$. Denote the crossing event of $A(z_0, r, R)$ by an open percolation path as $\mathscr{S}(A(z_0, r, R))$. Similarly define a discrete annulus $\tilde{A}(v, n, m)$ as the set of sites whose distance on the lattice to v is not less than n or greater than m. Notice that $\tilde{A}(v, n, m)$ is the set between two concentric hexagonal boundary arcs of side lengths n and m lattice steps. Let $\mathscr{S}(\tilde{A}(v, n, m))$ be the crossing event of $\tilde{A}(v, n, m)$, that is, the event that there is an open path connecting the two boundary arcs.

Corollary 6.4. *(Crossing of discrete annuli) For any* $m \in \mathbb{Z}_{>0}$, *there exists* $\varepsilon \in (0,1)$ *such that for every* $n \in \mathbb{Z}_{>0}$

$$\varepsilon \leq \mathsf{P}_{p_c}[\mathscr{S}(\tilde{A}(v, n, mn))] \leq 1 - \varepsilon.$$

[7] Such a curve is called a *crossing* and a crossing that doesn't contain a proper subcrossing is called a *minimal crossing*.

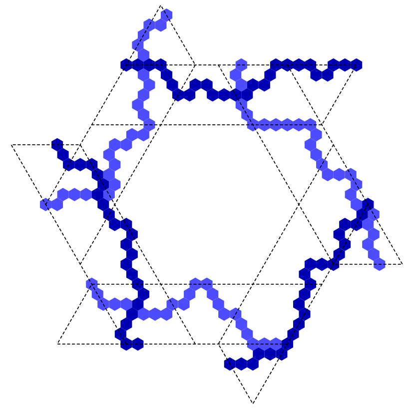


Fig. 6.4 Construction of a non-trial cycle in an annulus from six “left-to-right” crossing in rhombi.

Proof. For the lower bound, superimpose the annulus with a (long) rhombus, whose ends lie fully in the complement of the annulus, one inside and one outside. The bound follows from Corollary 6.3.

For the upper bound, notice that at p_c we can flip the state of each site (“open” between “closed”). Notice also that either there is a closed crossing of the annulus or an open path, which forms a loop that is non-contractible in the annulus. Next we construct such a non-trivial loop from open crossings of six rhombi by arranging them as in Figure 6.4.[8] The joint occurrence of those crossing events is bounded from below by the product of their probabilities by the FKG inequality. Thus the upper bound of the claim follows. □

Let’s call $A(z_1,r_1,R_1)$ a subannulus of $A(z_0,r,R)$ if $A(z_1,r_1,R_1) \subset A(z_0,r,R)$ and $A(z_1,r_1,R_1)$ separates the connected components of $\mathbb{C} \setminus A(z_0,r,R)$.

Corollary 6.5. *(Crossing of annuli) There exist constant K_1 and $\Delta_1 > 0$ such that for any $z_0 \in \mathbb{C}$ and $1 < r < R$,*

$$\mathsf{P}_{p_c}[\mathscr{S}(A(z_0,r,R))] \leq K_1 \left(\frac{r}{R}\right)^{\Delta_1} .$$

Proof. We will establish the bound when $R/r > 2$. For the complementary range the bound follows easily by choosing $K_1 > 2^{\Delta_1}$ which we can always do.

Since $r > 1$, there exists a lattice site v within the distance r from z_0 thus $A(v,2r,R/2)$ is a subannulus of $A(z_0,r,R)$. Here we used $R/r > 2$. Denote by $\tilde{B}(v,n)$ the discrete ball of radius r, i.e., the filled hexagon centered at v with sides on the lattices with length n. Let $n_0 = \lceil 4r/\sqrt{3} \rceil$ and k be the maximal integer such that $2^k(n_0+1) - 2 \leq R/2$, that is, $k = \lfloor \log_2((R+4)/(n_0+1)) \rfloor - 1$. Then since $\tilde{B}(v,n) \subset \overline{B(v,n)} \leq \tilde{B}(v, \lceil \frac{2n}{\sqrt{3}} \rceil)$, the discrete annuli $\tilde{A}_j := \tilde{A}(v, 2^j(n_0+1) - 2, 2^j(n_0+1) - 1)$,

[8] A small calculation shows that the long side of the rhombi has length $(2m-1)n$ and the short side $(m-1)n$.

$j = 0, 1, \ldots, k$, are subannuli of $A(z_0, r, R)$ and they are disjoint. By independence of percolation in disjoint sets,

$$\mathsf{P}_{p_c}[\mathscr{S}(A(z_0,r,R))] \leq \mathsf{P}_{p_c}[\bigcap_{j=0}^{k} \mathscr{S}(\tilde{A}_j)] = \prod_{j=0}^{k} \mathsf{P}_{p_c}[\mathscr{S}(\tilde{A}_j)] \leq (1-\varepsilon)^{k+1}. \tag{6.23}$$

Notice $n_0 + 1 \leq 4r/\sqrt{3} + 2 < 5r$ and $k + 1 \geq \log_2((R+4)/(n_0+1)) \geq \log_2(R/(5r))$.

$$(1-\varepsilon)^{k+1} \leq e^{\frac{\log \frac{1}{1-\varepsilon}}{\log 2}(\log 5 + \log \frac{r}{R})} = K_1 \left(\frac{r}{R}\right)^{\Delta_1}$$

where $\Delta_1 = \log_2 \frac{1}{1-\varepsilon}$ and $K_1 = \exp((\log_2 \frac{1}{1-\varepsilon})(\log 5))$ □

6.3.2.3 BK inequality and multiple crossings of quadrilaterals and annuli

Let A and B two events in a percolation model. We will denote by $A \square B$ the event that A and B both occur and they *occur disjointly*. We define it so that $\omega \in A \square B$ if and only if there exist disjoint sets F and G of sites (which might depend on ω) such that knowledge of ω restricted to F implies that $\omega \in A$ and knowledge of ω restricted to G implies that $\omega \in B$. For more details see [4], Section 4.3.

We will use below the following inequality. We omit the proof which can be found in [3] or [4].

Theorem 6.6 (van den Berg–Kesten (BK) inequality). *For increasing events A, B in a percolation model, it holds that* $\mathsf{P}[A \square B] \leq \mathsf{P}[A]\mathsf{P}[B]$.

Remark 6.7. For all events A, B, it holds that $A \square B \subset A \cap B$. It is natural to interpret the BK inequality as a counterpart for the FKG inequality.

Proposition 6.6. *(Disjoint open crossings of annuli) For each $C > 1$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{Z}_{>0}$ such that*

$$\mathsf{P}[\mathscr{S}_n(A(z_0,r,R))] \leq \varepsilon \tag{6.24}$$

for all $n \geq n_0$, $z_0 \in \mathbb{C}$, $r, R \in \mathbb{R}_{>0}$ such that $R/r \geq C$. Consequently, there exist non-negative constants Δ_n and K_n such that Δ_n tends to infinity as n tends to infinity and for every n, $z_0 \in \mathbb{C}$ and $0 < r < R$,

$$\mathsf{P}[\mathscr{S}_n(A(z_0,r,R))] \leq K_n \left(\frac{r}{R}\right)^{\Delta_n}.$$

Remark 6.8. We can choose Δ_n to be non-decreasing and by this result $\Delta_n > 0$ for all $n \geq n_0$ for some $n_n \in \mathbb{Z}_{>0}$.

Proof. The claim follows easily from Theorem 6.6 and Corollary 6.5 when $r > 1$. For $r \leq 1$, we need to notice that for $n \geq 10$, say, $\mathsf{P}[\mathscr{S}_n(A(z_0,r,R))] = 0$. □

6.3.3 Distortion estimates of annuli

Throughout this subsection U is a simply connected domain and $\phi : \mathbb{D} \to U$ is a conformal map. We study how annuli are distorted under conformal maps. We claim that the conformal image of an annulus has a subannulus (here we naturally extend the concept of a subannulus slightly) such that their conformal moduli, i.e., logarithms of ratios of the radii, are proportional. First we study annuli fully contained in the domain.

Lemma 6.9 (Distortion of annuli contained in $\mathbb{D}$). *For any $\rho' > 1$ sufficiently large, there exists $\rho > 1$ such that the following holds. Suppose U and ϕ are as above and that $A = A(z_0, r, R) \subset \mathbb{D}$ with $R/r > \rho$ and $R \leq \frac{1}{2}(1 - |z_0|)$. Then there exists an annulus $A' = A(z_0', r', R')$ with $R'/r' > \rho'$ such that $A' \subset \phi(A)$ and A' separates the components of $\phi(A)$ in $\mathbb{C}$. Furthermore the dependency of ρ and ρ' can be made linear.*

Proof. Let $A = A(z_0, r, R)$ be such that $|z_0| < 1$ and $0 < \rho^{-1} R < r < R < \frac{1}{2}(1 - |z_0|)$. Let ψ be a Möbius selfmap of $\mathbb{D}$ that sends z_0 to 0. As usual, the map ψ can be explicitly written and one can verify that $|\psi(z_0 + \tilde{r} e^{\mathrm{i}\theta})| = \tilde{r} / |1 - |z_0|^2 - \bar{z}_0 \tilde{r} e^{\mathrm{i}\theta}|$ and thus

$$\frac{1}{3} \frac{\tilde{r}}{1 - |z_0|} \leq |\psi(z_0 + \tilde{r} e^{\mathrm{i}\theta})| \leq \frac{\tilde{r}}{1 - |z_0|} \tag{6.25}$$

for all $\tilde{r} \in (0, 1 - |z_0|)$. This shows that there exists an annulus $A_1 = A_1(0, r_1, R_1)$ such that $A_1 \subset \psi(A)$ and $R_1/r_1 = (1/3)(R/r)$.

When $R \leq (1 - |z_0|)/2$, by the right-hand side of (6.25), we can choose $R_1 \leq \frac{1}{2}$. A similar argument to the above one using the Koebe distortion theorem, Theorem 3.9, shows that $\phi \circ \psi^{-1}$ distorts the two boundary components of A_1 proportionally. Consequently we can find $A' = A'(\phi(0), r', R')$ such that $A' \subset \phi(A)$ separates the boundary components of $\phi(A)$ and $r'/R' = \text{const.}(R_1/r_1)$ where the universal constant comes from the Koebe distortion theorem. □

Next we will apply extremal length (see also Section 3.4.2) to show a distortion estimate for annuli intersecting the boundary.

Lemma 6.10 (Distortion of annuli not fully contained in $\mathbb{D}$). *For any $\rho' > 1$ sufficiently large, there exists $\rho > 1$ such that the following holds. Suppose U and ϕ are as above and that $A = A(z_0, r, R)$ with $R/r > \rho$ is such that $1 - |z_0| < r$ and $R < 1$ (that is, $\partial\mathbb{D}$ crosses A). Then there exists an annulus $A' = A(z_0', r', R')$ with $R'/r' > \rho'$ such that there exists a connected component O of $U \cap A'$ such that $O \subset \phi(A \cap \mathbb{D})$ and O separates the components of $\phi((\partial A) \cap \mathbb{D})$ in U.*

Proof. The proof is based on the extremal length. The extremal length of the curve family connecting the components of $(\partial A) \cap \mathbb{D}$ in $A \cap \mathbb{D}$ is at least the extremal length of the curve family connecting the boundary components of A in A. The latter one is equal to $\frac{1}{2\pi} \log(R/r)$.

Let $Q = \phi(A \cap \mathbb{D})$. Then Q is a topological quadrilateral which then has four "marked" sides S_1, S_2, S_3, S_4. Suppose that $S_1 \cup S_3 = \phi((\partial A) \cap \mathbb{D})$. Let d_1 be the Euclidian distance inside Q from S_1 to S_3.

Let γ_1 be a path connecting S_2 and S_4 in Q whose length is (strictly) less than $2d_1$. Let z_0' to be its mid point (with respect to its length). Then $\gamma_1 \subset B(z_0, d_1)$.

Let γ be a path connecting S_1 and S_3 in Q whose diameter is d. Calculate a lower bound for the extremal length of the curve family connecting S_1 to S_3 in Q by using a metric ρ equal to 1 in a d_1-neighborhood of γ. Then the ρ-length of any curve connecting S_1 to S_3 in Q is at least d_1 and the ρ-area of Q is at most $(d+2d_1)^2$. Consequently the extremal length of the curve family is at least $d_1^2/(d+2d_1)^2$. By the reciprocity of the complementary extremal lengths in a topological quadrilateral, it holds now that

$$\frac{d_1^2}{(d+2d_1)^2} \leq \frac{2\pi}{\log(R/r)} =: m^{-2}. \tag{6.26}$$

Thus the diameter of γ satisfies $d \geq (m-2)d_1$. In particular, either for S_1 or S_3, it holds that any path from γ_1 to that arc has to have diameter at least $(d-d_1)/2 \geq ((m-3)/2)d_1$. Otherwise we could construct a path connecting S_1 or S_3 that has diameter less than d. Therefore γ has to intersect the complement of $B(z_0, ((m-3)/2)d_1)$.

Thus we have shown that any path connecting S_1 to S_3 in Q has to make at least one crossing of $A' = A'(z_0', d_1, ((m-3)/2)d_1)$. Let γ now be the path that makes the minimal number of such crossings.[9] Then each connected component of $U \cap A'$, whose closure contains a minimal subcrossing of γ, separates S_1 and S_3 in U. Choose any one them and denote it by O. Then O has the claimed properties. □

The same proof can be used for the following result. Let $\tilde{\phi} : \mathbb{H} \to U$ be a conformal, onto map.

Lemma 6.11 (Distortion of annuli not fully contained in $\mathbb{H}$). *For any $\rho' > 1$ sufficiently large, there exists $\rho > 1$ such that the following holds. Suppose U and ϕ are as above and that $A = A(z_0, r, R)$ with $R/r > \rho$ is such that $|\mathrm{Im}\, z_0| < r$ and $R > 0$ (that is, $\partial\mathbb{H}$ crosses A). Then there exists an annulus $A' = A(z_0', r', R')$ with $R'/r' > \rho'$ such that there exists a connected component O of $U \cap A'$ such that $O \subset \tilde{\phi}(A \cap \mathbb{H})$ and O separates the components of $\tilde{\phi}((\partial A) \cap \mathbb{H})$ in U.*

Let us now use Lemmas 6.9 and 6.10 to estimate the probability of the n-arms event in an annulus $A = A(z_0, r, R)$ by open paths of the percolation configuration transformed to $\mathbb{D}$ conformally. We call a path $x_k \in \mathbb{D}$, $k \in [\![0, n]\!]$, an open crossing of A in $\mathbb{D}$ if $\phi(x_k) \in \Omega$ are open sites of the triangular lattice of the percolation configuration in Ω, they form a lattice path (that is, $|\phi(x_{k+1}) - \phi(x_k)| = \delta$ where δ is the lattice mesh) and $|x_0 - z_0| \leq r$ and $|x_n - z_0| \geq R$. If there are n disjoint open crossings of A we say that (monochromatic) n-arms event occurs in A. Let $C > 1$ and define $m = \lfloor (\log(R/r))/(\log C) \rfloor$ and annuli

[9] The minimum number of crossings is finite since there are even smooth crossings such as any "radial" path $t \mapsto \phi(z_0 + te^{i\theta})$, $t \in (r, R)$ and $\theta \in \mathbb{R}$.

$$A_k = A\left(z_0, C^{k-\frac{2}{3}} r, C^{k-\frac{1}{3}} r\right), \qquad \hat{A}_j = A\left(z_0, C^{\frac{j-1}{3}} r, C^{\frac{j}{3}} r\right),$$

for $k \in [\![1,m]\!]$ and $j \in [\![1,3m]\!]$.

Let D_j be the infimum of $\operatorname{diam}(\phi(\gamma))$ where $\gamma \subset \mathbb{D}$ is a simple curve (which is open or close) that separates the components of $\mathbb{C} \setminus \hat{A}_j$. Similarly let L_j be the infimum of $\operatorname{diam}(\gamma)$ where $\gamma \subset \mathbb{D}$ is a simple curve that connects the components of $\mathbb{C} \setminus \hat{A}_j$. Then by a simple argument shows that $D_j \geq \delta$; otherwise there couldn't be $n \geq 2$ open crossings of $\hat{A}_j$. An argument using the extremal length similar to the proof of Lemma 6.10, shows that for any $M > 0$, there is C_0 such that if $C > C_0$, then $L_j/D_j \geq M$. Choose $M = 10$, say, and notice that this implies that $\phi(\mathbb{D} \cap \hat{A}_j)$ contains a path of neighboring hexagons that separate the components of $\phi(\mathbb{D} \setminus \hat{A}_j)$. Consequently, the crossing events in A_k for different $k \in [\![1,m]\!]$ are independent. Thus $\mathsf{P}[\mathscr{S}_n^{\mathbb{D}}(A)] \leq \prod_k \mathsf{P}[\mathscr{S}_n^{\mathbb{D}}(A_k)]$.

By Lemmas 6.9 and 6.10, there exists a constant $C' > 0$ such that we can select annuli $A'_k = A(z'_k, r'_k, C' r'_k)$ such that A'_k is a subannulus of $\phi(A_k)$. Let $\varepsilon > 0$. Apply Proposition 6.6 and in particular the inequality (6.24) to select n_0 such that $\mathsf{P}[\mathscr{S}_n^{\mathbb{D}}(A'_k)] \leq \varepsilon$ for all $n \geq n_0$. Then $\mathsf{P}[\mathscr{S}_n^{\mathbb{D}}(A)] \leq \varepsilon^m$.

Proposition 6.7 (Probability bound on multiple open crossings in $\mathbb{D}$). *There exist non-negative constants Δ_n and K_n such that Δ_n tends to infinity as n tends to infinity and for every n, $z_0 \in \mathbb{C}$ and $0 < r < R$,*

$$\mathsf{P}\left[\mathscr{S}_n^{\mathbb{D}}(A(z_0, r, R))\right] \leq K_n \left(\frac{r}{R}\right)^{\Delta_n}.$$

6.3.4 *Analysis of tortuosity*

6.3.4.1 Tortuosity and the connection to Hölder continuity

We follow here Aizenman's and Burchard's seminal paper [1]. Let's make the following definition for a curve $\gamma : [0,1] \to \mathbb{C}$. Define $[\gamma]$ to be the equivalence class of all reparameterizations of γ.[10] Define $M(\gamma, l)$ to be the minimum number of segments of γ with diameters less or equal to l that are needed to cover γ, which is a reparameterization invariant. A bound of the form

$$M(\gamma, l) \leq \frac{1}{\eta(l)}, \tag{6.27}$$

where $\eta : (0,1] \to (0,1]$ is a non-degreasing function, a *tortuosity* bound.

The following lemma establishes in one direction the connection between tortuosity bounds and Hölder continuity. For the other direction, see [1]. The proof of this lemma is given in Appendix D.

[10] That is, define $[\gamma] = \{\gamma \circ \phi : \phi : [0,1] \to [0,1] \text{ increasing homeomorphism}\}$.

Lemma 6.12. *If γ satisfies* (6.27)*, then it can be parametrized so that the parametrization $\tilde{\gamma} \in [\gamma]$ satisfies $\tilde{\eta}(|\tilde{\gamma}(t) - \tilde{\gamma}(s)|) \leq |t-s|$ where $\tilde{\eta}(y) = \frac{\eta(y/4)}{2\log(8/(y))^2}$.*

Remark 6.9. If $\eta(y) = Cy^{\alpha}$, then for each $\varepsilon > 0$ there exists a constant $\tilde{C}_\varepsilon$ such that $|\tilde{\gamma}(t) - \tilde{\gamma}(s)| \leq \tilde{C}_\varepsilon |t-s|^{\frac{1}{\alpha}-\varepsilon}$.

Definition 6.5. Suppose that $\varepsilon > 0$, $r_0 > 0$ and $k \in \mathbb{Z}_{>0}$. We say that γ has (ε, r_0, k)-*tempered crossing property* if for all $r \in (0, r_0)$ it holds that γ doesn't cross k or more times any annulus of the form $A = A(z_0, r^{1+\varepsilon}, r)$.

Define $N(\gamma, l)$ to be the minimal number of sets of diameter less or equal to l needed to cover γ. Then obviously $N(\gamma, l) \leq M(\gamma, l)$. The following result gives a complementary inequality.

Lemma 6.13. *If γ has the (ε, r_0, k)-tempered crossing property, then for any $l \in (0, r_0)$, $M(\gamma, 2l) \leq kN(\gamma, l^{1+\varepsilon})$.*

Proof. Cover γ by segments of diameter less than $2l$ recursively by choosing points $x_0, x_1, \ldots, x_n$ so that x_{k+1} is the first point after x_k that lies on the boundary of $B(x_k, l)$ and x_0 and x_n are the endpoints of γ. Then $M(\gamma, 2l) \leq n$.

Cover also γ with $N(\gamma, l^{1+\varepsilon})$ balls of diameter $l^{1+\varepsilon}$. Let $B = \overline{B(z_0, l^{1+\varepsilon}/2)}$ be one of those balls. Then since γ doesn't make k or more crossings of $A(z_0, (l/2)^{1+\varepsilon}, l/2)$, it holds that at most k of the points $x_0, x_1, \ldots, x_n$ can be contained in B. Therefore the claim follows. ☐

If the diameter of γ is at most R, then it can be covered with $\lceil R\sqrt{2}/l \rceil^2$ squares of diameter l. Therefore

$$N(\gamma, l) \leq 3R^2 l^{-2} \tag{6.28}$$

for all $l \in (0, R)$. Therefore the following result holds by Lemmas 6.12 and 6.13 and the remark after Lemma 6.12.

Proposition 6.8. *Let $\varepsilon > 0$. If γ is bounded, explicitly, $\gamma \subset B(z_0, R)$, and has (ε, r_0, k)-tempered crossing property for some $r_0 > 0$ and $k \in \mathbb{Z}_{>0}$, then for all $l \in (0, r_0)$ it holds that*

$$M(\gamma, l) \leq \tilde{C} k R^2 l^{-2(1+\varepsilon)}. \tag{6.29}$$

Here $\tilde{C}$ is an absolute constant.

Here is an interesting corollary, which we don't directly use, but which clarifies the role of tortuosity bounds.

Corollary 6.6. *Let $\varepsilon > 0$. If γ is bounded, explicitly, $\gamma \subset B(z_0, R)$, and has (ε, r_0, k)-tempered crossing property for some $r_0 > 0$ and $k \in \mathbb{Z}_{>0}$, then for each $\alpha \in (0, 1/(2+2\varepsilon))$, γ can be parametrized as $\gamma : [0,1] \to \mathbb{C}$ such that for all $s, t \in [0,1]$*

$$|\tilde{\gamma}(t) - \tilde{\gamma}(s)| \leq \tilde{C} |t-s|^{\alpha}. \tag{6.30}$$

Here $\tilde{C}$ doesn't depend directly on γ, but can depend on ε, r_0, k, R and α.

6.3.4.2 The percolation interface satisfies a tortuosity bound

Define γ to be the percolation interface in a discrete domain (U,a,b) at criticality, that is, when $p = p_c$. Let $\phi : \mathbb{D} \to U$ be a conformal, onto map such that $\phi(-1) = a$ and $\phi(+1) = b$. Define $\hat{\gamma} = \phi^{-1} \circ \gamma$. That is, γ is the interface on the original domain and $\hat{\gamma}$ is its conformal image on the unit disc.

Proposition 6.9. *There exists universal constants $\tilde{K}_n$ and $\tilde{\Delta}_n$ for each n such that $\tilde{\Delta}_n \to \infty$ as $n \to \infty$ and the following holds. Let $\hat{\gamma}$ be as above and $z_0 \in \mathbb{D}$ and $0 < r < R < 1$. Then*

$$\mathsf{P}[\hat{\gamma} \textit{ makes } n \textit{ crossings of } A(z_0,r,R)] \le \tilde{K}_n \left(\frac{r}{R}\right)^{\tilde{\Delta}_n}. \tag{6.31}$$

Proof. We may assume that R/r is sufficiently large that we can apply Lemma 6.9 and Lemma 6.10 below. Namely, for $R/r \le M$, we can arrange so that the inequality (6.31) holds by choosing $\tilde{K}_n$ larger than $M^{\tilde{\Delta}_n}$.

Let $d = 1 - |z_0|$. One of the following occurs: (i) $d < r$, (ii) $d \in [r,R]$ or (iii) $d > R$. In the cases (i) and (iii), set $r_1 = r$ and $R_1 = R$. In the case (ii), set $r_1 = \sqrt{rR}$ and $R_1 = R$, if $d < \sqrt{rR}$, and $r_1 = r$ and $R_1 = \sqrt{rR}$, otherwise. Then $r \le r_1 < R_1 \le R$, $r_1/R_1 \le \sqrt{r/R}$ and either $d \le r_1$ or $d \ge R_1$.

Let us then consider the crossing events of $\hat{\gamma}$. Suppose first that $d \le r_1$ and $\hat{\gamma}$ makes exactly n crossings of $A_1 := A(z_0,r_1,R_1)$. Then there are at least either $\lfloor n/2 \rfloor$ disjoint open crossing or $\lfloor n/2 \rfloor$ disjoint closed crossing, which are in addition disjoint from the boundary. We can suppose that they are open. Since the crossing are disjoint from the boundary we can change the state of the boundary sites, and we may as well suppose that all the boundary sites of U are closed. Then the crossings are automatically disjoint from the boundary. This leaves the probability unchanged. Thus

$$\mathsf{P}[\hat{\gamma} \text{ makes } n \text{ crossings of } A(z_0,r,R)] \le \mathsf{P}[\lfloor n/2 \rfloor \text{ open crossings of } A_1 \cap \mathbb{D}].$$

The same upper bound holds easily also when $d \ge R_1$; we don't have to worry about the boundary in that case.

Next we apply the conformal transformation ϕ. When $d \le r_1$, by Lemma 6.10, we can find z_2, r_2 and R_2 such that $R_2/r_2 \ge \textit{const}.(R_1/r_1)$ and $A_2 := A(z_2,r_2,R_2)$ is such that there exists a connected component O of $U \cap A_2$ such that $O \subset \phi(A \cap \mathbb{D})$ and O separates the components of $\phi((\partial A) \cap \mathbb{D})$ in U. It follows that if there are at least $\lfloor n/2 \rfloor$ open crossings of $A_1 \cap \mathbb{D}$, then there are at least $\lfloor n/2 \rfloor$ open crossings of A_2. We can remove all the closed boundary sites in A_2 and we get an upper bound by using the probability of at least $\lfloor n/2 \rfloor$ open crossings in the whole A_2 (without any boundary sites).

On the other hand, if $d \ge R_1$ apply Lemma 6.9, to show that there exists z_3, r_3 and R_3 such that $R_3/r_3 \ge \textit{const}.(R_1/r_1)$ and $A_3 := A(z_3,r_3,R_3)$ is a subannulus of $\phi(A(z_0,r_1,R_1))$. It follows that if there are at least $\lfloor n/2 \rfloor$ open crossings of $A_1 \cap \mathbb{D}$, then there are at least $\lfloor n/2 \rfloor$ open crossings of A_3.

The claim follows now from Proposition 6.6. □

Proposition 6.10. *Let $\varepsilon > 0$ and $k \in \mathbb{Z}_{>0}$ such that $\frac{\varepsilon}{1+\varepsilon}\Delta_k > 2$. Then there exists a random variable $r_0 > 0$ such that $\hat{\gamma}$ has (ε, r_0, k)-tempered crossing property. Furthermore, $\mathsf{P}[r_0 < r] \leq C r^{\varepsilon \Delta_k - 2(1+\varepsilon)}$ with some constant C.*

Proof. Let $\varepsilon > 0$ and $k \in \mathbb{Z}_{>0}$ be as in the statement of the proposition and let $I_n = ([-1,1] \cap 2^{-n}(\mathbb{Z} + \frac{1}{2}))^2$ for any $n \in \mathbb{Z}_{>0}$.

For any $r > 0$, let

$$n_r = \left\lfloor \log_2 \frac{1}{r^{1+\varepsilon}} \right\rfloor - 2.$$

Notice that holds that $2^{-n_r-3} < r^{1+\varepsilon} \leq 2^{-n_r-2}$. For any $z_0 \in \overline{\mathbb{D}}$, we can choose $z_1 \in I_{n_r}$ such that $|z_0 - z_1| < 2^{-n_r-1/2}$. Thus for any z such that $|z - z_1| \geq 2^{-n_r}$, it holds that $|z - z_0| > r^{1+\varepsilon}$. Similarly, $|z - z_0| \leq r$ for any z such that $|z - z_1| \leq \frac{1}{16} 2^{-\frac{n_r}{1+\varepsilon}}$ and for small enough r. Consequently, $A(z_1, 2^{-n_r}, \frac{1}{16} 2^{-\frac{n_r}{1+\varepsilon}})$ is a subannulus of $A(z_0, r^{1+\varepsilon}, r)$ for small enough r.

Therefore if we set

$$\eta = \sup \left\{ n \in \mathbb{Z}_{>0} : \begin{array}{c} A(z_1, 2^{-n}, (1/16) 2^{-n/(1+\varepsilon)}) \text{ contains} \\ k\text{-fold crossing for some } z_1 \in I_n \end{array} \right\}$$

then for any r such that $r^{1+\varepsilon} \leq 2^{-\eta-3}$ it holds that $A(z_0, r^{1+\varepsilon}, r)$ doesn't contain k-fold crossings for any $z_0 \in \mathbb{C}$. Then η is almost surely finite and has exponential tails, since

$$\begin{aligned} \mathsf{P}[\eta \geq n] &\leq \sum_{l=n}^{\infty} \sum_{z_1 \in I_l} \mathsf{P}[A(z_1, 2^{-l}, (1/16) 2^{-\frac{l}{1+\varepsilon}}) \text{ contains } k\text{-fold crossing}] \\ &\leq C_1 \sum_{l=n}^{\infty} 2^{(2 - \frac{\varepsilon}{1+\varepsilon}\Delta_k) l} = C_2 2^{(2 - \frac{\varepsilon}{1+\varepsilon}\Delta_k) n} \end{aligned}$$

when $\frac{\varepsilon}{1+\varepsilon}\Delta_k > 2$. This implies the claim. □

6.3.5 Regularity of the percolation interface in the capacity parametrization and existence of subsequent scaling limits

6.3.5.1 The speed of approach to the tip

Consider the simple curves in $\mathbb{D}$; more specifically, consider the collection

$$\left\{ \gamma \in C(\mathbb{R}_{\geq 0}, \mathbb{C}) : \begin{array}{c} \gamma(0) = -1, \gamma \text{ is simple}, \\ \gamma(t) \in \mathbb{D} \text{ for all } t > 0 \text{ and } \lim_{t \to \infty} \gamma(t) = +1 \end{array} \right\}. \qquad (6.32)$$

Remember that for a curve γ in the set (6.32), the standard way to transform γ to $\mathbb{H}$ is to define $\psi(z) = \mathrm{i}(z+1)/(1-z)$ and $\gamma_{\mathbb{H}} = \psi \circ \gamma$.

In this subsection, we study the following event.

Definition 6.6. Fix a (small) constant $\rho > 0$ and set $\tilde{B}_\rho = \psi^{-1}(\mathbb{H} \setminus B(0, \frac{1}{\rho}))$. Define a subset $E(r,R)$ of the set (6.32) such that there exists $s,t \in \mathbb{R}_{\geq 0}$ such that $s < t$ and the following statements are satisfied

- $\mathrm{diam}(\gamma[s,t]) \geq R$ and
- there exists a crosscut C in the domain $\mathbb{D} \setminus \gamma(0,s]$ such that $\mathrm{diam}(C) \leq r$ and C separates $\gamma(s,t]$ from $\tilde{B}_\rho$ in $\mathbb{D} \setminus \gamma(0,s]$.

Remark 6.10. The set $\tilde{B}_\rho$ is the intersection of the unit disc and a closed ball of radius $2\rho/(1-\rho^2)$. It is easy to verify that $B(1, \frac{2\rho}{1+\rho}) \subset \tilde{B}_\rho \subset B(1, \frac{2\rho}{1-\rho^2})$. We skip the details of this Möbius function calculation.

Remark 6.11. In fact, if $\gamma \in E(r,R)$, we can choose a pair (s,t) such that $\gamma(s)$ is one of the endpoints of C and $|\gamma(t) - \gamma(s)| \geq R/2$. This follows from the next lemma, by which $\gamma(u) \in \overline{C}$ for some $u \in [0,t]$ and then we can choose s to be maximal such u.

Lemma 6.14. *Let $R \geq 2r$ and $r < \min\{2, \rho\}$. Then $\overline{C} \cap \gamma[0,t] \neq \emptyset$.*

Proof. Assume the opposite, that is, that $\{x_1, x_2\} := \overline{C} \setminus C$ is a subset of $\partial\mathbb{D}$.

Write $\mathbb{D} \setminus C = D_1 \cup D_2$ where D_k are the connected components. such that $\mathrm{Length}(\partial D_1 \cap \partial\mathbb{D}) < \pi$ and $\mathrm{Length}(\partial D_2 \cap \partial\mathbb{D}) = 2\pi - \mathrm{Length}(\partial D_1 \cap \partial\mathbb{D}) > \pi$. Notice then that $C \subset \overline{B(x_1,r)}$ and thus $D_1 \subset B(x_1,r)$.

Since $B(1,\rho) \cap C = \emptyset$, it follows that $B(1,\rho) \cap \mathbb{D}$ is a subset of either D_1 or D_2. If $(B(1,\rho) \cap \mathbb{D}) \subset D_1$, then $\rho \leq r$, which is a contradiction. Therefore $(B(1,\rho) \cap \mathbb{D}) \subset D_2$. Since $\gamma[0,t] \cap C = \emptyset$, $\gamma[0,t]$ is in a similar manner a subset of either D_1 or D_2. Since C separates $\gamma[0,t]$ and $B(1,\rho)$ in $\mathbb{D}$, $\gamma[0,t] \subset D_1$.

Therefore $\mathrm{diam}(\gamma[0,t]) < 2r$. This is a contradiction and the claim follows. □

The following result connects the above annulus-crossing-type event $E(r,R)$ (we will clarify this later) to the speed of convergence of radial limit of a conformal map towards the tip of $\gamma_{\mathbb{H}}$.

Proposition 6.11. *There exists a constant $K > 0$ and an increasing function $\mu : [0,1] \to \mathbb{R}_{\geq 0}$ such that $\lim_{y \to 0} \mu(r) = 0$ and that the following holds. Let $r < \min\{2, \rho\}$, $R \geq 2r$ and γ be in (6.32). Assume that $\gamma(0,t] \subset \mathbb{D} \setminus \tilde{B}_{2\rho}$. If γ is not in $E(r,R)$, then*

$$\sup_{y \in (0,\mu(r)]} |\gamma_{\mathbb{H}}(t) - F(t,y)| \leq K \rho^{-2} R. \tag{6.33}$$

Remark 6.12. To apply the result, let $r = R^{1+\varepsilon}$ for some $\varepsilon > 0$ and let $y_0 = \mu(r)$. Then $K\rho^{-2} R = K\rho^{-2} r^{\frac{1}{1+\varepsilon}} = K\rho^{-2} (\mu^{-1}(y_0))^{\frac{1}{1+\varepsilon}} =: \lambda(y_0)$. Thus we can write (6.33) in the form

$$\sup_{y \in (0,y_0]} |\gamma_{\mathbb{H}}(t) - F(t,y)| \leq \lambda(y_0). \tag{6.34}$$

Proof. Let $\mu(r) = \exp\left(-\frac{2\pi^2}{r^2}\right)$. Fix $t \in \mathbb{R}_{\geq 0}$ and let $C_y = \{\psi^{-1} \circ f_t(W_t + ye^{i\theta}) : \theta \in (0,\pi)\}$ and $z_y = \psi^{-1} \circ f_t(W_t + iy)$.

By Lemma 4.6, for each $r > 0$, there exists $y_r \in [\mu(r), \sqrt{\mu(r)}]$ such that C_{y_r} has diameter less than r. Then by the assumption that $\gamma \notin E(r,R)$, the path with least diameter from z_{y_r} to $\gamma(t)$ has diameter at most $r + R < 2R$. By the Gehring–Hayman theorem, see [10], the diameter of $J := \psi^{-1} \circ f_t(\{W_t + iy : y \in (0, y_0]\})$ is less than KR, where K is an absolute constant.

Next notice that $\text{dist}(\mathbb{D} \setminus \tilde{B}_{2\rho}, \tilde{B}_\rho) > \rho$ and thus $J \subset \mathbb{D} \setminus \tilde{B}_\rho$. It is easy to verify that $|\psi'(z)| = 2/|1-z|^2$ and thus $\frac{1}{2} < |\psi'(z)| < \frac{1}{4}(1+\frac{1}{\rho})^2 < \frac{1}{\rho^2}$. Consequently, the diameter of the set $\psi(J) = f_t(\{W_t + iy : y \in (0, y_0]\})$ is at most $K\rho^{-2}R$. Hence (6.33) holds. ☐

The following definition enables the use of crossing probability bounds to establish a bound for the speed of convergence of the radial limit to the tip.

Definition 6.7. For $\tilde{\varepsilon} > 0$, r_0, $\rho > 0$, we say that the path $\hat{\gamma}$ has the $(\tilde{\varepsilon}, r_0, \rho)$-*tempered* 6-*fold crossing property* if for any $R < r_0$, we cannot find for any pair $(r,R) = (R^{1+\tilde{\varepsilon}}, R)$, parameters s,t so that the property of Definition 6.6 would hold.

Theorem 6.7. *For the site percolation interface $\hat{\gamma}$ (transformed to $\mathbb{D}$) for any $\tilde{\varepsilon} > 0$, there exists a tight random variable r_0 such that $\hat{\gamma}$ has the $(\tilde{\varepsilon}, r_0, \rho)$-tempered* 6*-fold crossing property.*

Proof. Let $r > 0$ and $R > 12r$. Let σ_k be defined by $\sigma_0 = 0$ and recursively by $\sigma_{k+1} = \sup\{t \geq \sigma_k : \text{diam}(\sigma[\sigma_k, t]) < R/4\}$. Then there exists finite random N such that $\sigma_{N-1} < \infty$, but $\sigma_N = \infty$. Let $J_k = \gamma[\sigma_{k-1}, \sigma_k]$, $k = 1,2,3,\ldots,N$ and let J_k, $k > N$, be a partition of $\partial\mathbb{D}$ into arcs of diameter at most $R/4$ — the number of such arcs can be chosen to be at most a constant times $1/R$. Observe that if the curve is divided into pieces that have diameter at most $R/4 - \varepsilon$, $\varepsilon > 0$, then none of these pieces can contain more than one of the $\gamma(\sigma_k)$. Therefore $N \leq \inf_{\varepsilon > 0} M(\gamma, R/4 - \varepsilon) \leq M(\gamma, R/8)$ where M is as in Section 6.3.4.1. By Propositions 6.8 and 6.10, N is a tight random variable, which we will use below.

Define also stopping times $\tau_{j,k} = \inf\{t \in [\sigma_{k-1}, \sigma_k] : \text{dist}_t(\gamma(t), J_j) \leq r\}$. for $k \in [\![1,N]\!]$ and $j \in [\![1,k-1]\!]$ or $j > N$. Here $\text{dist}_t(\gamma(t), A)$ is the infimum of the numbers l such that $\gamma(t)$ can be connected to the set A by a path of diameter less than l in $\mathbb{D} \setminus \gamma(0,t]$.

Suppose that the event $E(r,R)$ occurs. Take any C,s,t as in the definition of $E(r,R)$.

Let j,k be such that the end points of C are on J_j and J_k. Notice that $j \neq k$ and j,k can't both be larger than N and hence we can suppose that $k \in [\![1,N]\!]$ and $j \in [\![1,k-1]\!]$ or $j > N$. Also notice that $\text{dist}(J_j, J_k) \leq r$ and hence $\tau_{j,k}$ is finite.

Let $\tilde{C}$ be a path of diameter less than $2r$ in $\mathbb{D} \setminus \gamma(0, \tau_{j,k}]$ connecting $\gamma(\tau_{j,k})$ to J_j. We claim that $\tilde{C}$ disconnects $\gamma(t)$ from $+1$ in $\mathbb{D} \setminus \gamma(0, \tau_{j,k}]$ and that $|\gamma(t) - \gamma(\tau_{j,k})| > R/2$.

Let $\tilde{J}_j$ and $\tilde{J}_k$ be the subpaths of J_j and J_k, respectively, that connect an endpoint of C to an endpoint of $\tilde{C}$. Let Γ be the concatenation of C, $\tilde{J}_j$, $\tilde{C}$ and $\tilde{J}_k$ which closes to a loop. Then the points disconnected by C from $+1$ in $\mathbb{D}\setminus\gamma(0,s]$ but not by $\tilde{C}$ in $\mathbb{D}\setminus\gamma(0,\tau_{j,k}]$, form a subset of (closure of) interior of Γ.

Since Γ is contained in $\overline{B(\gamma(s),R/4+3r)}$ and $|\gamma(t)-\gamma(s)|\geq R$, $\gamma(t)$ is not in the closure of the interior of Γ. Consequently $\gamma(t)$ is disconnected by $\tilde{C}$ from $+1$ in $\mathbb{D}\setminus\gamma(0,\tau_{j,k}]$. In addition, by triangle inequality, $|\gamma(t)-\gamma(\tau_{j,k})|>R-(R/4+3r)>R/2$.

Let $A_{j,k}=A(\gamma(\tau_{j,k}),2r,R/r)$, $\tilde{A}_{j,k}=(A_{j,k}\cap(\mathbb{D}\setminus\gamma(0,\tau_{j,k}]))$ and

$$V_{j,k}=\left\{z\in\tilde{A}_{j,k}:\begin{array}{c}\text{connected component of } z \text{ in } \tilde{A}_{j,k} \text{ is disconnected}\\ \text{from } +1 \text{ by } \tilde{C} \text{ in } \mathbb{D}\setminus\gamma(0,\tau_{j,k}]\end{array}\right\}.$$

Notice that each component of $V_{j,k}$ has percolation boundary sites of only one type, either they are all open or all closed. Then on the event $E(r,R)$, when $C,\tilde{C},j,k$ are as above, then $\hat{\gamma}(u)$, $u>\tau_{j,k}$, crosses the annulus $A_{j,k}$ using the set $V_{j,k}$. As usual, this implies an open or closed percolation path crossing of $A_{j,k}$ in a component of $V_{j,k}$, whose boundary sites have the opposite state. By the RSW estimate of Corollary 6.5, using similarly Lemma 6.9 and Lemma 6.10 as in the proof of Proposition 6.9. we can show that

$$\mathsf{P}[\exists t\geq\tau_{j,k} \text{ s.t. } \gamma(\tau_{j,k},t] \text{ crosses } A_{j,k} \text{ in } V_{j,k}\,]\leq K_1\left(\frac{r}{R}\right)^{\Delta_1}.$$

By Proposition 6.8 and Proposition 6.10, summing over pairs (j,k) we find that

$$\mathsf{P}[E(r,R)]\leq const.\left(R^{\varepsilon\Delta_k-2(1+\varepsilon)}+R^{-4(1+\varepsilon)}\left(\frac{r}{R}\right)^{\Delta_1}\right)$$

If we choose $r=cR^{1+\tilde{\varepsilon}}$, $\tilde{\varepsilon}\in(0,\frac{4(1+\varepsilon)}{\Delta_1})$, then $\mathsf{P}[E(cR^{1+\tilde{\varepsilon}},R)]\leq CR^{\alpha}$ for some $\alpha>0$. If we set $R=2^{-n}$ and sum over n, we see that by the Borel-Cantelli lemma, there exists a random variable $r_0>0$ such that $\hat{\gamma}\notin E(R^{1+\tilde{\varepsilon}},R)$ for $R\in(0,r_0)$. □

6.3.5.2 Regularity of the driving process

Let's start by recalling some facts from the proof of Proposition 4.1. Notice firs that all γ in the set (6.32) once mapped to $\mathbb{H}$ conformally such that $+1$ is mapped on ∞ are eligible for description as Loewner chains. Therefore the conclusions in the proof of Proposition 4.1 apply to them.

Consider a simple curve γ of $\mathbb{H}$ of the Loewner type parametrized by the capacity. Let $0\leq s\leq t$ and define

$$\tilde{\gamma}_s(t)=g_s(\gamma(t))$$

Then by the proof of Proposition 4.1,

$$\max_{u\in[s,t]}\operatorname{Im}\tilde{\gamma}_s(u)\leq 2\sqrt{|t-s|},\qquad \max_{u\in[s,t]}|\operatorname{Re}\tilde{\gamma}_s(u)-W_s|\leq\max_{u\in[s,t]}|W_u-W_s|.$$

Therefore if $\gamma_s(u)$ exits the rectangle $[W_s - L, W_s + L] \times [0, 2\sqrt{|t-s|}]$ from the sides $\mathrm{Re}\, z = W_s \pm L$, then $\max_{u\in[s,t]} |W_u - W_s| \geq L$. Also a kind of a converse is true as shown next.

Lemma 6.15. $\frac{1}{2}\sup_{u\in[s,t]} |W(u) - W(s)| - 2\sqrt{|s-t|} \leq \sup_{u\in[s,t]} |\mathrm{Re}\,\tilde{\gamma}_s(u) - W(s)| \leq \sup_{u\in[s,t]} |W(u) - W(s)|$

Proof. The upper bound follows from the above considerations.

For the lower bound, assume without loss of generality that $s = 0$ and $W(0) = 0$. Then if $M = \|\mathrm{Re}\,\gamma\|_{\infty,[0,t]}$ and $R = \sqrt{M^2 + (2\sqrt{t})^2}$, then $\gamma[0,t] \subset \overline{B(0,R)}$. Consequently $W_u \in [g_u(-R), g_u(R)]$ by monotonicity of Loewner maps on the boundary. Next notice that $g_u(R) \leq \phi(R)$ and $g_u(-R) \geq \phi(-R)$ where $\phi(z) = z + R^2 z^{-1}$. Thus $-2R < g_u(-R) < W_u < g_u(R) < 2R$ and hence $\frac{1}{2}|W_u| \leq R \leq M + 2\sqrt{t}$. □

Proposition 6.12. *For percolation interface $\hat{\gamma}$, it holds that for each $\alpha < \frac{1}{2}$, there exists a random variable C_α such that $|W(t) - W(s)| \leq C_\alpha |t-s|^\alpha$ for $t,s \in [0,T]$. Furthermore, C_α is a tight random variable.*

Proof. Let us first show that there are constants K and ε such that

$$\mathsf{P}\left[\sup_{u\in[s,t]} |W(u) - W(s)| \geq L\right] \leq K \exp\left(-\varepsilon \frac{L}{\sqrt{|s-t|}}\right). \tag{6.35}$$

If $\sup_{u\in[s,t]} |W(u) - W(s)| \geq L$, then by Lemma 6.15 $\sup_{u\in[s,t]} |\mathrm{Re}\,\tilde{\gamma}_s(u) - W(s)| \geq \frac{L}{2} - 2\sqrt{|s-t|}$. On the other hand, $\sup_{u\in[s,t]} \mathrm{Im}\,\gamma_s(u) \leq 2\sqrt{|s-t|}$ always by (4.18).

Choose $\rho' > 0$ such that when $R/r > \rho'$ then the right-hand side of the inequality in Corollary 6.5 is less than e^{-1}. Then choose $\rho > 0$ such that ρ and ρ' are as in Lemma 6.11. Let $C_k = [x_k, x_k + \mathrm{i}2\sqrt{|s-t|}]$ where $x_k = 2\rho k\sqrt{|s-t|}$. By above if $\sup_{u\in[s,t]} |W(u) - W(s)| \geq L$, then $\gamma_s(u)$, $u \in [s,t]$, hits either all $C_1, C_2, \ldots, C_m$ or all $C_{-1}, C_{-2}, \ldots, C_{-m}$ where $m = \lfloor L/(2\rho\sqrt{|s-t|})\rfloor$. When $L/\sqrt{|s-t|}$ is large enough, then $m \geq L/(4\rho\sqrt{|s-t|})$. Thus $\mathsf{P}[\sup_{u\in[s,t]} |W(u) - W(s)| \geq L] \leq 2e^{-m}$ by Corollary 6.5 and Lemma 6.11, and (6.35) follows for some constant $\varepsilon > 0$.

Choose any $\alpha \in (0, 1/2)$ and set $\delta_n = T2^{-n}$ and $L_n = 2^{-\alpha n}$ for $n \in \mathbb{Z}_{>0}$. Then

$$\begin{aligned} &\mathsf{P}\left[\sup_{u\in[(k-1)\delta_n, k\delta_n]} |W(u) - W((k-1)\delta_n)| \geq L_n \text{ for some } k \in [\![1, 2^n]\!]\right] \\ \leq& 2^n K \exp\left(-\frac{\varepsilon}{\sqrt{T}} 2^{(\frac{1}{2}-\alpha)n}\right). \end{aligned} \tag{6.36}$$

Since the upper bound is summable over n, by the Borel–Cantelli lemma, there exists a random variable N such that for all $n \geq N$ and for all $k \in [\![1, 2^n]\!]$ it holds that

$$\sup_{u\in[(k-1)\delta_n, k\delta_n]} |W(u) - W((k-1)\delta_n)| < L_n. \tag{6.37}$$

Thus if $\delta_{n+1} < |u-v| \leq \delta_n$, then by the triangle inequality, $|W(u)-W(v)| \leq 3L_n = 3\,(2^{-n})^\alpha < 2^{2+\alpha}T^{-\alpha}|u-v|^\alpha$. Thus $|W(u)-W(v)| \leq 2^{2+\alpha}T^{-\alpha}|u-v|^\alpha$ for all $u,v \in [0,T]$ such that $|u-v| \leq \delta_N$.

Notice also that $\sup_{u\in[0,T]} |W(u)| \leq 2^N L_N$ which is finite. Thus the first claim of the proposition follows.

For the second claim, use (6.36) to give uniform estimates for $\mathsf{P}[N > n]$ as $n \to \infty$. □

6.3.5.3 Tightness of the capacity parametrized random curves

Remember that $\gamma_{\mathbb{D}}$ was the percolation interface transformed to $\mathbb{D}$. Let $\gamma_{\mathbb{H}}$ be $\phi(\gamma_{\mathbb{D}})$, where $\phi(z) = \mathrm{i}\frac{1+z}{1-z}$, reparameterized with capacity.

Remember also that a metric on $C(\mathbb{R}_{\geq 0}, \mathbb{C})$ is given by

$$\mathrm{d}(\gamma_1,\gamma_2) = \sum_{n=1}^{\infty} 2^{-n}(1 \wedge \|\gamma_1 - \gamma_2\|_{[0,n],\infty})$$

Theorem 6.8. *Let $\mathscr{D}$ be a collection of quadruplets (U,a,b,δ) and let $\mathscr{M}$ be the collection of all probability laws $\mu_{(U,a,b,\delta)}$ of $\gamma_{\mathbb{H}}$, where (U,a,b,δ) runs over all quadruplets in $\mathscr{D}$ and γ is the percolation interface in (U,a,b) with lattice mesh $\delta > 0$. Then the collection $\mathscr{M}$ is tight.*

Proof. Let $\varepsilon > 0$. We will find a relatively compact set E such that $\mu_{(U,a,b,\delta)}[E] \geq 1-\varepsilon$.

Let $n \in \mathbb{Z}_{>0}$. We will first consider $\gamma_{\mathbb{H}}(t)$, $t \in [0,n]$, and its driving term $W(t)$, $t \in [0,n]$. By Proposition 6.12, for any $\alpha > 0$, there exists a random variable $C_{n,\alpha}$ such that $|W(t)-W(s)| \leq C_{n,\alpha}\,|t-s|^\alpha$ for all $s,t \in [0,n]$. Notice also that $W(0) = 0$. Choose $m_{n,1} > 0$ such that $\mu_{(U,a,b,\delta)}[C_{n,\alpha} \leq m_{n,1}] \geq 1-\varepsilon 2^{-n-1}$ for all $(U,a,b,\delta) \in \mathscr{M}$. Then $\mu_{(U,a,b,\delta)}[C_{n,\alpha} \leq m_{n,1} \text{ for all } n] \geq 1-\varepsilon\sum_{n=1}^{\infty} 2^{-n-1} \geq 1-\varepsilon 2^{-1}$.

Let $f : (t,z) \to \mathbb{H}$ be the inverse (for fixed t) of the Loewner map $g : (t,z) \to \mathbb{H}$ as usual and $F(t,y) = f(t, W(t)+\mathrm{i}y)$. By Proposition 6.11 and Theorem 6.7, there exists a function $\mu : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a random variable $\tilde{C}_n$ such that $\lim_{y\to 0}\mu(y) = 0$ and $|F(t,y) - \gamma_{\mathbb{H}}(t)| \leq \tilde{C}_n \mu(y)$ for all $y \in \mathbb{R}_{>0}$ and $t \in [0,n]$. Choose $m_{n,2}$ such that $\mu_{(U,a,b,\delta)}[\tilde{C}_n \leq m_{n,2}] \geq 1-\varepsilon 2^{-n-1}$ for all $(U,a,b,\delta) \in \mathscr{M}$. Then it holds that $\mu_{(U,a,b,\delta)}[\tilde{C}_n \leq m_{n,2} \text{ for all } n] \geq 1-\varepsilon\sum_{n=1}^{\infty} 2^{-n-1} \geq 1-\varepsilon 2^{-1}$.

Let E be the event that $\gamma_{\mathbb{H}}$ satisfies $C_{n,\alpha} \leq m_{n,1}$ and $\tilde{C}_n \leq m_{n,2}$ for all n. By above, $\mu_{(U,a,b,\delta)}[E] \geq 1-\varepsilon$. The claim of the theorem follows if we manage to show that E is relatively compact.

Let γ_k be a sequence in E. Since W_k are all α-Hölder continuous with the Hölder norm bounded by $m_{n,1}$, we can extract a converging subsequence W_{k_j} by the Arzelà–Ascoli theorem. Furthermore, by a standard diagonal argument we can suppose that W_{k_j} converges uniformly on each $[0,n]$. In particular, W_{k_j} is a Cauchy sequence on each $[0,n]$.

Let $\tilde{\varepsilon} > 0$. Let $\delta > 0$ and $n_0 \in \mathbb{Z}_{>0}$ be such that $\lambda(\delta) < \tilde{\varepsilon}$ and $2^{-n_0} < \tilde{\varepsilon}$, where $\lambda(\delta)$ is as in Lemma 6.4. By passing to a subsequence, we can suppose that for any $i, j \geq n_0$ and $n \leq n_0$ it holds that $\|W_{k_i} - W_{k_j}\|_{[0,n],\infty} \leq (C(n,\delta))^{-1}\tilde{\varepsilon}$ where $C(n,\delta)$ is as in Lemma 6.4. By Lemma 6.4,

$$\|\gamma_{k_i} - \gamma_{k_j}\|_{\infty,[0,n]} \leq C(n,\delta)\|W_{k_i} - W_{k_j}\|_{[0,n],\infty} + 2\lambda(\delta).$$

Therefore

$$\mathrm{d}(\gamma_{k_i}, \gamma_{k_j}) \leq 2^{-n_0} + \sum_{n=1}^{n_0} 2^{-n}(C(n,\delta)\|W_{k_i} - W_{k_j}\|_{[0,n],\infty} + 2\lambda(\delta)) \leq 4\tilde{\varepsilon}.$$

Hence γ_{k_j} is a Cauchy sequence and converges in $C(\mathbb{R}_{\geq 0}, \mathbb{C})$ which is a complete metric space. Thus E is a relatively compact set. The claim follows. □

6.3.6 Cardy–Smirnov formula of a crossing probability

In this section, we present the full argument showing the convergence of the percolation interface γ_{δ_n} to a SLE(6) random curve γ. We use Smirnov's very readable original papers [11, 12] and Beffara's equally excellent note [2] on the convergence of so called discrete martingale observables of the site percolation model. We take the liberty to omit some details, but we will state explicitly when we do so. We are careful in particular on the mode of convergence of the observable which is needed in passing to the limit with a martingale property.

6.3.6.1 Introduction: boundary value problems and martingales

We have already developed the theory of regularity of random curves, which ensures the convergence of the random curves along subsequences by a compactness argument. The convergence of the entire sequence is thus equivalent to the uniqueness of the limit. Hence to complete the approach we need for each $t \in \mathbb{R}_{\geq 0}$, a random variable

$$\gamma \mapsto X(t,z;\gamma)$$

where $z \in \mathbb{C}$ is a free variable which we can vary, such that $X(t,z;\gamma)$ depends non-trivially on z and $\gamma[0,t]$ and its law can be efficiently written and analyzed. Not surprisingly, if you compare this approach to that of the characteristic function of a real-valued random variable, this is sufficient for characterizing the law of γ.

In many cases including the case of the percolation interface, the *observable* $X(t,z;\gamma)$ can be written as a solution of a boundary value problem as $X(t,z;\gamma) = h(z)$ where h is defined in the following way. First of all, h is the scaling limit $h = \lim_{\delta_n \to 0} H_{\delta_n}$ of a *discrete observable* H_{δ_n}, which solves a corresponding discrete boundary value problem and is a natural percolation quantity, namely, a crossing probability. Secondly, the continuum boundary value problem solved by h can be formulated in the following way. Denote by U_t the connected component containing a neighborhood of the arc bc in $U \setminus \gamma(0,t]$ and by a_t the point $\gamma(t)$. The continuum boundary value problem for percolation is the following

$$\begin{cases} \Delta h = 0 & \text{, in } U_t \\ h(a_t) = 1 & \\ h = 0 & \text{, in } bc \\ \partial_u h = 0 & \text{, in } (a_t b) \cup (c a_t) \end{cases} \tag{6.38}$$

where ∂_u is the directional derivative in direction $u = (\tau/\nu)^{\frac{1}{3}}\,\nu$ where ν is the outward normal and τ is the tangent of ∂U_t at the boundary point; the boundary is oriented from a_t to bc on both arcs $a_t b$ and $c a_t$ and the third root is defined such that $(e^{\mathrm{i}\theta})^{\frac{1}{3}} = e^{\frac{\mathrm{i}\theta}{3}}$ when $\theta \in (-\pi,\pi)$.

An essential property of $X(t,z;\gamma)$ that we need below, is that it satisfies a martingale property. For the discrete observable $X_n(\tau_n(t),z) = H_{\delta_n}(z)$, as we shall see, it holds that

$$\mathsf{E}_n[X_n(\tau_n(t),z) \,|\, \mathscr{F}_s] = X_n(\tau_n(s),z) \tag{6.39}$$

where $\tau_n(t)$ is the least discrete time such that the path $\gamma_{\mathbb{H}}$ has capacity greater or equal to t.

Let's extend the martingale property to the scaling limit of the observable. For that we need some assumptions on the mode of its convergence. Let E be an event such that

$$\mathsf{P}_n[E] > 1 - \varepsilon \tag{6.40}$$

for all $n \geq n_0$ and suppose that on the event E it holds that

$$\sup_{\gamma \in \operatorname{supp}(\mathsf{P}_n) \cap E} |X_n(\tau_n(t), z_n; \gamma) - X(t,z;\gamma)| < \varepsilon \tag{6.41}$$

for all $n \geq n_0$. We can furthermore set $E_n = \operatorname{supp}(\mathsf{P}_n) \cap E$ and then (6.40) holds when E is replaced by E_n.

Let $u \in [s,\infty)$ and f be a continuous, bounded $\mathscr{F}_s$-measurable random variable. By scaling we may assume that $|f| \leq 1$, $|X_n| \leq 1$ and $|X| \leq 1$. Then

$$\begin{aligned} &\int X_n(\tau_n(u),z)\, f \,\mathrm{d}P_n = \int \mathbb{1}_{E_n} X_n(\tau_n(u),z)\, f \,\mathrm{d}P_n + \text{error} \\ &= \int \mathbb{1}_{E_n} X(u,z)\, f\,\mathrm{d}P_n + \text{error} = \int X(u,z)\, f\,\mathrm{d}P_n + \text{error} \\ &= \int X(u,z)\, f\,\mathrm{d}P + \text{error}. \end{aligned} \tag{6.42}$$

On each line, the error is at most of the order ε (bounded from above by a universal constant times ε). The first and third equality uses (6.40). The second one uses (6.41) and the fourth one follows from the convergence of P_n to P in the sense of weak convergence of probability measures. The martingale property (6.39) is by definition equivalent to the fact that

$$\int X_n(\tau_n(t),z)\, f\, \mathrm{d}P_n = \int X_n(\tau_n(s),z)\, f\, \mathrm{d}P_n \tag{6.43}$$

holds for all continuous, bounded $\mathscr{F}_s$-measurable random variables h. Using the estimate (6.42) for both $u=t$ and $u=s$, we find that

$$\left| \int X(t,z)\, f\, \mathrm{d}P - \int X(s,z)\, f\, \mathrm{d}P \right| \le C \|f\|_\infty \varepsilon \tag{6.44}$$

holds for all continuous, bounded $\mathscr{F}_s$-measurable random variables f. Here C is a absolute constant. Since ε is arbitrary it holds that,

$$\mathsf{E}[X(t,z) \,|\, \mathscr{F}_s] = X(s,z). \tag{6.45}$$

That is, $X(t,z)$ as a stochastic process is a martingale.

In next subsections, we define the observable more carefully and establish most important properties, namely, the martingale property (6.39) and the uniform convergence (6.41).

6.3.6.2 The discretization of the domain

We aim to define a curve γ in a domain U from a boundary point a to another boundary point b. We need further third boundary point c and a point z in $\overline{U}$ to characterize the law of γ. We proceed in the following steps:

- Let $\mathbb{L}_{\mathrm{tri}}$ be the triangular lattice and $\mathbb{L}_{\mathrm{hex}}$ be its dual, the hexagonal lattice. Let
$$\tau = e^{\mathrm{i}2\pi/3}.$$
- Let U be any bounded simply connected domain and a,b its distinct boundary points.
- Choose a sequence $(U_{\delta_n}, a_{\delta_n}, b_{\delta_n})$ where δ_n is the lattice mesh tending to 0 as n tends to ∞ and U_{δ_n} a bounded, simply connected domain such that ∂U_{δ_n} is a lattice path on $\delta_n \mathbb{L}_{\mathrm{hex}}$ and $a_{\delta_n}, b_{\delta_n})$ its boundary points which are assumed to be sites on $\delta_n \mathbb{L}_{\mathrm{hex}}$.
- Map next (U,a,b) conformally onto $(\mathbb{H},0,\infty)$.
- Pick $-l \in \mathbb{R}_{<0}$ and $m \in \mathbb{H} \cup \mathbb{R}_{>0}$. Then map back to U using the inverse of the conformal map. Denote the image of $-l$ by c and m by z.
- Choose c_{δ_n} in the arc $b_{\delta_n} a_{\delta_n}$ and z_{δ_n} in the union of the arc $a_{\delta_n} b_{\delta_n}$ and U_n such that they are sites on $\delta_n \mathbb{L}_{\mathrm{hex}}$.
- Let ϕ_{δ_n} and ϕ be the conformal and onto maps from $(\mathbb{D}, 1, \tau, \tau^2)$ onto the domains $(U_{\delta_n}, a_{\delta_n}, b_{\delta_n}, c_{\delta_n})$ and (U,a,b,c), respectively. Those maps are unique. Next we

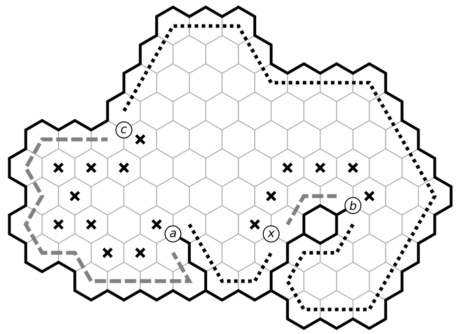


Fig. 6.5 The microscopic details of discrete domain. Notice that we arrange so that a and b are boundary points — this is needed, if we want the interface to be well-described by a chordal Loewner chain. The points c, z or x are interior points, but c and z, when $z = x$, are points "next to the boundary."

make a critical assumption that $(U_{\delta_n}, a_{\delta_n}, b_{\delta_n}, c_{\delta_n})$ converges to (U, a, b, c) in the Carathéodory sense, that is, $\phi = \lim_{n\to\infty} \phi_{\delta_n}$. Assume also that $z = \lim_{n\to\infty} z_{\delta_n}$.

- Let V_{δ_n} and V_{1,δ_n} be as in Section 6.3.1.1. Then V_{δ_n} is the set of sites where we consider the percolation configuration and V_{1,δ_n} is the set of boundary sites where we apply the chosen boundary conditions.

Let us shorten the notation by dropping n from δ_n.

6.3.6.3 The crossing event of Cardy–Smirnov formula

Let U, a, b, c, z, U_δ and $a_\delta, b_\delta, c_\delta, z_\delta$ be as above. Assume first that z_δ is a vertex of a hexagon in V_δ.

Definition 6.8. Define an event $E_{a,\delta}(z_\delta)$ that there exists a simple open path on V_δ that separates a_δ and z_δ from b_δ and c_δ. More specifically, this means that there exists a simple path $\pi = (x_0, x_1, \ldots, x_n, x_{n+1})$ such that (i) $x_1, \ldots, x_n$ are in V_δ and their state is open, (ii) x_0 and x_1 are in $V_{1,\delta}$, the edge $\{x_0, x_1\}$ crosses the arc $a_\delta b_\delta$ and the edge $\{x_n, x_{n+1}\}$ crosses the arc $c_\delta a_\delta$, and (iii) the union of the hexagons with centers $x_0, x_1, \ldots, x_n, x_{n+1}$ disconnects z_δ from $b_\delta c_\delta$ in U_δ. Define similarly $E_{b,\delta}(z)$ and $E_{c,\delta}(z)$ by cyclically permuting the points $a_\delta, b_\delta, c_\delta$.

Define $H_{a,\delta}(z_\delta)$ as the probability of the event $E_{a,\delta}(z_\delta)$ and similarly $H_{b,\delta}(z_\delta)$ and $H_{c,\delta}(z_\delta)$. Then $H_{a,\delta} = 0$ on the boundary arc $b_\delta c_\delta$ and as we shall see $H_{a,\delta}(a_\delta) \approx 1$.

Suppose that the set of z_δ's that we have defined $H_{a,\delta}$ is denoted by W. When z_δ is a vertex of some hexagon whose center is in $V_{1,\delta}$ and none whose center is in V_δ, define $H_{a,\delta}(z_\delta)$ to be equal to the value of $H_{a,\delta}$ at the neighboring site in W, if an edge like that exists (there is at most one). If the edge doesn't exists, fix an arbitrary rule (which can depend for instance on the local shape of the boundary) for calculating the value as a convex combination of the values of $H_{a,\delta}$ on the sites of W in the hexagon of z_δ. Threat z_δ in this case as a generalized boundary point (as a limit of a sequence of interior points in a topology that separates the different sides of the boundary). As a consequence $H_{a,\delta}$ is now defined on all vertices of the hexagons whose centers are in $V_\delta \cup V_{1,\delta}$ and we are in good shape to define it in all points of $\overline{U_\delta}$ as we will do in the next subsection.

Define functions

$$H_\delta = H_{a,\delta} + \tau H_{b,\delta} + \tau^2 H_{c,\delta}, \qquad S_\delta = H_{a,\delta} + H_{b,\delta} + H_{c,\delta}. \tag{6.46}$$

Suppose now that z is on the boundary arc $a_\delta b_\delta$ (similar statement holds for $c_\delta a_\delta$). Then $H_{a,\delta}(z)$ is equal to the probability that in V_δ there exists an open path from the arc $c_\delta a_\delta$ to the arc $z b_\delta$. The next lemma follows since for instance, on $a_\delta b_\delta$, $E_{a,\delta}(z)$ and $E_{b,\delta}(z)$ are almost complementary events.

6.3.6.4 A continuous extension of a discrete function

Use the following (one of many) construction that extends a function f defined on a set of sites of a simply connected subgraph of a planar lattice to the closed set of points consisting of the union of the closed faces of the graph. For a neighboring pair of sites x, y, extend f on the edge (line segment) between them linearly, that is, $f(tx+(1-t)y) = t\,f(x) + (1-t)\,f(y)$ for any $t \in [0,1]$. After this step, f is defined on all edges of the graph and thus on the boundary of any face of the graph.

Extend f inside each of the faces using only the values on the boundary of that face. Explicitly, we can use the *harmonic extension* of f inside each face. A property of the chosen extension is that $|f(x) - f(y)|$ is maximized over a face when x and y are sites of the lattice. Since the diameter of a hexagon in $\delta\mathbb{L}_{\text{hex}}$ is equal to $2\delta/\sqrt{3}$, we get the following lemma.

Lemma 6.16. *For any $r > 0$ and $z_0 \in \mathbb{C}$,*

$$\sup_{x,y \in B(z_0,r)} |f(y) - f(x)| \le \max_{v,w \in V(\delta\mathbb{L}_{\text{hex}}) \cap B(z_0, r+2\delta/\sqrt{3})} |f(v) - f(w)|.$$

6.3.6.5 Equicontinuity of discrete observables and uniform convergence

Next we will establish the convergence of $H_{a,\delta}, H_\delta, S_\delta, \ldots$ as δ tends to 0. First we will establish equicontinuity. For a domain (U,a,b,c) and $z \in U$, define

$$d_{(U,a,b,c)}(z) := \max\{\text{dist}(z,AB), \text{dist}(z,BC), \text{dist}(z,CA)\}. \tag{6.47}$$

Lemma 6.17. *Let $0 < r < R$ and $0 < m < \frac{1}{100}$. For any $\varepsilon > 0$, there exists $\eta > 0$ such that the following holds. If $(U,a,b,c;\delta)$ is a discrete domain such that $\delta < mr$,* $\text{diam}(U) < R$ *and* $\inf\{d_{(U,a,b,c)}(z) : z \in \overline{U}\} > r$, *and $\phi : \mathbb{D} \to U$ a conformal and* *onto map such that* $\phi(1) = a$, $\phi(\tau) = b$ *and* $\phi(\tau^2) = c$, *then*

$$|H_{A,\delta} \circ \phi(z) - H_{A,\delta} \circ \phi(w)| < \varepsilon \tag{6.48}$$

for all $z, w \in \overline{\mathbb{D}}$ *such that* $|z - w| < \eta$.

Proof. Let $\tilde{\varepsilon} > 0$ be much smaller than r and let $0 < \eta < \frac{1}{2}$ be such that

$$2\pi R / \sqrt{\log(1/\eta)} < \tilde{\varepsilon}.$$

Then by Lemma 4.6, it holds that Length$[\phi(\mathbb{D} \cap \partial B(z,r))] < \tilde{\varepsilon}$ for some $\rho \in [\eta, \frac{1}{2}]$. Denote $\phi(\mathbb{D} \cap \partial B(z,\rho))$ by C. Let $z_0 \in C$. Then $C \subset B(z_0, \tilde{\varepsilon})$.

It is sufficient to establish that when $\tilde{\varepsilon} > \delta$, there are universal constants K and Δ such that

$$|H_{A,\delta} \circ \phi(z) - H_{A,\delta} \circ \phi(w)| \le K \left(\frac{\tilde{\varepsilon}}{r} \right)^{\Delta}. \tag{6.49}$$

Namely, if $\tilde{\varepsilon}$ is chosen to be less than $r(\varepsilon/K)^{1/\Delta}$ and $\delta < \tilde{\varepsilon}$, the claim follows directly. On the other hand, if $\delta \ge \tilde{\varepsilon}$, then C is fully contained in the union of at most three neighboring hexagons (meeting at a common vertex). In that case, there are two options: either C is a closed loop and $\phi(z)$ and $\phi(w)$ are in its interior or C is an open path with endpoints on the boundary of the domain (and on boundaries of hexagons) and $\phi(z)$ and $\phi(w)$ are contained in the interior of the concatenation of C and the shortest path along the hexagonal boundaries connecting the endpoints. In both cases, it follows that $\phi(z)$ and $\phi(w)$ are in the union of those three hexagons mentioned above. Consequently, C can be replaced by a lattice path with identical properties, except that its length is of the order δ. If (6.49) holds for z and w replaced by (points mapped to) vertices of the hexagonal lattice (of mesh size δ) and $\tilde{\varepsilon}$ by δ on its right, then using harmonic extension property, the claim follows for the case $\delta \ge \tilde{\varepsilon}$. The bound (6.49) follows in this case in the same way as we will show it below for the other case. We leave filling the full details for the case $\delta \ge \tilde{\varepsilon}$ to the reader.

Since $d_{(U,a,b,c)}(z_0) \ge r$, we can assume that either $\mathrm{dist}(z_0, AB) \ge r$ or $\mathrm{dist}(z_0, BC) \ge r$. The case $\mathrm{dist}(z_0, CA) \ge r$ is symmetric to the former one.

Suppose that $\mathrm{dist}(z_0, AB) \ge r$. Then in the annulus $A(z_0, \tilde{\varepsilon}, r)$ there is a closed non-trivial loop in the full-plane percolation configuration with high probability. This closed path disconnects $\phi(z)$ and $\phi(w)$ from AB. Consequently on that event, either $E_{A,\delta}(\phi(z))$ and $E_{A,\delta}(\phi(w))$ both occur or they both fail to occur. Thus the bound (6.49) follows from the RSW estimate for open crossings in annuli.

Suppose next that $\mathrm{dist}(z_0, BC) \ge r$. There is an open non-trivial loop in $A(z_0, \tilde{\varepsilon}, r)$ with high probability and thus similarly as above, (6.49) follows. □

The very same proof gives the following lemma which is a key lemma for verifying the boundary conditions of $H_{a,\delta}$, H_δ and S_δ.

Lemma 6.18. *For each* r, R, m *as in the previous lemma, there exists a constant* C *such that the following holds. Let* $(U, a, b, c; \delta)$ *be as in the previous lemma. Then*

$0 \le H_{a,\delta} \circ \phi(z) \le C(\log \frac{1}{\varepsilon})^{-1/2}$ *for all* $z \in \overline{\mathbb{D}}$ *such that* $|z| > 1-\varepsilon$ *and* $\arg z \in (2\pi/3 - \varepsilon, 4\pi/3 + \varepsilon)$ *and* $1 - C(\log \frac{1}{\varepsilon})^{-1/2} \le H_{a,\delta} \circ \phi(z) \le 1$ *for all* $z \in \overline{\mathbb{D}}$ *such that* $|z| > 1-\varepsilon$ *and* $\arg z \in (-\varepsilon, +\varepsilon)$.

This implies the next result.

Lemma 6.19. *On the boundary,* $S_\delta = 1 + o(1)$ *uniformly as* $\delta \to 0$. *Values of* H_δ *at boundary points are at most at the distance* $o(1)$ *from the boundary of equilateral triangle* $(1, \tau, \tau^2)$ *and the values are in the same order as the boundary points and include the three vertices of the triangle.*

Remark 6.13. Define the set of points W_δ such that $\phi_\delta(z)$ is a boundary point of the union of the hexagons with centers in V_δ for any $z \in W_\delta$. Then for $z \in W_\delta$, $|z| = 1 + o(1)$ uniformly as $\delta \to 0$, for consecutive points, $|z_{k+1} - z_k| = o(1)$ and H_δ is monotonic on the boundary in the following sense: if z_k, $k = 1, 2, \dots, n$ corresponds to $a_\delta b_\delta$, then $H_{a,\delta} \circ \phi_\delta(z_k)$ is monotonic decreasing and $H_{b,\delta} \circ \phi_\delta(z_k)$ is monotonic increasing for $k = 1, 2, \dots, n$. Similarly for the other boundary arcs.

Finally, the next theorem gives the convergence of the observables.

Theorem 6.9. *Let* $R > 0$. *Let* (U, a, b, c) *is a bounded domain and* $(U_\delta, a_\delta, b_\delta, c_\delta; \delta)$ *is a discrete domain such that* $U, U_\delta \subset B(0, R)$ *If* $(U_{\delta_n}, a_{\delta_n}, b_{\delta_n}, c_{\delta_n})$ *converges to* (U, a, b, c) *and* $\phi_n : \mathbb{D} \to U$ *are the corresponding conformal and onto maps (such that* $\phi_n(1) = a_{\delta_n}$, $\phi_n(\tau) = b_{\delta_n}$ *and* $\phi_n(\tau^2) = c_{\delta_n}$*), then* $H_{a,\delta_n} \circ \phi_n$ *converges uniformly on* $\overline{\mathbb{D}}$ *to* h_a *which is equal to* $(2\,\mathrm{Re}\,\Phi(z) + 1)/3$.

Proof. Let (U, a, b, c) is a bounded domain $U \subset B(0, R)$. Then there is $r \in (0, R)$ such that $\operatorname{diam}(U) < R$ and $\inf\{d_{(U,a,b,c)}(z) : z \in \overline{U}\} > r$. Therefore for a sequence $(U_{\delta_n}, a_{\delta_n}, b_{\delta_n}, c_{\delta_n})$ of discrete domains such that $U_{\delta_n} \subset B(0, R)$ it holds for n large enough that $\inf\{d_{(U_{\delta_n}, a_{\delta_n}, b_{\delta_n}, c_{\delta_n})}(z) : z \in \overline{U_{\delta_n}}\} > r$.

By Lemma 6.17, the sequence $H_{a,\delta_n} \circ \phi_n$ is an equicontinuous family of functions on the compact set $\overline{\mathbb{D}}$ and it is also uniformly bounded as $0 \le H_{a,\delta_n} \le 1$. Thus by Arzelà–Ascoli theorem, there exists a subsequence $H_{a,\delta_{n_k}} \circ \phi_{n_k}$ such that it converges to a continuous function h_a uniformly on $\overline{\mathbb{D}}$. By the same argument, we can suppose that $H_{b,\delta_{n_k}} \circ \phi_{n_k}$ and $H_{c,\delta_{n_k}} \circ \phi_{n_k}$ converge uniformly to continuous functions h_b and h_c, respectively.

By Lemma 6.18, $h_a(1) = 1$ and $h_n(e^{\mathrm{i}\theta}) = 0$, $\theta \in [2\pi/3, 4\pi/3]$. Moreover $s = \lim_{k\to\infty} S_{\delta_{n_k}}$ and $h = \lim_{k\to\infty} H_{\delta_{n_k}}$ satisfy $s = 1$ on $\partial\mathbb{D}$ and $\theta \mapsto h(e^{\mathrm{i}\theta})$, $\theta \in [0, 2\pi]$, is closed loop whose trace is the equilateral triangle $(1, \tau, \tau^2)$ and which winds around the origin once in the counterclockwise direction.

By the argument presented in Beffara's note [2], the functions s and h are holomorphic in $\mathbb{D}$. Consequently, $s = 1$ everywhere and h is the conformal map from $\mathbb{D}$ onto the equilateral triangle $(1, \tau, \tau^2)$ such that $h(\tau^k) = \tau^k$ for $k = 0, 1, 2$. We skip presenting that argument by Smirnov, instead we emphasize the importance of that combinatorial result which implies that S_δ and H_δ are approximately holomorphic.

We deduce that $h=\Phi$. Therefore from $h_a+h_b+h_c=1$ and $h_a+\tau h_b+\tau^2 h_c=\Phi$. Thus $\operatorname{Re}\Phi=h_a-\frac{1}{2}(h_b+h_c)=\frac{1}{2}(3h_a-1)$ or equivalently $h_a=\frac{1}{3}(2\operatorname{Re}\Phi+1)$. Thus the limit is independent of the choice of subsequence. Consequently, $\lim_{n\to\infty}H_{a,\delta_n}=\frac{1}{3}(2\operatorname{Re}\Phi+1)$. □

Corollary 6.7. *The convergence is uniform over domains in the following sense. Let C be a crosscut in $\mathbb{D}$ connecting the arc $1\,\tau$ to the arc $\tau^2\,1$, V_C is the connected component of $\overline{\mathbb{D}}\setminus\overline{C}$ that contains the arc $\tau\,\tau^2$. Then for any converging sequence of discrete domains $(U_n,a_n,b_n,c_n;\delta_n)$ for any sequence of curves γ_n such that $\gamma_n\subset V_C$, γ_n converges in the Carathéodory sense and $\phi_n\circ\gamma_n$ is a lattice path, the sequence $H_{a,\delta_n}\circ\phi_n$ converges uniformly on $\overline{\mathbb{D}}$ to h_a which is equal to $(2\operatorname{Re}\Phi(z)+1)/3$ and the rate of convergence (as a function of δ_n) is uniform over all sequences γ_n.*

6.3.7 The characterization of the percolation interface scaling limit

Theorem 6.10. *Let μ_n be the law of the percolation interface in $(U_{\delta_n},a_{\delta_n},b_{\delta_n})$ and ϕ_n be a conformal map from $(U_{\delta_n},a_{\delta_n},b_{\delta_n})$ onto $(\mathbb{H},0,\infty)$ and assume that $\phi_n^*\mu_n$ converges in the weak sense to μ in the topology described in Theorem 6.8. Then μ is the law of SLE(6) in $\mathbb{H}$.*

Remark 6.14. Together with Theorem 6.8 this shows that any sequence of laws μ_n of the percolation interface in $(U_{\delta_n},a_{\delta_n},b_{\delta_n})$ converges to the law of SLE(6).

Proof. Take any subsequence of laws of percolation interfaces that converges in the sense of Theorem 6.8.

Let $\hat{h}_A$ be the scaling limit of the discrete observable transformed to the upper half-plane with $a=1$, $b=\infty$ and $c=0$. Then $\hat{h}_A=0$ on $\mathbb{R}_{<0}$ and $\hat{h}_A(1)=1$ and on $(0,1)$ and $(1,\infty)$ it satisfies the correct Neumann-type boundary condition (derivative to the direction of the tangent rotated by $\pm\pi/6$ vanishes). Then we can write the martingale observable $X(t,z)$ as

$$X(t,z)=\hat{h}_A\left(\frac{g_t(z)-g_t(c)}{W_t-g_t(c)}\right). \tag{6.50}$$

By the discussion in Section 6.3.6.1, $(X(t,x))_{t\in\mathbb{R}_{\geq 0}}$ is a martingale for any subsequent scaling limit of percolation laws.

It is easy to verify based on Example 3.2 that the observable can be written in the form

$$\hat{h}_A(z)=L\ \operatorname{Re}\left[-\mathrm{i}\tau^2\int_0^z(\zeta-1)^{-\frac{2}{3}}\,\zeta^{-\frac{2}{3}}\,\mathrm{d}\zeta\right]. \tag{6.51}$$

Here the branches of the integrand are chosen so that for $\zeta>1$ it is real and positive and then extended continuously to $\overline{\mathbb{H}}\setminus\{0,1\}$. The constant L is positive and we omit here it's value, which can be written explicitly.

Let $\lambda\in\mathbb{R}_{>0}$. Expand for parameters $z=l$, $c=-\lambda l$, where $l>0$, the expression

$$\frac{g_t(z)-g_t(c)}{W_t-g_t(c)} = \frac{l(1+\lambda)+\frac{2t}{l}\left(1+\frac{1}{\lambda}\right)+\dots}{l\lambda+W_t+\frac{2t}{l}\frac{1}{\lambda}+\dots}$$
$$= \frac{1+\lambda}{\lambda}\left(1-\frac{W_t}{l\lambda}+\frac{1}{l^2\lambda^2}(W_t^2+(\lambda-1)2t)+\dots\right) \tag{6.52}$$

in powers of l, as $l\to\infty$. Write $\hat{\Phi}(z)=\int_0^z(\zeta-1)^{-\frac{2}{3}}\zeta^{-\frac{2}{3}}\,\mathrm{d}\zeta$. Notice then that for $x>1$, it holds that $\mathrm{Re}[-\mathrm{i}\tau^2\hat{\Phi}'(x)]\neq 0$ and $\hat{\Phi}''(x)/\hat{\Phi}'(x)=-\frac{2}{3}\frac{2x-1}{x(x-1)}$. If we combine these with (6.50), (6.52) and (6.51), we get after an easy calculation that

$$X(t,z)=C_1+C_2\left(\left(-\frac{1+\lambda}{\lambda^2}\right)\frac{W_t}{l}+\left(\frac{1-\lambda^2}{3\lambda^3}\right)\frac{W_t^2-6t}{l^2}\right)+\mathscr{O}(l^{-3}) \tag{6.53}$$

where $C_1=\mathrm{Re}[-\mathrm{i}\tau^2\hat{\Phi}(1+\lambda^{-1})]$ and $C_2=\mathrm{Re}[-\mathrm{i}\tau^2\hat{\Phi}'(1+\lambda^{-1})]$.

Since $H_t(z)$ as a process in the time variable t is a martingale, we deduce from (6.53) that the processes W_t and W_t^2-6t are martingales. From Lévy's martingale characterization theorem (Theorem 2.6) we deduce that $W_t=\sqrt{6}B_t$ for some standard Brownian motion $(B_t)_{t\in\mathbb{R}_{\geq 0}}$. □

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