Projection estimates for harmonic measure

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Abstract

Stochastic proofs of the Beurling projection theorem and the Hall projection theorem for harmonic measure are given. Some d-dimensional versions (for all $d>1$) which follow from this approach are pointed out.

1. Introduction

If U is an open subset of the complex plane $\mathbf{C}$, $a \in U$ and E is a Borel subset of ∂U, we let $\lambda_a(E)=\lambda_a^U(E)$ denote the harmonic measure of E w.r.t. U (at the point a). (We assume that ∂U, the boundary of U, has positive logarithmic capacity i.e. ∂U is not polar.) It is well known that λ_a can be described in terms of Brownian motion, as follows: If $b_\omega^a(t)$, $\omega \in \Omega$, $t \geqq 0$ denotes Brownian motion starting at a with probability law P^a, then

$$\lambda_a^U(E) = P^a\big(b_\omega^a(T_U) \in E\big),$$

where $T_U = \inf\{t>0;\, b_\omega^a(t) \notin U\}$ is the first exit time of U. In Sections 2 and 3 we use Brownian motion to give proofs of the Beurling projection theorem and the Hall projection theorem for harmonic measure. With natural modifications the proofs can be applied to give extensions of these projections theorems to $\mathbf{R}^d$, for all $d>1$. In Section 4 we formulate some such d-dimensional projection theorems which are not so easily available via extensions of the classical proofs.

We refer the to [1], [2], [3] and [5] for proofs of the Beurling and Hall projection theorems and more information about harmonic measure. A survey of the stochastic potential theory can be found in [7] and [8].

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2. The Beurling projection theorem

Let us first introduce some notation. If G is a set G^0 denotes the interior of G and G' denotes the reflection of G about the x-axis. The *circular projection* of a plane set E about a point x_0 is defined as follows:

$$E^* = E^*(x_0) = \{|z-x_0|;\ z\in E\}.$$

Let $0\leqq R_1<R_2\leqq\infty$ and let A denote the annulus

$$A = \{z;\ R_1 \leqq |z| \leqq R_2\}$$

Theorem 1. (Beurling projection theorem.) *Let K be a compact subset of A and suppose $-R_2<a<-R_1$. Put $K^*=K^*(0)$ and define*

$$U = A^0\setminus K, \quad V = A^0\setminus K^*$$

Then

$$\lambda_a^U(K) \geqq \lambda_a^V(K^*).$$

To prove Theorem 1, we first establish the following, which will also be needed in the proof of the Hall projection theorem.

Lemma 1. (Reflection lemma.) *Let $U\subset \mathbf{C}$ be open such that $U'=U$ and let $a\in I=\mathbf{R}\cap U$. Let $K\subset U$ be compact and put*

$$K^+ = \{z\in K;\ \operatorname{Im} z \geqq 0\}$$

$$K^- = \{z\in K;\ \operatorname{Im} z < 0\}$$

$$\tilde{K} = (K^+)'\cup K^-$$

Then

$$\lambda_a^{U\setminus K}(K) \geqq \lambda_a^{U\setminus \tilde{K}}(\tilde{K}).$$

Proof. Put $V=U\setminus K$, $W=U\setminus\tilde{K}$. Consider $H=K^+\cup(K^-\setminus(K^+)')$. Then $H\subseteq K$ and $\tilde{H}=\tilde{K}$. Therefore it is enough to prove the result for the case when $(K^+)'\cap K^-=\emptyset$. Put $D=U\setminus(K\cup K')$. Then by the strong Markov property

(1) $$P^a(b(T_V)\in K) = P^a(b(T_D)\in K)+\int_{K'\setminus K} P^y(b(T_V)\in K)\,d\nu_a(y),$$

where ν_a is the distribution of $b^a(T_D)$ on ∂D, i.e. $\nu_a(B)=P^a(b(T_D)\in B)$, for B a Borel set.

Now let $G=U\setminus I$ and let μ_y be the distribution of $b^y(T_G)$ on ∂G. Then we clearly have, again using the strong Markov property,

(2) $$P^y(b(T_V)\in K) \geqq \int_I P^x(b(T_V)\in K)\,d\mu_y(x).$$

So combining (1) and (2) we obtain

(3) $$P^a(b(T_V)\in K) \geqq P^a(b(T_D)\in K)+\int_{K'\setminus K}\Big(\int_I P^x(b(T_V)\in K)\,d\mu_y(x)\Big)\,d\nu_a(y).$$

Repeating the argument (1)—(3) n times, we obtain

$$(4)\qquad P^a(b(T_V)\in K) \geqq P^a(b(T_D)\in K)$$

$$+\sum_{k=1}^{n}\int_{K'\setminus K}\Big(\int_I\cdots\int_{K'\setminus K}\Big(\int_I P^{x_k}(b(T_D)\in K)\,d\mu_{y_k}(x_k)\Big)\,d\nu_{x_{k-1}}(y_k)\ldots\Big)\,d\nu_a(y_1)$$

$$+\int_{K'\setminus K}\Big(\int_I\cdots\int_{K'\setminus K}\Big(\int_I P^{x_n}(b(T_V)\in K,\ b(T_D)\in K'\setminus K)\,d\mu_{y_n}(x)\Big)\ldots\Big)\,d\nu_a(y_1).$$

Since $\nu_x(K'\setminus K)\leqq\frac{1}{2}$ for all $x\in I$, the last term in (4) tends to zero as $n\to\infty$ and the series converges.

We now apply the procedure (1)—(4) to $P^a(b(T_W)\in\tilde{K})$ and obtain similarly (except with equality instead of inequality):

$$(5)\qquad P^a(b(T_W)\in\tilde{K}) = P^a(b(T_D)\in\tilde{K})$$

$$+\sum_{k=1}^{\infty}\int_F\int_I\cdots\int_F\Big(\int_I P^{x_k}(b(T_D)\in\tilde{K})\,d\mu_{y_k}(x_k)\Big)\,d\nu_{x_{k-1}}(y_k)\ldots d\nu_a(y_1),$$

where $F=(K\cup K')\setminus\tilde{K}=K^+\cup(K^-)'$ (since we have assumed $(K^+)'\cap K^-=\emptyset$). By symmetry $P^x(b(T_D)\in\tilde{K})=P^x(b(T_D)\in K)$ for all $x\in I$, and since $K'\setminus K=(K^+)'\cup(K^-)'$ we get by symmetry that each term in (5) is equal to the corresponding term in the sum in (4). That completes the proof of Lemma 1.

We now proceed to prove Theorem 1:

If H is a set, we let $H^{(1)}$ be the reflection of H about the y-axis J_1. Put

$$W_1 = A^0\cap\{z;\ \mathrm{Re}\,z<0\}.$$

Then

$$(7)\qquad P^a(b(T_U)\in K) \geqq \int_{J_1} P^y(b(T_U)\in K)\,d\sigma_a(y)$$

where σ_a is the distribution of $b(T_{W_1})$.

By Lemma 1 with $U=A^0$, reflecting about J_1, we have

$$(8)\qquad P^y(b(T_U)\in K) \geqq P^y(b(T_{\tilde{U}^{(1)}})\in\tilde{K}^{(1)}),$$

where $\tilde{K}^{(1)}=(K\cap W_1)^{(1)}\cup[K\setminus W_1)]$, $\tilde{U}^{(1)}=A^0\setminus\tilde{K}^{(1)}$.
Therefore,

$$(9)\qquad P^a(b(T_U)\in K) \geqq \int_J P^y(b(T_{\tilde{U}^{(1)}})\in\tilde{K}^{(1)})\,d\sigma_a(y) = P^a(b(T_{\tilde{U}^{(1)}})\in\tilde{K}^{(1)}).$$

We now repeat the process, at the n'th step first reflecting about the line

$$J_n = \{re^{i\theta};\ r\in\mathbf{R},\ \theta=2^{-n}\pi\}$$

and then about the line

$$J_n = \{re^{i\theta};\ r\in\mathbf{R},\ \theta=-2^{-n}\pi\}$$

and each time using Lemma 1 with $U=A^0$.

In the limit we obtain

$$P^a(b(T_U)\in K) \geqq P^a(b(T_V)\in K^*)\quad\text{as asserted.}$$

3. The Hall projection theorem

We will prove the following version of the Hall projection theorem:

Theorem 2 (Hall projection theorem). *Let A and a be as in Theorem* 1. *Suppose $R_1<r_1<r_2<R_2$. Then there exists a constant $c>0$ such that for all compact $K\subset\{z;\ r_1<|z|<r_2\}$ we have*

$$\lambda_a^M(K) \geqq c\cdot m_1(K^*),$$

where $M=A^0\setminus K\setminus[0,\infty)$ and m_1 denotes 1*-dimensional Lebesgue measure on* $\mathbf{R}$. *(c does not depend on K.)*

To prove Theorem 2, we first establish the following:

Lemma 2. *Suppose that — in addition to the hypothesis of Theorem* 2 *— there exists $0<\delta\leqq\dfrac{\pi}{4}$ such that*

$$K\subset\{re^{i\theta};\ r\geqq 0,\ \delta\leqq\theta\leqq 2\delta\}.$$

Then there exists a constant c_0, independent of δ and K, such that

$$\lambda_a^M(K)\geqq c_0 m_1(K^*).$$

Proof of Lemma 2. Put $I_m=\{\mathrm{re}^{i\theta};\ r\geqq 0,\ \theta=\delta(1+2^{-m})\}$, and $L_m=\{\mathrm{re}^{i\theta};\ r\geqq 0,\ \theta=\delta(2+2^{-m+1})\}$, $m=0,1,2,\dots$. For $x_{m-1}\in I_{m-1}$ let $\nu_m^{x_{m-1}}$ be the distribution on I_m of $b^{x_{m-1}}(T_{Z_m})$, where

$$Z_m = A\cap\{re^{i\theta};\ r\geqq 0,\ \delta(1+2^{-m})\leqq\theta\leqq\delta(1+2^{-m+2})\}.$$

Let $U_m=\{\mathrm{re}^{i\theta};\ R_1<r<R_2,\ 0<\theta<\delta(2+2^{-m+1})\}$ and $S_m=\{\mathrm{re}^{i\theta};\ r\geqq 0,\ \delta(1+2^{-m})\leqq\theta\leqq\delta(1+2^{-m+1})\}$. Let ν^a be the distribution of the first exit of $b^a(t)$ from $A\setminus I_1$. Then

$$P^a\big(b(T_M)\in K\big)\geqq\int_{I_1}P^{x_1}\big(b(T_M)\in K\big)\,d\nu^a(x_1)\geqq\int_{I_1}P^{x_1}\big(b(T_0)\in K\big)\,d\nu^a(x_1),\tag{1}$$

where T_0 is the first exit time of $b^{x_1}(t)$ from the set $A^0\setminus K\setminus L_1\setminus\mathbf{R}$. We now apply Lemma 1 with $U=U_1$, reflecting about I_1:

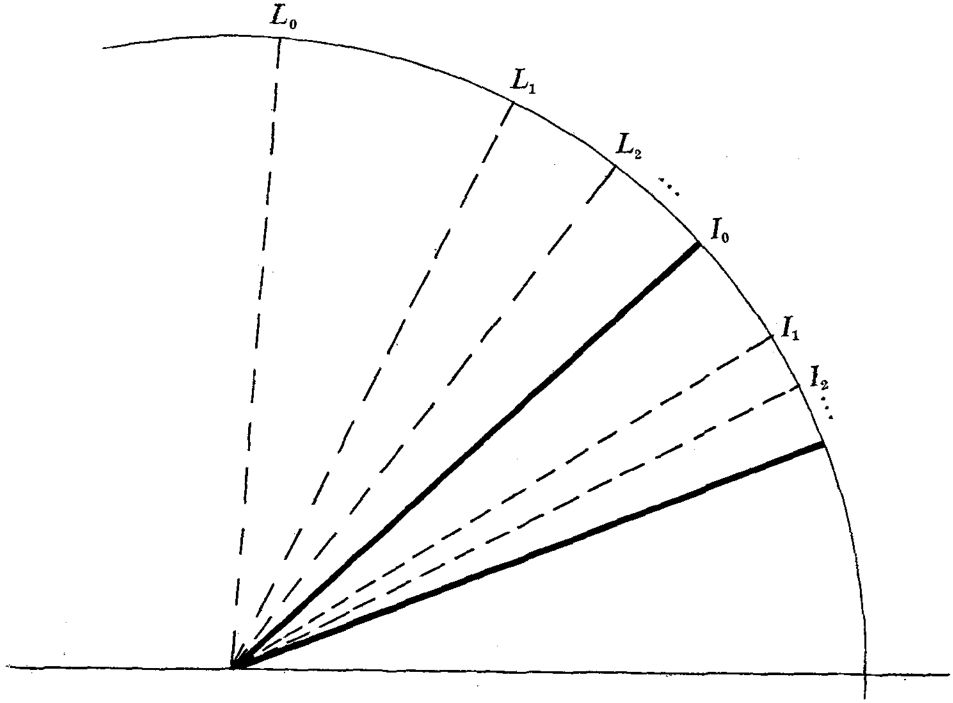
$$K_1=(K\cap S_1)^{(1)}\cup(K\setminus S_1),$$

where for all m $E^{(m)}$ denotes reflection of a set E about I_m, and let T_1 be the first exit time of $b(t)$ from $A^0\setminus K_1\setminus L_1\setminus\mathbf{R}$. Then

$$P^{x_1}\big(b(T_0)\in K\big)\geqq P^{x_1}\big(b(T_1)\in K_1\big)\quad\text{for all}\quad x\in I_1.\tag{2}$$

Therefore

$$P^a\big(b(T_M)\in K\big)\geqq\int_{I_1}P^{x_1}\big(b(T_1)\in K_1\big)\,d\nu^a(x_1).\tag{3}$$



Repeating the argument, we get

$$(4) \qquad P^a(b(T_M)\in K) \geqq \int_{I_1} \Big(\int_{I_2} P^{x_2}(b(T_2)\in K_2)\, d\nu^{x_1}(x_2)\Big)\, d\nu^a(x_1),$$

where $K_2=(K_1\cap S_2)^{(2)}\cup(K_1\setminus S_2)$, T_2 is the first exit time from $A^0\setminus K_2\setminus L_2\setminus \mathbf{R}$ and $\nu_2^{x_1}$ is the first exit distribution of $b^{x_1}(t)$ from $A^0\setminus L_1\setminus I_2$. After n repetitions we have

$$(5) \qquad P^a(b(T_M)\in K) \geqq \int_{I_1} \Big(\int_{I_2} \Big(\dots \int_{I_n} P^{x_n}(b(T_n)\in K_n)\, d\nu_n^{x_{n-1}}(x_n)\Big)\dots\Big)\, d\nu^a(x_1),$$

where $K_j=(K_{j-1}\cap S_j)^{(j)}\cup(K_{j-1}\setminus S_j)$, T_j is the first exit time from $A^0\setminus K_j\setminus L_j\setminus \mathbf{R}$, $j=1, 2, \dots, n$ and $\nu_j^{x_{j-1}}$ is the first exit distribution of $b^{x_{j-1}}(t)$ from $A^0\setminus L_{j-1}\setminus I_j$.

We now replace the measures $\nu_m^{x_{m-1}}$ by the measures $\mu_m^{x_{m-1}}$ defined as follows:

For x_{m-1} between I_0 and I_m we let $\mu_m^{x_{m-1}}$ be the distribution of the first exit of $b^{x_{m-1}}(t)$ from the set of points in A between I_0 and I_m. Then clearly

$$(6) \qquad \nu_m^{x_{m-1}} \geqq \mu_m^{x_{m-1}} \quad \text{on} \quad I_m,$$

and therefore, by (5)

$$(7) \qquad P^a(b(T_M)\in K) \geqq \int_{I_1} \Big(\int_{I_2} \dots \Big(\int_{I_n} P^{x_n}(b(T_n)\in K_n)\, d\mu_n^{x_{n-1}}(x_n)\Big)\dots\Big)\, d\nu^a(x_1)$$
$$= \int_{I_1} \Big(\int_{I_n} P^{x_n}(b(T_n)\in K_n)\, d\mu_n^{x_1}(x_n)\Big)\, d\nu^a(x_1),$$

by the strong Markov property.

We may assume that K is a finite union of closed discs, $K=\bigcup_{i=1}^{N}\Delta_i$. Choose n so large that $2^{-n}\delta r_2$ is less than the smallest of the radii of these discs.

Then if

$$\tilde{K}=\bigcup_{i=1}^{N}\frac{1}{2}\Delta_i,$$

we have

(8) $$P^{x_n}(b(T_n)\in K_n)=1$$

for all $x_n\in E=\{re^{i\theta};\ r\in\tilde{K}^*,\ \theta=\delta(1+2^{-n})\}$.

Combined with (7) this gives

(9) $$P^a(b(T_M)\in K)\geqq\int_{I_1}\mu_n^x(E)\,dv^a(x).$$

Chop the interval $[r_1,r_2]$ into $D=2^k$ disjoint intervals $S_1,S_2,\dots,S_D$ of length

$$\varrho=2^{-k}(r_2-r_1),$$

where k is chosen so large that

$$\tfrac{1}{2}\delta<\varrho\leqq\delta.$$

Put $G_i=\{re^{i\theta};\ r\in S_i\}$. Then if $x\in I_1\cap G_i$ we have

(10) $$\mu_n^x(E)\geqq\mu_n^x(E\cap G_i)\geqq c_1\cdot\frac{m_1(E\cap G_i)}{\varrho}$$

and therefore

$$\int_{I_1}\mu_n^x(E)\,dv^a(x)=\sum_{i=1}^{D}\int_{I_1\cap G_i}\mu_n^x(E)\,dv^a(x)\geqq\sum_{i=1}^{D}c_1\cdot\frac{m_1(E\cap G_i)}{\varrho}\cdot v(I_1\cap G_i)$$

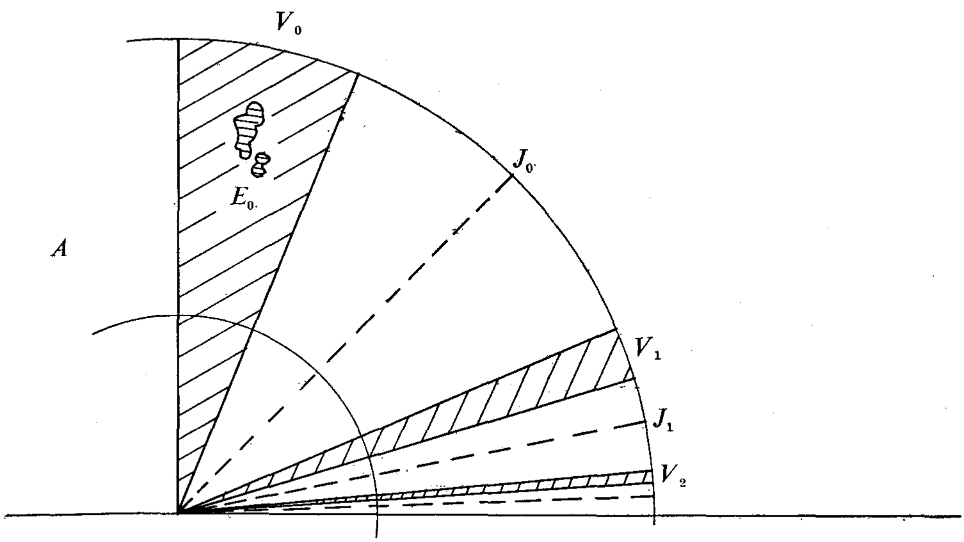
$$\geqq c_2\sum_{i=1}^{D}m_1(E\cap G_i)=c_2m_1(E)\geqq c_3\cdot m_1(K^*),$$

where the constant c_3 is independent of K and δ. That completes the proof of Lemma 2.

Proof of Theorem 2. We may assume that K is situated in the union of the sectors

$$V_n=\left\{re^{i\theta};\ r\geqq0,\ \frac{3}{4}\cdot4^{-n}\frac{\pi}{2}\leqq\theta\leqq4^{-n}\frac{\pi}{2}\right\},\quad n=0,1,2,\dots$$

Put $E_n=K\cap V_n$, $J_n=\left\{re^{i\theta}\in A;\ r\geqq0,\ \theta=\frac{1}{2}\cdot4^{-n}\frac{\pi}{2}\right\}$, $n=0,1,2,\dots$



For $n=1, 2, \ldots$ and $x \in J_{n-1}$ define

$$W_n = \left\{ re^{i\theta} \in A;\ r \geqq 0,\ 0 \leqq \theta \leqq 4^{-n} \frac{\pi}{2} \right\}$$

$$D_{n,x} = \left\{ re^{i\theta} \in W_n;\ \left| r - |x| \right| \geqq 4^{-n} \right\}$$

$$B_{n,x} = A \setminus [0, \infty) \setminus D_{n,x}.$$

Then there exists a constant $d_1 < 1$, independent of $x \in J_{n-1}$ and n, such that

$$\lambda_x^{B_{n,x}}(\partial D_{n,x}) \leqq d_1 \tag{1}$$

Moreover, there exists a constant $\eta > 0$, such that for all compact $F \subset W_n$, all $x \in J_{n-1}$ and all n we have

$$|\lambda_x^B(F) - \lambda_{x'}^B(F)| \leqq \tfrac{1}{3}(1-d_1) \tag{2}$$

for all $x' \in J_{n-1}$ with $|x-x'| \leqq \eta \cdot 4^{-n}$, where $B = A \setminus [0, \infty) \setminus F$. For $n=0,1,2,\ldots$ define

$$\sigma_n = \inf\{t > 0;\ b_\omega^a(t) \in \partial A \cup J_n\}$$

$$\tau_n = \inf\{t > 0;\ b_\omega^a(t) \in \partial A \cup E_n \cup J_n\}$$

and let α_n, β_n be the distribution of $b_\omega^a(\sigma_n)$ and $b_\omega^a(\tau_n)$, respectively.

Then (as before $M=A^0\setminus K\setminus[0,\infty)$)

(3) $$P^a(b(T_M)\in K) = P^a(b(\tau_0)\in E_0)+\int_{J_0} P^x(b(T_M)\in K)\,d\beta_0(x)$$

Put $R_n=\bigcup_{k\geqq n}E_k$ and

(4) $$H_0 = \{x\in J_0;\ P^x(b(T_M)\in R_1) > d_2\},$$

where $d_2=1-\frac{1}{3}(1-d_1)$, and d_1 is given by (1).

Then if

(5) $$G_0 = \{x\in J_0;\ \operatorname{dist}(x, H_0) \leqq \eta\cdot 4^{-1}\},$$

we have by (2)

(6) $$P^x(b(T_M)\in R_1) > d_3$$

for all $x\in G_0$, with $d_3=1-\frac{2}{3}(1-d_1)>d_1$.

Put

(7) $$F_1 = R_1\setminus\{z\in K;\ \big||z|-|w|\big| \leqq 4^{-1}\quad\text{for some}\quad w\in G_0\}.$$

Then if $M_1=A^0\setminus F_1\setminus[0,\infty)$ we have by (1) and (3)

(8) $$\begin{aligned} P^a(b(T_M)\in K) &\geqq \beta_0(E_0)+\int_{J_0} P^x(b(T_M)\in R_1)\,d\beta_0 \\ &\geqq \beta_0(E_0)+\int_{G_0} P^x(b(T_M)\in R_1)\,d\beta_0+\int_{J_0\setminus G_0} P^x(b(T_{M_1})\in F_1)\,d\beta_0 \\ &\geqq \beta_0(E_0)+d_3\beta_0(G_0)+\int_{J_0} P^x(b(T_{M_1})\in F_1)\,d\beta_0-\int_{G_0} P^x(b(T_M)\in F_1)\,d\beta_0 \\ &\geqq \beta_0(E_0)+(d_3-d_1)\beta_0(G_0)+\int_{J_0} P^x(b(T_{M_1})\in F_1)\,d\beta_0 \\ &= \beta_0(E_0)+(d_3-d_1)\beta_0(G_0)+\int_{J_0} P^x(b(T_{M_1})\in F_1)\,d\alpha_0-\int_{J_0} P^x(b(T_{M_1})\in F_1)\,d(\alpha_0-\beta_0) \\ &\geqq \beta_0(E_0)+(d_3-d_1)\beta_0(G_0)+\int_{J_0} P^x(b(T_{M_1})\in F_1)\,d\alpha_0-d_2\beta_0(E_0). \end{aligned}$$

Since $\alpha_0(J_0)=\beta_0(J_0)+\beta_0(E_0)$, this gives

(9) $$\begin{aligned} P^a(b(T_M)\in K) &\geqq (1-d_2)\beta_0(E_0)+(d_3-d_1)\beta_0(G_0)+\int_{J_0} P^x(b(T_{M_1})\in F_1)\,d\alpha_0 \\ &\geqq c_1\beta_0(E_0\cup G_0)+P^a(b(T_{M_1}\in F), \end{aligned}$$

where $c_1=\frac{1}{3}(1-d_1)$.

By Lemma 2 we conclude that

(10) $$P^a(b(T_M)\in K) \geqq c_2[m_1(E_0^*)+m_1(G_0^*)]+P^a(b(T_{M_1})\in F_1)$$

Since $m_1(G_0^*)\geqq\eta\cdot m_1((R_1\setminus F_1)^*)$, we get

(11) $$P^a(b(T_M)\in K) \geqq c_3[m_1(E_0^*)+m_1(R_1^*\setminus F_1^*)]+P^a(b(T_{M_1})\in F_1)$$

We now start with the term $P^a(b(T_{M_1})\in F_1)$ and repeat the process (3)—(11) above etc.

After sufficiently many iterations we get

(12) $$P^a(b(T_M)\in K) \geqq \tfrac{1}{2}\, c_3\cdot m_1(K^*),$$

and the proof is complete.

4. Projection estimates in $\mathbf{R}^d$

Both the classical proofs and the proofs given in § 2 and § 3 apply to certain d-dimensional situations, for any $d \geqq 2$. Here we mention some d-dimensional projection theorems which follow naturally from the proofs given in § 2 and § 3, but would be harder to obtain by extending the classical proofs. (For $r>0$ let Λ_r denote r-dimensional Hausdorff measures.)

Theorem 3. *Define* $R\colon \mathbf{R}^d \to \mathbf{R}^{d-1}$ by

$$R(x_1, \ldots, x_d) = \left(\sqrt{x_1^2+x_d^2}, x_2, \ldots, x_{d-1}\right).$$

Suppose $B \subset \mathbf{R}^+ \times \mathbf{R}^{d-2}$ *is open (where* $\mathbf{R}^+ = \{x \in \mathbf{R};\, x \geqq 0\}$*) and let* $A = R^{-1}(B)$, $a = (-a_1, 0, \ldots, 0) \in A$ $(a_1 > 0)$. *Then if* K *is a compact subset of* A, *we have*

(i) $\lambda_a^{A \setminus K}(K) \geqq \lambda_a^{A \setminus R(K)}(R(K))$ *(Extension of Beurling theorem).*

If $\operatorname{dist}(K, \partial A) = \delta > 0$ and $L = \{(x, 0, \ldots, 0) \in \mathbf{R}^d;\, x \geqq 0\}$, *then*

(ii) $\lambda_a^{A \setminus K \setminus L}(K) > c \cdot \Lambda_{d-1}(R(K))$,

where c *only depends on* δ. *(Extension of Hall theorem.)*

Similarly one can prove the following:

Theorem 4. *Define* $P\colon \mathbf{R}^n \to \mathbf{R}^{d-1}$ *by*

$$P(x_1, \ldots, x_d) = (x_1, \ldots, x_{d-1}).$$

Suppose $B \subset \mathbf{R}^{d-1}$ *is open and let* $A = P^{-1}(B)$, $a = (0, \ldots 0, a_d) \in A$ *where* $a_d > 0$. *Then if* K *is a compact subset of* A *such that*

$$(x_1, \ldots, x_d) \in K \Rightarrow x_d < 2a_d,$$

we have

(i) $\lambda_a^{A \setminus K}(K) \geqq \lambda_a^{A \setminus P(K)}(P(K))$ *(Extension of Beurling theorem).*

Furthermore, if $S \leqq 0 < a < R$ *and*

$$\hat{A} = \{(x_1, \ldots, x_d) \in A;\; S < x_d < R\}$$

we have

$$\text{(ii)} \qquad \lambda_{\hat{a}}^{\hat{A} \setminus K}(K) \geqq c \cdot \Lambda_{d-1}(P(K)),$$

where c *only depends on* S, R *and the distance from* K *to* ∂A. *(Extension of Hall theorem.)*

The proofs of Theorem 3 and 4 follow the same line as the proof given in the two preceding paragraphs and are omitted.

Finally we mention that one can also obtain the following:

Theorem 5. (Radial projection theorem.) *Let* B *be the unit ball in* $\mathbf{R}^d$, $d \geqq 2$ *and let* $K \subset B$ *be compact,* $0 \notin K$. *Let* $K^* = \left\{\dfrac{x}{|x|};\, x \in K\right\}$ *be the radial projection* *of K into the boundary S of B. Then there exists a constant $c>0$ depending only on the dimension d such that*

$$\lambda_0^{B\setminus K}(K) \geqq c\cdot\lambda_{d-1}(K^*) \tag{1}$$

Remarks:

1) In the case $d=2$ this result has been obtained independently by T. Lyons and J. Taylor, who have proved that such an estimate holds for all symmetric spaces of rank 1 (private communication).
2) It would be interesting to find the best constant c in (1).

Proof. We apply the same procedure as in Sections 2—3, except that we replace reflection about hyperplanes by reflection (inversion) about spheres. Thus, the reflection lemma (Lemma 1) is replaced by the following:

Lemma 3. (Spherical reflection lemma.) *Define* $I\colon \mathbf{R}^d\to\mathbf{R}^d$ *by* $I(x)=\dfrac{x}{|x|^2}$; $x\in\mathbf{R}^d\setminus\{0\}$. *Let* $U\subset\mathbf{R}^d$ *be open such that* $I(U)=U$, $\operatorname{dist}(0,U)>0$, *and let* $a\in S$, *the unit sphere in* $\mathbf{R}^d$. *Let* $K\subset U$ *be compact and put*

$$K_1=K\cap\bar B,\quad \textit{where}\quad \bar B=\{x\in\mathbf{R}^d;\ |x|\leqq 1\},$$

$$K_2=K\setminus\bar B,$$

$$\tilde K=I(K_1)\cup K_2.$$

Then

$$\lambda_a^{U\setminus K}(K)\geqq\delta^{2-d}\cdot\lambda_a^{U\setminus\tilde K}(\tilde K),$$

where $\delta=\operatorname{dist}(0,K)$.

Proof. Proceeding as in the proof of Lemma 1, we put $D=U\setminus(K\cup I(K))$, $G=U\setminus S$ and obtain

$$\lambda_a^{U\setminus K}(K)\geqq P^a(b(T_D)\in K) \tag{2}$$
$$+\sum_{n=1}^{\infty}\int_E\Big(\int_S\Big(\cdots\int_E\Big(\int_S P^{x_n}(b(T_D)\in K)\,d\mu_{y_n}(x_n)\Big)\,d\nu_{x_{n-1}}(y_n)\cdots\Big)\,d\nu_a(y_1)\Big)$$

where ν_x is the distribution of $b^x(T_D)$, μ_y is the distribution of $b^y(T_G)$ and $E=I(K)\setminus K=I(K_1)\cup I(K_2)$ (assuming $I(K_1)\cap K_2=\emptyset$). Similarly,

$$\lambda_a^{U\setminus\tilde K}(\tilde K)=P^a(b(T_D)\in K) \tag{3}$$
$$+\sum_{n=1}^{\infty}\int_F\Big(\int_S\Big(\cdots\int_F\Big(\int_S P^{x_n}(b(T_D)\in\tilde K)\,d\mu_{y_n}(x_n)\Big)\,d\nu_{x_{n-1}}(y_n)\cdots\Big)\,d\nu_a(y_1)\Big),$$

where $F=(K\cup I(K))\setminus\tilde K=K_1\cup I(K_2)$.

So the proof of Lemma 3 will be complete if we can show that

(4) $$P^x\big(b(T_D)\in K_1\big) \geqq \delta^{2-d}\cdot P^x\big(b(T_D)\in I(K_1)\big) \quad \text{for all} \quad x\in S$$

and

(5) $$\int_S f(x)\,d\mu_y(x) \leqq \int_S f(x)d\mu_{I(y)}(x) \quad \text{for all} \quad y\in K_1,$$

and all continuous functions f.

Statements (4) and (5) are consequences of the following result:

Lemma 4. *Let* V *be an open set with* $I(V)=V$, dist $(0, V)>0$. *Then if* $X\subset V$ *is compact and* $y\in V$ *we have*

$$\lambda_y^{V\setminus X}(X) \leqq \left(\frac{|y|}{\delta}\right)^{d-2}\cdot \lambda_{I(y)}^{V\setminus I(X)}(I(X)), \quad \delta = \operatorname{dist}(0, X).$$

Proof of Lemma 4. Let H denote the Kelvin transformation on the space $C(\overline{W})$ of (real) continuous functions on the closure of $W=V\setminus X$ defined by

(6) $$(Hf)(x) = \frac{1}{|x|^{d-2}}\cdot f\left(\frac{x}{|x|^2}\right); \quad f\in C(\overline{W}).$$

Then if g is harmonic in W, Hg is harmonic in $I(W)$. If $y\in W$ we define the measure β_y on ∂W by

(7) $$\int f(x)\,d\beta_y(x) = \int (Hf)(x)\,d\lambda_{I(y)}^{I(W)}(x)$$

Then for g harmonic in $I(W)$ (and continuous in $I(\overline{W})$)

$$\int g\,d\beta_y = \int Hg\cdot d\lambda_{I(y)}^{I(W)} = Hg(I(y)) = |y|^{2-d}g(y)$$

Therefore $|y|^{d-2}\beta_y$ represents the point y for the functions $g\in C(\overline{W})$ harmonic in W. Since β_y is carried by ∂W, we must by uniqueness have

$$|y|^{d-2}\beta_y = \lambda_y^W,$$

i.e.

(8) $$\int f\,d\lambda_y^W = |y|^{d-2}\int f\,d\beta_y = |y|^{d-2}\int Hf\cdot d\lambda_{I(y)}^{I(W)}; \quad f\in C(\overline{W})$$

This implies that, letting $f=\chi_X$ (the characteristic function of X)

$$\lambda_y^W(X) = |\lambda|^{d-2}\int H\chi_X\cdot d\lambda_{I(y)}^{I(W)} \leqq \left(\frac{|y|}{\delta}\right)^{d-2}\cdot \lambda_{I(y)}^{I(W)}(I(X)),$$

as asserted in Lemma 4.

We now observe that if in Lemma 4 we put $V=D$ and let $y\in S$, we obtain (3). If we put $V=G$ and $y\in K_1$ (so that $|y|\leqq 1$) we obtain

$$\int_S f(x)\,d\mu_y(x) \leqq |y|^{d-2}\int f(x)\,d\mu_{I(y)}(x) \leqq \int f(x)\,d\mu_{I(y)}(x),$$

which is (4). That completes the proof of Lemma 3.

We are now ready to complete the proof of Theorem 5:

First of all we note that it has been proved by T. Lyons ([6], Theorem 4.1) that the estimate (1) holds if the compact K has a distance $\geqq\delta$ to S (with C depending on δ). Lyons used Brownian motion in his proof. Subsequently an alternative, non-probabilistic proof has been found by F. Fuglede ([4], Lemma 2). Based on this result, we may assume that

$$\delta = \operatorname{dist}(0, K) \geqq \tfrac{1}{2}. \tag{9}$$

Second, we note that it is sufficient to establish the estimate (1) under the extra assumption that there exists $\varrho<1$ such that

$$K\subset\{x;\ \varrho^2 \leqq |x| \leqq \varrho\}, \tag{10}$$

with the constant c not depending on ϱ, $\varrho^2\geqq\frac{1}{2}$. (This will be the analogue of Lemma 2 of Section 3.) For once this has been established, one obtains the result for general K (satisfying (9)) by adopting the same technique as in the proof of the Hall projection theorem in Section 3, with obvious modifications.

So we assume that (10) holds for some ϱ, $\frac{1}{2}\leqq\varrho^2<1$. To establish the estimate (1) we apply the proof of Lemma 3 in Section 3, except that the iterated use of the reflection lemma (Lemma 1) is replaced by iterated use of the spherical reflection lemma (Lemma 3), at the k'th iteration reflecting (inverting) about the sphere of radius $\varrho_k=\varrho^{1+2^{-k}}$. The k'th iteration gives an extra factor of $\left(\frac{\varrho_k}{\delta_k}\right)^{d-2}$, where δ_k is the distance from 0 to the k'th reflected set $K^{(k)}$. Since $\delta_k\geqq\varrho_{k-1}$, this gives a total factor of

$$\prod_{k=1}^{\infty}\left(\frac{\varrho_k}{\delta_k}\right)^{d-2} \leqq \prod_{k=1}^{\infty}\varrho^{(2-d)2^{-k}} = \varrho^{2-d} \leqq 2^{\frac{d}{2}-1}.$$

This establishes the estimate (1) under the assumption (10) and thus completes the proof of Theorem 5.

References

1. AHLFORS, L.: *Conformal Invariants*. McGraw-Hill 1973.
2. BEURLING, A.: *Études sur un problème de majoration*. Thèse, Uppsala 1933.
3. DUREN, P.: *Theory of H^p spaces*. Academic Press 1970.
4. FUGLEDE, B.: Asymptotic paths for subharmonic functions and polygonal connectedness of fine domains. In Séminaire de Théorie du Potential Paris, No. 5. *Springer Lecture Notes in Math.* **814** (1980), 97—115.
5. HALL, T.: Sur la mesure harmonique de certains ensebles. *Arkiv Mat. Astr. Fys.* **25 A** (1937), No. 28 (8 pp.).
6. LYONS, T.: Finely holomorphic functions. *J. Functional Analysis* **37** (1980), 1—18.
7. PORT, S. and STONE, C.: *Brownian Motion and Classical Potential Theory*. Academic Press 1979.
8. RAO, M.: Brownian Motion and Classical Potential Theory. *Aarhus Universitet Mat. Inst. Lecture Notes* Series No. **47**, 1977.

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