Physica A 168 (1990) 529–535
North-Holland

# UPPER BOUNDS FOR THE GROWTH RATE OF DLA

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We show that the maximal length of the arms in DLA can grow at most at a rate $n^{2/(d+1)}$, where $n$ denotes the volume or mass of the aggregate, and $d$ is the dimension. (A logarithmic correction factor is needed if $d=3$). We also give a lower bound for the number of sites in a sphere which are eventually occupied by the aggregate.

## 1. Statement of results

We consider the standard DLA model of Witten and Sander [1]. $A_1$ consists of the orign of $\mathbb{Z}^d$. At time $n$, the aggregate $A_n$ is a connected set in $\mathbb{Z}^d$, consisting of $n$ sites and containing the origin. $A_{n+1} = A_n \cup \{y_n\}$ with the new point $y_n$ a point of $\partial A_n$, where $\partial A_n$, the boundary of $A_n$, consists of the sites of $\mathbb{Z}^d$ which are adjacent to $A_n$ but not in $A_n$. $y_n$ is chosen according to the harmonic measure on $\partial A_n$, i.e., $y_n$ has the distribution of the first hitting place of $\partial A_n$ by a simple random walk "starting at $\infty$" and conditioned to hit $\partial A_n$ at some time (see Kesten [2] for more details). We define the radius of $A_n$ as

$$r(n) := \max\{|x| : x \in A_n\}$$

($|x|$ is the Euclidean norm of $x$). We show the following upper bound for $r(n)$.

*Theorem 1.* There exist constants $C(d)$ such that with probability 1

$$r(n) \leqslant C(d)\, n^{2/(d+1)} \quad \text{eventually if } d \geqslant 2 \quad \text{but } d \neq 3\,, \tag{1}$$

$$r(n) \leqslant C(3)\, (n \log n)^{1/2} \quad \text{eventually if } d = 3\,. \tag{2}$$

It is useful to express this result somewhat differently. Let

$$T(r) = \text{number of cells in the aggregate at the first time its radius reaches } r\,.$$

[1] Research supported by the NSF through a grant to Cornell University.

Then (1) is equivalent to

$$T(r) \geq C^* r^{(d+1)/2} \qquad \text{eventually} \tag{3}$$

for some constant $C^* > 0$. If there exists a fractal dimension $\bar{d}$ such that

$$\lim_{r\to\infty} \frac{\log T(r)}{\log r} = \bar{d} ,$$

then (3) (and hence also (1)) imply

$$\bar{d} \geq \tfrac{1}{2}(d+1) . \tag{4}$$

As another measure of the density of A one may consider

$$N(r) := \text{number of cells in the ball } \{x: |x| \leq r\} \text{ which are occupied by the aggregate A eventually} .$$

It is reasonable to conjecture that there exists an exponent $\bar{e}$ such that

$$\lim_{r\to\infty} \frac{\log N(r)}{\log r} = \bar{e} .$$

Clearly

$$N(r) \geq T(r) \quad \text{and} \quad \bar{e} \geq \bar{d} \tag{5}$$

(provided the exponents $\bar{d}$ and $\bar{e}$ exist). As far as we know no clear distinction has been made between $\bar{d}$ and $\bar{e}$ in the literature so far.

*Theorem 2.* There exist constants $C(d)$ such that w.p. 1

$$N(r) \geq C(d) r^{d-1} \qquad \text{for infinitely many } r .$$

*Corollary.* If $\bar{e}$ exists, then $\bar{e} \geq d - 1$.

*Remarks.* (i) For $d = 2$, (1) was proved already by Kesten [2, 3], while for $d \geq 3$, (1) and (2) improve refs. [2, 3]. Lawler [4] will contain a proof of $r(n) \leq C(3)\, n^{1/2}(\log n)^{1/4}$ eventually for $d = 3$. This slightly improves (2). Here we only consider (1), or rather (3), for $d \geq 4$. Only a slight modification of our argument is needed to prove (2).

(ii) A number of non-rigorous arguments have been given to bound $\bar{d}$ or $\bar{e}$. Ball and Witten [5] argued that $\bar{d} \geq d-1$, and various non-rigorous arguments have been used [6–8] to arrive at $\bar{d} = (d^2+1)/(d+1)$. Hentschel [9] claims $\bar{d} = (d^2+4d+8)/(d-2)$ for large $d$. The lower bound $(d+1)/2$ for $\bar{d}$ appears also in the paper of Family [10] (see his eq. (8) with $\eta = 1$).

(iii) At this stage it is unclear whether $\bar{e} > \bar{d}$ or $\bar{e} = \bar{d}$. There is weak numerical evidence in the paper of Tolman and Meakin [11] that $\bar{e} = \bar{d}$.

(iv) The proof of theorem 2 gives some information about the density of $r$'s for which $N(r) \geq C(d)\, r^{d-1}$. In fact, for a suitable $C > 0$, each interval $[C\sqrt{R}, R]$ with $R$ large enough contains an $r$ with $N(r) \geq C(d)\, r^{d-1}$. We can also obtain $\bar{d} \geq \bar{e}/(\bar{e}+2-d)$ from our methods.

## 2. Proofs

*Throughout we take* $d \geq 4$. $S_t$, $t = 0, 1, \dots$, denotes a symmetric nearest neighbor random walk on $\mathbb{Z}^d$. $P^x$ denotes the probability governing random walk paths starting at $x$. $P$, without a superscript, will be used for the probability measure governing the Markov chain $A_0, A_1, \dots$. $E^x$ and $E$ denote expectation with respect to $P^x$ and $P$, respectively. Finally, $C_i$ will be a strictly positive finite constant whose precise value is of no significance for us.

Define

$$M_n(x) = \sum_{y \in A_n} G(x, y)\,,$$

where $G$ is the Green function for $S_n$, i.e.

$$G(x, y) = \sum_{t=0}^{\infty} P^x\{S_t = y\}\,.$$

It is well known (cf. Spitzer [12], prop. 26.1 or ref. [4], theorem 1.5.3) that

$$\frac{C_1}{1+|x-y|^{d-2}} \leq G(x, y) \leq \frac{C_2}{1+|x-y|^{d-2}}\,.$$

Also introduce the escape probabilities from B,

$$e(y, B) = P^y\{S_t \not\in B \quad \text{for } t \geq 1\}\,,$$

and the capacity of B,

$$C(\mathrm{B}) = \sum_{y \in \mathrm{B}} e(y, \mathrm{B}) .$$

It was shown by Spitzer (ref. [12], sect. 25, 26) (see also ref. [2]) that

$$P\{y_n = y | \mathrm{A}_1, \ldots, \mathrm{A}_n\} = \frac{e(y, \partial \mathrm{A}_n)}{C(\partial \mathrm{A}_n)} , \qquad y \in \partial \mathrm{A}_n .$$

Note that the random walk $\mathrm{S}_t$ cannot move from $\mathrm{A}_n$ to the complement of $\mathrm{A}_n$, or vice versa, without passing through a point of $\partial \mathrm{A}_n$. From this it is easy to see that (with $\bar{\mathrm{A}}_n = \mathrm{A}_n \cup \partial \mathrm{A}_n$)

$$e(y, \partial \mathrm{A}_n) = e(y, \bar{\mathrm{A}}_n) \quad \text{for } y \in \bar{\mathrm{A}}_n , \qquad C(\partial \mathrm{A}_n) = C(\bar{\mathrm{A}}_n) .$$

Also, if $x \in \bar{\mathrm{A}}_n$

$$\begin{aligned} \sum_{y \in \bar{\mathrm{A}}_n} G(x, y)\, e(y, \bar{\mathrm{A}}_n) &= \sum_{y \in \bar{\mathrm{A}}_n} \sum_{t=0}^{\infty} P^x\{\mathrm{S}_t = y \quad \text{but} \quad \mathrm{S}_k \not\in \bar{\mathrm{A}}_n \quad \text{for all} \quad k > t\} \\ &= P^x\{\mathrm{S}_t \text{ leaves } \bar{\mathrm{A}}_n \text{ eventually}\} = 1 . \end{aligned}$$

Therefore, if $x \in \bar{\mathrm{A}}_n$

$$\begin{aligned} &E\{M_{n+1}(x) - M_n(x) | \mathrm{A}_1, \ldots, \mathrm{A}_n\} = E\{G(x, y_n) | \mathrm{A}_n\} \\ &\quad = \sum_{y \in \partial \mathrm{A}_n} G(x, y) \frac{e(y, \partial \mathrm{A}_n)}{C(\partial \mathrm{A}_n)} = \sum_{y \in \bar{\mathrm{A}}_n} G(x, y) \frac{e(y, \bar{\mathrm{A}}_n)}{C(\bar{\mathrm{A}}_n)} = [C(\bar{\mathrm{A}}_n)]^{-1} . \end{aligned}$$

Thus, on $\{x \in \bar{\mathrm{A}}_k\}$,

$$Z_n(x) := M_n(x) - \sum_{k}^{n-1} [C(\bar{\mathrm{A}}_j)]^{-1}, \qquad n \geqslant k ,$$

is a martingale. Since (ref. [12], prop. 1.3)

$$0 \leqslant M_{n+1}(x) - M_n(x) = G(x, y_n) \leqslant G(0, 0) < \infty ,$$

the increments of $Z_n$ are bounded. Moreover, its square function

$$\Sigma_n := \sum_{l=k}^{n-1} \operatorname{var}\{M_{l+1} - M_l | \mathrm{A}_1, \ldots, \mathrm{A}_l\}$$

satisfies

$$\Sigma_n \leq \sum_k^{n-1} E\{(M_{l+1} - M_l)^2 | \mathrm{A}_1, \ldots, \mathrm{A}_l\} = \sum_k^{n-1} E\{G^2(x, y | \mathrm{A}_l\}$$
$$\leq G(0,0) \sum_k^{n-1} E\{G(x, y) | \mathrm{A}_l\} = G(0,0) \sum_k^{n-1} [C(\bar{\mathrm{A}}_l)]^{-1} .$$

The standard exponential estimates (Neveu [13], sect. VII.2) for martingales now yield the following result.

*Lemma 1.* There exists a constant $\lambda$ such that

$$P\Big\{\text{there exists some } x \in \bar{\mathrm{A}}_k \text{ and an } n \geq k \text{ with}$$
$$\Big| M_n(x) - M_k(x) - \sum_k^{n-1} [C(\bar{\mathrm{A}}_l)]^{-1} \Big| \geq a + \tfrac{1}{2} \sum_k^{n-1} [C(\bar{\mathrm{A}}_l)]^{-1} \Big\} \leq 2dk\, \mathrm{e}^{-\lambda a} .$$

The arguments of refs. [2, 3] are easily adapted to yield the following estimate.

*Lemma 2.* For any $r \geq 1$

$$P\Big\{ \sum_{l=T(r)}^{T(2r)-1} [C(\bar{\mathrm{A}}_l)]^{-1} \leq \frac{r}{8d} \Big\} \leq C_3\, \mathrm{e}^{-C_4 r} .$$

Lemma 1 with $a = (3/\lambda) \log k$ shows that eventually

$$\sum_k^{n-1} [C(\bar{\mathrm{A}}_l)]^{-1} \leq \frac{6}{\lambda} \log k + 2[M_n(x) - M_k(x)] , \qquad x \in \mathrm{A}_k , \quad n \geq k .$$

Combined with lemma 2 and the observation $T(r) \leq C_5 r^d$, $\log T(r) \leq d \log r + C_6$, this yields our fundamental estimate.

*Proposition 3.* W.p. 1 one has for all sufficiently large $r$ and all $x \in \mathrm{A}_{T(r)}$

$$\frac{r}{16d} \leq \frac{r}{8d} - \frac{6d}{\lambda} \log r - \frac{6}{\lambda} C_6 \leq \sum_{T(r)}^{T(2r)-1} [C(\bar{\mathrm{A}}_l)]^{-1} - \frac{6}{\lambda} \log T(r)$$
$$\leq 2\{M_{T(2r)} - M_{T(r)}\} = 2 \sum_{y \in \mathrm{A}_{T(2r)} \setminus \mathrm{A}_{T(r)}} G(x, y)$$
$$\leq C_7 \sum_{y \in \mathrm{A}_{T(2r)} \setminus \mathrm{A}_{T(r)}} \frac{1}{1 + |x - y|^{d-2}} . \tag{6}$$

The remaining arguments are purely deterministic. We merely have to investigate how small $|A_{T(2r)} \backslash A_{T(r)}|$ can be and still satisfy (6) for all $x \in A_{T(r)}$. ($|B|$ denotes the cardinality of B.) Let

$$B(x, r) = \{x : |x| \le r\}, \qquad C(x, r) = B(x, 2r) \backslash B(x, r)$$

be the ball of radius $r$ with center at $x$, and the shell between $B(x, 2r)$ and $B(x, r)$, respectively. Then, for suitable $C_8$ and $C_9$ (cf. ref. [2], p. 181)

$$C_7 \sum_{y \in B(x, C_8\sqrt{r})} \frac{1}{1 + |x - y|^{d-2}} \le C_9 C_8^2 r \le \frac{r}{32d}.$$

Consequently (6) yields

$$\begin{aligned} r &\le C_{10} \sum_{\frac{1}{2}C_8\sqrt{r} \le 2^k \le r} \; \sum_{y \in A_{T(2r)} \cap C(x, 2^k)} \frac{1}{1 + |x - y|^{d-2}} \\ &\le C_{10} \sum_{\frac{1}{2}C_8\sqrt{r} \le 2^k \le r} 2^{-k(d-2)} |A_{T(2r)} \cap C(x, 2^k)|. \end{aligned} \tag{7}$$

We claim that (7) implies for a suitable $C_{11} > 0$ (independent of $x$ and $r$) that there exists a $k(x)$ with

$$\tfrac{1}{2} C_8 \sqrt{r} \le 2^{k(x)} \le r$$

and

$$|A_{T(2r)} \cap C(x, 2^{k(x)})| \ge C_{11} r^{(d-1)/2} 2^{k(x)}. \tag{8}$$

Indeed, if (8) fails for some $x$, then the right-hand side of (7) for this $x$ is at most $2(C_8/2)^{-(d-3)/2} C_{10} C_{11} r$, which cannot be for small $C_{11}$.

*Proof of theorem 1.* Choose a number of more or less disjoint (see (9) below) balls $B(x, 2^{k(x)})$ with $x \in A_{T(r)}$ as follows. Let $(v_0, v_1, \ldots, v_\nu)$ be a path in $A_{T(r)}$, whose initial point $v_0$ is the origin and whose endpoint $v_\nu$ satisfies $|v_\nu| = r$. Choose $B_0 = B(0, 2^{k(0)+1})$. Once $B_0, \ldots, B_l$ have been chosen of the form $B(v_{j_i}, 2^{k(v_{j_i})+1})$, $0 \le i \le l$, with $j_0 < j_1 \cdots < j_l$, we take

$$j_{l+1} = \max\{j > j_l : B(v_j, 2^{k(v_j)+1}) \text{ intersects } B_l\}.$$

Thus, by definition

$$B_{l+1} \cap B_l \ne \emptyset, \qquad \text{but } B_m \cap B_l = \emptyset \quad \text{for } m > l + 1. \tag{9}$$

In particular, no vertex $v$ can be in more than two of the balls $B_j$. Thus, if $B_0, \ldots, B_\sigma$, with $v_{j_\sigma} = v_\nu$ are all the balls selected in this way until we are forced to stop, then we have

$$T(2r) = \left|A_{T(2r)}\right| \geq \frac{1}{2} \sum_0^\sigma \left|A_{T(2r)} \cap B_l\right| \geq \frac{1}{2} \sum_0^\sigma \left|A_{T(2r)} \cap C(v_{j_l}, 2^{k(v_{j_l})})\right|$$
$$\geq \tfrac{1}{2} C_{11} r^{(d-1)/2} \sum_0^\sigma 2^{k(v_{j_l})} \qquad \text{(by (8))}. \tag{10}$$

On the other hand,

$$r = \left|v_\nu\right| \leq \sum_0^{\sigma-1} \left|v_{j_{l+1}} - v_{j_l}\right| \leq \sum_0^{\sigma-1} (2^{k(v_{j_{l+1}})+1} + 2^{k(v_{j_l})+1}) \leq 4 \sum_0^\sigma 2^{k(v_{j_l})}. \tag{11}$$

(10) and (11) yield (3) with $r$ replaced by $2r$.

*Proof of theorem 2.* This is much easier. Indeed, assume

$$N(2^{k+1}) \leq \frac{1}{3C_{10}} 2^{k(d-1)} \qquad \text{for all } 2^k \in [\tfrac{1}{2} C_8 \sqrt{r}, r]. \tag{12}$$

Then the right-hand side of (7), with $x$ the origin, is at most

$$C_{10} \sum_{2^k \leq r} \frac{1}{3C_{10}} 2^k < r.$$

Thus, (12) must fail for all sufficiently large $r$.

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