

HITTING PROBABILITIES OF RANDOM WALKS ON \mathbb{Z}^d

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Let S_0, S_1, \dots be a simple (nearest neighbor) symmetric random walk on \mathbb{Z}^d and $H_B(x, y) = P\{S$ visits B for the first time at $y \mid S_0 = x\}$. If $d = 2$ we show that for any connected set B of diameter r , and any $y \in B$, one has

$$\limsup_{|x| \rightarrow \infty} H_B(x, y) \leq C(2)r^{-1/2}.$$

If $d \geq 3$ one has for any connected set B of cardinality n ,

$$\limsup_{|x| \rightarrow \infty} H_B(x, y) \leq C(d)n^{-1+2/d}.$$

These estimates can be used to give bounds on the maximal growth rate of diffusion limited aggregation, a fashionable growth model for various physical phenomena.

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1. Introduction and statement of results

Witten and Sander (1981) described a stochastic growth model which they called “diffusion limited aggregation” (DLA for short). Simulations show that it mimicks several physical phenomena well. As examples of such phenomena Meakin (1986) mentions “dendritic growth, fluid–fluid displacement, colloidal aggregation and dielectric breakdown”. The model is quite fashionable, as one can see from the many articles devoted to DLA in Stanley and Ostrowski (1986), as well as the many talks on this model at the 1986 Statistical Physics meeting in Boston (Proceedings forthcoming). The aggregates or clusters formed in this model tend to be rather thin, with large empty regions and long arms. Fig. 1 from Witten and Sander (1981) shows one simulation.

It is believed that aggregates of a very large number of particles, n , consist almost entirely of $2d$ arms along the positive and negative coordinate axes. The length of these arms should be of greater order than $n^{1/d}$, since the aggregates are much

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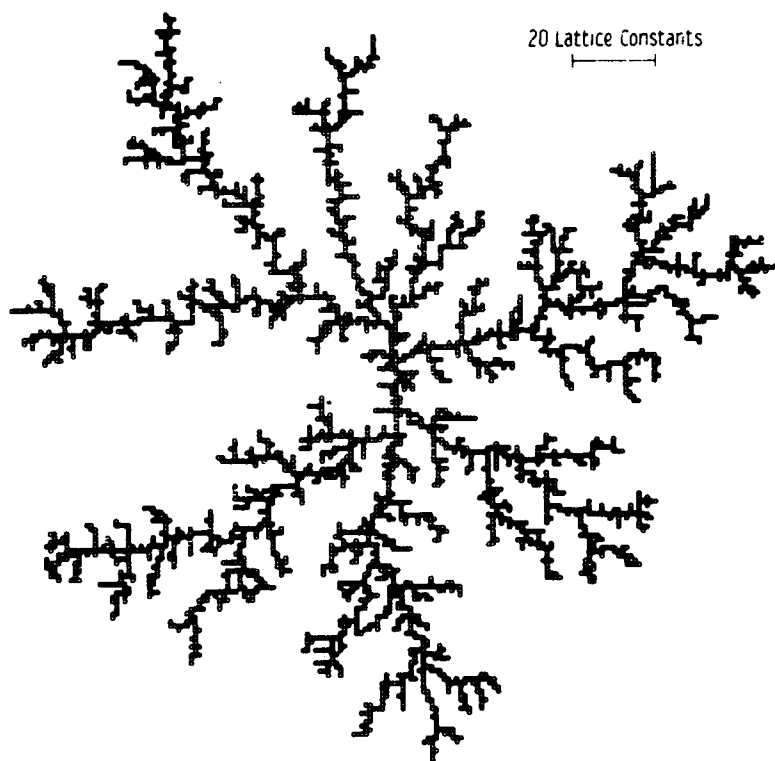


Fig. 1 (reproduced from Witten and Sander). Random aggregate of 3600 particles on \mathbb{Z}^2 .

thinner than solid balls. In fact, if r_n is the radius of the aggregate (see (1.5) for the precise definition) then simulations indicate that

$$\lim_{n \rightarrow \infty} \frac{\log r_n}{\log n} \quad \text{exists and is approximately equal to } ((0.85)d)^{-1}$$

(see Table 1 in Meakin (1986)). In Kesten (1986) we showed how a certain estimate for hitting probabilities of a simple random walk on \mathbb{Z}^d implies $r_n = O(n^{2/3})$ for $d = 2$ and $r_n = O(n^{2/d})$ for $d \geq 3$. It is the purpose of this paper to prove the required estimates for the hitting probabilities.

Several variants of the DLA model have been considered. We deal only with the original, and simplest model. A_1 consists of the origin. A_n is a connected set of n vertices on \mathbb{Z}^d . A_{n+1} is formed by adding to A_n a site y_{n+1} in the boundary of A_n . The boundary of a set A of vertices is defined as

$$\partial A := \{y \in \mathbb{Z}^d : y \text{ is adjacent to some site in } A, \text{ but } y \notin A\}.$$

The site y_{n+1} , to be added to A_n , is chosen according to a chance mechanism, which can intuitively be described as follows. A particle is released at ∞ and performs a simple symmetric random walk on \mathbb{Z}^d . y_{n+1} is the position where the random walk first hits ∂A_n . A more formal description of the distribution of y_{n+1} (or rather, the conditional distribution of y_{n+1} , given A_n) runs as follows. Let S_0, S_1, \dots be a simple (nearest neighbor) symmetric random walk on \mathbb{Z}^d , starting at S_0 . Denote by P_x the conditional distribution of $\{S_n\}_{n \geq 0}$, given that $S_0 = x$, and by E_x expectation with respect to P_x . For any set B define its hitting time as

$$\tau = \tau(B) := \inf\{n \geq 0 : S_n \in B\}.$$

The distribution of the hitting position S_τ is described by

$$H_B(x, y) := P_x\{\tau(B) < \infty \text{ and } S_{\tau(B)} = y\}.$$

For $d = 2$, and finite nonempty B , $\tau(B) < \infty$ w.p.1,

$$\mu_B(y) := \lim_{|x| \rightarrow \infty} H_B(x, y) \text{ exists,} \tag{1.1}$$

and

$$\sum_{y \in B} \mu_B(y) = 1 \tag{1.2}$$

(cf. Spitzer (1976, Theorem 14.1)). For $d = 2$ the conditional distribution of y_{n+1} , given A_n , is taken to be

$$P\{y_{n+1} = y | A_n\} = \mu_{\partial A_n}(y), \quad y \in \partial A_n.$$

(The limit $|x| \rightarrow \infty$ in (1.1) corresponds to “releasing the particle at ∞ ”.) For $d \geq 3$, $\lim_{|x| \rightarrow \infty} H_B(x, y) \equiv 0$, since the random walk is transient (cf. Spitzer (1976, Proposition 25.3)). We must now condition on $\tau(B) < \infty$ to obtain a nontrivial limit. For $d \geq 3$ we define

$$\begin{aligned} \mu_B(y) &= \lim_{|x| \rightarrow \infty} \frac{H_B(x, y)}{\sum_{z \in B} H_B(x, z)} \\ &= \lim_{|x| \rightarrow \infty} P_x\{S_{\tau(B)} = y | \tau(B) < \infty\}, \quad y \in B. \end{aligned} \tag{1.3}$$

Again this limit exists and satisfies (1.2). In fact, Spitzer (1976, Proposition 26.2 and Definition 25.1), identifies the limit in (1.3) as

$$\frac{e_B(y)}{\sum_{z \in B} e_B(z)}, \tag{1.4}$$

where

$$e_B(y) := P_y\{S_n \notin B, n \geq 1\}$$

is the escape probability of B from y .

We can now state the required bound for the hitting probabilities.

Theorem. *Let B be a connected set of vertices in \mathbb{Z}^d which contains the origin. Let $|B|$ denote its cardinality and let*

$$r(B) := \max\{|x|: x \in B\} \tag{1.5}$$

be its “radius”. Then there exists constants $C(d) < \infty$, depending on d only, such that, for all $y \in B$,

$$\mu_B(y) \leq C(2)[r(B)]^{-1/2} \quad \text{if } d = 2, \tag{1.6}$$

$$\mu_B(y) \leq C(d)[|B|]^{(2-d)/d} \quad \text{if } d \geq 3. \tag{1.7}$$

Corollary. *In the DLA model there exist constants $C^*(d) < \infty$ such that w.p. 1*

$$\limsup_{n \rightarrow \infty} n^{-2/3} r(A_n) \leq C^*(2) \quad \text{if } d = 2, \tag{1.8}$$

$$\limsup_{n \rightarrow \infty} n^{-2/d} r(A_n) \leq C^*(d) \quad \text{if } d \geq 3. \tag{1.9}$$

To prove (1.6) we more or less have to find B and y which maximize $\mu_B(y)$, when $r(B)$ is held fixed. Known results for Brownian motion (or harmonic measures) suggest that one maximizes $\mu_B(y)$ when B is a straight line segment of length $2r(B)$ and y is one of its endpoints. As it is, we can only show that for any y and B , $\mu_B(y)$ is at most a constant times the value obtained for B a straight line segment and y one of its endpoints. The proof, which imitates classical estimations for harmonic measure, uses somewhat involved estimates for escape probabilities (by time reversal these are equivalent to estimates for hitting probabilities). Fortunately the proof of (1.7) is much easier. Since Spitzer (1976) has shown that $\mu_B(y)$ is bounded by the inverse of the capacity of B , (1.7) reduces to a simple estimate of this capacity.

2. The two-dimensional case

We begin with the proof of (1.6). Its proof is based on the fact that there exist for any $y \in B$ vertices $u_0 = y, u_1, \dots, u_k = z$ in B such that u_0, \dots, u_k is a path on \mathbb{Z}^d (i.e., u_{i+1} is adjacent to $u_i, 0 \leq i \leq k-1$) starting at y , and ending at some $z \in B$ with $|z - y| \geq r(B)/2$. Indeed, if $u \in B$ is such that $|u| = r(B)$, then $|y - \mathbf{0}| \geq r(B)/2$ or $|y - u| \geq r(B)/2$. In the former case we take $z = \mathbf{0}$ and connect y to $\mathbf{0}$ by a path in B ; such a path exists since B is connected. In the latter case, take $z = u$ and connect y to u . If P is the set $\{u_0, u_1, \dots, u_k\}$, then $P \subset B$, and trivially $H_B(x, y) \leq H_P(x, y), x \notin B$, and therefore $\mu_B(y) \leq \mu_P(y)$. The main estimate is to show that $\mu_P(y)$ is maximized (up to a constant factor) when the path P is along a straight line (see Lemma 9). More specifically, let

$$C = ([\lfloor r(B)/2 \rfloor, 0] \times \{0\}) \cap \mathbb{Z}^2$$

be a line segment along the negative x -axis of length $\lfloor r(B)/2 \rfloor$. (Here $\lfloor a \rfloor =$ greatest integer $\leq a$.) We shall then prove (1.6) by carrying out the following two steps.

- (i) $\mu_C((-k, 0)) \leq C_1[(k+1)r(B)]^{-1/2}, \quad 0 \leq k \leq r(B)/4,$
- (ii) $\mu_B(y) \leq \mu_P(y) \leq C_2[r(B)]^{-1/2}$

for suitable constants C_1, C_2 , independent of y and B . The analogue of the first step for Brownian motion is easy, since the harmonic measure for an interval can be explicitly calculated. In fact the map $z \rightarrow w = a(z + z^{-1}), a > 0$, maps the exterior of the unit disc conformally to the plane slit along the real axis from $-2a$ to $2a$. The analogue of (ii) for Brownian motion is almost immediate from Beurling's

circular projection theorem (cf. Ahlfors (1973, Theorem 3.6)). Unfortunately, we have been unable to use the Brownian motion results directly for our proof. Instead they merely form a guideline for proofs to be done separately for random walks. Step (i) is largely computational, while the proof of (ii) closely mimicks the proof of Beurling's theorem as given in Ahlfors.

Proof of (i). We need some known facts about the potential kernel of two dimensional random walk, which we collect as a lemma.

Lemma 1. *The series*

$$a(x) = \sum_{n=0}^{\infty} [P_0\{S_n = \mathbf{0}\} - P_0\{S_n = x\}]$$

converges for each $x \in \mathbb{Z}^2$, and the function $a(\cdot)$ has the following properties (where $a \wedge b = \min\{a, b\}$):

$$a(x) \geq 0 \text{ for all } x, \quad a(\mathbf{0}) = 0, \tag{2.1}$$

$$a((0, \pm 1)) = a((\pm 1, 0)) = 1,$$

$$E_x\{a(S_1)\} - a(x) = \delta(x, \mathbf{0}), \tag{2.2}$$

so $a(S_{n \wedge \tau(v)} - v)$ is a nonnegative martingale, for any $v \in \mathbb{Z}^d$ ($\tau(v) = \tau(\{v\})$),

$$\left| a(x) - \frac{1}{2\pi} \log|x| - C_0 \right| = O(|x|^{-2}), \tag{2.3}$$

as $|x| \rightarrow \infty$ for a suitable C_0 .

Proof. The convergence of $a(x)$ and (2.1), (2.2) are in Spitzer (1976, Propositions 12.1, 11.7, 13.3, and p. 148), while (2.3) is in Stöhr (1950, part III). (A less precise form of (2.3) is in Spitzer (1976, Proposition 12.3); we will not need the full force of (2.3).) \square

In the sequel C_i will denote a strictly positive finite constant whose value is unimportant for our purposes; its value does not have to be the same at different appearances. $\mathcal{C}(s)$ will be the "circle" of radius s and center at $\mathbf{0}$. This means

$$\mathcal{C}(s) = \{x \in \mathbb{Z}^d : |x| > s, \text{ but } x \text{ is the endpoint of an edge whose other endpoint, } x', \text{ satisfies } |x'| \leq s\}.$$

In particular,

$$s < |x| \leq s + 1 \text{ for all } x \in \mathcal{C}(s). \tag{2.4}$$

Lemma 2. Let $r = r(B)$ and D a collection of vertices of \mathbb{Z}^2 contained in the disc $\{u: |u| \leq r\}$. Set

$$\lambda = \lambda(D) = \min\{n \geq 1: S_n \in D\}. \quad (2.5)$$

Then there exists a constant C_3 , independent of r, D such that

$$\begin{aligned} \mu_D(y) &\leq C_3 \limsup_{R \rightarrow \infty} \frac{1}{R} P_y\{\tau(\mathcal{C}(R)) < \lambda\} \\ &\quad \cdot \max_{v \in \mathcal{C}(R)} [P_v\{S \text{ visits } D \text{ before returning to } \mathcal{C}(R)\}]^{-1} \\ &\leq C_3 \sum_{z \in \mathcal{C}(2r)} P_y\{\tau(\mathcal{C}(2r)) < \lambda, S_{\tau(\mathcal{C}(2r))} = z\} \\ &\quad \cdot \limsup_{R \rightarrow \infty} \frac{1}{R} P_z\{\tau(\mathcal{C}(R)) < \lambda\} \\ &\quad \cdot \max_{v \in \mathcal{C}(R)} [P_v\{S \text{ visits } D \text{ before returning to } \mathcal{C}(R)\}]^{-1}, \quad y \in D. \quad (2.6) \end{aligned}$$

Proof. We use time reversal and symmetry, i.e., the fact that the probability of the random walk taking successively the steps x_1, x_2, \dots, x_n is the same as the probability of taking successively the steps $-x_n, -x_{n-1}, \dots, -x_1$. This implies for any set D (compare Spitzer (1976, Proposition 10.2)),

$$P_x\{S_n = y, S_j \notin D \text{ for } 0 \leq j \leq n-1\} = P_y\{S_n = x, S_j \notin D \text{ for } 0 < j \leq n\}.$$

In particular, for $y \in D, x \notin D$,

$$\begin{aligned} H_D(x, y) &= \sum_{n=0}^{\infty} P_y\{S_n = x, S_j \notin D \text{ for } 0 < j \leq n\} \\ &= E_y\{\text{number of visits to } x \text{ during the time interval } [0, \lambda)\}. \end{aligned}$$

Since we already know that the limit in (1.1) exists we can write ($|\mathcal{C}|$ = cardinality of \mathcal{C})

$$\begin{aligned} \mu_D(y) &= \lim_{R \rightarrow \infty} \frac{1}{|\mathcal{C}(R)|} \sum_{x \in \mathcal{C}(R)} H_D(x, y) \\ &\leq \limsup_{R \rightarrow \infty} \frac{C_3}{R} \sum_{x \in \mathcal{C}(R)} E_y\{\text{number of visits to } x \text{ during } [0, \lambda)\} \\ &= C_3 \limsup_{R \rightarrow \infty} \frac{1}{R} E_y\{\text{number of visits to } \mathcal{C}(R) \text{ during } [0, \lambda)\}. \quad (2.7) \end{aligned}$$

(2.6) is now obvious if we take into account that any path from D to $\mathcal{C}(R)$ must first hit $\mathcal{C}(2r)$, and then $\mathcal{C}(R)$ before λ . Once it reaches $\mathcal{C}(R)$, in w say, then the

conditional expectation of the number of visits to $\mathcal{C}(R)$ before λ is

$$\sum_{k=0}^{\infty} P_w\{S. \text{ returns at least } k \text{ times to } \mathcal{C}(R) \text{ before } \lambda\} \\ \leq \sum_{k=0}^{\infty} \left[\sup_{v \in \mathcal{C}(R)} P_v\{S. \text{ returns at least once to } \mathcal{C}(R) \text{ before } \lambda\} \right]^k,$$

which is just the last factor in the right hand side of (2.6). \square

We now estimate the three factors in the right hand side of (2.6) in a sequence of lemmas.

Lemma 3. *Let $D \subset \{u: |u| \leq r\}$ contain the origin. Then for $R > 2r + 1$ sufficiently large one has uniformly in $v \in \mathcal{C}(R)$ and in D*

$$P_v\{S. \text{ visits } D \text{ before returning to } \mathcal{C}(R)\} \geq C_4 [R \log R]^{-1}. \tag{2.8}$$

Proof. Let $v \in \mathcal{C}(R)$ and w a vertex of \mathbb{Z}^2 such that $|w| \leq R - 1$, $|v - w| \leq 4$. Then the left hand side of (2.8) is at least

$$P_v\{S_4 = w, S_j \notin \mathcal{C}(R) \text{ for } 1 \leq j \leq 4\} \cdot P_w\{S. \text{ visits } D \text{ before } \mathcal{C}(R)\}. \tag{2.9}$$

The first factor in (2.9) is at least 4^{-6} , since we can go in one step from v to some v' with $|v'| \leq R$ (by definition of $\mathcal{C}(R)$), and from there to w in at most 5 steps, staying in the disc of radius R . To give a lower bound for the second factor in (2.9) we use the martingale $Y_n := a(S_{n \wedge \tau(0)})$ (cf. (2.2)). Starting at w we have for $\sigma = \tau(D) \wedge \tau(\mathcal{C}(R)) \leq \tau(0)$

$$a(w) = E_w\{a(S_\sigma); \tau(\mathcal{C}(R)) < \tau(D)\} + E_w\{a(S_\sigma); \tau(D) < \tau(\mathcal{C}(R))\} \\ \geq P_w\{\tau(\mathcal{C}(R)) < \tau(D)\} E_w\{a(S_\sigma) | \tau(\mathcal{C}(R)) < \tau(D)\}. \tag{2.10}$$

Now, by (2.3),

$$a(w) = \frac{1}{2\pi} \log |w| + C_0 + O(|w|^{-2}) \\ \leq \frac{1}{2\pi} \log R - \frac{1}{2\pi R} + C_0 + O(R^{-2}).$$

Similarly, since on $\{\tau(\mathcal{C}(R)) < \tau(D)\} S_\sigma \in \mathcal{C}(R)$, we have by (2.4)

$$E_w\{a(S_\sigma) | \tau(\mathcal{C}(R)) < \tau(D)\} \geq \frac{1}{2\pi} \log R + C_0 + O(R^{-2}).$$

Substitution of these estimates into (2.10) yields

$$P_w\{\tau(D) < \tau(\mathcal{C}(R))\} = 1 - P_w\{\tau(\mathcal{C}(R)) < \tau(D)\} \\ \geq \frac{1}{R \log R} + O\left(\frac{1}{R \log^2 R}\right),$$

and (2.8) now follows from (2.9). \square

The next lemma does no longer apply to a general D but only to the specific set C .

Lemma 4. For $R > 2r + 1$ sufficiently large we have, uniformly in $z \in \mathcal{C}(2r)$,

$$(\log R)P_z\{\tau(\mathcal{C}(R)) < \lambda(C)\} \leq C_5.$$

Proof. This time we consider the martingale

$$Z_n := \sum_{v \in C} a(S_{n \wedge \tau(C)} - v).$$

We start at $z \in \mathcal{C}(2r)$. Again with $\sigma = \tau(C) \wedge \tau(\mathcal{C}(R)) \leq \tau(\mathbf{0})$ we have

$$\begin{aligned} \sum_{v \in C} a(z - v) &= P_z\{\tau(\mathcal{C}(R)) < \tau(C)\} \sum_v E_z\{a(S_\sigma - v) \mid \tau(\mathcal{C}(R)) < \tau(C)\} \\ &\quad + (1 - P_z\{\tau(\mathcal{C}(R)) < \tau(C)\}) \sum_v E_z\{a(S_\sigma - v) \mid \tau(C) < \tau(\mathcal{C}(R))\}. \end{aligned} \tag{2.11}$$

By (2.3) we have ($|C|$ = cardinality of C)

$$\sum_{v \in C} a(z - v) = \frac{1}{2\pi}|C| \log(r + 1) + O(|C|) \tag{2.12}$$

uniformly for $z \in \mathcal{C}(2r)$. Also, on $\{\tau(\mathcal{C}(R)) < \tau(C)\}$ we have $S_\sigma \in \mathcal{C}(R)$ and hence

$$\sum_v a(S_\sigma - v) = \frac{1}{2\pi}|C| \log R + O(|C|). \tag{2.13}$$

Finally, on $\{\tau(C) < \tau(\mathcal{C}(R))\}$ we must have $S_\sigma = (-k, 0)$ for some $0 \leq k \leq |C| - 1 \leq r/2$. If $S_\sigma = (-k, 0)$, then, uniformly in $0 \leq k < |C|$,

$$\begin{aligned} \sum_{v \in C} a(S_\sigma - v) &= \sum_{0 \leq j < |C|} a((j - k, 0)) \\ &= \sum_{\substack{0 \leq j < |C| \\ j \neq k}} \left\{ \frac{1}{2\pi} \log|j - k| + C_0 + O(|j - k|^{-2}) \right\} \\ &\geq \frac{1}{2\pi}|C| \log|C| + O(|C|). \end{aligned} \tag{2.14}$$

Since $|C| \sim \frac{1}{2}r$ and $\tau(C) = \lambda(C)$ for $S_0 = z \notin C$, the lemma follows by substitution of (2.12)-(2.14) in (2.11). \square

Combining Lemmas 2-4 we find

$$\mu_C(y) \leq C_6 P_y\{\tau(\mathcal{C}(2r)) < \lambda(C)\}, \quad y \in C, \tag{2.15}$$

and we proceed to estimate the right hand side of (2.15) by means of the imbedded random walk on the x -axis. This imbedded random walk is defined as follows. Set $\sigma_0 = 0$,

$$\sigma_{k+1} = \inf\{n > \sigma_k : S_n \text{ belongs to the } x\text{-axis}\},$$

and

$$T_k = x\text{-coordinate of } S_{\sigma_k}, \quad Y_k = T_{k+1} - T_k.$$

If S_0 lies on the x -axis, then the random variables $\{Y_k, k \geq 0\}$ are i.i.d. The common distribution is calculated in Spitzer (1976, pp. 155-156). For $S_0 = (j, 0)$, $Y_1 = T_1 - j$ takes the values $+1$ and -1 , each with probability $\frac{1}{4}$ (if the first step at S is horizontal), and with probability $\frac{1}{2}$ $y_1 = T_1 - j$ is a random variable with the characteristic function $\varphi(\theta)$ of Spitzer (1976, formula (15.7)) (if the first step of S is vertical). Therefore

$$E_{(j,0)} e^{i\theta Y_1} = \left(\frac{1}{2} \cos \theta\right) + \frac{1}{2} \{2 - \cos \theta - [(1 - \cos \theta)(3 - \cos \theta)]^{1/2}\}, \tag{2.16}$$

$$1 - E_{(j,0)} e^{i\theta Y_1} \sim \frac{1}{2} |\theta|, \quad \theta \rightarrow 0, \tag{2.17}$$

and the Y_k are symmetric (when $S_0 = (j, 0)$).

It is an easy matter (e.g. by the invariance principle) to show that uniformly in $z \in \mathcal{C}(2r)$ ($r = r(B)$)

$$P_z\{S \text{ hits } (-\infty, -r(B)/2) \text{ before } C\} \geq C_7 > 0$$

(recall that $\mathcal{C}(2r)$ is essentially the circle with radius $2r$ and center at $\mathbf{0}$, while $C \subset [-r(B)/2, 0] \times \{0\}$). Therefore, for $y \in C$,

$$\begin{aligned} &P_y\{S \text{ visits } (-\infty, -r/2) \text{ before it returns to } C\} \\ &\geq \sum_{z \in \mathcal{C}(2r)} P_y\{\tau(\mathcal{C}(2r)) < \lambda \text{ and } S_{\tau(\mathcal{C}(2r))} = z\} \\ &\quad \cdot P_z\{S \text{ hits } (-\infty, -r/2) \text{ before } C\} \\ &\geq C_7 P_y\{\tau(\mathcal{C}(2r)) < \lambda(C)\}. \end{aligned} \tag{2.18}$$

If we define the hitting time of the negative half line for T by

$$\rho = \min\{k \geq 1: T_k \leq 0\},$$

then the left hand side of (2.18) can also be written as $P_y\{T_\rho < -r/2\}$. Combining (2.15) with (2.18) we therefore have

$$\mu_C(y) \leq C_8 P_y\{T_\rho < -r/2\}. \tag{2.19}$$

Finally, we define the Green function for the random walk T stopped at ρ as

$$\begin{aligned} G(j, l) &= E_{(j,0)}\{\text{number of visits by } T \text{ to } l \text{ before } \rho\} \\ &= E_{(j,0)}\left\{\sum_{0 \leq n < \rho} I[T_n = l]\right\}. \end{aligned}$$

Lemma 5. For $j, l > 0$,

$$G(j, l) = \sum_{n=1}^{j \wedge l} v(j-n)v(l-n) \tag{2.20}$$

for some numbers $v(\cdot)$ satisfying

$$v(n) \geq 0, \tag{2.21}$$

$$V(n) := \sum_0^n v(k) \sim C_9 \sqrt{n}, \quad n \rightarrow \infty, \tag{2.22}$$

$$V(T_{n \wedge \rho}) \text{ is a nonnegative martingale under } P_{(j,0)}, j > 0. \tag{2.23}$$

Proof. The representation (2.20) for some v satisfying (2.21) and (2.23) is proved in Spitzer (1976, Proposition 19.3, 19.5). (Note that we stop the random walk when it enters $-\infty, 0]$, whereas Spitzer stops it upon entrance of $-\infty, -1]$. Therefore the sum in (2.20) starts at $n = 1$ rather than $n = 0$ as in Spitzer. Also Spitzer's $u(\cdot)$ and $v(\cdot)$ can be taken equal by virtue of the symmetry of the Y 's.) Moreover (Spitzer (1976, Definition 18.2)), for $|z| < 1$,

$$\sum_0^\infty v(n)z^n = \exp\left\{\sum_1^\infty \frac{1}{k} [E_0\{z^{T_k}; T_k > 0\} + \frac{1}{2}P_0\{T_k = 0\}]\right\}.$$

Thus, by Karamata's Tauberian theorem (cf. Feller 1971, Theorem XIII.5.5)) it suffices for (2.22) to prove that

$$\lim_{r \uparrow 1} \sqrt{1-r} \exp\left\{\sum_1^\infty \frac{1}{k} [E_0\{r^{T_k}; T_k > 0\} + \frac{1}{2}P_0\{T_k = 0\}]\right\} = C_{10} > 0. \tag{2.24}$$

However, the calculations on p. 184 of Spitzer (1976) show that the left hand side of (2.24) equals

$$\lim_{r \uparrow 1} \exp\left\{\frac{1}{4\pi} \int_{-\pi}^{+\pi} \frac{1-r^2}{1+r^2-2r \cos \theta} \log \frac{|1-e^{i\theta}|}{1-\Psi(\theta)} d\theta\right\}, \tag{2.25}$$

where

$$\Psi(\theta) = E_0 e^{i\theta Y_1}.$$

(2.24) (with $C_{10} = \sqrt{2}$), and hence (2.22), now follows from (2.25) and (2.17) and standard results about radial limits of Poisson integrals (cf. Hille (1962, Theorem 17.5.1)). \square

Lemma 6. *Uniformly for $y = (-k, 0)$, $0 \leq k \leq r/4 = r(B)/4$, $r \geq 1$, one has*

$$P_y\{T_\rho < -r/2\} \leq C_{11}[(k+1)r]^{-1/2}. \tag{2.26}$$

Proof. If $y = (-k, 0)$, $0 \leq k \leq r/4$, then

$$\begin{aligned} P_y\{T_\rho < -r/2\} &= P_y\{T_1 < -r/2\} + \sum_{j=1}^\infty P_y\{T_1 = j \text{ and } T_\rho < -r/2\} \\ &\leq P_y\{Y_1 < -r/4\} + \sum_{j=1}^\infty P_y\{Y_1 = j+k\} \sum_{l=1}^\infty G(j, l) P_{(l,0)}\{T_1 < -r/2\}. \end{aligned}$$

Under P_y , Y_1 has the characteristic function given in (2.16), and hence belongs to the domain of attraction of the Cauchy distribution. (2.17) implies

$$P_y\{Y_1 < -n\} \sim \frac{1}{2n}, \quad n \rightarrow \infty, \tag{2.27}$$

and also

$$P_{(l,0)}\{T_1 < -r/2\} = P_{(l,0)}\{Y_1 < -(l + \frac{1}{2}r)\} \sim \frac{1}{2l+r}.$$

By virtue of Lemma 5 we therefore have

$$P_y\{T_\rho < -r/2\} \leq C_{12} \left[r^{-1} + \sum_{j=1}^{\infty} P_y\{Y_1 = j+k\} \sum_{n=1}^j v(j-n) \sum_{l=n}^{\infty} v(l-n) \frac{1}{l+r} \right]. \tag{2.28}$$

Also, by (2.22),

$$\sum_{n=1}^j v(j-n) \leq C_{13} \sqrt{j}$$

and

$$\begin{aligned} \sum_{l=n}^{\infty} v(l-n) \frac{1}{l+r} &= \sum_{m=0}^{\infty} V(m) [(m+n+r)(m+n+1+r)]^{-1} \\ &\leq C_{13} (n+r)^{-1/2} \leq C_{13} r^{-1/2}. \end{aligned}$$

Therefore the right hand side of (2.28) is at most

$$C_{14} \left[r^{-1} + r^{-1/2} \sum_{j=1}^{\infty} P_y\{Y_1 = j+k\} j^{1/2} \right].$$

Another summation by parts and an appeal to (2.27) now establishes (2.26). \square

(i) now is immediate from (2.19) and (2.26). We remark that by symmetry between left and right (i) implies

$$\begin{aligned} \mu_C((-k, 0)) &\leq C_1 [\{(k+1) \wedge (r(B)/2) - k + 1\} r(B)]^{-1/2} \\ &\text{for } 0 \leq k \leq r(B)/2. \end{aligned} \tag{2.29}$$

Proof of (ii). As pointed out the first inequality in (ii) is immediate from $P \subset B$. For the second inequality we follow Ahlfors (1973, pp. 43, 44). Let the Green function with respect to $\mathcal{C}(R)$, $g(\cdot, \cdot)$, be defined by

$$\begin{aligned} g(u, v) &= g_R(u, v) = E_u \{ \text{number of visits by } S. \text{ to } v \text{ before } \tau(\mathcal{C}(R)) \} \\ &= E_u \left\{ \sum_{0 \leq n < \tau(\mathcal{C}(R))} I[S_n = v] \right\}. \end{aligned}$$

Lemma 7. For $R \geq 2r \geq 2$ we have

$$\left| g_R(u, v) - \frac{1}{2\pi} \log R + \frac{1}{2\pi} \log(|u-v| \vee 1) \right| \leq C_3 \left[\frac{r}{R} + (|u-v| \vee 1)^{-2} \right], \tag{2.30}$$

uniformly in $|u|, |v| \leq r$ (where $a \vee b = \max\{a, b\}$).

Proof. The formula for g immediately shows

$$g(u, v) = P_u\{\tau(v) < \tau(\mathcal{C}(R))\} \cdot [1 - P_v\{\lambda(v) < \tau(\mathcal{C}(R))\}]^{-1}, \tag{2.31}$$

where

$$\lambda(v) = \inf\{n \geq 1: S_n = v\}.$$

Similarly to (2.10) we consider now the martingale $a(S_{n \wedge \tau(v) \wedge \tau(\mathcal{C}(R))} - v)$, and take $S_0 = u$.

$$a(u - v) = P_u\{\tau(\mathcal{C}(R)) < \tau(v)\} \cdot E_u\{a(S_{\tau(\mathcal{C}(R))} - v) \mid \tau(\mathcal{C}(R)) < \tau(v)\}.$$

By (2.3) and $|v| \leq r$ we have

$$\begin{aligned} \left| E_u\{a(S_{\tau(\mathcal{C}(R))} - v) \mid \tau(\mathcal{C}(R)) < \tau(v)\} - \frac{1}{2\pi} \log R - C_0 \right| &\leq C_4 \left(\frac{r}{R} + \frac{1}{R^2} \right) \\ &\leq C_4 \frac{r+1}{R}, \end{aligned}$$

uniformly in $|u| \leq R$. It follows that

$$P_u\{\tau(v) < \tau(\mathcal{C}(R))\} = 1 - \left[\frac{1}{2\pi} \log R + C_0 + \theta C_4 \frac{r+1}{R} \right]^{-1} a(u - v) \quad (2.32)$$

for some $\theta = \theta(u, v, R) \in [-1, +1]$. This also implies, for some other $\theta' \in [-1, +1]$,

$$\begin{aligned} 1 - P_v\{\lambda(v) < \tau(\mathcal{C}(R))\} &= \frac{1}{4} \sum_{\substack{w \text{ adjacent} \\ \text{to } v}} P_w\{\tau(\mathcal{C}(R)) < \tau(v)\} \\ &= \left[\frac{1}{2\pi} \log R + C_0 + \theta' C_4 \frac{r+1}{R} \right]^{-1}. \end{aligned} \quad (2.33)$$

Here we used $a(w - v) = 1$ for w adjacent to v ; see (2.1). (2.30) follows from (2.31)-(2.33) and (2.3). \square

In the proof of (ii) we may—without loss of generality—assume that $y = \mathbf{0}$ and that P is a path from $\mathbf{0}$ to a point z with $|z| \geq r(B)/2$; if this is not the case originally, we merely have to shift P by $-y$. We can pick a subset Q of P such that

$$\mathbf{0} \in Q \subset \{v: |v| \leq r(B)/2\} \quad (2.34)$$

and

$$\text{for } 1 \leq k \leq r(B)/2, Q \text{ contains exactly one point, } q_k \text{ say, with } k-1 < |q_k| \leq k. \quad (2.35)$$

Finally, we may assume

$$(1, 0) \notin Q, \quad (2.36)$$

since we can always reflect Q in the y -axis. Since $Q \subset P$

$$\mu_P(y) \leq \mu_Q(y) = \mu_Q(\mathbf{0}),$$

so that it suffices to prove

$$\mu_Q(\mathbf{0}) \leq C_2[r(B)]^{-1/2} \tag{2.37}$$

To prove (2.37) we define

$$\omega_R(u) = P_u\{\tau(C) < \tau(\mathcal{C}(R))\}$$

and

$$\psi_R(u) = P_u\{\tau(Q) < \tau(\mathcal{C}(R))\}.$$

We first compare $\omega_R((1, 0))$ with $\psi_R((1, 0))$. This requires a representation formula for ω_R , which is the analogue of formula (3.6) of Ahlfors (1973). The latter is just a form of Green's theorem, and, thanks to some help of R. Durrett, our analogue is proved just as quickly.

Lemma 8

$$\omega_R(u) = - \sum_{v \in C} g_R(u, v) \Delta \omega_R(v), \quad |u| \leq R, \tag{2.38}$$

where Δ is the discrete Laplacian defined by

$$\Delta \omega(v) = \frac{1}{4}[\omega(v + (1, 0)) + \omega(v + (-1, 0)) + \omega(v + (0, 1)) + \omega(v + (0, -1))] - \omega(v).$$

Moreover,

$$\omega_R(u) = 1 \text{ if } u \in C, \quad \omega_R(u) = 0 \text{ if } u \in \mathcal{C}(R)$$

and

$$\Delta \omega_R(v) \leq 0 \text{ for } v \in C.$$

Proof. The boundary conditions for ω_R are immediate from the definition. Since $\omega_R(u) \leq 1$ for all u , these also imply $\Delta \omega_R(v) \leq 0$ on C . Now let $S_0 = u$. Then $Z_0 = 0$ and

$$Z_n := \sum_{1 \leq k \leq n \wedge \tau(\mathcal{C}(R))} [\omega_R(S_k) - E_u\{\omega_R(S_k) | S_0, \dots, S_{k-1}\}], \quad n \geq 1,$$

defines a martingale with respect to P_u . Moreover, for $|S_{k-1}| \leq R$ (i.e., "inside $\mathcal{C}(R)$ ")

$$E_u\{\omega_R(S_k) | S_0, \dots, S_{k-1}\} = \omega_R(S_{k-1}) + \Delta \omega_R(S_{k-1}),$$

and if in addition $S_{k-1} \notin C$, then $\Delta \omega_R(S_{k-1}) = 0$, by the definition of ω_R . Thus

$$\begin{aligned} Z_n &= \sum_{1 \leq k \leq n \wedge \tau(\mathcal{C}(R))} [\omega_R(S_k) - \omega_R(S_{k-1})] \\ &\quad - \sum_{1 \leq k \leq n \wedge \tau(\mathcal{C}(R))} I[S_{k-1} \in C] \Delta \omega_R(S_{k-1}). \end{aligned}$$

Taking expectations with respect to P_u yields

$$\begin{aligned} 0 &= E_u\{\omega_R(n \wedge \tau(\mathcal{C}(R)))\} - \omega_R(u) \\ &\quad - \sum_{v \in C} \Delta \omega_R(v) E_u\{\text{number of visits to } v \text{ during } [0, n \wedge \tau(\mathcal{C}(R)) - 1]\}, \end{aligned}$$

and (2.38) follows by taking the limit $n \rightarrow \infty$. \square

Lemma 9. For $R \geq r^2 \geq 4$,

$$\psi_R((1, 0)) \geq \omega_R((1, 0)) - C_5 r^{-1/2} (\log R)^{-1}.$$

Proof. We define the new function

$$\theta_R(u) = - \sum_{0 \leq k \leq r/2} g_R(u, q_k) \Delta \omega_R((-k, 0)),$$

where $q_0 = \mathbf{0}$, and q_k for $1 \leq k \leq r/2 = r(B)/2$ is the unique point in Q with $k-1 < |q_k| \leq k$ as in (2.35). We shall show separately that

$$\theta_R((1, 0)) \geq \omega_R((1, 0)) - C_6 [r^{3/2} R^{-1} + r^{-1/2}] (\log R)^{-1} \tag{2.39}$$

$$\psi_R(u) \geq \theta_R(u) - C_6 [r^{3/2} R^{-1} + r^{-1/2}] (\log R)^{-1} \quad \text{for } |u| \leq R. \tag{2.40}$$

Clearly these inequalities will imply the lemma.

We start with (2.39). To simplify the notation let us write e_k for the point $(k, 0)$ on the x -axis ($k \in \mathbb{Z}$). Then by the definitions of θ , ω and C , and (2.38)

$$\theta_R(e_1) - \omega_R(e_1) = - \sum_{1 \leq k \leq r/2} [g_R(e_1, q_k) - g_R(e_1, e_{-k})] \Delta \omega_R(e_{-k}).$$

By virtue of (2.30),

$$\begin{aligned} &g_R(e_1, q_k) - g_R(e_1, e_{-k}) \\ &\geq \frac{1}{2\pi} \log \frac{|e_1 - e_{-k}|}{|e_1 - q_k|} - C_3 \left[\frac{2r}{R} + |e_1 - e_{-k}|^{-2} + |e_1 - q_k|^{-2} \right]. \end{aligned}$$

Since $|e_1 - e_{-k}| = k+1 \geq |e_1 - q_k| \geq (k-2) \vee 1$ (recall $k-1 < |q_k| \leq k$ and $e_1 \neq q_k$ by (2.36)) we see that

$$g_R(e_1, q_k) - g_R(e_1, e_{-k}) \geq -C_7 \left[\frac{r}{R} + k^{-2} \right]. \tag{2.41}$$

Furthermore, as pointed out in Lemma 8, $-\Delta \omega_R \geq 0$, so that

$$\theta_R(e_1) - \omega_R(e_1) \geq C_7 \sum_{1 \leq k \leq r/2} \left[\frac{r}{R} + k^{-2} \right] \Delta \omega_R(e_{-k}).$$

To obtain (2.39) it therefore suffices to prove

$$-\Delta \omega_R(e_{-k}) \leq C_8 r^{-1/2} \left[\left(k \wedge \left(\frac{r}{2} - k \right) \right) + 1 \right]^{-1/2} (\log R)^{-1}. \tag{2.42}$$

This, however, is essentially proved in Lemmas 4 and 6. Indeed, by the definitions of Δ and ω_R we have

$$-\Delta \omega_R(e_{-k}) \leq \max_{|z - e_{-k}|=1} [1 - \omega_R(z)] = \max_{|z - e_{-k}|=1} P_z \{ \tau(\mathcal{C}(R)) < \tau(C) \}.$$

Of course the probability in the right hand side equals 0 if $z \in C$. On the other hand, if we write y for e_{-k} , and let z be a neighbor of y such that $z \notin C$, then (since there

is a probability $\frac{1}{4}$ that a random walk starting at y will move to z at the first step; cf. (2.5) for λ)

$$\begin{aligned} & \frac{1}{4}P_z\{\tau(\mathcal{C}(R)) < \tau(C)\} \leq P_y\{\tau(\mathcal{C}(R)) < \lambda(C)\} \\ &= \sum_{v \in \mathcal{C}(2r)} P_y\{\tau(\mathcal{C}(2r)) < \lambda(C) \text{ and } S_{\tau(\mathcal{C}(2r))} = v\} \\ & \quad \cdot P_v\{\tau(\mathcal{C}(R)) < \lambda(C)\} \quad (\text{compare (2.6)}) \\ & \leq C_5(\log R)^{-1}P_y\{\tau(\mathcal{C}(2r)) < \lambda(C)\} \quad (\text{by Lemma 4}) \\ & \leq C_8r^{-1/2} \left[\left(k \wedge \left(\frac{r}{2} - k \right) \right) + 1 \right]^{-1/2} (\log R)^{-1} \quad (\text{by (2.18), (2.19) and (2.29)}). \end{aligned}$$

Thus (2.42) and hence (2.39) hold.

We turn to (2.40) which is based on similar estimates as (2.39) plus the maximum principle. It is immediate from its definition that $u \rightarrow g_R(u, q)$ is harmonic, i.e., $(\Delta g_R(\cdot, q))(u) = 0$, on the set $\{u: |u| \leq R, u \neq q\}$. Therefore $\psi_R(\cdot) - \theta_R(\cdot)$ is harmonic on $\{u: |u| \leq R\} \setminus Q$. On $\mathcal{C}(R)$

$$\psi_R(u) - \theta_R(u) = 0, \quad u \in \mathcal{C}(R),$$

since $\psi_R(u)$ and $g_R(u, q)$ vanish on $\mathcal{C}(R)$. Lastly, we claim that

$$\psi_R(q_l) - \theta_R(q_l) \geq -C_6[r^{3/2}R^{-1} + r^{-1/2}](\log R)^{-1} \tag{2.43}$$

for all points q_l of Q . These facts together will imply (2.40), by virtue of the well known maximum principle, that a harmonic function on a bounded region must take its maximum (and minimum) on the boundary of the region. To prove the remaining estimate (2.43) we observe that for each $q_l \in Q$, $\psi_R(q_l) = 1 \geq \omega_R(e_{-l})$ so that

$$\begin{aligned} \psi_R(q_l) - \theta_R(q_l) & \geq \omega_R(e_{-l}) - \theta_R(q_l) \\ & = - \sum_{0 \leq k \leq r/2} [g_R(e_{-l}, e_{-k}) - g_R(q_l, q_k)] \Delta \omega_R(e_{-k}). \end{aligned}$$

Essentially as in (2.41) (we now have $|q_l - q_k| \geq |e_{-l} - e_{-k}| - 1$),

$$\begin{aligned} & g_R(e_{-l}, e_{-k}) - g_R(q_l, q_k) \\ & \geq \frac{1}{2\pi} \log \left[\frac{|q_l - q_k| \vee 1}{|e_{-l} - e_{-k}| \vee 1} \right] - C_3 \left[\frac{2r}{R} + (|q_l - q_k| \vee 1)^{-2} + (|e_{-l} - e_{-k}| \vee 1)^{-2} \right] \\ & \geq -C_7 \left[\frac{r}{R} + (|l - k| \vee 1)^{-1} \right]. \end{aligned}$$

(2.43), and hence (2.40), now follows again from (2.42). \square

It is now easy to complete the proof of (ii). Indeed, as we saw before it suffices to prove (2.37). Now the first inequality in (2.6) and Lemma 3 with $D = Q$ yield for large R

$$\mu_Q(\mathbf{0}) \leq C_5 \limsup_{R \rightarrow \infty} (\log R) P_{\mathbf{0}}\{\tau(\mathcal{C}(R)) < \lambda(Q)\}.$$

But, for $R \geq r^2$,

$$P_0\{\tau(\mathcal{C}(R)) < \lambda(Q)\} = \frac{1}{4} \sum_{\substack{|z|=1 \\ z \notin Q}} P_z\{\tau(\mathcal{C}(R)) < \tau(Q)\},$$

and if $z = e_1$, then by Lemma 9

$$P_z\{\tau(\mathcal{C}(R)) < \tau(Q)\} = 1 - \psi_R(e_1) \leq P_{e_1}\{\tau(\mathcal{C}(R)) < \tau(C)\} + C_5 r^{-1/2} (\log R)^{-1} \tag{2.44}$$

If $z = e_{-1}$ or $(0, +1)$ or $(0, -1)$ but $z \notin Q$, then we can rotate the plane over 180° or 90° , taking z to e_1 and Q to some set Q' which satisfies (2.34)–(2.36). Applying Lemma 9 to Q' rather than Q shows that (2.44) remains valid for z any neighbor of $\mathbf{0}$ which does not belong to Q . Thus

$$\mu_Q(\mathbf{0}) \leq C_6 r^{-1/2} + \limsup_{R \rightarrow \infty} (\log R) P_{e_1}\{\tau(\mathcal{C}(R)) < \tau(C)\}.$$

Finally, as in (2.6)

$$\begin{aligned} (\log R) P_{e_1}\{\tau(\mathcal{C}(R)) < \tau(C)\} &\leq P_{e_1}\{\tau(\mathcal{C}(2r)) < \tau(C)\} \\ &\cdot (\log R) \sup_{v \in \mathcal{C}(2r)} P_v\{\tau(\mathcal{C}(R)) < \tau(C)\} \\ &\leq C_5 P_{e_1}\{\tau(\mathcal{C}(2r)) < \tau(C)\} \quad (\text{by Lemma 4}) \\ &\leq 4C_5 P_0\{\tau(\mathcal{C}(2r)) < \lambda(C)\} \quad (\text{since the first step can go from } \\ &\quad \mathbf{0} \text{ to } e_1 \text{ with probability } \frac{1}{4}). \end{aligned} \tag{2.45}$$

The last probability is just the right hand side of (2.15) for $y = \mathbf{0}$, and the estimates for (2.15) (in particular Lemma 6) show that the right hand side of (2.45) is at most $C_6 r^{-1/2}$. Thus $\mu_Q(\mathbf{0}) \leq 2C_6 r^{-1/2}$ and step (ii) is complete.

3. The case $d \geq 3$

Fortunately the proof of (1.7) is much simpler than that of (1.6). We already know from (1.3) and (1.4) that

$$\mu_B(y) \leq \left[\sum_{z \in B} e_B(z) \right]^{-1} = [C(B)]^{-1}, \quad y \in B, \tag{3.1}$$

where $C(B)$ is the ‘‘capacity of B ’’ (cf. Spitzer (1976, Definition 25.3)). Also, by Spitzer (1976, Proposition 25.10) (with $\psi(x) \equiv 1/\Lambda$ on B), if

$$\begin{aligned} G(x, y) &= E_x(\text{number of visits to } y) \\ &= \sum_{n=0}^{\infty} P_x\{S_n = y\}, \end{aligned}$$

and $\Lambda > 0$ is such that

$$\frac{1}{\Lambda} \sum_{y \in B} G(x, y) \leq 1 \quad \text{for all } x \in B, \tag{3.2}$$

then

$$C(B) \geq \frac{1}{\Lambda} |B|. \tag{3.3}$$

Therefore, we only have to show that

$$\sum_{y \in B} G(x, y) \leq C(d) |B|^{2/d} \quad \text{for all } x \in B,$$

for any set $B \subset \mathbb{Z}^d$. This, however, is easy since it is well known (cf. Spitzer (1976, pp. 308-311 and 75-81)) or Ito and McKean (1960, pp. 121-123) that

$$G(x, y) = \sum_{n=0}^{\infty} P_0\{S_n = y - x\} \sim C_1 |y - x|^{2-d}$$

as $|y - x| \rightarrow \infty$. Therefore, uniformly in x ,

$$\begin{aligned} \sum_{y \in B} G(x, y) &\leq C_2 \sum_{k=0}^{\infty} [1+k]^{2-d} \left[\# \text{ of vertices } v = (v_1, \dots, v_d) \right. \\ &\qquad \qquad \qquad \left. \text{in } B \text{ with } \sum_1^d |v_i - x_i| = k \right] \\ &\leq C_3 \sum_{k=0}^{\infty} \left[\# \text{ of vertices } v \text{ in } B \text{ with } \sum_1^d |v_i - x_i| \leq k \right] [1+k]^{1-d} \\ &\leq C_4 \sum_{k=0}^{\infty} [(1+k)^d \wedge |B|] [1+k]^{1-d} \\ &\leq C_5 |B|^{2/d}. \end{aligned}$$

Thus (3.2) holds for $\Lambda = C_5 |B|^{2/d}$, and (1.7) now follows from (3.1) and (3.3).

4. Comments on the corollary

For typographical reasons we abbreviate $r(A_n)$ to $r(n)$. For $d = 2$ it was shown in Kesten (1987) that (1.6) implies that w.p. 1

$$r(2^{k+1}) - r(l) \leq C(2) \frac{2^{k+4}}{\sqrt{r(l)}} + 2^{k/2} \tag{4.1}$$

for all $2^k \leq l \leq 2^{k+1}$ and k sufficiently large. The only missing step for (1.8) is therefore the purely deterministic result, that if $r(l)$ is a sequence of numbers such that

$$r(1) = 0, \quad r(\cdot) \text{ is increasing}, \quad r(l+1) - r(l) \leq 1, \tag{4.2}$$

and such that (4.1) holds for all $k \geq k_0$, for some $k_0 < \infty$, then

$$\limsup n^{-2/3} r(n) \leq C^*(2) < \infty. \quad (4.3)$$

We give here the simple proof of this fact.

By (4.2), $r(l) \leq l$, so that after multiplication of (4.1) by $[r(l)]^{1/2}$ we obtain for $k \geq k_0$

$$[r(l)]^{1/2} [r(2^{k+1}) - r(l)] \leq C_3 2^k, \quad 2^k \leq l \leq 2^{k+1}, \quad (4.4)$$

with $C_3 = (16C(2) + \sqrt{2})$. For the remainder we take $k \geq k_0$. We consider two cases. First assume that

$$r(2^{k+1}) \leq 2r(2^k). \quad (4.5)$$

Then by (4.4) with $l = 2^k$

$$\begin{aligned} [r(2^{k+1}) - r(2^k)]^{3/2} &\leq \frac{3}{2} [r(2^{k+1})]^{1/2} [r(2^{k+1}) - r(2^k)] \\ &\leq \frac{3}{\sqrt{2}} [r(2^k)]^{1/2} [r(2^{k+1}) - r(2^k)] \leq \frac{3}{\sqrt{2}} C_3 2^k, \end{aligned} \quad (4.6)$$

and, consequently,

$$r(2^{k+1}) - r(2^k) \leq \left[\frac{3}{\sqrt{2}} C_3 \right]^{2/3} 2^{2k/3}. \quad (4.7)$$

If (4.5) fails, let L be the smallest l in $[2^k, 2^{k+1}]$ for which $r(l)$ exceeds the right hand side of (4.7). If no such l exists then (4.7) still holds, since $r(2^{k+1}) - r(2^k) \leq r(2^{k+1})$. Otherwise

$$\left[\frac{3}{\sqrt{2}} C_3 \right]^{2/3} 2^{2k/3} \leq r(L) \leq \left[\frac{3}{\sqrt{2}} C_3 \right]^{2/3} 2^{2k/3} + 1.$$

Together with (4.4) this gives

$$r(2^{k+1}) \leq r(L) + C_3 2^k [r(L)]^{-1/2} \leq \left[1 + \frac{\sqrt{2}}{3} \right] r(L).$$

Just as in (4.6) this now yields

$$[r(2^{k+1}) - r(L)]^{3/2} \leq \frac{3}{2} \left[1 + \frac{\sqrt{2}}{3} \right]^{1/2} [r(L)]^{1/2} [r(2^{k+1}) - r(L)] \leq C_4 2^k$$

with $C_4 = \frac{3}{2} (1 + 2^{1/2}/3)^{1/2} C_3$. Thus also

$$\begin{aligned} r(2^{k+1}) - r(2^k) &\leq r(L) + r(2^{k+1}) - r(L) \\ &\leq \left[\frac{3}{\sqrt{2}} C_3 \right]^{2/3} 2^{2k/3} + 1 + C_4^{2/3} 2^{2k/3} \\ &\leq C_5 2^{2k/3} \end{aligned}$$

with

$$C_5 = \left[\frac{3}{\sqrt{2}} C_3 \right]^{2/3} + 1 + (C_4)^{2/3}.$$

Thus in all cases

$$r(2^{k+1}) - r(2^k) \leq C_5 2^{2k/3}, \quad k \geq k_0,$$

from which (4.3) is immediate (since r is increasing). Thus (4.3) and (1.8) hold.

Finally a brief comment to (1.9). The same kind of argument (but simpler) as used in Kesten (1987) to derive (4.1) from (1.6) shows directly from (1.7) that

$$r(A_n) \leq 8d^2 C(d) n^{2/d} \quad \text{eventually, w.p. 1.} \tag{4.8}$$

One merely has to observe that $r(A_n) > 8d^2 C(d) n^{2/d}$ can occur only if some path without double points $u_0 = \mathbf{0}, \dots, u_k$, with $k = \text{smallest integer } \geq 8d^2 C(d) n^{2/d}$, is “filled in order” during the time interval $[1, n]$ (see Kesten (1987) for explanation of the terminology). There are at most $(2d)^k$ such paths, and (1.7) applied to $B = A_l \cup \partial A_l$ shows that for any vertex u

$$\begin{aligned} P\{u \text{ is added to } A_l \text{ to form } A_{l+1} \mid A_l\} &\leq C(d) |B|^{(2/d)-1} \\ &\leq C(d) l^{(2/d)-1}. \end{aligned}$$

From this it follows as in Kesten (1987) (again use the exponential bounds of Freeman (1973, Theorem 4b) that the probability of filling up a fixed path u_0, \dots, u_k in order during $[1, n]$ is at most

$$\left[\frac{1}{k} e C(d) \sum_1^n l^{(2/d)-1} \right]^k \leq Z \left[\frac{e}{8d} \cdot (1 + n^{-2d}) \right]^k.$$

The probability of filling up any path u_0, \dots, u_k in order during $[1, n]$ is therefore at most

$$\left[\frac{e}{4} (1 + n^{-2d}) \right]^k,$$

and (4.8) follows from the Borel-Cantelli lemma.

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