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Hausdorff dimension of the hairs without endpoints for $\lambda \exp z$

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Abstract. We consider the complex exponential maps $E_\lambda(z) = \lambda\,\mathrm{e}^z$, where $z \in \mathbb{C}$ and $\lambda \in \left(0, \frac{1}{e}\right)$. It is known that the Julia set of E_λ is a *Cantor bouquet* of curves ("hairs") extending from the set of their endpoints to ∞. We prove that the Hausdorff dimension of the set of these curves without the endpoints is equal to 1; in particular, it is smaller than the Hausdorff dimension of the set of endpoints alone (which is known to be equal to 2, *see* [5]). © Académie des Sciences/Elsevier, Paris

La dimension de Hausdorff des cheveux sans leurs extrémités pour $\lambda \exp z$

Résumé. *On considère les applications exponentielles complexes* $E_\lambda(z) = \lambda\,\mathrm{e}^z$, *où* $z \in \mathbb{C}$ *et* $\lambda \in \left(0, \frac{1}{e}\right)$. *On sait que l'ensemble de Julia de* E_λ *est un bouquet de Cantor de courbes (« cheveux ») qui, partant de leurs extrémité, vont jusqu'à l'*∞. *Nous prouvons que la dimension de Hausdorff des cheveux privés de leurs extrémités est égale à* 1 *; en particulier, elle est plus petite que la dimension de Hausdorff de l'ensemble des extrémités seules (que l'on sait être égale à* 2, voir [5]*).* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Soit $E_\lambda : z \mapsto \lambda \exp z$, où $z \in \mathbb{C}$ et $\lambda \in \left(0, \frac{1}{e}\right)$. L'ensemble de Julia de cette application a été étudié par Devaney et Krych [1], puis par Devaney et Tangerman (*voir* [3]) qui ont montré qu'il était constitué d'un bouquet de Cantor de courbes.

On note p_λ le point fixe attractif de E_λ et q_λ son point fixe répulsif. Comme tous les points du demi-plan $\{\mathrm{Re}\, z < q_\lambda\}$ sont attirés vers le puits p_λ, l'ensemble de Julia J_λ est contenu dans le demi-plan $H = \{\mathrm{Re}\, z \geq q_\lambda\}$. Devaney et Krych ont montré que si l'orbite d'un point z est entièrement contenue dans H, alors z appartient à J_λ ([1], *voir* aussi [2]).

Soit

$$P(k) = \{z \in H : (2k-1)\pi \leq \mathrm{Im}\, z < (2k+1)\pi\}, \quad \text{où } k \in \mathbb{Z}.$$

Note présentée par Adrien DOUADY.

À chaque point $z \in J_\lambda$ on peut faire correspondre une suite $s = (s_0, s_1, \ldots)$, appelée *itinéraire*, telle que $s_j = k$ si $E_\lambda^j(z) \in P(k)$. Une suite donnée s est associée à l'orbite d'un point $z \in J_\lambda$ si et seulement si il existe $x \in \mathbb{R}$ tel que $E_\lambda^j(x) \geq (2|s_j| + 1)\pi$ pour tous $j = 0, 1, \ldots$ (les suites qui vérifient cette condition sont appelées *admissibles*). L'ensemble des points qui ont une suite s admissible comme itinéraire forme une courbe X_s (*voir* [1], [2]) qui peut être paramétrée par $[0, +\infty)$. La courbe X_s est appelée *cheveu* et le point de X_s qui correspond à la valeur 0 du paramètre est appelé *extrémité*. Nous désignerons par $\mathcal{C}_\lambda$ l'ensemble de toutes ces extrémités.

Soit $\Sigma_N = \{s = (s_0, s_1, \ldots) : \ \forall j, \ |s_j| \leq N\}$, où $N \in \mathbb{N}$. Nous définissons le N-bouquet de Cantor $C_N = \bigcup_{s \in \Sigma_N} X_s$. Alors $J_\lambda = \operatorname{cl}\left(\bigcup_{N>0} C_N\right) = \bigcup_s X_s$, où la dernière union est prise pour toutes les suites admissibles. Mc Mullen [6] a prouvé que la dimension de Hausdorff de J_λ est 2.

Dans [2], Devaney et Goldberg prouvent que les cheveux ne sont pas accessibles depuis le bassin d'attraction à l'exception de leurs extrémités $\mathcal{C}_\lambda$, i.e. le seul point accessible de chaque X_s est son extrémité. Nous avons démontré dans [5] que l'ensemble des extrémités a une dimension de Hausdorff égale à 2. Cela signifie que l'ensemble des extrémités est « très dense » dans J_λ. Le but de cette Note est de prouver que, de façon inattendue, les cheveux sans leurs extrémités forment un ensemble qui est beaucoup plus « petit » (en terme de dimension de Hausdorff) que l'ensemble des extrémités :

THÉORÈME 1. – *Pour E_λ comme ci-dessus la dimension de Hausdorff de l'ensemble des points qui ne sont pas accessibles depuis le bassin d'attraction est égale à* 1.

Considérons le recouvrement de J_λ par des boîtes définies comme suit :

$$B_k^j = \left\{ z \in H : \operatorname{Re} z \in [q_\lambda + j\pi, q_\lambda + (j+1)\pi] \ \text{ et } \ \operatorname{Im} z \in \left[-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi \right] \right\},$$

où $j \in \mathbb{N}$ et $k \in \mathbb{Z}$. Puisque $\operatorname{HD}(J_\lambda \setminus \mathcal{C}_\lambda) \geq 1$ et que ces boîtes forment un recouvrement dénombrable de J_λ, il est suffisant de prouver que pour tout $\varepsilon \in (0, 1)$ et toute boîte B,

$$\operatorname{HD}((J_\lambda \setminus \mathcal{C}_\lambda) \cap B) \leq 1 + \varepsilon.$$

Nous définissons pour chaque entier n la famille suivante d'ensembles :

$$\mathcal{K}_n = \left\{ K_n : \ K_n \ \text{ est une composante de } \ E_\lambda^{-n}\left(B_k^j\right) \ \text{ pour un couple } (j,k) \right\}.$$

Soit $R\left(E_\lambda(B_k^j)\right)$ le rayon extérieur du demi-anneau $E_\lambda(B_k^j)$. Nous prouvons que si z est un point inaccessible de J_λ, alors il existe une suite $\{K_n\}_{n=1}^\infty$ d'ensembles $K_n \in \mathcal{K}_n$ dont l'intersection est exactement $\{z\}$ possédant en outre la propriété suivante :

$$\text{il existe } k \in \mathbb{N} \text{ tel que pour tout } n > k, \quad E_\lambda^n(K_n) \subset P(\ell), \ \text{ où } |\ell| \leq 1 + \frac{1}{2\pi}\left(R(E_\lambda^n(K_{n-1})\right)^{\varepsilon/2}. \quad (*)$$

Nous fixons le boîte B et appelons Y_k l'ensemble des points de B qui ont la propriété ci dessus pour un k donné. Alors $(J_\lambda \setminus \mathcal{C}_\lambda) \cap B$ est contenu dans $\bigcup_{k \in \mathbb{N}} Y_k$. Soit $\mathcal{A}_n$ la famille des ensembles K_n $(n > k)$ vérifiant (*).

L'étape suivante est de prouver que pour k fixé, il existe $n_0 > k$ tel que pour tout $n > n_0$ et tout $K_n \in \mathcal{A}_n$:

$$\sum_{K_{n+1} \in \mathcal{G}(K_n)} (\operatorname{diam} K_{n+1})^{1+\varepsilon} \leq (\operatorname{diam} K_n)^{1+\varepsilon},$$

où $\mathcal{G}(K_n)$ est l'ensemble de tous les $K_{n+1} \in \mathcal{A}_{n+1}$ qui ont une intersection non vide avec K_n.

Ainsi, $\sum_{K_n \in \mathcal{A}_n} (\operatorname{diam} K_n)^{1+\varepsilon}$ est fini. Comme $Y_k \subset \bigcap_{n>k} \bigcup_{K_n \in \mathcal{A}_n} K_n$ et comme $\max_{K_n \in \mathcal{A}_n} \operatorname{diam} K_n$ tend vers 0 quand $n \to \infty$, il en résulte que

$$\mathrm{HD}(Y_k) \leq 1 + \varepsilon,$$

ce qui prouve le théorème. □

1. Introduction

Let $E_\lambda : z \mapsto \lambda \exp z$, where $z \in \mathbb{C}$ and $\lambda \in \left(0, \frac{1}{e}\right)$. The Julia set for this map has been studied by Devaney and Krych in [1] and then by Devaney and Tangerman (*see* [3]) who showed that it was a *Cantor bouquet* of curves.

Let us denote by p_λ the unique attracting fixed point of E_λ and by q_λ the repelling one. The half-plane $\{\operatorname{Re} z < q_\lambda\}$ is mapped by E_λ into itself and in fact all points of this half-plane are attracted to the sink p_λ. Hence the Julia set $J_\lambda = J(E_\lambda)$ is contained in the right half-plane $H = \{\operatorname{Re} z \geq q_\lambda\}$. Devaney and Krych proved that if the orbit of a point z is completely contained in H, then z belongs to the Julia set ([1], *see* also [2]).

Let

$$P(k) = \{z \in H : (2k-1)\pi \leq \operatorname{Im} z < (2k+1)\pi\}, \quad \text{where } k \in \mathbb{Z}.$$

For every point $z \in J_\lambda$ one can assign a sequence $s = (s_0, s_1, \ldots)$ called itinerary such that $s_j = k$ if $E_\lambda^j(z) \in P(k)$. A given sequence corresponds to an orbit of a point $z \in J_\lambda$ if and only if there exists $x \in \mathbb{R}$ such that $E_\lambda^j(x) \geq (2|s_j| + 1)\pi$ for each $j = 0, 1, \ldots$ (sequences satisfying this condition are called *allowable*). The set of points which have an allowable s as itinerary forms a curve X_s (*see* [1], [2]): there is a continuous map $\phi_s : [0, +\infty) \to \mathbb{C}$ such that $\phi_s([0, +\infty)) = X_s$ and $\phi_s(t) \to \infty$ as $t \to \infty$. The curve X_s is called *hair* and the point $\phi_s(0) \in X_s$ is called the *endpoint*. We shall denote by $\mathcal{C}_\lambda$ the set of all endpoints. Let $\Sigma_N = \{s = (s_0, s_1, \ldots) : \forall j \ |s_j| \leq N\}$, where $N \in \mathbb{N}$. We define the Cantor N-bouquet $C_N = \bigcup_{s \in \Sigma_N} X_s$. Then $J_\lambda = \operatorname{cl}\left(\bigcup_{N>0} C_N\right) = \bigcup_s X_s$, where the last union is taken over all allowable sequences. It was shown by Mc Mullen that the Hausdorff dimension of J_λ is equal to 2 (*see* [6]).

In [2], Devaney and Goldberg prove that the hairs are not accessible from the basin of attraction except for their endpoints $\mathcal{C}_\lambda$, i.e. the only accessible point in every curve X_s is its endpoint. We proved in [5] that the set of endpoints has Hausdorff dimension equal to 2. It means that the set of endpoints is "very dense" in J_λ. The aim of this Note is to prove that surprisingly the hairs without endpoints form a set which is much "smaller" (in terms of Hausdorff dimension) than the set of endpoints:

THEOREM 1.1. – *For E_λ as above the Hausdorff dimension of the set of points which are not accessible from the basin of attraction is equal to* 1.

(By the Hausdorff dimension of an unbounded set we mean the supremum of the Hausdorff dimensions of its bounded subsets.)

2. Trajectories of inaccessible points

Let $S = \bigcup_{k \in \mathbb{Z}} S_k$, where $S_k = \left\{z \in H : -\frac{\pi}{2} + 2k\pi \leq \operatorname{Im} z \leq \frac{\pi}{2} + 2k\pi\right\}$. Note that $J_\lambda \subset S$ (if $z \notin S$, then $E_\lambda(z) \notin H$ and the trajectory of z is attracted to the sink p_λ). We consider the cover of S with boxes B_k^j defined as follows:

$$B_k^j = \{z \in S_k : q_\lambda + j\pi \le \operatorname{Re} z \le q_\lambda + (j+1)\pi)\},$$

where $j \in \mathbb{N}$ and $k \in \mathbb{Z}$. For every n we define the following family of sets:

$$\mathcal{K}_n = \{K_n : K_n \text{ is a component of } E_\lambda^{-n}(B_k^j) \text{ for some } j, k\}.$$

The image of every box B_k^j is a half-annulus $E_\lambda(B_k^j)$; let us denote by $R(E_\lambda(B_k^j))$ the outer radius of this annulus. We shall denote by L_k the branch of the inverse function of E_λ mapping H into $P(k)$.

PROPOSITION 2.1. – *Let z be an inaccessible point in J_λ. Then there exists a sequence $\{K_n\}_{n=1}^\infty$ of sets $K_n \in \mathcal{K}_n$ such that $\{z\} = \bigcap_{n=1}^\infty K_n$ with the following property:*

$$\forall \kappa \in (0,1),\ \exists k \in \mathbb{N},\ \forall n > k,\ E_\lambda^n(K_n) \subset P(\ell),\ \textit{where } |\ell| \le 1 + \frac{1}{2\pi}\big(R(E_\lambda^n(K_{n-1}))\big)^\kappa.$$

The above condition says that starting from some k the trajectory of an inaccessible point remains in the "middle part" of $E_\lambda^n(K_{n-1})$.

Proof. – Let $z \in J_\lambda$ and let $\{s_i\}_{i=1}^\infty$ be the itinerary of z. Since $J_\lambda \subset H$, $z \in \bigcap_{n=1}^\infty K_n$ for some $K_n \in \mathcal{K}_n$ and in fact $\{z\} = \bigcap_{n=1}^\infty K_n$ (because for $z \in H$, $|E'_\lambda(z)| \ge q_\lambda > 1$). Assume that there exists $\kappa \in (0,1)$ and a sequence $n_k \to \infty$ such that

$$E_\lambda^{n_k}(K_{n_k}) \subset P(\ell), \quad \text{where } |\ell| > 1 + \frac{1}{2\pi}\big(R(E_\lambda^{n_k}(K_{n_k-1}))\big)^\kappa. \tag{1}$$

We shall prove that z is accessible from the basin of attraction using a result by Devaney and Goldberg: in [2] they prove that there is only one accessible point z_s whose itinerary is s, and they give a construction of this point. It easily follows from this construction that $z_s = \lim_{n\to\infty} L_{s_0} \circ \cdots \circ L_{s_n}(q_\lambda)$. Hence, in order to show that z is accessible it suffices to show that: $z = \lim_{i\to\infty} L_{s_0} \circ \cdots \circ L_{s_{n_i}}(q_\lambda)$ for some sequence n_i tending to ∞.

Let us denote by q_{n_i+1} the (n_i+1)-th preimage of q_λ in the above composition, i.e. $L_{s_0} \circ \cdots \circ L_{s_{n_i}}(q_\lambda)$. We take $n_k > 2$ such that condition (1) holds, and we define $R = R(E_\lambda^{n_k}(K_{n_k-1}))$. By our assumptions

$$R^\kappa \le |E_\lambda^{n_k}(z)| \le R \quad \text{and} \quad 2\pi|s_{n_k}| \ge R^\kappa.$$

Let $a = E_\lambda^{n_k-1}(q_{n_k+1}) = L_{s_{n_k-1}}(q_\lambda + 2\pi i s_{n_k})$ and $b = L_{s_{n_k-1}}(E_\lambda^{n_k}(z))$. Since $|q_\lambda + 2\pi s_{n_k}| \le R + 2\pi < 2R$ we see that

$$|\operatorname{Re} b - \operatorname{Re} a| = \left|\log\frac{|E_\lambda^{n_k}(z)|}{\lambda} - \log\frac{|q_\lambda + 2\pi i s_{n_k}|}{\lambda}\right| \le (1-\kappa)\log R + \log 2.$$

Additionally, $\min\{\operatorname{Re} a, \operatorname{Re} b\} \ge \kappa \log R$. Therefore, the distance between the preimages of the points a and b does not depend on R. Indeed,

$$\operatorname{dist}\big(L_{s_{n_k-2}}(a), L_{s_{n_k-2}}(b)\big) \le \sup_{x\in\overline{ab}} |x^{-1}| \operatorname{dist}(a,b) \le \frac{1-\kappa}{\kappa} + \frac{2\pi + \log 2}{\kappa}.$$

We have proved that for every $n_k > 2$ the distance between $E_\lambda^{n_k-2}(z)$ and $E_\lambda^{n_k-2}(q_{n_k+1})$ is bounded by a constant (independently on s and n_k). Since E_λ is expanding in H we see that $\operatorname{dist}(z, q_{n_k+1}) \to 0$ as $k \to \infty$. □

3. Estimates of the Hausdorff dimension

Since the boxes B_j^k form a countable cover of J_λ it is sufficient to estimate the Hausdorff dimension of $(J_\lambda \setminus \mathcal{C}_\lambda) \cap B$ for an arbitrarily chosen box $B = B_j^k$. Our aim is to prove that $\operatorname{HD}((J_\lambda \setminus \mathcal{C}_\lambda) \cap B)) \le 1 + \varepsilon$ for every $\varepsilon > 0$. Fix $\varepsilon \in (0,1)$. We shall denote by Y_k the set of points in B which have the property described in Proposition 2.1 with k chosen for $\kappa = \varepsilon/2$, i.e. $z \in Y_k$ if $\{z\} = \bigcap_{n\in\mathbb{N}} K_n$ for some sequence of sets $K_n \in \mathcal{K}_n$ such that for $n > k$ the following condition holds:

$$E_\lambda^n(K_n) \subset P(\ell), \quad \text{where } |\ell| \le 1 + \frac{1}{2\pi}\big(R(E_\lambda^n(K_{n-1}))\big)^{\varepsilon/2}. \tag{2}$$

We have proved that $(J_\lambda \setminus \mathcal{C}_\lambda) \cap B \subset \bigcup_{k\in\mathbb{N}} Y_k$; therefore, we need only to show that for every $k \in \mathbb{N}$, $\operatorname{HD}(Y_k) \le 1 + \varepsilon$.

Note that for every $z \in J_\lambda \setminus \mathcal{C}_\lambda$ and for every $C > 0$ there exists $m \in \mathbb{N}$, arbitrarily large, such that $\operatorname{Re} E_\lambda^m(z) > C$. Otherwise, z would be an accessible point (it follows from the proof of Proposition 2.1: if for every n large enough $\operatorname{dist}(q_\lambda + 2\pi i s_n, E_\lambda^n(z)) \le C$, then $\operatorname{dist}(q_{n+1}, z) \to 0$). Since Y_k increases with k we can assume that for $z \in Y_k$, $\operatorname{Re} E_\lambda^k(z) > C$. From now on k is fixed. One may check, since $\operatorname{Re} E_\lambda^k(z) > C$, that for C sufficiently large, for every $z \in Y_k$ and for every $j \in \mathbb{N}$:

$$\operatorname{Re} E_\lambda^{k+j}(z) \ge \frac{1}{2}\big|E_\lambda^{k+j}(z)\big| \ge \widetilde{E}_\lambda^j(C), \tag{3}$$

where $\widetilde{E}_\lambda(z) = E_\lambda(z)/2$.

For every $n > k$ we define $\mathcal{A}_n$ to be the family of sets $K_n \in \mathcal{K}_n$ satisfying condition (2). Every $\mathcal{A}_n$ is a covering of Y_k. Let C be a large constant such that condition (3) holds. We can assume (by a small change of C) that

$$\bigcup_{K_n\in\mathcal{A}_n} E_\lambda^n(K_n) \subset \big\{\operatorname{Re} z > \widetilde{E}_\lambda^{n-k}(C)\big\}. \tag{4}$$

Since E_λ is expanding in H we see that $\max_{K_n\in\mathcal{A}_n} \operatorname{diam} K_n \to 0$ as $n \to \infty$. Therefore, it is sufficient to prove that $\lim_{n\to\infty} \sum_{K_n\in\mathcal{A}_n} (\operatorname{diam} K_n)^{1+\varepsilon}$ is finite. We shall show that, in fact, for n sufficiently large,

$$\sum_{K_{n+1}\in\mathcal{A}_{n+1}} (\operatorname{diam} K_{n+1})^{1+\varepsilon} \le \sum_{K_n\in\mathcal{A}_n} (\operatorname{diam} K_n)^{1+\varepsilon}. \tag{5}$$

Let $K_n \in \mathcal{A}_n$. We define $\mathcal{G}(K_n)$ to be the collection of all $K_{n+1} \in \mathcal{A}_{n+1}$ which have nonempty intersection with K_n. Then (5) is the consequence of the following proposition:

PROPOSITION 3.1. – *There exists $n_0 > k$ such that for every $n > n_0$ and for every $K_n \in \mathcal{A}_n$,*

$$\sum_{K_{n+1}\in\mathcal{G}(K_n)} (\operatorname{diam} K_{n+1})^{1+\varepsilon} \le (\operatorname{diam} K_n)^{1+\varepsilon}.$$

Proof. – Let $n > k$ and $K_n \in \mathcal{A}_n$. For $i = 0, \ldots, n$, we shall denote by B_{s_i} the box which has nonempty intersection with $E_\lambda^i(K_n)$, $B_{s_0} = B$ and $B_{s_n} = E_\lambda^n(K_n)$ (the box B_{s_i} lies in $P(s_i)$). Let $R_{i+1} = R\big(E_\lambda(B_{s_i})\big)$.

We first observe that the distortion of E_λ^i is universally bounded on each $K_i \in \mathcal{K}_i$ (it does not depend on i by Koebe distortion theorem, *see* [4]). So, there exists a constant $d > 1$ such that $\operatorname{diam} K_n \geq d^{-1}\big|(E_\lambda^{-n})'(z)\big|$, where z is an arbitrary point in B_{s_n}. Thus we can write

$$\operatorname{diam} K_n \geq d^{-1}|z|^{-1}\prod_{i=1}^{n-1}\big|L_{s_i}\circ\cdots\circ L_{s_{n-1}}(z)\big|^{-1} \geq d^{-1}\prod_{i=1}^{n}\frac{1}{2R_i}.$$

Now we take $K_{n+1} \in \mathcal{G}(K_n)$, $z \in E_\lambda^{n+1}(K_{n+1})$ and we estimate the diameter of K_{n+1} from above:

$$\operatorname{diam} K_{n+1} \leq d|z|^{-1}\prod_{i=1}^{n}\big|L_{s_i}\circ\cdots\circ L_{s_n}(z)\big|^{-1} \leq d\prod_{i=1}^{n+1}\frac{a}{R_i},$$

where $a = 2\exp\pi$ ($R_i/\exp\pi$ is the inner radius of $E_\lambda\big(B_{s_{i-1}}\big)$).

The number of sets K_{n+1} in $\mathcal{G}(K_n)$ is the number of boxes which intersect $E_\lambda^{n+1}(K_n)$ and $\bigcup_\ell P(\ell)$, where $|\ell| \leq 1 + (2\pi)^{-1}(R_{n+1})^{\varepsilon/2}$. Hence it is smaller than $R_{n+1}\big(2\pi + R_{n+1}^{\frac{\varepsilon}{2}}\big) \leq 4\pi(R_{n+1})^{1+\frac{\varepsilon}{2}}$.

Therefore, there exists a constant $c > 0$ such that

$$\sum_{K_{n+1}\in\mathcal{G}(K_n)}(\operatorname{diam} K_{n+1})^{1+\varepsilon} \leq c(a^{n+1})^{1+\varepsilon}(R_{n+1})^{1+\frac{\varepsilon}{2}}\prod_{i=1}^{n+1}\frac{1}{R_i^{1+\varepsilon}}.$$

It follows from (4) that $R_{n+1} \geq E_\lambda\big(\widetilde{E}_\lambda^{n-k}(C)\big)$. Since R_{n+1} grows superexponentially fast (as $n \to \infty$), for n sufficiently large $R_{n+1}^{\varepsilon/2} \geq c\big(a^{n+1}2^n d\big)^{1+\varepsilon}$.

Therefore,

$$\sum_{K_{n+1}\in\mathcal{G}(K_n)}(\operatorname{diam} K_{n+1})^{1+\varepsilon} \leq \big(2^{-n}d^{-1}\big)^{1+\varepsilon}\prod_{i=1}^{n}\frac{1}{R_i^{1+\varepsilon}} \leq (\operatorname{diam} K_n)^{1+\varepsilon}.$$

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References

[1] Devaney R., Krych M., Dynamics of $\exp(z)$, Ergod. Th. Dynam. Sys. 4 (1984) 35–52.
[2] Devaney R.L., Goldberg L., Uniformization of attracting basins for exponential maps, Duke Math. J. 2 (1987) 253–266.
[3] Devaney R.L., Tangerman F., Dynamics of entire functions near the essential singularity, Ergod. Th. Dynam. Sys. 6 (1986) 489–503.
[4] Duren P.L., Univalent functions, Springer-Verlag, New York, 1983.
[5] Karpińska B., Area and Hausdorff dimension of the set of accessible points of the Julia sets of $\lambda\exp z$ and $\lambda\sin z$, Fund. Math. 159 (1999) 269–287.
[6] Mc Mullen C., Area and Hausdorff dimension of Julia sets of entire function, Trans. Amer. Math. Soc. 300 (1987) 329–34.