MAT 627, Spring 2022, Stony Brook University

TOPICS IN COMPLEX ANALYSIS

CONFORMAL FRACTALS, PART III: MARTINGALES AND MAKAROV'S THEOREMS

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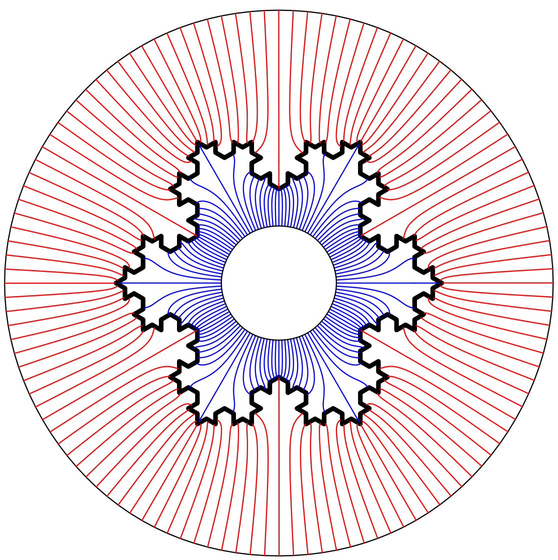
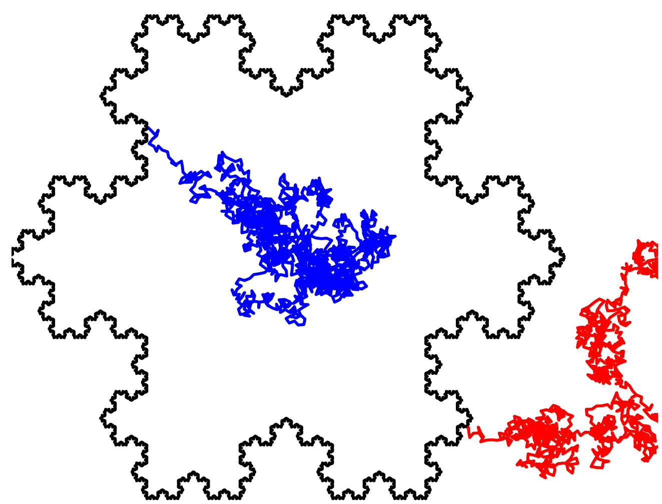




1. Dyadic Martingales

A filtration is a increasing sequence of σ-algebras $\{\mathcal{F}_n\}$.

Example: $\mathcal{F}_n$ generated by nth generation dyadic intervals in $[0, 1]$.

A martingale with respect to a filtration is a sequence of functions $\{M_n\}$ so that M_n is $\mathcal{F}_n$ measurable and $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n$.

Brownian motion stopped at time n.

A simple random walk on $\mathbb{Z}$.

An atom of a σ-field is an element with no non-empty subset in the field.

Every finite σ-field has a finite number N of atoms and has 2^N elements.

A filtration $\{\mathcal{F}_n\}$ is called dyadic of for all n, $\mathcal{F}_n$ contains 2^n atoms and each of these is a union of two atoms in $\mathcal{F}_{n+1}$.

If $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale, and $\{\mathcal{F}_m\}$ is dyadic with $\mathbb{P}(A) = 2^{-n}$ for every atom A, then $\{M_n\}$ is a dyadic martingale.

A dyadic martingale M_n on $[0,1]$ is constant on each nth generation dyadic interval. The average of M_{n+1} over the two children of I is equal to M_n on I.

Example: average of a L^1 function over dyadics.

Example: average of a measure over dyadics.

Example: Simple random walk.

A stopping time τ with respect to $\mathcal{F}_n$ is a function with values in $\mathbb{N} \cup \{\infty\}$ so that $\{\tau \leq n\} \subset \mathcal{F}_n$ for all n.

Less precisely, we can decide if $\tau(x) \leq n$ by only checking if x is contained in certain elements of $\mathcal{F}_n$.

Recall $a \wedge b = \min(a, b)$.

Lemma 1.1. *(Easy Optimal Stopping) If $\{M_n\}$ is a martingale for $\mathcal{F}_n$, then*

(a) $\{M_{\tau\wedge n}\}$ is also a martingale

(b) $\mathbb{E}M_{\tau\wedge n} = \mathbb{E}M_0$.

Proof. (a)

$$\begin{aligned} \mathbb{E}[M_{\tau\wedge(n+1)} - M_{\tau\wedge n}|\mathcal{F}_n] &= \mathbb{E}[(M_{n+1} - M_n)1_{\tau>n}|\mathcal{F}_n] \\ &= 1_{\tau>n}|\mathbb{E}[(M_{n+1} - M_n)|\mathcal{F}_n] \\ &= 0 \end{aligned}$$

(b) Since $\mathbb{E}M_{\tau\wedge(n+1)}|\mathcal{F}_n = M_{\tau\wedge n}$ almost everywhere, taking expectations of both sides gives $\mathbb{E}M_{\tau\wedge(n+1)} = \mathbb{E}M_{\tau\wedge n}$. □

The martingale increment is defined as

$$\Delta_{n+1} = M_{n+1} - M_n.$$

The conditional variance for $n \geq 1$ is

$$\mathrm{Var}(\Delta_n) = \sigma_n^2 = \mathbb{E}[\Delta_n^2 | \mathcal{F}_{n-1}]$$

This is just the L^2 norm squared.

Δ_n is constant on atoms of $\mathcal{F}_n$, but $|\Delta_n|$ is constant on atoms of $\mathcal{F}_{n-1}$.

The square function is

$$S_n = \sum_{k=1}^{n} \sigma_k^2.$$

S_n is predictable for $\{\mathcal{F}_n\}$, i.e., is $\mathcal{F}_{n-1}$-measurable.

Let $S_\infty = \lim_n S_n$ (may be $= +\infty$ at some points).

Theorem 1.2. *Let* $\{M_n\}$ *be a martingale with increments* $\{\Delta_n\}$*.*

(a) Almost everywhere on the set $\{\lim_n S_n < \infty\}$ *the limit of* M_n *exists and is finite.*

(b) If $\sup_n |\Delta_n| \leq C < \infty$*, then almost everywhere on* $\{\lim_n S_n = \infty\}$ *we have* $\limsup_n M_n = +\infty$ *and* $\liminf_n M_n = -\infty$ *and*

Corollary 1.3. *If* $\sup_n |\Delta_n| \leq C < \infty$*, then* $\liminf M_n < \infty$ *a.e..*

Corollary 1.4. *If* $\sum a_n^2 < \infty$ *then* $\sum \epsilon_n a_n < \infty$ *a.e.* $\epsilon_n \in \{\pm 1\}^{\mathbb{N}}$*. Otherwise the sum diverges almost surely.*

To prove theorem we need some preliminary results.

Lemma 1.5. *For a $\{\mathcal{F}_n\}$-martingale $\{M_n\}$ with $M_0 = 0$, if $S_n \in L^1$ for all n, then the process $Z_n = M_n^2 - S_n$ is an $\{\mathcal{F}_n\}$-martingale.*

Proof.

$$\begin{aligned}
\mathbb{E}[M_{n+1}^2 - M_n^2|\mathcal{F}_n] &= \mathbb{E}[(M_n + \Delta_{n+1})^2 - M_n^2|\mathcal{F}_n] \\
&= \mathbb{E}[2\Delta_{n+1}M_n - \Delta_{n+1}^2|\mathcal{F}_n] \\
&= \mathbb{E}[\Delta_{n+1}^2|\mathcal{F}_n] \\
&= \sigma_{n+1}^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}[Z_{n+1} - Z_n|\mathcal{F}_n] &= \mathbb{E}[M_{n+1}^2 - S_{n+1} - (M_n^2 - S_n)|\mathcal{F}_n] \\
&= \sigma_{n+1}^2 - (S_{n+1} - S_n) \\
&= 0.
\end{aligned}$$

$\square$

Lemma 1.6. *(Doob's maximal inequality on L^2) For a $\{\mathcal{F}_n\}$-martingale $\{M_k\}_{k=0}^n$, write*

$$M_k^* = \max_{0\le k\le n} |M_k - M_0|.$$

Then for $\lambda > 0$

$$\mathbb{P}(M_n^* \le \lambda) \le \lambda^{-2}\mathbb{E}[(M_n - M_0)^2].$$

Proof. Without loss of generality we may assume $M_0 = 0$.

Fix n and define the stopping time

$$\gamma = \gamma_{\lambda,n} = \inf\{k \ge 1 : |M_k - M_0| \ge \lambda\} \wedge n,$$

(let $\gamma = n$ if the set in the infimum is empty). Then

$$\lambda^2\mathbb{P}(M_n^* \ge \lambda) \le \mathbb{E}(|M_\gamma - M_0|^2 1_{M_n^*\ge\lambda}) \le \mathbb{E}|M_\gamma - M_0|^2.$$

Next we claim that $M_n - M_\gamma$ and $M_\gamma - M_0$ are orthogonal. First for all $k \in [0, n]$,

$$\mathbb{E}(M_n - M_\gamma)1_{\gamma=k}(M_\gamma - M_0)|\mathcal{F}_k] = 0.$$

Since $\gamma \leq n$, we have $\cup_{k=0}^n \{\gamma = k\}$ is the whole space. Thus summing the equality over k gives

$$\mathbb{E}[(M_n - M_\gamma)(M_\gamma - M_0)] = 0.$$

Thus

$$\mathbb{E}[(M_n - M_0)^2] = \mathbb{E}[(M_n - M_\gamma)^2] + \mathbb{E}[(M_\gamma - M_0)^2].$$

and so

$$\mathbb{E}[(M_n - M_0)^2] \geq \mathbb{E}[(M_\gamma - M_0)^2] \geq \lambda^2 \mathbb{P}(M_n^* \geq \lambda).$$

$\square$

Proposition 1.7. *(Martingales bounded in L^2 converge almost surely) If $\{M_n\}_{n=0}^{\infty}$ is a $\{\mathcal{F}_n\}$-martingale $V_\infty = \sup_n \mathbb{E}M_n^2 < \infty$, then $\lim_n M_n$ exists and is finite almost surely.*

Proof. Note that

$$\mathbb{E}[M_n^2] = \mathbb{E}[(M_n - M_{n-1})^2 + \cdots + \mathbb{E}[(M_1 - M_0)]^2 + \mathbb{E}[M_0^2]$$

is non-decreasing. Given $\epsilon, \delta > 0$ there exists ℓ so that

$$V_\infty - \mathbb{E}M_\ell^2 \leq \epsilon^2 \delta.$$

For each $n \geq 1$ applying Doob's maximal inequality to $\{M_{\ell+n}\}_{k=0}^{n}$ gives

$$\mathbb{P}[\max_{0 \leq k \leq n} |M_{\ell+k} - M_\ell| \geq \epsilon] \leq (\epsilon^2 \delta)/\epsilon^2 = \delta.$$

These events increase with n so

$$\mathbb{P}[\sup_{k\geq 0} |M_{\ell+k} - M_\ell| \geq \epsilon] \leq \delta.$$

Thus for every $\epsilon > 0$

$$\mathbb{P}[\limsup M_n - \liminf M_n \geq 2\epsilon| \geq 2\delta.$$

Since this holds for all $\delta > 0$, we have

$$\mathbb{P}[\limsup M_n - \liminf M_n \geq 2\epsilon] = 0.$$

Taking $\epsilon \searrow 0$ shows $\lim M_n$ exists almost surely. $\square$

Now we are ready to prove Theorem 1.2, which we restate for convenience

Theorem 1.8. *Let* $\{M_n\}$ *be a martingale with increments* $\{\Delta_n\}$*.*

(a) Almost everywhere on the set $\{\lim_n S_n < \infty\}$ *the limit of* M_n *exists and is finite.*

(b) If $\sup_n |\Delta_n| \le C < \infty$*, then almost everywhere on* $\{\lim_n S_n = \infty\}$ *we have* $\limsup_n M_n = +\infty$ *and* $\liminf_n M_n = -\infty$ *and*

Proof of part (a):

Proof. We may assume $M_0 = 0$. Given $R > 0$ define

$$T = T_R = \inf\{n \geq 1 : S_{n+1} \geq R\}$$

where we take the infimum of the empty set to be $+\infty$.

Note T is a stopping time since

$$\{T \leq k\} = \{\max_{1 \leq j \leq k+1} S_j \geq R\} \subset \mathcal{F}_k.$$

By Lemma 1.5 $\{M_{T\wedge n}^2 - S_{T\wedge n}\}$ is a martingale so

$$\mathbb{E} M_{T\wedge n}^2 = \mathbb{E} S_{T\wedge n} \leq R.$$

Thus $\{M_{T\wedge n}\}$ is a martingale bounded in L^2.

Hence for every R, $\lim_n M_{T\wedge n}$ exists almost surely. On the set $\{S_\infty < R\}$ this implies $\lim_n M_n$ exists. Taking the limit as $R \nearrow \infty$ we see $\lim_n M_n$ exists almost surely on $\{S_\infty < R\}$. $\square$

Proof of part (b):

Proof. As before, we may assume $M_0 = 0$.

Given $R, Q > 0$ define the stopping time

$$\tau = \tau_{R,Q} = \inf\{n \geq 0 : M_n \notin (-R, Q)\}.$$

As before, the infimum over the empty set is $+\infty$.

Recall that C is the upper bound for the L^∞ norm of the increments. By Lemma 1.5, the process

$$\begin{aligned} Y_n &= (R + C + M_n) * (Q + C - M_n) + S_n \\ &= (R + C)(Q + C) + (Q - R)M_n - (M_n^2 - S_n)) \end{aligned}$$

is a $\{\mathcal{F}_n\}$-martingale.

Since $|\Delta_n| \leq C$ we have $M_{\tau\wedge n} \in (R - C, Q + C)$. Hence

$$(R + C + M_n) * (Q + C - M_n) \geq 0.$$

The optimal stopping lemma (Lemma 1.1) then gives

$$\begin{aligned} (R + C)(Q + C) &= \mathbb{E}(Y_0) = \mathbb{E}Y_{\tau\wedge n} \\ &= \mathbb{E}[(R + C + M_n) * (Q + C - M_n) + S_{\tau\wedge n}] \\ &\geq \mathbb{E}S_{\tau\wedge n}. \end{aligned}$$

Thus

$$\begin{aligned}
&\Rightarrow (R+C)(Q+C) \geq \mathbb{E} S_\tau \geq \mathbb{E}[S_\infty \cdot 1_{\tau=\infty}] \\
&\Rightarrow \mathbb{P}(\{S_\infty = \infty\} \text{ and } \{\tau = \infty\}) = 0 \\
&\Rightarrow \{S_\infty = \infty\} \subset \{\tau < \infty\} \text{ almost surely} \\
&\Rightarrow \mathbb{P}(S_\infty = \infty) \leq \mathbb{P}(\tau < \infty)
\end{aligned}$$

Note

$$0 = M_0 = \mathbb{E} M_{\tau\wedge n} \leq (Q+C)\mathbb{P}(M_{\tau\wedge n} \geq -R) - R \cdot \mathbb{P}(M_{\tau\wedge n} \leq -R)$$

so

$$\mathbb{P}(M_{\tau\wedge n} \leq -R) \leq \frac{Q+C}{R}\mathbb{P}(M_{\tau\wedge n} \geq -R) \leq \frac{Q+C}{R}.$$

This implies

$$\mathbb{P}(\tau \le n) \le \mathbb{P}(M_{\tau\wedge n} \le -R) + \mathbb{P}(M_{\tau\wedge n} \ge Q) \le \frac{Q+C}{R} + \mathbb{P}(M_{\tau\wedge n} \ge Q).$$

Finally,

$$\begin{aligned}\mathbb{P}(\sup_{\ell} M_\ell \ge Q) &\ge \mathbb{P}(\tau < \infty) - \mathbb{P}(M_{\tau\wedge n} \le -R)\\ &\ge \mathbb{P}(\tau < \infty) - \frac{Q+C}{R}\\ &\ge \mathbb{P}(S_\infty = \infty) - \frac{Q+C}{R}.\end{aligned}$$

Now let $R \nearrow \infty$. We get

$$\mathbb{P}(\sup_{\ell} M_\ell \ge Q) \ge \mathbb{P}(S_\infty = \infty).$$

Intersecting the nested events over all integers $Q > 0$ gives

$$\mathbb{P}(\sup_{\ell} M_{\ell} = \infty) \geq \mathbb{P}(S_{\infty} = \infty).$$

Part (a) tells us that

$$\mathbb{P}(\sup_{n} M_{n} = \infty | S_{\infty} < \infty) = 0$$

so

$$\mathbb{P}(\sup_{n} M_{n} = \infty | S_{\infty} = \infty) = 1.$$

For any real sequence, $\sup_n M_n = \infty$ iff $\limsup_n M_n = \infty$, so this prove the "limsup" part of (b).

Finally, apply this argument to $-M_n$ to show

$$\mathbb{P}(\liminf_{n} M_{n} = -\infty | S_{\infty} = \infty) = 1.$$

$\square$

Bounded increments are needed for part (b).

Example 1: Let

$$\Delta_n = \begin{cases} -1 & \text{with probability } 1 - 2^{-n} \\ 2^n - 1 & \text{with probability } 2^{-n} \end{cases}$$

This implies $\mathbb{E}(M_n) = 0$, where $M_n = \sum_{k=0}^{n} \Delta_k$ is a martingale. In this case $S_n = \infty$ almost surely, but by Borel-Cantelli, with probability 1 we have that $\Delta_n = 1$ for all large enough n. Thus $M_n \to -\infty$ almost surely.

Example 2: Let

$$\Delta_n = \begin{cases} 2^n & \text{with probability } 2^{-n} \\ 2^n & \text{with probability } 2^{-n} \\ 0 & \text{with probability } 1 - 2^{-n+1} \end{cases}.$$

Then $M_n = \sum_1^n \Delta_k$ is a martingale with $S_n \to \infty$ almost surely, but again by Borel-Cantelli Δ_k is eventually all zeros, so M_n is eventually constant, hence finite, almost everywhere.

2. Bloch Functions and Bloch Martingales

For g analytic on the disk we define

$$\|g\|_{\mathcal{B}} = |g(0)| + \sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|^2),$$

which is called the Bloch norm of g. The collection of analytic functions with finite Bloch norm is called the Bloch space $\mathcal{B}$.

Note that these are exactly the holomorphic Lipschitz functions from the hyperbolic metric on the disk to the Euclidean metric on the plane.

Any bounded holomorphic function is Bloch by the Cauchy estimate for f'.

We proved earlier that $\log f'$ is Bloch for any conformal map f.

Another example of a Bloch function is the lacunary series

$$\varphi(z) = \sum_{n=1}^{\infty} z^{2^n}.$$

To prove this is Bloch, fix a point $z \in \mathbb{D}$ and choose n so that

$$(n-1)^{-1} \geq 1 - |z| > n^{-1}.$$

Split the sum defining φ at n and use the fact that $(1-1/n)^n < e^{-1}$ to get

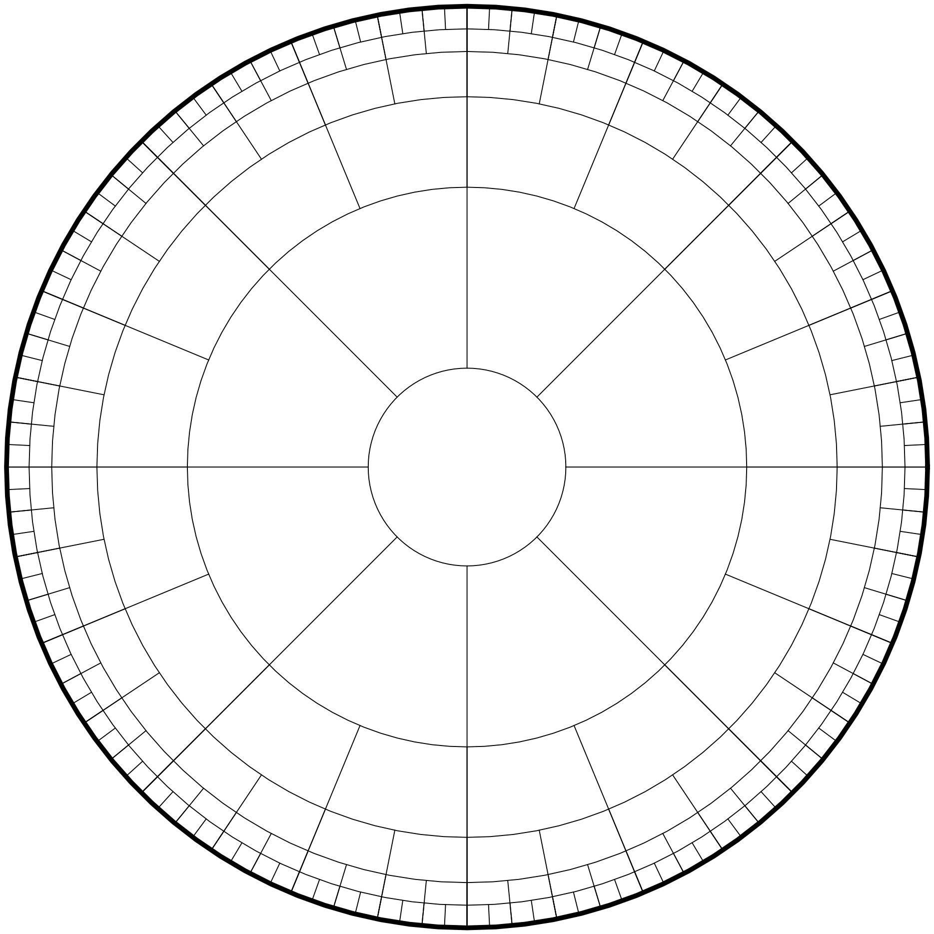
$$\begin{aligned}
|\varphi'(z)| &\leq \sum_{k:2^k \leq n} 2^k |z|^{2^k-1} + \sum_{k:2^k > n} 2^k |z|^{2^k-1} \\
&\leq \sum_{k:2^k \leq n} 2^k + \sum_{j>0} n2^j |(1-\frac{1}{n})^{n-1}|^{2^j} \\
&\leq 2n + n\sum_{j>0} 2^j e^{-2^j} \\
&\leq Cn \\
&\leq \frac{C}{1-|z|}.
\end{aligned}$$

Of course, a similar computation works for $\sum z^{b^n}$ for any integer $b \geq 2$.

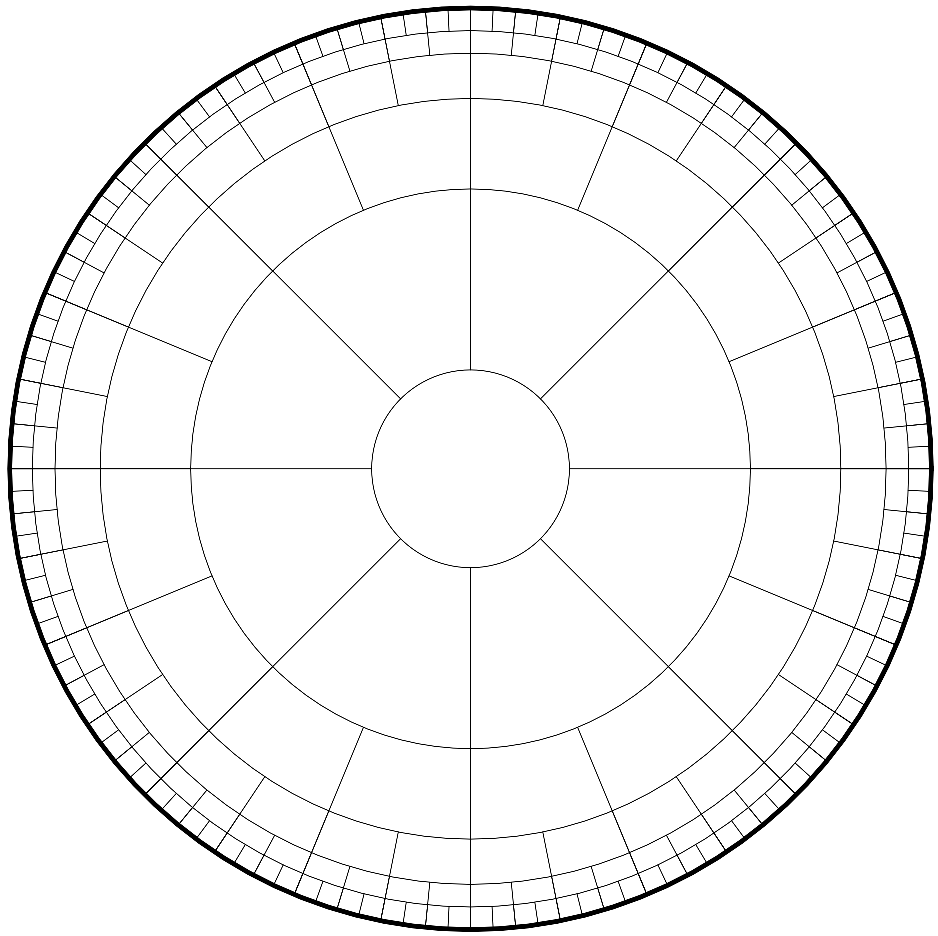
The example is suggestive because $\{z^{2^n}\}$ looks roughly like a sum if independent random variables. In fact, we will later show that all Bloch functions look like martingales.

Consider a Whitney decomposition of the disk, as illustrated on the next page.

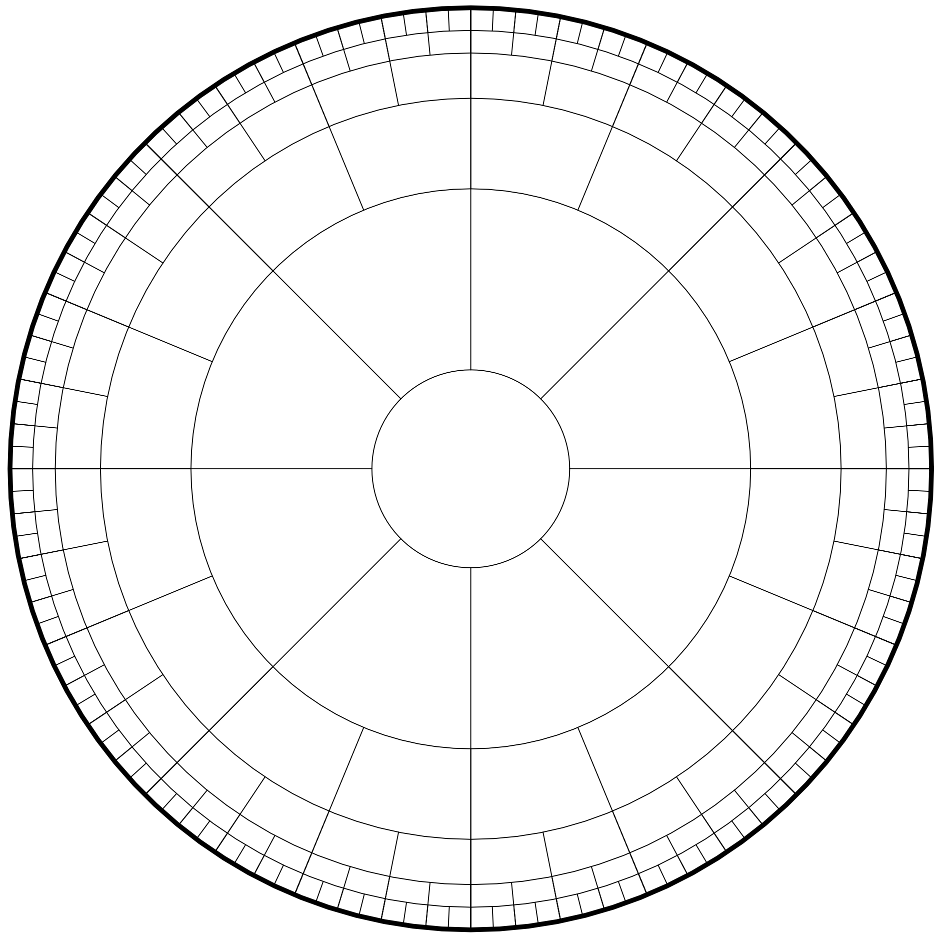
The innermost part of the decomposition is a central disk of radius $1/4$. Outside of the central disk, the annulus $A_1 = \{\frac{1}{4} < |z| < \frac{1}{2}\}$ is divided into eight equal sectors, the annulus $A_2 = \{\frac{1}{2} < |z| < \frac{3}{4}\}$ into sixteen sectors.



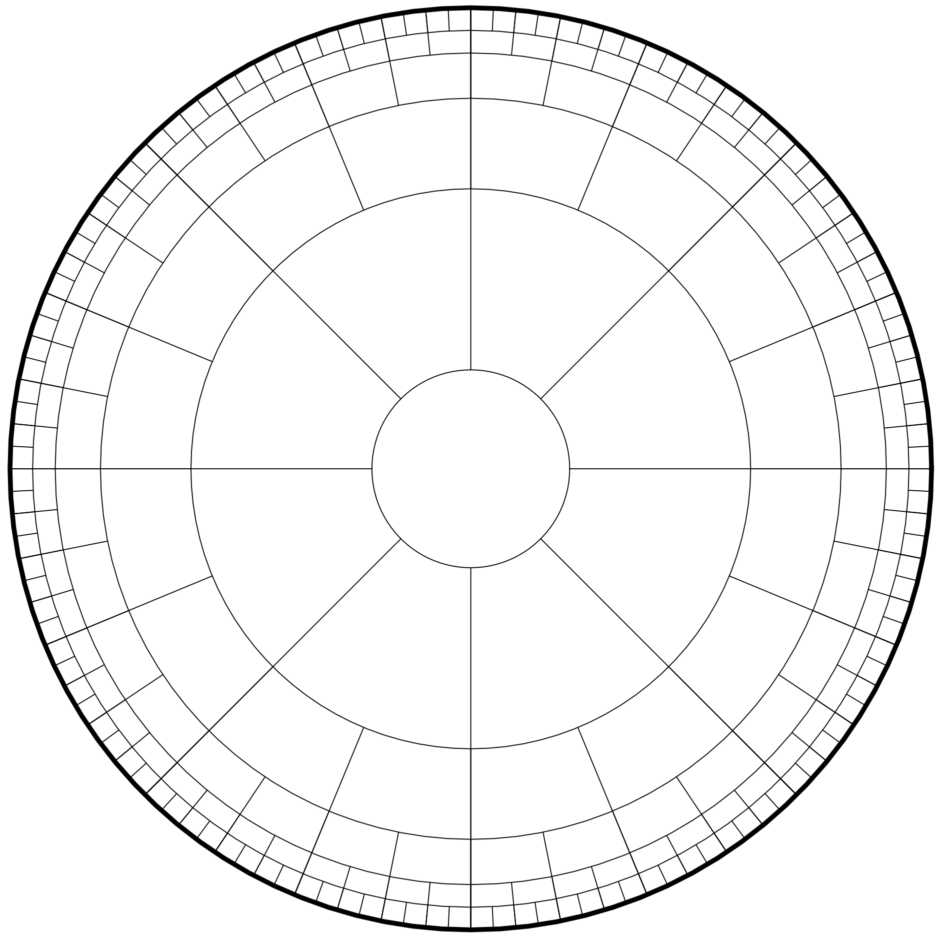
Each Whitney box has two radial sides and two circular arc sides concentric with the origin. The circular arc closer to the origin is called the top of the box and the arc further from the origin is called the bottom.



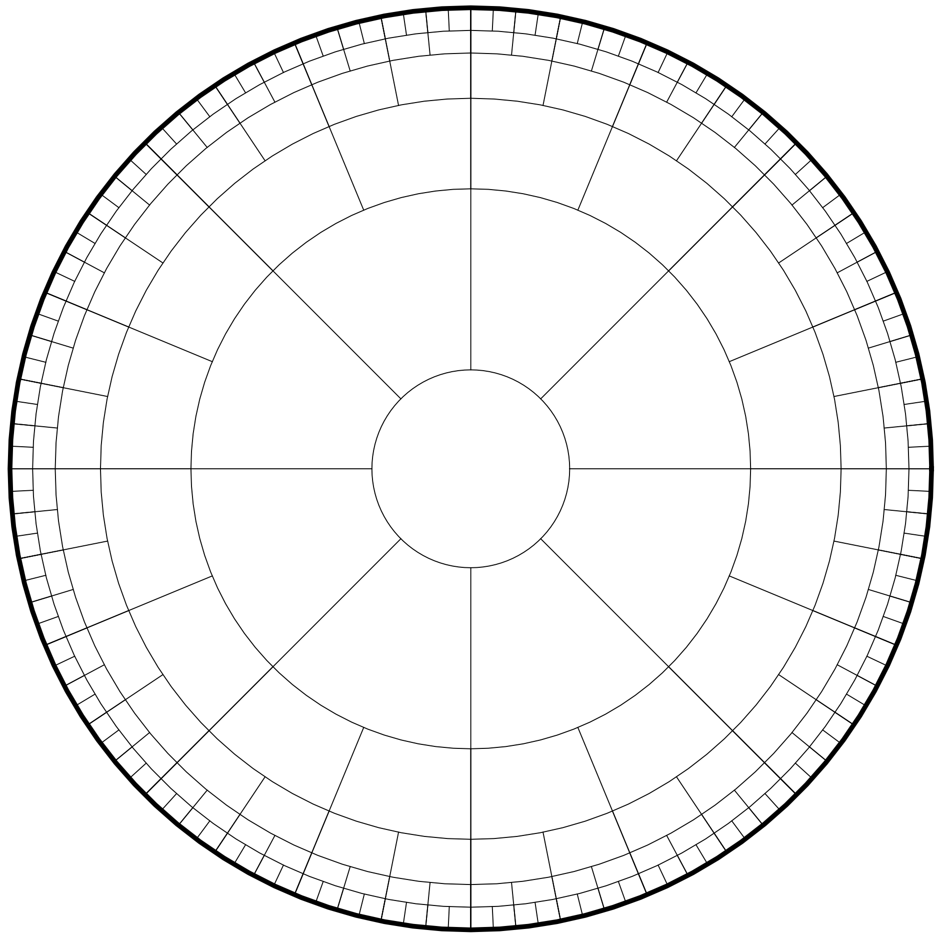
Each bottom arc is divided into two pieces by the tops of the Whitney boxes below it ("below" means between the given box and the unit circle). We call these the left and right sides of the bottom arc (left is the one further clockwise).



The sides and bottoms of Whitney boxes we will call the Whitney edges, their endpoints we call Whitney vertices. The union of these edges and vertices forms an infinite graph in $\mathbb{D}$ which we call the Whitney graph.



The radial projection of a closed Whitney box B onto the unit circle, $\mathbb{T}$, is a closed arc that we denote B^* (this is sometimes called the "shadow" of B, thinking of a light source at the origin).



The union of a closed Whitney box B and all the closed Whitney boxes B' so that $(B')^* \subset B^*$ is called the Carleson square with base $I = B^*$.

A **dyadic martingale** on the unit circle is a sequence $\{f_n\}$ of functions so that each f_n is constant on the interiors of the nth level dyadic intervals and so that value of f_n on any such interval I is the sum of values of f_{n+1} on the two children of I.

We shall use f_n to denote the martingale as a function on the circle and f_I to denote the value taken by f_n on I, if I is a nth generation dyadic interval. Thus we can write

$$f_n(x) = \sum_I f_I \mathbf{1}_I(x),$$

where the sum is over all nth generation dyadic intervals.

The dyadic martingale on the circle is called a Bloch martingale if

$$\sup_n \sup_{I,J} |f_I - f_J| < \infty$$

whenever I, J are adjacent dyadic intervals of the same length.

This implies the increments of the martingale are bounded, but is stronger. The latter condition only says $|f_I - f_J|$ is bounded if I and J have the same parent.

Bounded increments implies difference is $O(k)$ if I, J are adjacent but only have a kth generation ancestor in common.

Lemma 2.1. *For any harmonic Bloch function u on the disk, there is a Bloch martingale $\{f_n\}$ on the circle so that $\|\{f_n\}\|_{\mathcal{B}} \leq C\|u\|_{\mathcal{B}}$ and*

$$\sup_{I \in \mathcal{D}_n} |u(z_I) - f_n(I)| \leq C\|u\|_{\mathcal{B}}.$$

There is a converse we will prove later: given a Bloch martingale, there is a corresponding Bloch harmonic function u.

Proof. Suppose u is a harmonic Bloch function. Without loss of generality we may assume its Bloch norm is 1.

Suppose $I \subset \mathbb{T}$. We claim that the limit

$$u_I = \lim_{r \nearrow 1} \frac{1}{|I|} \int_I u(re^{i\theta}) d\theta \tag{2.1}$$

exists and satisfies

$$|u_I - u(z_I)| = O(\|u\|_{\mathcal{B}}). \tag{2.2}$$

If so, then $\{u_I\}$ defines a Bloch martingale, as I ranges over all dyadic subintervals of $\mathbb{T}$ defined the desired martingale.

We apply Green's theorem over the truncated Carleson box

$$Q_r = \{se^{i\theta} : r < 1 - s < |I|, e^{i\theta} \in I\},$$

for $r \ll |I|$. Taking $v = \log \frac{1}{|z|}$, Green's theorem

$$\iint_Q (v\Delta u - u\Delta v) dxdy = \int_{\partial Q} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} ds$$

says that since both u and v are harmonic in Q_r, the boundary integral

$$\int_{Q_r} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} ds = 0.$$

Thus the integral over the "bottom" of the truncated box is the negative of the integral over the other three sides.

The integral over the top side is
$$\frac{1}{|I|}\int_I [u(|I|e^{i\theta}) - u(z_I)]d\theta = O(1)$$
since u itself varies by less than $O(1)$ over this arc.

To handle the sides of the box, note that a Bloch function satisfies for $s < r < 1$
$$|u(rx) - u(sx)| \leq \int_s^r \frac{dt}{1-t^2} \leq 2\int_s^r \frac{dt}{1-t} \leq 2\log\frac{1-s}{1-r}$$

Thus the integrand over each radial side of Q_r is bounded by

$$|(u - u(z_I))\frac{\partial v}{\partial n}| + |v\frac{\partial u}{\partial n}| \le (\log\frac{1-|I|}{1-t})(1) + (1-t)\frac{1}{1-t}.$$

This is integrable on $[1-|I|, 1)$ and the integral is bounded by $O(1)$.

Hence the limits over the radial sides as $r \nearrow 1$ exists and are $O(1)$. Thus the limit in (2.1) exists and satisfies (2.2) as desired. $\square$

Lemma 2.2. *If* $\{f_n\}$ *is a real-valued Bloch martingale, then*

$$\liminf_{n\to\infty} f_n(\theta) < \infty,$$

for almost every θ.

Proof. By Corollary 1.3, Bloch martingales satisfy $\liminf f_n < \infty$ a.e. $\square$

Lemma 2.3. *If $f : \mathbb{D} \to \Omega$ is a conformal map then*

$$\liminf_{r \nearrow 1} |f'(re^{i\theta})| < \infty,$$

for almost every θ.

Proof. Let $\{f_n\}$ be the dyadic martingale associated to the real-valued harmonic Bloch function $u = \mathrm{Re}(\log f')$. By Lemma 2.2, the martingale as finite liminf almost everywhere, and by Lemma 2.1, so does u. Since $|f'| = \exp(u)$, the lemma follows. □

This also follows from Plessner's theorem: almost everywhere on the circle, a holomorphic function on the disk either has a non-tangential limit or is non-tangentially dense.

Plessner's theorem follows from Fatou's theorem: a harmonic function on the disk has non-tangential limits a.e. where it is non-tangentially bounded.

For a Bloch martingale, $|f_n(\theta)| = O(n)$ for every θ by definition. We need the following slight improvement of this.

Lemma 2.4. *If $\{f_n\}$ is a Bloch martingale, then for almost every θ, we have $|f_n(\theta)| = o(n)$.*

A much stronger result is true. By the LIL for martingales $|f_n(\theta)| = O(\sqrt{n \log\log n})$. We will prove this later.

Proof. Let C be the Bloch norm of $\{f_n\}$. Since $\{f_k - f_{k+1}\}$ are orthogonal,

$$\int f_n^2 d\theta = \sum_{k=0}^{n-1} |f_{k+1} - f_k|^2 d\theta \leq C^2 n,$$

so by Chebyshev's inequality

$$\{f_n > \lambda\} \leq \frac{1}{\lambda}\|f_n\|_1 \leq \frac{1}{\lambda}\|f_n\|_2^{1/2} \leq \frac{C\sqrt{n}}{\lambda}.$$

Taking $\lambda = \epsilon n$ we get

$$\{f_n > \epsilon n\} = O(\frac{1}{\epsilon\sqrt{n}}).$$

Taking $n = m^3$, this becomes

$$\{f_{m^3} > \epsilon m^3\} = O(\frac{1}{\epsilon m^{3/2}}),$$

which is summable over m, so by Borel-Cantelli

$$\limsup_{m\to\infty} \frac{f_{m^3}(\theta)}{m^3} \leq \epsilon,$$

holds almost everywhere.

For $m^3 < n < (m+1)^3$, we have $n - m^3 = O(m^2)$, so the bounded difference condition for Bloch martingales implies, for almost every θ,

$$f_n(\theta) \leq f_{m^3}(\theta) + O(m^2) = (\epsilon + \frac{1}{m})O(m^3) = (\epsilon + n^{-1/3})O(n).$$

Since $\epsilon > 0$ was arbitrary, this proves the lemma. $\square$

Corollary 2.5. *If $f : \mathbb{D} \to \Omega$ is a conformal map and $\epsilon > 0$, then*

$$\liminf_{r \to 1} \frac{|f'(re^{i\theta})|}{(1-r)^\epsilon} \geq 1$$

for almost every θ.

Proof. Lemmas 2.4 and Lemma 2.1 imply

$$\operatorname{Re} \log f'(z) = o(\log(1 - |z|))$$

almost everywhere, and this implies the lemma. $\square$

Similarly,

$$\limsup_{r \to 1} |f'(re^{i\theta})|(1-r)^\epsilon \leq 1$$

3. Makarov's theorem: $\dim(\omega) \leq 1$

The dimension of a measure μ is defined to be

$$\dim(\mu) = \inf\{\dim(A) : A \text{ has full } \mu \text{ measure}\}.$$

Note that if $E = \cup E_n$ has full measures for μ then $\dim(\mu) \leq \sup_n \dim(E_n)$.

Suppose $\Omega \subset \mathbb{R}^d$ is open and suppose ω is harmonic measure on $\partial\Omega$. What can we say about $\dim(\omega)$? Harmonic measure depends on a choice of base point in Ω, but the different points all give mutually absolutely continuous measures, so this question does not depend on the base point.

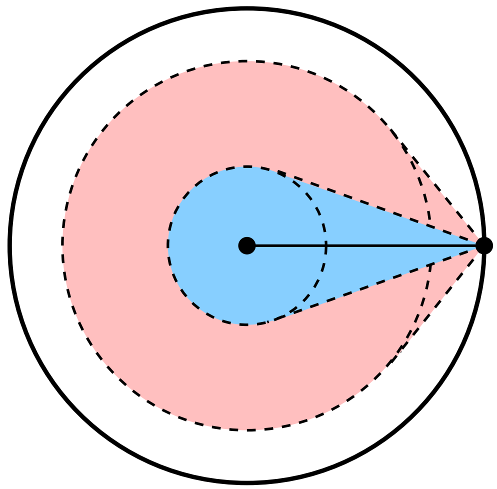
For $d \geq 3$ there are a few results, but still many open questions. For $d = 2$, things are much better understood. One of the key results is due to Makarov who proved that if $\Omega \subset \mathbb{R}^2$ is simply connected then $\dim(\omega) = 1$.

Vitali Covering Theorem:

Theorem 3.1. *If $E \subset \mathbb{R}^n$ and $\{Q_k\}$ is a family of cubes so that each $x \in E$ is contained in cubes from the family with arbitrarily small diameter, then there is a pairwise disjoint subcollection of the cubes that covers Lebesgue almost every point of E.*

Proof in most analysis textbooks.

A Stolz cone $\Gamma_\alpha(x)$ is a α-neighborhood of the radial segment from 0 to $x \in \mathbb{T}$ (an "ice cream cone").



We say f has a non-tangential limit at x if the limit of $f(z)$ exists as z approaches x through any Stolz cone.

Similarly, a function is non-tangentially bounded at x if it is bounded on each Stolz cone at x (the bound may depend on α).

A Bloch function is radially bounded iff it is non-tangentially bounded at x.

Fatou's theorem says that a bounded harmonic functions on the disk has non-tangential limits almost everywhere on the circle.

Local version says that if f is bounded on some Stolz cone at each point of E, then f has non-tangential limits almost everywhere on E.

Proof is a corollary of the Hardy-Littlewood maximum theorem; sometimes covered in 1st year analysis course.

Plessner's theorem: If f is holomorphic on the disk, then as almost every $x \in \mathbb{T}$, f either has a non-tangential limit or is non-tangentially dense.

Non-tangentially dense means that every non-degenerate Stolz cone has dense image in the plane.

Cor: If f is holomorphic on $\mathbb{D}$ then at almost every $x \in \mathbb{T}$, f' either has a finite limit or $f'(r_n x) \to 0$ for some $r_n \nearrow 1$.

Can prove Makarov's theorem upper bound using this, but we will avoid Plessner's theorem by using martingales.

Theorem 3.2 (Makarov's upper bound)**.** *Suppose* Ω *is a simply connected plane domain with a locally connected boundary. Then there exists* $E \subset \partial\Omega$ *with full harmonic measure and* σ*-finite* $\mathcal{H}^1$ *measure.*

Proof. Let $f : \mathbb{D} \to \Omega$ be conformal (injective and holomorphic). Let $\{\varphi_n\}$ be the Bloch martingale associated to the Bloch harmonic function $\varphi = \log f'$.

By Theorem 1.8, a.e. on $\mathbb{T}$, φ_n either has a finite limit or the liminf is $-\infty$.

Thus almost everywhere f' is non-tangentially bounded, or $|f'|$ has liminf $= 0$.

Divide the unit circle into three disjoint sets E_1, E_2, E_3 with the properties

(1) f' has a non-tangential limit at all $e^{i\theta} \in E_1$.

(2) $\liminf_{z \to e^{i\theta}, z \in \Gamma(e^{i\theta})} |f'(z)| = 0$ for all $e^{i\theta} \in E_2$.

(3) $\mathcal{H}^1(E_3) = 0$.

By the conformal invariance of harmonic measure, the harmonic measure for Ω is supported on $f(E_1)$ and $f(E_2)$.

First we will show that there is a subset $F \subset f(E_2)$ so that $\omega(F) = \omega(f(E_2))$ and $\mathcal{H}^1(f(F)) = 0$. Fix an integer k and for each z in the disk where $|f'(z)| \leq 2^{-k}$ let I_z denote the largest dyadic arc on the unit circle with containing $z/|z|$ and length $\leq 1 - |z|$.

Each point of E_2 is in infinitely many such arcs (with arbitrarily small size) so by the Vitali covering theorem, we can choose a disjoint subcollection of the arcs $\{I_j^k\}_{j=1}^{\infty}$ so that $\mathcal{H}^1(E \setminus \cup_j I_j^k) = 0$.

Let $\{z_j^k\}$ be the points in the disk corresponding to the chosen arcs. Also set

$$w_j^k = f(z_j^k), \qquad d_j^k = \operatorname{dist}(w_j^k, \partial\Omega),$$

$$D_j^k = \{|w - w_j^k| \leq Ck^2 d_j^k\}, \qquad G_k = \cup_j D_j^k,$$

$$F_n = \cup_{k \geq n} G_k, \qquad F = \cap_n F_n.$$

Where C is as in Beurling's estimate.

Then

$$\omega(w_j^k, D_j^k \cap \partial\Omega, \Omega) \geq 1 - \frac{1}{k} > 0,$$

and so

$$\omega(z_j^k, f^{-1}(G_k), \mathbb{D}) \geq 1 - \frac{1}{k},$$

which implies

$$|I_j^k \setminus G_k| \leq O(1/k).$$

Thus for an interval I_j^k,

$$|(E \cap I_j^k) \setminus G_k| \leq |I_j^k \setminus G_k| \leq O(|I_j^k|/k).$$

Therefore

$$|E \setminus F_n| \leq \inf_{k \geq n} |E \setminus G_k| = 0.$$

Since $\{F_k\}$ are nested decreasing, $\{E \setminus F_k\}$ is nested increasing and their measures converge to the measure of $E \setminus F$, which therefore must be zero.

Finally, we just have to show $\mathcal{H}^1(F) = 0$.

By Koebe's theorem $|D_j^k| \sim k^2 |f'(z_j^k)| |I_j^k|$,

$$\begin{aligned}
\mathcal{H}^1(F) &\leq \inf_n \sum_{k>n} \sum_j |D_j^k| \\
&\leq C \inf_n \sum_{k>n} \sum_j k^2 |f'(z_j^k)| |I_j^k| \\
&\leq C \inf_n \sum_{k>n} 2^{-k} k^2 \sum_j |I_j^k| \\
&\leq C \inf_n \sum_{k>n} 2^{-k} k^2 \sum_j 2\pi \\
&= 0.
\end{aligned}$$

Now we have to deal with E_1. For each integer $n \geq 1$, let E_1^n be the subset of E_1, where $|f'|$ is radially bounded by n. The union of these sets is all of E_1.

Choose a compact subset F_1^n so that $|E_1^n \setminus F_1^n| \leq 1/n$. By definition $\mathrm{Re} \log f'$ is in Bloch and so is bounded by $\log n + O(1)$ on any hyperbolic neighborhood of a radial segment ending in F_1^n, hence $|f'| = O(n)$ on the region W_F, the "sawtooth" region associated to F.

The boundary of W_F has length at most $2\pi^2$, so its image under f has length at most $O(n)$, and this includes the set $f(F_1^n)$. Since $\cup_n F_1^n$ is a full measure subset of E_1, the dimension of harmonic measure is ≤ 1. $\square$

Theorem 3.3 (Jones-Wolff). *For any compact planar set E with positive logarithmic capacity, harmonic measure for $\Omega = \mathbb{C} \setminus E$ with respect to ∞ gives full measure to a set of Hausdorff dimension at most* 1.

The proof uses properties of Green's function in place of conformal maps.

The proof becomes much easier if we make some regularity assumptions on the boundary: the capacity density condition. This means that for any $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$ the logarithmic capacity of $D(x, r) \cap \partial\Omega$ is comparable to the capacity of $D(x, r)$.

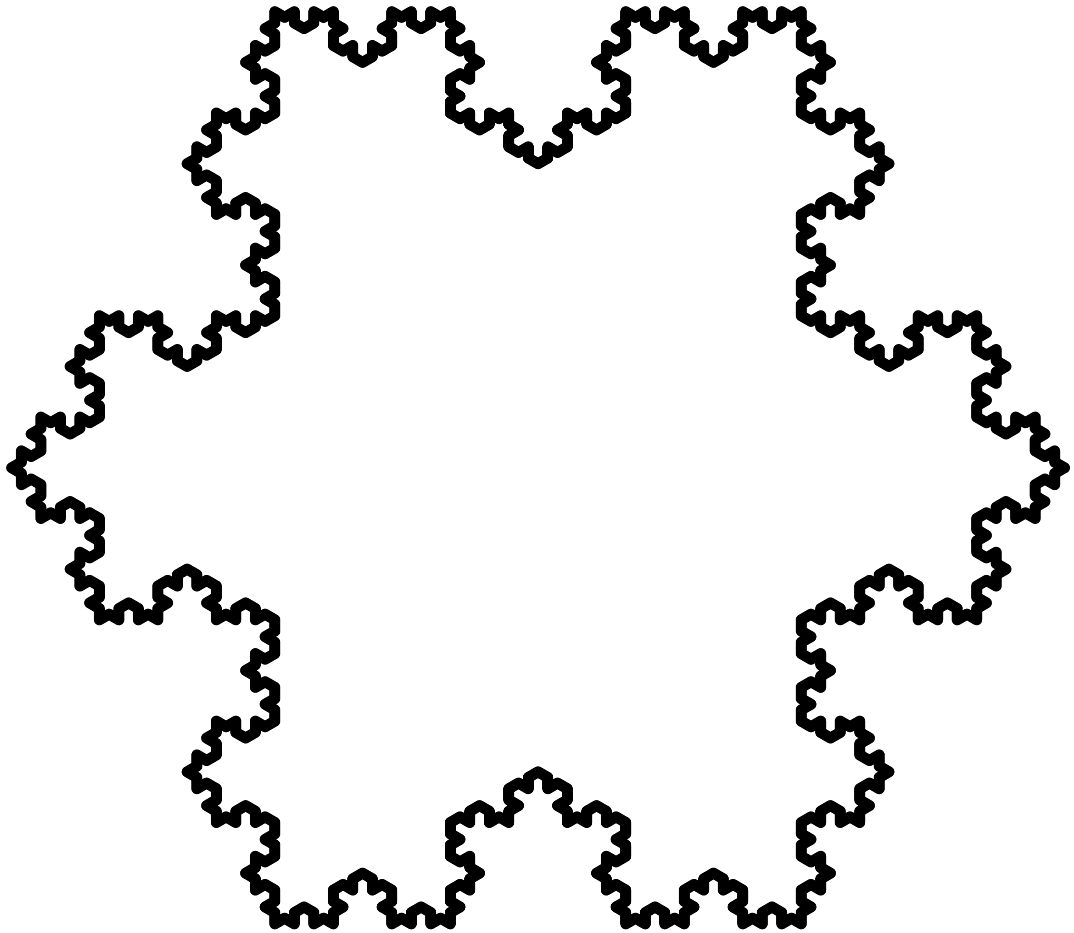
Tom Wolff later proved a stronger result: for any planar domain, there is a subset of the boundary of σ-finite length and full harmonic measure.

For domains in $\mathbb{R}^n$, Wolff showed the dimension of harmonic measure may be either strictly larger or strictly smaller than $n-1$.

4. Makarov's theorem: $\dim(\omega) \geq 1$

Next we prove that harmonic measure has dimension at least one. To avoid technicalities we will make a regularity assumption on the boundary of Ω; this assumption will be removed later.

We say that a closed Jordan curve γ is a **quasidisk** if there is a $M < \infty$ so that $\mathrm{diam}(\gamma(x,y)) \leq M|x-y|$, where $\gamma(x,y)$ is the subarc of γ between x and y of smaller diameter. Such curves are also called "bounded turning", or said to satisfy Ahlfors' 3-point condition.



The name “quasicircle” comes from the fact that these curves are exactly the images of the unit circle under quasiconformal mappings of the plane to itself.

Although we will not prove this, we will use the word “quasicircle” since this is the most common term for this class of curves. Similarly, a bounded domain whose boundary is a quasicircle is called a **quasidisk**. The definition is sufficiently general to include many fractal curves, such as the von Koch snowflake.

Theorem 4.1. *If Ω is a quasidisk and ω is harmonic measure for Ω, then* $\dim(\omega) = 1$.

Proof. We have already see $\dim(\omega) \leq 1$, so we only need to prove the other direction.

Fix $\epsilon > 0$. Suppose $X \subset \partial\Omega$ has positive harmonic measure. By Lemma 2.5 we can choose a compact set $E \subset [0, 2\pi]$ and $0 < s < 1$ so that $Y = f(E) \cap X$ has positive harmonic measure and

$$|f'(re^{i\theta})| \geq (1-r)^{\epsilon}$$

for all $r > s$ and $\theta \in E$. We claim that $\dim(Y) \geq 1 - \epsilon$.

Suppose $\{D_j\}$ is a cover of Y by disks. By the quasicircle assumption, we can associate to each disk an arc γ_j so that $D_j \cap \partial\Omega \subset \gamma_j$ and $\mathrm{diam}(\gamma_j) \simeq \mathrm{diam}(D_j)$. Each γ_j corresponds to an arc $I_j \subset \mathbb{T}$. By assumption I_j contains a point $e^{i\theta}$ of E, and by the Koebe $\frac{1}{4}$-theorem,

$$|f'(z_j)| \gtrsim |I_j|^{\epsilon},$$

where $z_j = z_{I_j}$.

Therefore

$$\mathrm{diam}(\gamma_j)^{1-\epsilon} \geq (|I_j| \cdot |f'(z_j)|)^{1-\epsilon} \geq |I_j|^{(1+\epsilon)(1-\epsilon)} \geq |I_j|^{(1-\epsilon^2)} \geq |I_j|.$$

Since $\{I_j\}$ covers the set $f^{-1}(Y)$, we deduce that

$$\sum_j |I_j| \geq |f^{-1}(Y)| > 0,$$

is bounded away from zero. Hence the $(1-\epsilon)$ Hausdorff content of Y is also bounded away from zero, so $\dim(X) \geq \dim(Y) \geq 1-\epsilon$. Since $\epsilon > 0$ was arbitrary, we have shown $\dim(X) \geq 1$ for any set X of positive harmonic measure. $\square$

5. The Law of the Iterated Logarithm for Dyadic Martingales (easy version)

For a dyadic martingale the increments Δ_n are $\{\mathcal{F}_{n+1}\}$ measurable and the square function can written

$$S_n = \sum_{k=1}^{n} \Delta_k^2.$$

Sharp version of the Law of the Iterated Logarithm for dyadic martingales:

Theorem 5.1. *For a dyadic martingale $\{M_n\}$ with $\sup_n \|\Delta_n\|_\infty \le C < \infty$, then*

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{M_n}{\sqrt{2S_n \log\log S_n}} = 1 | S_\infty = \infty\right) = 1 \tag{5.1}$$

There is an easier and less precise version that is sufficient for proving Makarov's theorems on harmonic measure.

The sharp version is proven later in these notes.

We will prove these two weaker versions first.

Theorem 5.2. *For a dyadic martingale* $\{M_n\}$ *with* $\sup_n \|\Delta_n\|_\infty \le C < \infty$, *then almost surely*

$$\limsup_{n\to\infty} \frac{M_n}{\sqrt{n \log \log n}} < \infty \tag{5.2}$$

Theorem 5.3. *For a dyadic martingale* $\{M_n\}$ *with* $\sup_n \|\Delta_n\|_\infty \le C < \infty$, *then almost surely on the set where* $\limsup_n \frac{1}{n} S_n > 0$,

$$\limsup_{n\to\infty} \frac{M_n}{\sqrt{n \log \log n}} > 0 \tag{5.3}$$

The idea is to deduce the LIL for dyadic martingales from the LIL for Brownian motion, as we did for simple random walks before.

For a random walk, we defined a stopping time $\tau_0 = 0$ and

$$\tau_n = \min\{t > \tau_{n-1} | \quad |B(t) - B(\tau_{n-1})| = 1\}, n \geq 1.$$

For a dyadic martingale (say on $[0, 1]$) we want to take

$$\tau_n = \min\{t > \tau_{n-1} | \quad |B(t) - B(\tau_{n-1})| = \Delta_n\}, n \geq 1.$$

But does this make sense? Δ_n is a function on $[0, 1]$ so when to stop Brownian motion depends on a choice of $x \in [0, 1]$.

We need a map from Brownian paths to $[0, 1]$ so we know where to evaluate $\Delta_n(x)$ and when to stop the path.

Proof. Let $\Omega = \{0,1\}^{\mathbb{N}}$ be the space where our dyadic martingale is defined. For our applications we can think of this as identified with $[0,1]$ (or the unit circle) via binary expansions.

Let $\mathcal{F}$ be the completion of the Borel sets. The are our measurable sets in Ω.

Let $\mathbb{P} = (\frac{1}{2}, \frac{1}{2})^{\mathbb{N}}$ be the probability measure on Ω. (This is the same as Lebesgue measure, if we replace Ω by $[0,1]$.)

Assume we have a dyadic filtration $\{\mathcal{F}_n\}$. This is generated by n bit-functions $b_k : \Omega \to \{0, 1\}$ that give the k co-ordinate of a sequence in Ω. Specifying values for $b_1, \ldots b_n$ specifies one of the 2^n atoms.

When Ω is identified with $[0, 1]$ these functions are the Radamacher functions.

The bit functions are iid with mean zero (fair coin flips) with respect to $\mathbb{P}$.

Suppose we are given a $\{\mathcal{F}_n\}$-adapted martingale $\{M_n\}$.

We assume $M_0 = 0$.

The values of M_n are determined from the values of $b_1, \ldots, b_n$. (Since M_n is constant on the atoms of $\mathcal{F}_n$.)

Recall the increment function $\Delta = M_n - M_{n-1}$.

Δ is constant on elements of $\mathcal{F}_n$.

But $|\Delta_n|$ is constant on elements of $\mathcal{F}_{n-1}$. (Only sign of Δ_n depends on b_n.)

Thus $|\Delta_n|$ is function of the bits $b_1, \ldots, b_{n-1}$.

We assume Brownian motion is defined on a probability space $(\Gamma, \mathcal{H}, \mathbb{P}^*)$.

Define a stopping time and map from Brownian paths to bits as follows.

Let $\tau_0 = 0$.

Define $\tau_1 = \min\{t \geq 0 : |B(t)| = |\Delta_1|\}$.

Since $|\Delta_1|$ is constant on the whole space, this is well defined.

Define a function b_1 on Ω (space of Brownian paths) so that

$$\Delta_1(b_1) = B(\tau_1).$$

In other words, $B(\tau_1) = \pm|\Delta_1|$ and b_1 is defined by whether it is $|\Delta_1|$ or $-|\Delta_1|$.

Inductively, assume $\{\tau_k\}_1^{n-1}$ and $\{b_k\}_1^{n-1}$ have been defined and let

$$\tau_k = \min\{t \geq \tau_{k-1} : |B(t) - B(\tau_k)| = |\Delta_k|\}.$$

and define b_k so that

$$\Delta(b_1, \dots b_k) = B(\tau_k) - B(\tau_{k-1}).$$

More formally, we have defined $b = (b_1, \dots b_k) : \Gamma \to \Omega$ so that

$$M_k(b_1, \dots, b_k) = B(\tau_k).$$

The functions $b_1, \dots b_k$ define a dyadic filtration $\{\mathcal{G}_n\}$ on $(\Gamma, \mathcal{H}, \mathbb{P}^*)$

Since $|\Delta_n|$ is a function of $b_1, \dots b_{n-1}$, we can think of it as a $\mathcal{G}_{n-1}$ measurable function on Γ (Brownian path space).

Hence the same is true for the square function

$$S_n = \sum_{k=1}^{n} |\Delta_n|^2.$$

Claim: $\{\tau_n - S_n\}_{n\geq 0}$ is a $\{\mathcal{G}_n\}$-martingale.

To prove this we just need to verify the definition of a martingale:

$$\mathbb{E}(\tau_n - S_n|\mathcal{G}_{n-1}) = \tau_{n-1} - S_{n-1}.$$

Since S_n is constant on atoms of $\mathcal{G}_{n-1}$, we have $\mathbb{E}(\tau_n - S_n|\mathcal{G}_{n-1}) = \mathbb{E}(\tau_n|\mathcal{G}_{n-1}) - S_n$, so also using the fact that $S_n - S_{n-1} = \Delta_n^2$, it suffices to show

$$\mathbb{E}(\tau_n|\mathcal{G}_{n-1}) = \tau_{n-1} + S_n - S_{n-1} = \tau_{n-1} + \Delta_n^2.$$

or

$$\mathbb{E}(\tau_n - \tau_{n-1}|\mathcal{G}_{n-1}) = \Delta_n^2.$$

However, $\tau_n - \tau_{n-1}$ is the time needed for a 1-dimensional Brownian motion to travel distance $|\Delta_n|$. By Wald's lemma the expectation of this is $|\Delta_n|^2$, so the claim is true.

Let

$$Z_n = (\tau_n - \tau_{n-1}) - \Delta_n^2 = (\tau_n - S_n) - (\tau_{n-1} - S_{n-1}.$$

Since these functions are increments of a martingale (the claim on the previous slide), they are orthogonal. They also have uniformly bounded L^2 norms because M_n has bounded increments.

Thus by the Strong Law of Large Numbers, almost surely

$$\frac{1}{N}\sum_{n=1}^{N} Z_n \to 0$$

$$\frac{1}{N}(\tau_N - S_N) \to 0 \tag{5.4}$$

Thus

$$\limsup_{n\to\infty} \frac{\tau_n}{n} = \limsup_{n\to\infty} \frac{S_n}{n} \leq C^2.$$

This implies, almost surely $\tau_n \leq 2C^2 n$ for all large n.

Hence with (as usual) $\psi(t) = \sqrt{2t \log \log t}$,

$$\limsup_{n \to \infty} \frac{\psi(\tau_n)}{\psi(n)} \leq \sqrt{2}C < \infty.$$

Therefore

$$\limsup_{n \to \infty} \frac{M_n}{\psi(n)} = \limsup_{n \to \infty} \frac{B(\tau_n)}{\psi(\tau_n)} \cdot \frac{\psi(\tau_n)}{\psi(n)} < \infty.$$

Thus a Bloch martingale almost surely satisfies the LIL upper bound

$$\limsup_{n \to \infty} \frac{M_n}{\sqrt{n \log \log n}} < \infty.$$

$\square$

Lemma 5.4. *If $\{M_n\}$ is a Bloch martingale, then*

$$\limsup_{n\to\infty} \frac{M_n}{\psi(n)} > 0,$$

almost surely on the set where $\limsup_{n\to\infty} \frac{1}{n} S_n > 0$.

Proof. If $S_n \geq \epsilon n \to \infty$ (5.4) implies

$$\tau_n - S_n = o(n) = o(S_n)$$

and hence

$$\frac{\psi(\tau_n)}{\psi(S_n)} = \frac{\psi(S_n + o(S_n))}{\psi(S_n)} \frac{\psi(S_n) + o(\psi(S_n))}{\psi(S_n)} = 1 + o(1)$$

Therefore

$$\limsup_{n\to\infty} \frac{M_n}{\psi(S_n)} = \limsup_{n\to\infty} \frac{B(\tau_n)}{\psi(\tau_n)} \cdot \frac{\psi(\tau_n)}{\psi(S_n)} = 1$$

Since $S_n \geq \epsilon n$, this proves the lemma. $\square$

Next we consider a case when the hypothesis of the previous lemma can be easily verified in practice.

Lemma 5.5. *Suppose the dyadic martingale $\{M_n\}$ satisfies the following: for some fixed δ and integer ℓ, and for every $b_1, \dots, b_n$ there are $b_{n+1}, \dots, b_{k+\ell}$ so that*

$$\Delta_{n+\ell}^2(b_1, \dots, b_{n+\ell}) \geq \delta^2.$$

Then almost surely

$$\liminf_{n\to\infty} \frac{1}{n} S_n \geq 2^{-\ell}\delta^2.$$

This will be useful later when we want to show some some fractal curves, like the von Koch snowflake, are sharp for Makarov's theorem.

Proof. Fix $0 \leq r < \ell$. Then

$$\{\Delta_n^2 - \mathbb{E}(\Delta_n^2|\mathcal{F}_{n-\ell} : n = ql + r\}_{q=1}^{\infty}$$

are orthogonal bounded variables, so by the Strong Law of Large Numbers, the averages summed over these ℓ-arithmetic sequences tend to zero almost surely

$$\frac{1}{N}\sum_{n=\ell+1}^{N}[\Delta_n^2 - \mathbb{E}(\Delta_n^2|\mathcal{F}_{n-\ell})] \to 0$$

Our hypothesis implies $\mathbb{E}(\Delta_n^2|\mathcal{F}_{n-\ell}) \geq 2^{-\ell}\delta^2$ which gives

$$\liminf_{n\to\infty}\frac{1}{n}S_n = \liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Delta_k^2 \geq 2^{-\ell}\delta^2 > 0.$$

$\square$

6. Makarov's Law of the Iterated Logarithm

Theorem 6.1 (Makarov's LIL for Bloch functions). *There is constant $C < \infty$ so that the following holds. Suppose u is a real-valued Bloch function and*

$$\psi(t) = \sqrt{\log \frac{1}{t} \log \log \frac{1}{t}}.$$

Then

$$\limsup_{r \nearrow 1} \frac{u(re^{i\theta})}{\psi(1-r)} \leq O(\|u\|_{\mathcal{B}}),$$

for almost every θ.

This is immediate from the LIL for Bloch martingales and the correspondence between Bloch martingales and Bloch harmonic functions.

Theorem 6.2 (Makarov's LIL for harmonic measure)**.** *There is constant $C < \infty$ so that the following holds. Suppose Ω is a quasidisk and $E \subset \partial\Omega$ has zero φ-measure for the guage function*

$$\varphi_C(t) = t \exp(C\sqrt{\log\frac{1}{t}\log\log\frac{1}{t}}).$$

Then E has zero harmonic measure in Ω.

This is sharp except for C.

For some fractal domains like the von Koch snowflake this fails for a different value of C.

The theorem is true with $C = 1$, but it is not known if this is best possible.

LIL proved for self-similar curves by Feliks Przytycki, Mariusz Urbański, Anna Zdunik.

Proof. The proof is essentially the same as Theorem 4.1, except that instead of the easy $o(n)$ upper bound for martingales, we use the more difficult LIL for martingales.

If $f : \mathbb{D} \to \Omega$ is a conformal map, then $g = \operatorname{Re} \log f'$ is a real Bloch harmonic function and therefore satisfies the LIL for Bloch functions: for a.e. $x \in \mathbb{T}$

$$\limsup_{r \to 1} \frac{|g(rx)|}{\sqrt{\log \frac{1}{1-r} \log \log \log \frac{1}{1-r}}} \leq C < \infty.$$

Let
$$\psi^{-1} = t\exp\left(-C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right)$$
Then for almost every $x \in \mathbb{T}$,
$$\liminf_{r\to 1}\frac{(1-r)|f'(rx)|}{\psi^{-1}(1-r)} \geq 1.$$
This implies $\omega \ll \mathcal{H}_\psi$.

A calculation shows
$$\psi(t) = O(\varphi(t)) = O\left(t\exp\left(C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right)\right)$$
and this implies $\omega \ll \mathcal{H}_\psi \ll \mathcal{H}_{\varphi_C}$. $\square$

7. Sharpness of Makarov's LIL

Theorem 7.1. *There is an $\epsilon > 0$, so that if φ is in Bloch with norm at most ϵ, then $\varphi = \log f'$ for some conformal map f onto a quasidisk.*

Proof. We will need the following inequality

$$\int_x^y ((\frac{1-x}{1-t})^\epsilon - 1)dt \leq \frac{\epsilon}{1-\epsilon}(y-x),$$

for $0 < \epsilon < 1$, $0 \leq x \leq y \leq 1$. This can be proved by observing that the left hand side is a convex function of y (for fixed x) and equals the the linear right hand side at $y = x$ and $y = 1$.

Given $z_1 \neq z_2$ in the disk we wish to show $f(z_1) \neq f(z_2)$. First consider the case when $z_1 = 0$ and $z_2 = r > 0$. Then for $0 < t < r$,

$$|\varphi(t)| = |\int_{r_1}^t \varphi'(s)ds| \leq \epsilon \int_0^t \frac{ds}{1-s} = \epsilon \log \frac{1}{1-t}.$$

Thus,

$$\begin{aligned}
|f(r_2) - f(r_1) - (r_2 - r_1)| &= |\int_0^r (f'(t) - 1)dt| \\
&\leq \int_0^r (e^{|\varphi(t)|} - 1)dt \\
&\leq \int_0^r ((\frac{1 - r_1}{1 - t})^\epsilon - 1)dt \\
&\leq \frac{\epsilon}{1 - \epsilon} r \\
&\leq \frac{1}{2} r,
\end{aligned}$$

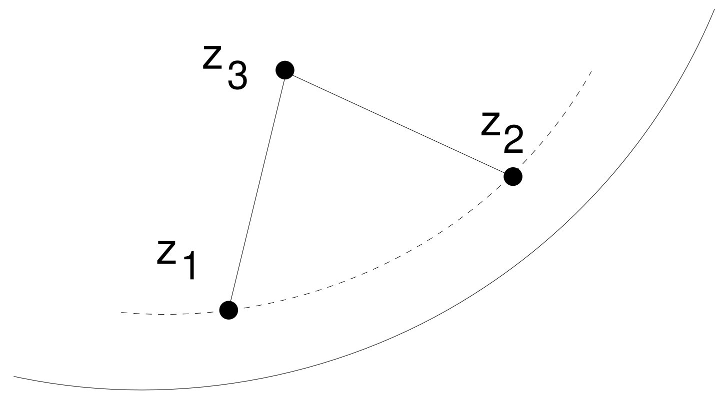
if $\epsilon \leq 1/3$. Thus $f(0) \neq f(r)$.

Now we consider the general case. It is easy to see that if f is not univalent then there are points z_1, z_2 such that $f(z_1) = f(z_2)$ and $|z_1| = |z_2|$. Without loss of generality we may take $z_1 = r$ and $z_2 = re^{i\theta}$ with $0 < \theta \leq \pi$. If $r < \theta$, then the previous estimate gives

$$
\begin{aligned}
|f(z_1) - f(z_2)| &\geq |z_2 - z_1| - |f(z_2) - f(0) - z_2| - |f(z_1) - f(0) - z_1| \\
&\geq |z_2 - z_1| - \frac{2\epsilon r}{1-\epsilon} \\
&\geq \frac{2}{3}|z_2 - z_1|,
\end{aligned}
$$

if $\epsilon < 1/4$.

Finally, if $r \geq \theta$ define a third point $z_3 = (r - \theta)e^{i\theta/2}$. This point is approximately "between" z_1 and z_2 and will play the role the origin did in the previous argument.



WLOG we may assume $f(z_3) = 0$ and $f'(z_3) = 1$ (so $\varphi(z_3) = 0$). Then if w lies on the line segment between z_3 and z_1, i.e., $w = (1-t)z_3 + tz_1$, then we have

$$|\varphi(w)| = |\int_{z_3}^{w} \varphi'(\zeta)d\zeta| \leq 2\epsilon \int_{|z_3|}^{|w|} \frac{d\zeta}{1-|\zeta|} \leq 2\epsilon \log(\frac{1-|z_3|}{1-|w|}).$$

Thus by repeating the argument from above,

$$\begin{aligned} |f(z_1) - f(z_3) - (z_3 - z_1)| &= |\int_{z_3}^{z_1} (f'(t) - 1)dt| \\ &\leq 2\int_{|z_3|}^{|z_1|} ((\frac{1-r_1}{1-t})^{2\epsilon} - 1)dt \\ &\leq \frac{4\epsilon}{1-2\epsilon}(|z_1| - |z_3|) \\ &\leq \frac{4\epsilon}{1-2\epsilon}|z_1 - z_3|. \end{aligned}$$

Of course, the same works with z_1 replaced by z_2.

Thus

$$\begin{aligned} |f(z_1) - f(z_2)| &\geq |z_2 - z_1| - |f(z_2) - (z_2 - z_3)| - |f(z_1) - (z_1 - z_3)| \\ &\geq |z_2 - z_1| - \frac{8\epsilon}{1-2\epsilon}|z_1 - z_2| \\ &\geq \frac{1}{2}|z_2 - z_1|, \end{aligned}$$

if ϵ is sufficiently small. $\square$

Lemma 7.2. *Given any Bloch martingale $\{f_n\}$ on the circle, there is harmonic Bloch function u on the disk, such that $\|u\|_{\mathcal{B}} \le C\|\{f_n\}\|_{\mathcal{B}}$ and*

$$\sup_{I \in \mathcal{D}_n} |u(z_I) - f_n(I)| \le C\|\{f_n\}\|_{\mathcal{B}}.$$

Proof. Suppose $\{f_n\}$ is a Bloch martingale of norm 1. Without loss of generality we may assume $f_0 = 0$ and hence all the elements have mean value zero.

Let u_n is the harmonic extension of f_n to the unit disk. By our assumption $u_n(0) = 0$ for all n.

$$\begin{aligned} u_n(z) &= \int_{\mathbb{T}} P_z(e^{i\theta}) f_n(e^{i\theta}) d\theta \\ &= \sum_{k=0}^{n-1} \int_{\mathbb{T}} P_z(e^{i\theta}) [f_{n+1}(e^{i\theta}) - f_n(e^{i\theta})] d\theta \\ &= \sum_{k=0}^{n-1} \int_{\mathbb{T}} P_z(e^{i\theta}) \Delta_n(e^{i\theta}) d\theta \end{aligned}$$

Note that Δ_n has means value zero over each dyadic interval of generation n, is bounded by 1 everywhere, and P_z differ from a constant by at most $O(\frac{2^{-n}}{1-|z|})$ on such an interval.

Thus if $n < m$,

$$|u_n(z) - u_m(z)| \le \sum_{k=n}^{m-1} \int_{\mathbb{T}} \frac{2^{-k}}{1-|z|} d\theta = O(\frac{2^{-n}}{1-|z|}),$$

which shows the sequence of harmonic functions converges uniformly on compact sets to a harmonic function u.

Next we want to prove

$$u(z) - f_I = O(1), z \in T(I).$$

This automatically proves that u is Bloch, since its variation over $T(I)$ is uniformly bounded.

Given a dyadic interval I we form a disjoint collection $\mathcal{C}$ of dyadic intervals J that cover the circle and so that $|I| \leq |J| \simeq \mathrm{dist}(J, I)$ and there are only a bounded number of intervals of any given size.

$$
\begin{aligned}
u_n(z) - f_I &= \int_{\mathbb{T}} P_z f_n - P_z f_I d\theta \\
&= \int_{\mathbb{T}} P_z (f_n - f_I) d\theta \\
&= \sum_{J \in \mathcal{C}} \int_J P_z (f_n - f_I) d\theta \\
&= \sum_{k=0}^{\infty} \sum_{J \in \mathcal{C}, |J| = 2^k |I|} \int_J P_z (f_n - f_I) d\theta \\
&= \sum_{k=0}^{\infty} \sum_{J \in \mathcal{C}, |J| = 2^k |I|} \int_J P_z (f_n - f_J) + (f_J - f_I) d\theta \\
&= \sum_{k=0}^{\infty} \sum_{J \in \mathcal{C}, |J| = 2^k |I|} \left(\int_J P_z (f_n - f_J) d\theta + \int_J P_z (f_J - f_I) d\theta \right)
\end{aligned}
$$

Note that $|J| = 2^k|I|$ and $\mathrm{dist}(J, I) \leq 2^k|I|$ implies that $|f_J - f_I| = O(k)$ by the Bloch condition

Thus the sum of the second integral $\int_J P_z(f_J - f_I)d\theta$ is bounded by

$$\sum_{k=0}^{\infty} \sum_{J\in\mathcal{C},|J|=2^k|I|} \int_J O(2^{-2k}|I|^{-1}k)d\theta \leq \sum_{k=0}^{\infty} \sum_{J\in\mathcal{C},|J|=2^k|I|} \sum_{k=0\infty} O(2^{-k})k = O(1).$$

Next we handle the first integral $\int_J P_z(f_n - f_J)d\theta$.

Let I be of generation m.

If $n > m$ then $f_n - f_I$ has mean zero on I and $f_n - f_J$ has mean zero on each $J \in \mathcal{C}$, since these are all in earlier generations and hence at least as long as I.

Since P_z varies by less than $2^{-n-2k}|I|^{-1}$ on intervals of length 2^{-n} in J_k, we get:

$$
\begin{aligned}
|\sum_{k=0}^{\infty}\sum_{J\in\mathcal{C},|J|=2^k|I|}\int_J P_z(f_n-f_J)d\theta| &= |\sum_{k=0}^{\infty}\sum_{J\in\mathcal{C},|J|=2^k|I|}\int_J P_z\sum_{j=m-k}^{n-1}\Delta_j d\theta| \\
&= |\sum_{k=0}^{\infty}\sum_{J\in\mathcal{C},|J|=2^k|I|}\int_J \sum_{j=m-k}^{n-1}|I|^{-1}2^{-2k-j}d\theta| \\
&\leq \sum_{k=0}^{\infty}\sum_{J\in\mathcal{C},|J|=2^k|I|} O(2^m 2^{-2k-m+k}) \\
&= O(\sum_{k=0}^{\infty}\sum_{J\in\mathcal{C},|J|=2^k|I|} 2^{-k}) \\
&= O(1)
\end{aligned}
$$

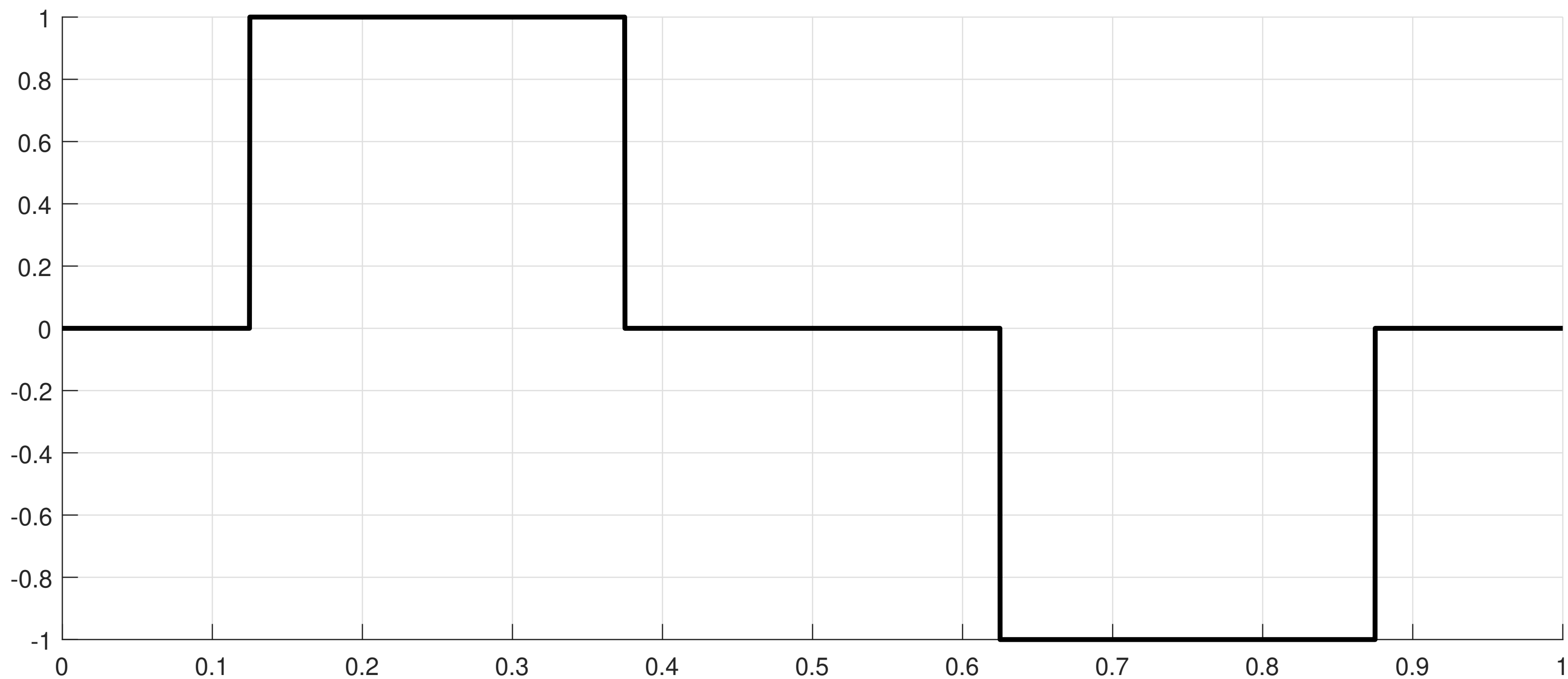
This completes the proof. $\square$

Theorem 7.3. *There is a $c > 0$, a quasidisk Ω and a set $E \subset \partial\Omega$ that has full harmonic measure, but zero Hausdorff φ-measure for*

$$\varphi(t) = t \exp(c\sqrt{\log \frac{1}{t} \log \log \frac{1}{t}}).$$

Proof. It suffices to show there is a dyadic Bloch martingale so that the square function grows linearly almost everywhere. Then a small multiple of this martingale will correspond to a harmonic Bloch function u of small norm, that has growth rate at least $\epsilon\sqrt{\log \frac{1}{1-r} \log \log \log \frac{1}{1-r}}$ almost everywhere on the circle, for some $\epsilon > 0$.

Then u will be of the form $u = \log f'$ for some conformal map, and using arguments from earlier, it is easy to see the image curve $\Gamma = f(\mathbb{T})$ has harmonic measure is supported on a set of zero Hausdorff φ-measure if c is chosen small enough. $\square$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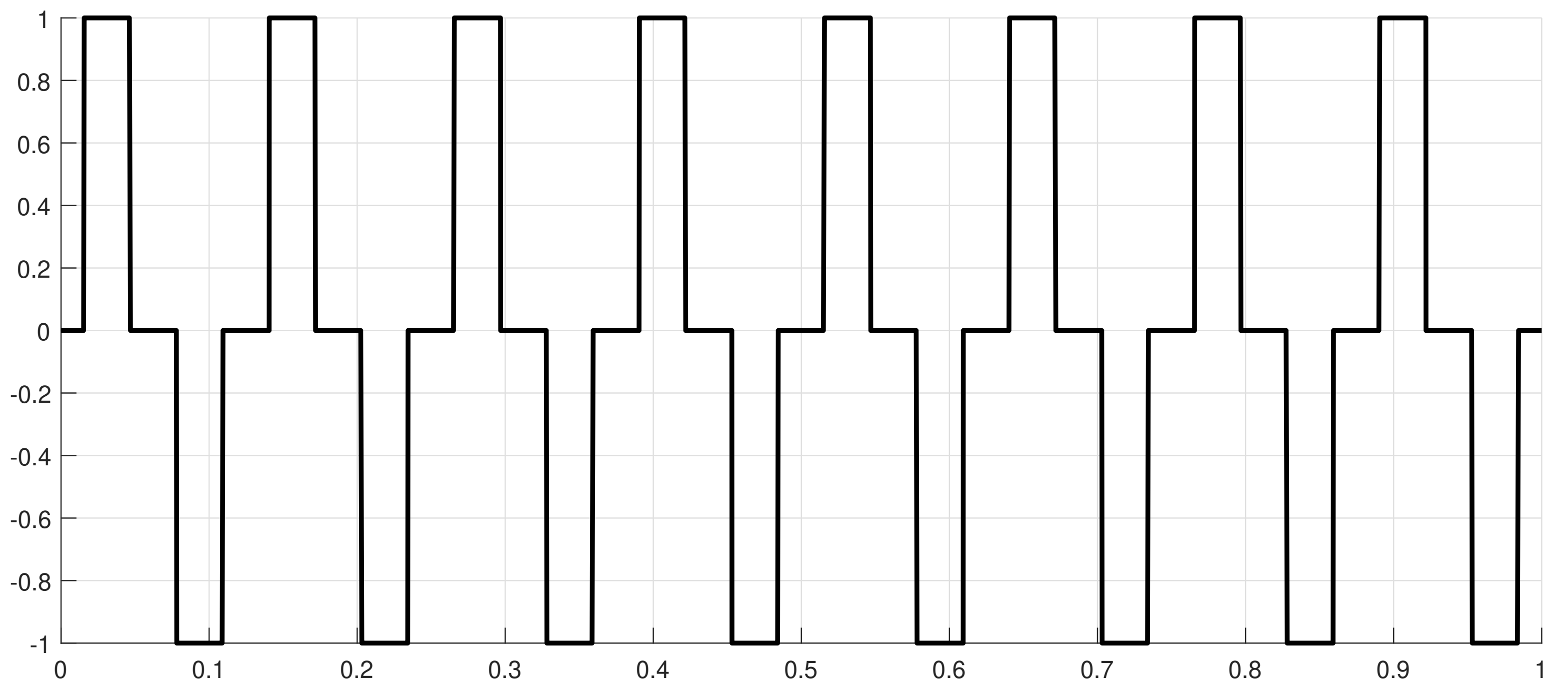
1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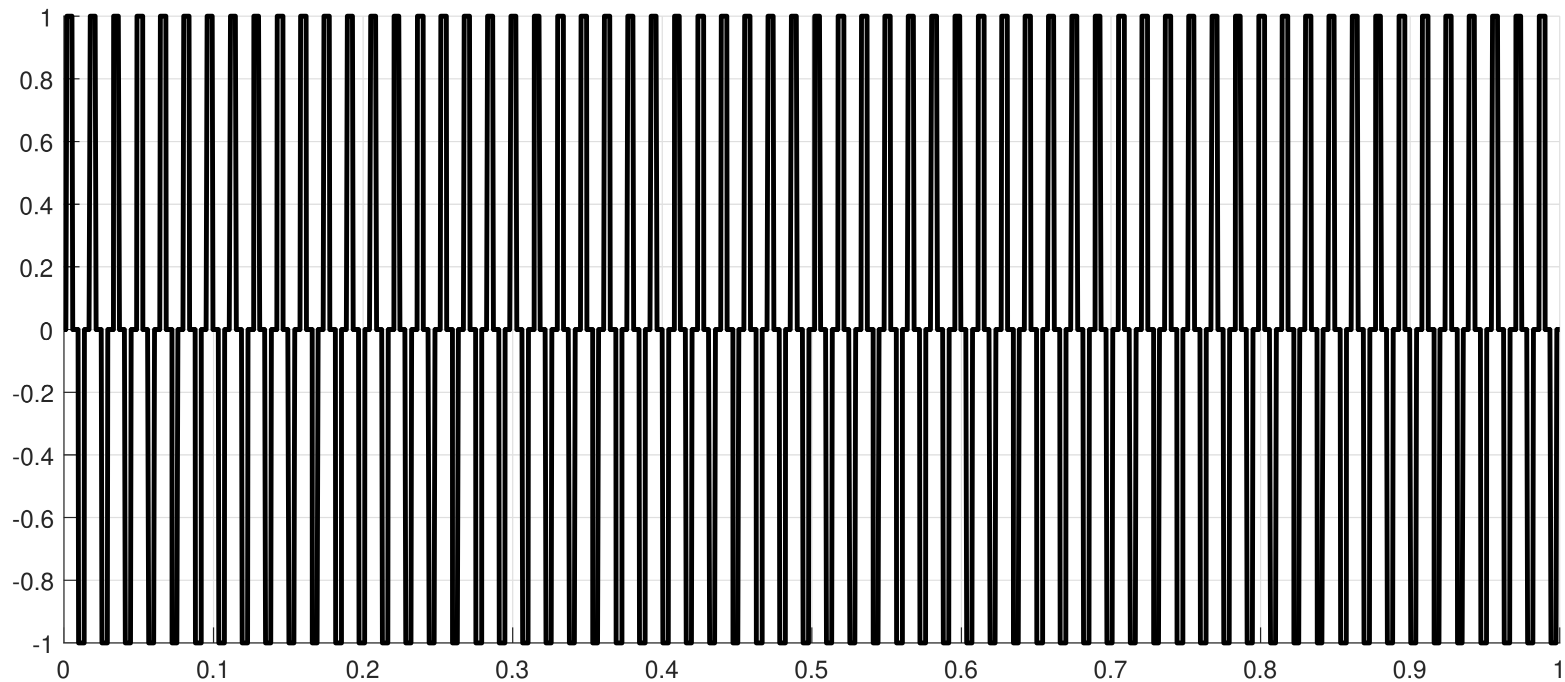
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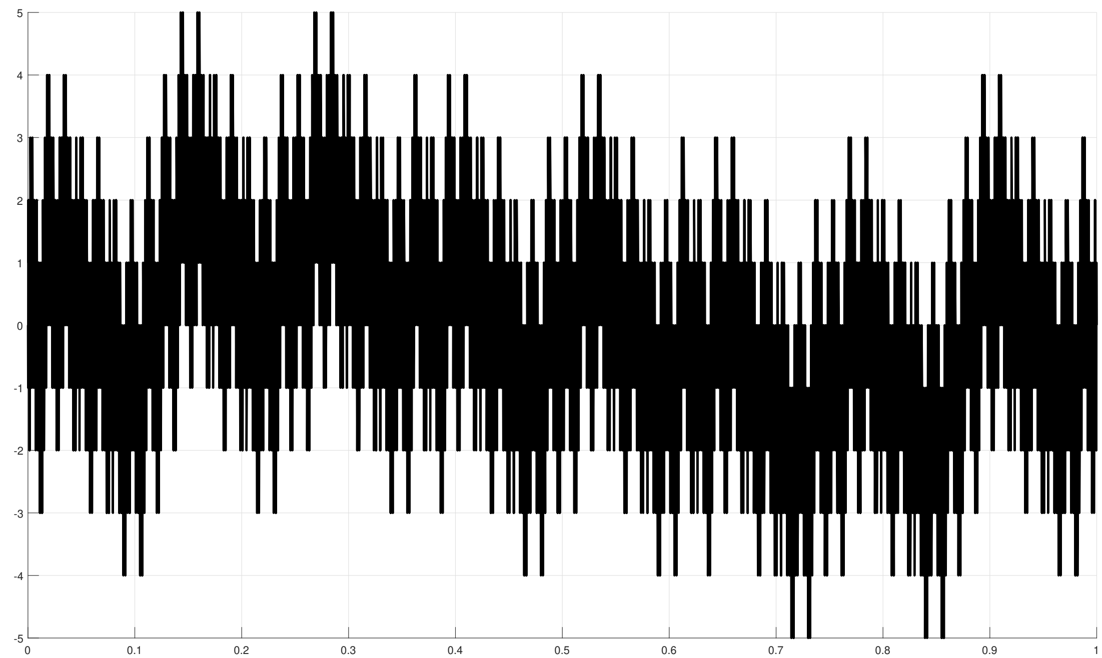
5
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3
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-1
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0.6
0.7
0.8
0.9
1

The von Koch snowflake provides an example where Makarov's LIL occurs. It requires some work to show that $\log f'$ is has the correct square function estimate.

Lemma 7.4. *Suppose $f : \mathbb{D} \to \Omega$ is the conformal map onto the interior of the von Koch snowflake. Given any $M > 0$, there is positive integer N so the following holds. Given any dyadic interval $I \subset \mathbb{T}$, there is a dyadic interval $J \subset I$ with $|J| \geq 2^{-N}|I|$ and so that*

$$\log|f'(z_I)| - \log|f'(z_J)| > M.$$

Assuming this, then by taking M large enough, we can deduce t that the dyadic martingale associated to $\log|f'|$ has values differing by 1 at I and J, so it satisfies the hypothesis of Lemma 5.5. Thus the square function of the martingale has linear growth a.e., and hence the martingale grows faster that $\epsilon\sqrt{n \log\log n}$ by Lemma 5.4. Thus the same is true for $\log|f'|$.

More generally, Peter Jones showed if the boundary of Ω deviates from a straight line (in a certain precise sense) at every point and every scale, then harmonic measure satisfies Makarov's LIL for some constant C.

Such domains give functions $\varphi = \log|f'|$ that are **maximal Bloch**: there is a constants $C < \infty$ and $\epsilon > 0$ so that every point $z \in \mathbb{D}$ is within hyperbolic distance c of a point w where

$$|\varphi'(w)| \leq \frac{\epsilon}{1-|w|}.$$

One can prove that maximal Bloch functions satisfy

$$\limsup \frac{|\varphi(rx)|}{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}} > 0$$

for a.e. $x \in \mathbb{T}$.

8. Law of the Iterated Logarithm for dyadic martingales (sharp version)

Recall the sharp version of the LIL we stated earlier.

Theorem 8.1. *For a dyadic martingale* $\{M_n\}$ *with* $\sup_n \|\Delta_n\|_\infty \le C < \infty$, *then*

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{M_n}{\sqrt{2S_n \log\log S_n}} = 1 | S_\infty = \infty\right) = 1 \tag{8.1}$$

I will give the proof of this for completeness, but we will not use this version.

Proof. We use the same definitions of the bit functions $\{b_k\}$, stopping times $\{\tau_n\}$ and σ-fields $\{\mathcal{G}_n\}$ as before.

Also as before we have (by Wald's lemma),

$$\mathbb{E}(\tau_n - \tau_{n-1}|\mathcal{G}_{n-1}) = \Delta_n^2.$$

It is left as an exercise to check that

$$\mathbb{E}(\tau_n - \tau_{n-1})^2|\mathcal{G}_{n-1}) \leq 4\Delta_n^4.$$

(Any multiple of $|\Delta_n|^2$ is OK for us; the expectation actually equals $\frac{2}{3}\Delta_n^4$.)

Define $\lambda_n = \tau_n$ for all n if $S_\infty = \lim_n S_n = \sum_1^\infty \Delta_n^2 = \infty$.

Otherwise, let $\lambda_n = n$.

The assumption that $|\Delta_n| \leq C$ implies $S_{n+1} = S_n + O(1)$ and hence that

$$\frac{S_{n+1}}{S_n} = 1 + O\left(\frac{1}{S_n}\right) \to 1$$

on the set $\{S_\infty = \infty\}$.

Therefore $\lambda_{n+1}/\lambda_n \to 1$ almost surely on whole space.

The corollary about the LIL for Brownian motion evaluated at random times that we proved earlier now implies

$$\frac{B(\lambda_n)}{\psi(\lambda_n)} \to 1$$

almost surely. Since $\lambda_n = \tau_n$ on $\{S_\infty = \infty\}$, we get

$$\frac{M_n}{\psi(\tau_n)} \to 1$$

almost surely on the set $\{S_\infty = \infty\}$.

We are done if we can prove that almost surely of $\{S_\infty = \infty\}$.

$$\lim_n \frac{\tau_n}{S_n} = 1. \tag{8.2}$$

This will follow from Lemma 8.2 below with $Y_n = \Delta_n^2$.

Thus the proof of the sharp LIL is complete, except for Lemma 8.2 $\square$

Suppose $\{\mathcal{G}_n\}_{n\geq 0}$ is a filtration.

A sequence $\{Y_n\}$ is called **predictable** if for all $n \geq 1$ Y_n is $\mathcal{G}_{n-1}$-measurable.

A sequence $\{\tau_n\}$ is called **adapted** if τ_n is is $\mathcal{G}_n$-measurable.

Lemma 8.2. *Suppose $\{\mathcal{G}_n\}$ is a filtration, $\{Y_n\}$ i s predictable with $0 \leq Y_n \leq C^2$, and $\{\tau_n\}$ is adapted and satisfy $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$. Define $S_n = \sum_{k=1}^n Y_k$ and $S_\infty = \lim_n S_n$. Suppose that there is an $A < \infty$ so that for all $n \geq 1$,*

$$\mathbb{E}(\tau_n - \tau_{n-1}|\mathcal{G}_{n-1}) = Y_n$$

$$\mathrm{Var}(\tau_n - \tau_{n-1}|\mathcal{G}_{n-1}) \leq AY_n^2.$$

Then almost surely on the set $\{S_\infty = \infty\}$,

$$\lim_{n\to\infty} \frac{\tau_n}{S_n} = 1.$$

Proof. We may assume $\mathbb{P}(S_\infty = \infty) > 0$, since otherwise the lemma is trivial.

Let $N_0 = 0$ and set

$$N_k = \min\{n > N_{k-1} : S_n - S_{N_{k-1}} \geq C^2\}$$

if the set is nonempty, and otherwise $N_k = N_{k-1}$.

If $N_k > N_{k-1}$ then $C^2 \leq S_{N_k} - S_{N_{k-1}} \leq 2C^2$.

Let $Z_k = \tau_{N_k} - S_{N_k}$. We claim $\{Z_k - Z_{k-1}\}$ are orthogonal functions with

$$\operatorname{Var}(Z_k - Z_{k-1}) \leq 2AC^4. \tag{8.3}$$

Assuming the claim, the Strong Law of Large Numbers implies (using a telescoping series) that $\frac{1}{k}Z_k \to 0$ almost surely. Hence

$$\frac{\tau_{N_k} - S_{N_k}}{k} \to 0,$$
$$\frac{\tau_{N_k} - S_{N_k}}{S_{N_k}} \cdot \frac{S_{N_k}}{k} \to 0,$$
$$\left(\frac{\tau_{N_k}}{S_{N_k}} - 1\right)\frac{S_{N_k}}{k} \to 0,$$

By definition $S_{N_k} \geq C^2 k$, so we must have

$$\frac{\tau_{N_k}}{S_{N_k}} \to 1$$

almost surely on the set $\{S_{N_k}\} \to \infty$.

This is the desired result, but only along the sequence $\{N_k\}$.

For n between N_{k-1} and N_k, we have
$$\frac{\tau_{N_{k-1}}}{S_{N_k}} \leq \frac{\tau_n}{S_{N_k}} \leq \frac{\tau_{N_k}}{S_{N_{k-1}}}$$
But right hand term satisfies
$$\frac{\tau_{N_k}}{S_{N_{k-1}}} = \frac{\tau_{N_k}}{S_{N_k}} \cdot \frac{S_{N_k}}{S_{N_{k-1}}} \leq \frac{\tau_{N_k}}{S_{N_k}} \cdot \frac{S_{N_{k-1}} + 2C^2}{S_{N_{k-1}}} \to 1.$$
Similarly for the lower bound.

This proves the lemma, except for verifying the claim. $\square$

Proof of the claim.

Note that $\{N_{k-1} < j\} \in \mathcal{G}_{j-1}$, since this only requires knowing S_k for $k < j$. Since σ-fields are closed under complements, we also have $\{N_{k-1} \geq j\} \in \mathcal{G}_{j-1}$. Thus

$$\begin{aligned}
&\mathbb{E}[(\tau_j - \tau_{j-1} - Y_j)^2 \cdot 1_{[N_{k-1}<j\leq N_k]}|\mathcal{G}_{j-1}] \\
&\qquad = 1_{[N_{k-1}<j\leq N_k]}\mathrm{Var}(\tau_j - \tau_{j-1}|\mathcal{G}_{j-1}) \\
&\qquad \leq AY_j^2 \leq AC^2Y_j
\end{aligned}$$

We can take expectations and sum over j to get

$$\begin{aligned}
\sum_{j=1}^{\infty}\mathbb{E}[(\tau_j - \tau_{j-1} - Y_j)^2 \cdot 1_{[N_{k-1}<j\leq N_k]}] &\leq AC^2\mathbb{E}\left[\sum_{j=1}^{\infty} Y_j \cdot 1_{[N_{k-1}<j\leq N_k]}\right] \\
&= AC^2\mathbb{E}(S_{N_k} - S_{N_{k-1}}) \\
&\leq 2AC^4.
\end{aligned}$$

But

$$\mathbb{E}[(\tau_j - \tau_{j-1} - Y_j) \cdot 1_{[N_{k-1} < j \leq N_k]} | \mathcal{G}_{j-1}] = 0,$$

since the indicator is $\mathcal{G}_{j-1}$-measurable.

Thus by the orthogonality of martingale increments,

$$\mathrm{Var}(Z_k - Z_{k-1}) = \mathbb{E}\left[\left(\sum_{j=1}^{\infty} (\tau_j - \tau_{j-1} - Y_j) \cdot 1_{[N_{k-1} < j < N_k]}\right)^2\right] \leq 2AC^4.$$

This proves the claim and finishes the proof of the LIL for martingales.

9. From Quasicircles to Jordan domains

Next we remove the quasidisk assumption from Makarov's theorem, and prove the result for all Jordan domains. An approximation argument shows that special case of Jordan domains implies the general case of all simply connected domains.

In the previous case we assumed that for any disk D, all the components of $D \cap \partial\Omega$ were contained in a single arc of $\partial\Omega$ whose diameter was comparable to the diameter of D.

In general, this is not true, so we consider the components of $2D \cap \partial\Omega$ separately. Although there may be infinitely many such components, using extremal length we can show that at most $O(-\log \operatorname{diam}(D))$ of these components account for most of the harmonic measure of $D \cap \partial\Omega$ and the extra logarithmic factor can be absorbed into the Makarov's guage function by changing the constant.

A **crosscut** of Ω is a Jordan arc σ in Ω with both endpoints on $\partial\Omega$.

By Beurling's estimate we can choose a δ so any disk of radius δ has harmonic measure less than $1/n$ (with respect to any base point distance ≥ 1 from the boundary).

The following lemma says that the diameter of a crosscut can be estimated in terms of the size of its preimage and and the estimates on $|f'|$.

Recall

$$\varphi_C(t) = t\exp\left(C\sqrt{\log\frac{1}{t}\log\log\frac{1}{t}}\right).$$

and let

$$E_n = \{e^{i\theta} : |f'(re^{i\theta})| \geq \frac{\varphi_{-a}(1-r)}{1-r}, 1-\frac{1}{n} \leq r < 1\},$$

Lemma 9.1. *Suppose σ is a crosscut on Ω contained in some disk D of radius $\leq \delta$. Let β be the subarc of $\partial\Omega$ separated from z_0 by σ, let D_σ be the region bounded by $\sigma \cup \beta$ and let $I \subset \mathbb{T}$ be the arc corresponding arc to β. Assume $I \cap E_n \neq \emptyset$. Then*

$$\operatorname{diam}(D_\sigma) \geq \operatorname{diam}(\sigma) \geq C\varphi_{-a}(|I|).$$

Proof. The left hand inequality is trivial since $\sigma \subset \partial D_\sigma$ implies

$$\operatorname{diam}(D_\sigma) = \operatorname{diam}(\partial D_\sigma) \geq \operatorname{diam}(\sigma).$$

To prove the right hand inequality, choose $e^{i\theta} \in I \cap E_n$ and let $z = (1 - |I|)e^{i\theta}$. By our choice of δ, $|I| \leq 1/n$, so

$$|f'(z)| \geq \varphi_{-a}(|I|)|I|^{-1}.$$

By the Koebe 1/4 theorem,

$$d = \operatorname{dist}(f(z), \partial\Omega) \geq C\varphi_{-a}(|I|),$$

and so by Beurling's estimate

$$\omega(f(z), \sigma, \Omega \setminus \sigma) \leq C(\frac{|\sigma|}{d})^{1/2},$$

for some $C < \infty$.

However, if $d \gg |\sigma|$ we will show this gives a contradiction.

Note that

$$\begin{aligned}\omega(f(z), \sigma, \Omega \setminus \sigma) &= \omega(z, f^{-1}(\sigma), \mathbb{D} \setminus f^{-1}(\sigma)) \\ &\geq \min\{\omega(z, I, \mathbb{D}), \omega(z, I^c, \mathbb{D})\},\end{aligned}$$

depending on which side of $f^{-1}(\sigma)$ the point z lies.

By the definition of z we see that both these terms are bigger than some absolute constant and so

$$|\sigma| \geq Cd \geq C\varphi_{-a}(|I|),$$

as required. $\square$

The following lemma says that a neighborhood on $\partial\Omega$ does not have too many preimages on the unit circle with large measure. The proof uses some simple facts about extremal length.

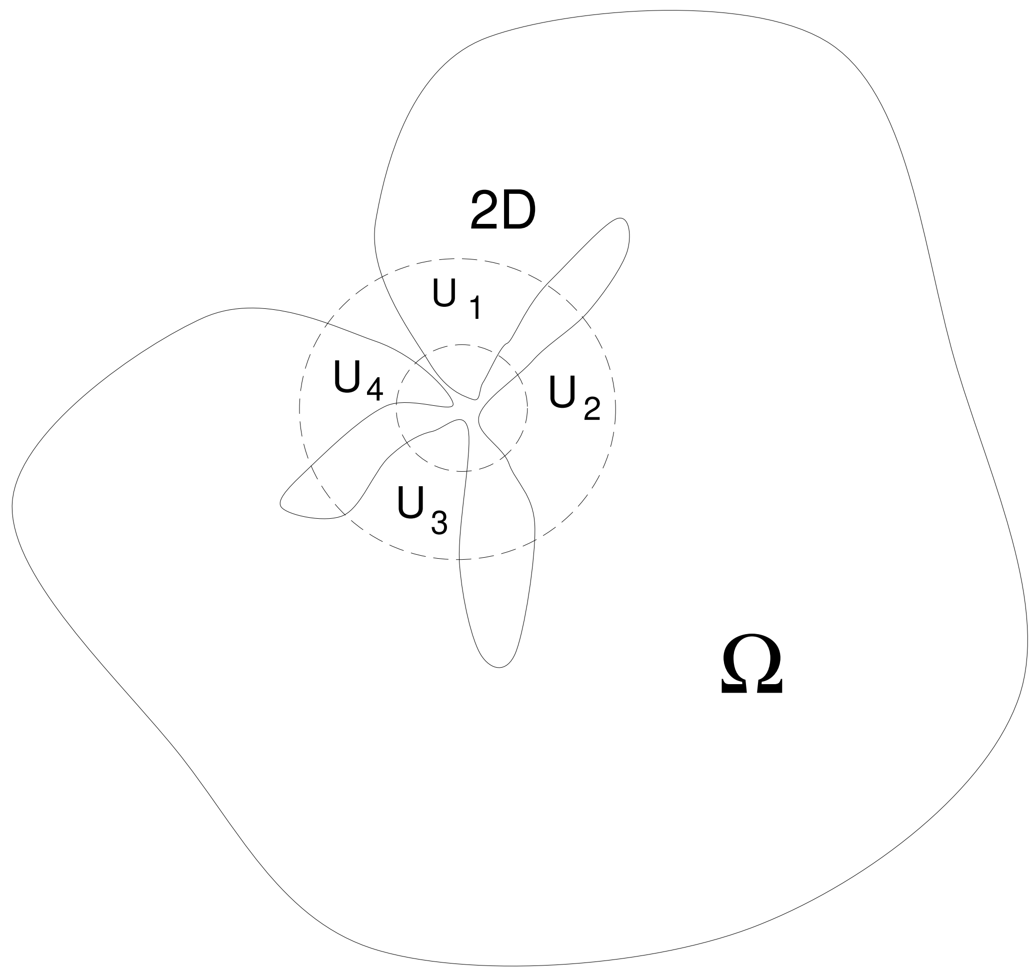
Lemma 9.2. *Suppose D is a disk of radius $r < r_0$ such that $\omega(D) \geq r$ and let $2D$ be the concentric disk with twice the radius. Then there are m crosscuts $\{\sigma_j\} \subset 2D$, $j = 1, \ldots, m$ with associated arcs $\{\beta_j\}$, $\{I_j\}$ such that*

$$m \leq \frac{2\pi}{\log 2} \log \frac{1}{r},$$

$$\sum_j |I_j| \geq \frac{3}{4}\omega(D).$$

Proof. To normalize the situation assume that $\operatorname{dist}(z_0, \partial\Omega) = 1$ and let K be a disk of radius $1/2$ centered at z_0.

Let Ω_0 be the component of $\Omega \setminus D$ containing $\{z_0\}$ and let $\{U_j\}$ be the components of $\Omega_0 \cap 2D$ whose boundary contains arcs on both ∂D and $\partial 2D$. Since $\omega(D) > 0$ this collection is nonempty.

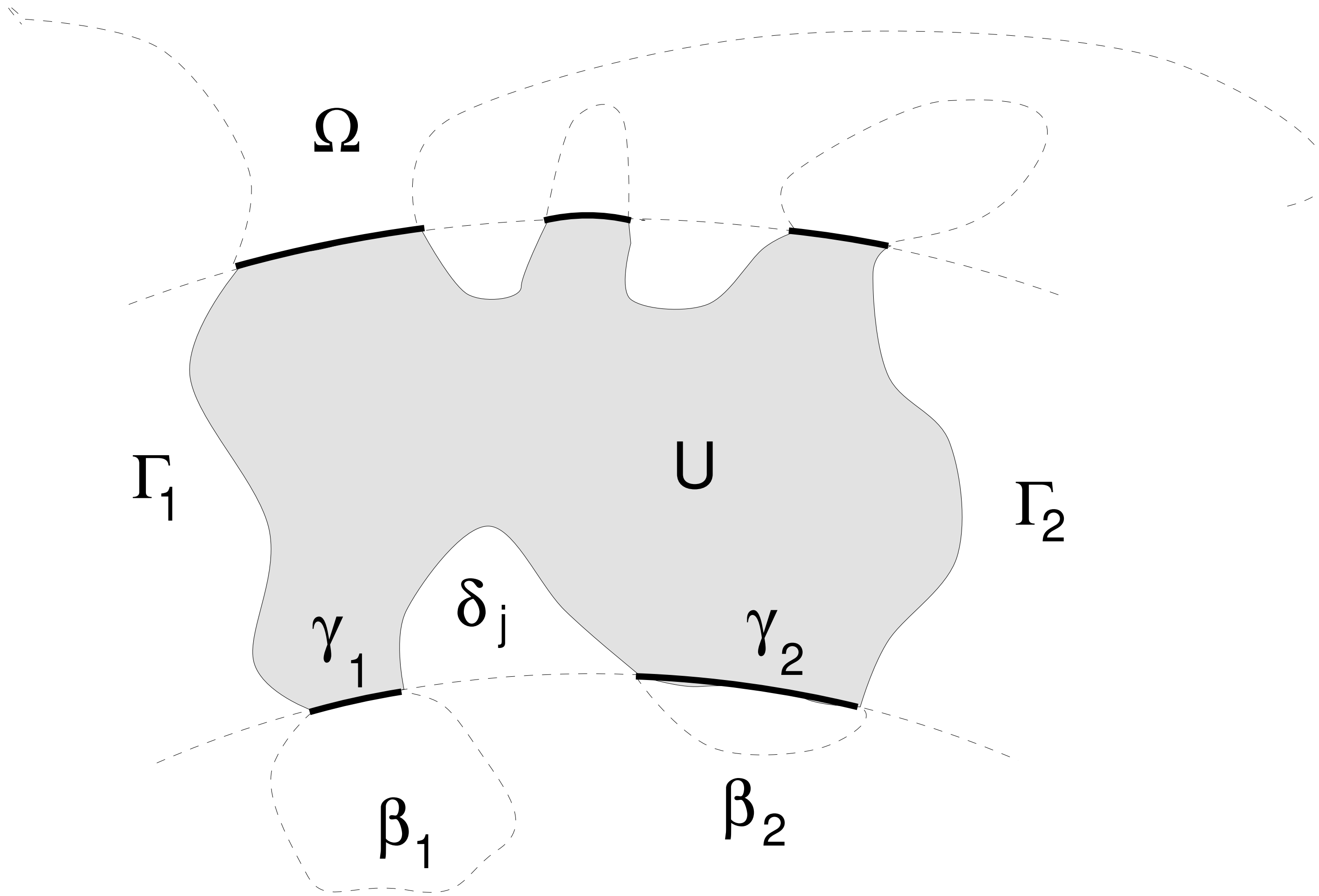


2D
U 1
U 4
U 2
U 3
Ω

Fix j and consider U_j.

It is a Jordan domain and $\partial U_j \cap (2D \setminus D)$ is a union of arcs exactly two of which Γ_1^j, Γ_2^j connect ∂D to $\partial 2D$. Their complement in ∂U_j consists of arcs, one of which, call it δ_j, hits ∂D.

Then the set $(\partial U_j \cap \partial D) \setminus \partial\Omega$ is a union of arcs $\{\gamma_k\}$ of ∂D each of which is a crosscut of Ω with associated arcs β_k of $\partial\Omega$. Let $E_j = \cup_k \gamma_k$.



Ω
Γ1
U
Γ2
γ1
δj
γ2
β1
β2

Let $\mathcal{F}_j$ be the family of all arcs separating K from E_j and let $\tilde{\mathcal{F}}_j$ be the family of all arcs in U_j connecting Γ_1^j to Γ_2^j and $\mathcal{F}$ the family of all arcs in $2D\backslash D$ separating the two boundary circles. Then by Pfluger's theorem relating extremal length to harmonic measure,

$$M(\mathcal{F}_j) \leq \frac{1}{\pi} \log(\frac{C}{\omega(E_j)}).$$

By monotonicity of extremal length (fewer arcs $\Rightarrow$ more metrics $\Rightarrow$ smaller modulus)

$$M(\tilde{\mathcal{F}}_j) \leq M(\mathcal{F}_j).$$

By the series rule

$$\sum_j \lambda(\tilde{\mathcal{F}}_j) \le \lambda(\mathcal{F}) = \frac{2\pi}{\log 2},$$

or

$$\sum_j M(\tilde{\mathcal{F}}_j)^{-1} \le M(\mathcal{F})^{-1} = \frac{2\pi}{\log 2},$$

so

$$\sum (\log \frac{C}{\omega(E_j)})^{-1} \le \frac{2}{\log 2}.$$

By Tchebyshev's inequality,

$$|\{j : \omega(E_j) \ge Cr^{\pi k}\}| \le \frac{2\pi}{\log 2} k \log \frac{1}{r}.$$

Hence

$$\begin{aligned}
\sum_{\omega(E_j)\le Cr^\pi} \omega(E_j) &\le \sum_{k=1}^{\infty}(\frac{2\pi}{\log 2}(k+1)\log\frac{1}{r})Cr^{\pi k} \\
&\le (\frac{2\pi C}{\log 2})\log\frac{1}{r}\sum_{k=1}^{\infty}(k+1)r^{\pi k} \\
&\le \frac{1}{4}r,
\end{aligned}$$

if $r < r_0$ is small enough. Thus

$$\sum_{\omega(E_j)>Cr^\pi} \omega(E_j) \ \ge\ \omega(D) - \frac{1}{4}r \ge \frac{3}{4}\omega(D),$$

and there are at most $2\pi\log\frac{1}{r}/\log 2$ such j's.

So if we take the σ_j to be a crosscut of Ω contained in U_j with

$$\beta_j = E_j \cup (\delta_j \cap \partial\Omega),$$

the lemma is proven. $\square$

Now that the technical lemmas are finished, we can complete the proof of Makarov's theorem for general Jordan curves.

Proof. By the LIL for conformal maps we can choose a universal $a > 0$ so that

$$\liminf_{r\to 1} |f'(re^{i\theta})|\left(\frac{1-r}{\varphi_{-a}(1-r)}\right) = +\infty,$$

for almost every θ. So if we define

$$E_n = \{e^{i\theta} : |f'(re^{i\theta})| \geq \frac{\varphi_{-a}(1-r)}{1-r}, 1 - \frac{1}{n} \leq r < 1\},$$

then $\cup_n E_n$ has full measure.

Thus it suffices to prove: if $E \subset f(E_n)$ and $\mathcal{H}_{\varphi_{2a}}(E) = 0$, then $\omega(E) = 0$.

Suppose $\{D_j\}$ is a covering of E by disks of radius $\{r_j\}$ such that $\max_j r_j \leq \delta$ and $\sum_j \varphi_{2a}(r_j) \leq \epsilon$. By considering points of density of E (with respect to harmonic measure) and taking δ to be small enough we may suppose

$$\omega(D_j) \leq 2\omega(D_j \cap E).$$

We may also assume $\omega(D_j) \geq r_j$, for the remaining disks satisfy

$$\Omega(\cup_j D_j) \leq \sum_j r_j \leq \sum_j \varphi_{2a}(r_j) \leq \epsilon.$$

Now for each j let $\{\sigma_k^j\}$ be the crosscuts given by the previous lemma which also satisfy $\beta_k^j \cap E \neq \emptyset$. Then

$$\sum_k \omega(\beta_k^j) \geq \frac{1}{4}\omega(D_j),$$

so by the two lemmas,

$$\omega(E) \leq \sum_j \omega(D_j) \leq 4 \sum_j \sum_k \omega(\beta_k^j) \leq \frac{8\pi}{\log 2} \sum_j \log \frac{1}{r_j} \varphi_{-a}^{-1}(\frac{2r_j}{C}).$$

Since $\varphi_{-a}(t) \le t$ and is convex up, $\varphi_{-a}^{-1}(t) \ge t$ and is concave down, so

$$\varphi_{-a}^{-1}(t) \le t \cdot t/\varphi_{-a}(t) = \varphi_a(t).$$

Thus

$$\varphi_{-a}^{-1}(t) \le \varphi_a(t) = \varphi_{2a}(t) \cdot \varphi_{-a}(t)/t,$$

and hence

$$\omega(E) \le \frac{2\pi}{\log 2} \sum_j \varphi_{2a}(\frac{2r_j}{C})(\log \frac{1}{r_j} \frac{\varphi_{-a}(2r_j/C)}{C/2r_j}).$$

Next note that

$$(\log \frac{1}{t})(\frac{\varphi_{-a}(t)}{t}) = \exp\left(\log\log\frac{1}{t} - a\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right)$$

This tends to zero as $t \to 0$, so is bounded for $t \in (0, \delta]$.

Hence

$$\omega(D) \leq C\varphi_{2a}(|D|),$$

so

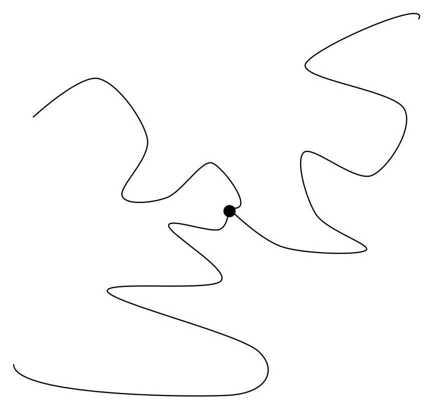
$$\omega(E) \leq C\sum_j \varphi_{2a}(2r_j/C) \leq C\sum_j \varphi_{2a}(r_j) \leq C\epsilon.$$

Since ϵ was arbitrary, we have proven the theorem. $\square$

A triod is a union of three closed Jordan arcs that are disjoint except for sharing a common endpoint. Homeomorphic to a "Y".

Theorem 9.3. *(Moore's triod theorem) any collection of disjoint triods in the plane is countable.*

Corollary: If $f : \mathbb{D} \to \Omega$ is conformal then there is a set $E \subset \mathbb{T}$ of full measure so that f has non-tangential limits on E and it at most 2-to-1 on E.



Proof. If there is an uncountable such collection. then must be integers n, m and distinct rationals r_1, r_2, r_3 with $|r_i - r_j| > 2/m$ all i, j and an uncountable subset of triods so that if x is the common endpoint of a triod T then the three arcs first intersect $\{|z-x| = 1/n\}$ within angle $1/m$ of the arguments r_1, r_2, r_3. Since any uncountable set has a finite accumulation point, it is easy to drive a contradiction. $\square$

Theorem 9.4. *Suppose $f : \mathbb{D} \to \Omega$ is a conformal map onto a simply connected domain. If $E \subset \mathbb{T}$ has positive length, then there is subset $F \subset E$ of positive length so that $f(W_F)$ is a Jordan domain.*

Proof. Let $E_1 \subset \mathbb{T}$ be where f has a radial limit and define $d : E \times E \to [0, \infty)$ as $d(z, w) = |f(x) - f(y)|$. Let

$$E_1 = \{z \in E : d(z, w) = 0 \text{ only if } w = z\},$$

$$E_2 = \{z \in E : d(z, w) = 0 \text{ for exactly one } w \neq z\}.$$

By Moore's triod theorem, $E \setminus (E_1 \cup E_2)$ is a countable set, hence has linear measure zero. For each n let

$$E_2^n = \{z \in E_2 : d(z, w) = 0, |z - w| < \frac{1}{n} \text{ implies } z = w\}.$$

Since $\cup_n E_2^n = E_2$ and these sets are nested, the length of E_2^n converges to the length of E_2.

If $X \subset \mathbb{T}$ has positive measure, then so does either $X \cap E_1$ or $X \cap E_2$. If the former has positive measure, then consider $X \cap E_1 \cap F_m$, where $\{F_m\}$ are of measure $\geq 1 - 1/m$ with f having nontangential limits on F_m, and m is chosen so large that $X \cap E_1 \cap F_m$ has positive measure. If we take F to be a compact, positive length subset of this set, then f is continuous on W_F and is 1-to-1 on the whole boundary, so $f(W_F)$ is a Jordan domain.

If $X \cap E_2$ has positive length, then so does $X \cap E_2^n$ for some n. Fix such an n and choose an interval I of length $1/2n$ so that $I \cap X \cap E_2^n$ also has positive length. Then the radial limits of f are 1-to-1 restricted to this set, and the proof is finished exactly as above. $\square$

Corollary 9.5. *Harmonic measure for a general simply connected domain gives full measure to a set of σ-finite 1-measure, and Makarov's LIL holds.*

Sketch of proof. If Ω is a simply connected domain, then any subset of $\partial\Omega$ of positive harmonic measure also has positive harmonic measure for some Jordan subdomain of Ω.

Thus harmonic measure for Ω dominated by a countable sum of harmonic measures for Jordan domains and thus gives full measure to σ-finite length.

On the the other hand if harmonic measure for Ω gave positive measure to a set of zero φ-measure (φ as in Makarov's LIL), then some Jordan domain would also give it positive measure, a contradiction. $\square$

10. The F. and M. Riesz Theorem

Lemma 10.1 (Jensen's Formula)**.** *If f is analytic on the unit disk with zeros $\{z_n\}_1^N$ in $D(0,r)$, and suppose $f(0) \neq 0$ and f has no zeros on the circle $\{|z| = r\}$. Then*

$$|f(0)| \prod_{n=1}^{N} \frac{r}{|z_n|} = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta\right).$$

Proof. Let
$$B(z) = \prod_{n=1}^{N} \frac{r^2 - \bar{z}_n z}{r(z_n - z)}.$$
then B and f have the same zeros and $|B| = 1$ on the circle $\{|z| = r\}$. Thus $g = f/B$ is analytic in $D(0,r)$, never vanishes in this disk. Thus $\log|g|$ is harmonic in $D(0,r)$, so by the mean value property
$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|d\theta.$$
Since $|g| = |f|$ on the circle of radius r and $|f(0)| = |g(0)||B(0)| = |g(0)| \prod_{n=1}^{N} \frac{r}{|z_n|}$, we get the desired equality. $\square$

Hardy space: for f holomorphic on the disk, and $1 \leq p < \infty$ let

$$\|f\|_p^p = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{-\theta})|^p d\theta.$$

This is a Banach space H^p. Some properties:

- Non-tangential limits exist a.e. on circle.
- Boundary limits $f^*$are in L^p
- $f(re^{i\theta}) \to f^*(e^{i\theta})$ in L^p norm
- f is the Poisson integral of its boundary values.
- Zeros of H^p function satisfy $\sum 1 - |z_n| < \infty$ (iff).
- H^∞ denotes bounded holomorphic functions on disk.

Lemma 10.2. *If $f \in H^1$ is not the constant zero function, then the boundary values f^* satisfy $|f^*(e^{i\theta})| > 0$ for almost every θ.*

This is a generalization of the much simpler fact that an analytic function on the disk cannot vanish on an interval of the circle.

Proof. Suppose $f \in H^1$ did have boundary values which vanish on a set of positive measure E on the boundary. By replacing $f(z)$ by $f(z)/z^k$ for some k, if necessary, we may assume $f(0) \neq 0$. Let

$$E_+ = \{|f^*| \geq 1\} = \{\log |f^*| \geq 0\},$$
$$E_- = \{|f^*| < 1\} = \{\log |f^*| < 0\}.$$

Then since $\log x \le x$ for $x \ge 1$, for any $0 < r < 1$,

$$\int_{E_+} \log|f(re^{i\theta})|d\theta| \le Mf(re^{i\theta})d\theta \le C\|f\|_1.$$

On the other hand, for any $\epsilon > 0$ there is an r_0 so that if $r > r_0$ then $|f(re^{i\theta})| < \epsilon$ on a set of θ's of measure $> |E|/2$. Thus

$$\int_{E_-} \log|f(re^{i\theta})|d\theta \le \frac{1}{2}H^1(E)\log\epsilon$$

Combining the two estimates we get

$$\lim_{r\to 1}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta| = -\infty,$$

which implies $|f(0)| = 0$, a contradiction, Thus $|f^*| > 0$ almost everywhere. $\square$

Theorem 10.3 (F. and M. Riesz, Version 1)**.** *Suppose μ is a finite measure on the unit circle. Then*

$$\int f(e^{i\theta})d\mu(\theta) = 0,$$

for every analytic function on the disk with continuous boundary values, iff $d\mu(\theta) = h(e^{i\theta})d\theta$ for some H^1 analytic function with $h(0) = 0$. In particular, μ is mutually absolutely continuous with respect to Lebesgue measure.

Proof. Suppose μ annihilates analytic functions. Let h be the Poisson integral of μ, then h is clearly harmonic and satisfies

$$\|h\|_{H^1} = \sup_r \int_0^{2\pi} |h(re^{i\theta})| d\theta < \|\mu\|.$$

In fact, h must be analytic since for all $n \geq 1$,

$$\begin{aligned}
\int_0^{2\pi} h(re^{i\theta})(re^{i\theta})^n d\theta &= \int_0^{2\pi} [\int_0^{2\pi} P_r(e^{i(\theta-\psi)}) d\mu(\psi)](re^{i\theta})^n d\theta \\
&= \int_0^{2\pi} [\int_0^{2\pi} P_r(e^{i(\theta-\psi)})(re^{i\theta})^n d\theta] d\mu(\psi) \\
&= r^n \int z^n d\mu(z) \\
&= 0,
\end{aligned}$$

Thus h is in the Hardy space $H^1(\mathbb{D})$, and so h is the Poisson integral of its boundary values, i.e., $d\mu = h^* d\theta$, as desired. Since μ kills constants, it must have mean value 0, hence $h(0) = 0$. The other direction follows easily from the Cauchy integral formula. $\square$

We say that a connected set is rectifiable if it has finite 1-dimensional measure. It is easy to check that if K is locally rectifiable, then it is locally connected. Thus if Ω is a simply connected domain with rectifiable boundary, $\partial\Omega$ is locally connected so by Carathéodory theorem any Riemann mapping of the disk onto Ω extends continuously to the boundary.

Theorem 10.4. *If Φ is univalent mapping of the unit disk onto a simply connected domain with rectifiable boundary, then $\Phi' \in H^1$. In particular, Φ' has finite, non-zero, non-tangential limits almost everywhere.*

Proof. Since $\int_0^{2\pi} |\Phi(re^{i\theta})| d\theta$ is the length of the image of circle $\{|z| = r\}$ we only have to check that these lengths remain uniformly bounded as $r \to 1$.

Since $\partial\Omega$ is rectifiable, it is locally connected, so Φ extends continuous to every boundary point. Thus every point in $\partial\Omega$ is the endpoint of a curve which is the image of a radius of the disk under Φ.

By the Moore triod theorem (Theorem 9.3) only a countable subset of $\partial\Omega$ can be the endpoints of three or more such rays.

Now cover $\{|z| = r\}$ by intervals $\{I_j\}$ of length $1 - r$ and centered at points $\{z_j\}$. Let $\{J_j\}$ be the radial projections of these intervals onto the unit circle.

Since $\omega(z_j, J_j, \mathbb{D})$ is clearly bounded away from zero, Beurling's estimate and Koebe's theorem implies

$$\mathcal{H}^1(\Phi(J_j)) \geq \mathrm{diam}(\Phi(J_j)) \geq C\mathrm{dist}(\Phi(z), \partial\Omega) \geq C(1-r)|\Phi'(z_j)|.$$

Moreover, Moore's theorem implies that $\sum_j \mathbf{1}_{\Phi(J_j)}(x) \leq 2$ except possibly on a countable set.

Since $\log f'$ is a Bloch function there is a uniform $C < \infty$ such that if f is univalent on the unit disk and $z_0 \in \mathbb{D}$, $D = D(z_0, \frac{1}{2}(1 - |z_0|)$, then

$$C^{-1} \leq \frac{\max_D |f'(z)|}{\min_D |f'(z)|} \leq C.$$

Therefore, if $d = 1 - |z_0|$ and I is the interval of length d centered at $z_0/|z_0|$,

$$\int_I |\Phi'(re^{i\theta})|d\theta \le C|I||\Phi'(z_j)|.$$

Using this and summing over the points $\{z_j\}$, we get

$$\begin{aligned}\int_0^{2\pi} |\Phi'(re^{i\theta})|d\theta &\le C(1-r)\sum_j |\Phi'(z_j)| \\ &\le C\sum_j \mathcal{H}^1(\Phi(J_j)) \\ &\le 2C\mathcal{H}^1(\partial\Omega). \quad \square\end{aligned}$$

Theorem 10.5 (F. and M. Riesz Theorem, Version 2)**.** *Suppose that Φ is a univalent map of $\mathbb{D}$ onto a simply connected domain Ω with rectifiable boundary. Suppose $E \subset \mathbb{T}$. Then $\mathcal{H}^1(E) = 0$ iff $\mathcal{H}^1(\Phi(E)) = 0$. In other words, harmonic measure on $\partial\Omega$ is mutually absolutely continuous to 1-dimensional Hausdorff measure.*

Proof. Since Φ is smooth inside the unit disk we have

$$\Phi(re^{i\theta_1}) - \Phi(re^{i\theta_2}) = \int_{\theta_1}^{\theta_2} \Phi'(re^{i\theta}) ire^{i\theta} d\theta,$$

for any $0 < r < 1$. Clearly the left hand side converges to

$$\Phi(e^{i\theta_1}) - \Phi(e^{i\theta_2}),$$

as $r \to 1$.

By Theorem 10.4 $\Phi' \in H^1$, so the radial maximal function of Φ' in is L^1. Thus we may use the Lebesgue dominated convergence theorem to deduce the left hand side converges to

$$\int_{\theta_1}^{\theta_2} \Phi'(e^{i\theta}) ire^{i\theta} d\theta,$$

Therefore,

$$\Phi(e^{i\theta_1}) - \Phi(e^{i\theta_2}) = \int_{\theta_1}^{\theta_2} \Phi'(re^{i\theta}) ie^{i\theta} d\theta,$$

for all θ_1, θ_2. This implies Φ is absolutely continuous on the unit circle. Thus if $E \subset \mathbb{T}$ has zero length we have

$$\mathcal{H}^1(\Phi(E)) \leq \int_E |\Phi'| d\theta = 0.$$

Conversely, if E has positive length, then the boundary values of Φ' are non-zero almost everywhere on E, so there is a subset $F \subset E$ so that Φ' only takes values in a ball $D_0 = D(x, |x|/2)$ on the set F.

Let W be the union on Stolz cones with vertices on F (and angle close to π) and let Γ be the boundary of W.

Then using the existence of non-tangential limits we can find a subarc of γ of Γ which hits F in positive measure and on which Φ' on takes values in D_0. Then Φ is bi-Lipschitz on this arc and so F is mapped to a set of positive length. This completes the proof. $\square$

11. McMillan's Twist Point Theorem

Suppose γ is an analytic Jordan curve defined on $[0,1]$ such that $\gamma(0) = 0$ and $\gamma(1) = 1$. If x is a point not on γ we can define the winding $w(x, \gamma)$ of γ around x by taking

$$\arg(0 - x) - \arg(1 - x),$$

where we take a continuous branch of $\arg(z - x)$ defined on γ.

Since the curve is analytic it has a well defined tangent at each endpoint, so we can also define the windings at the endpoints by truncating the curve and taking limits.

We can also define the change of argument of γ' as $\arg(\gamma'(0)) - \arg(\gamma'(1))$ where again we choose a continuous branch of arg.

Lemma 11.1.

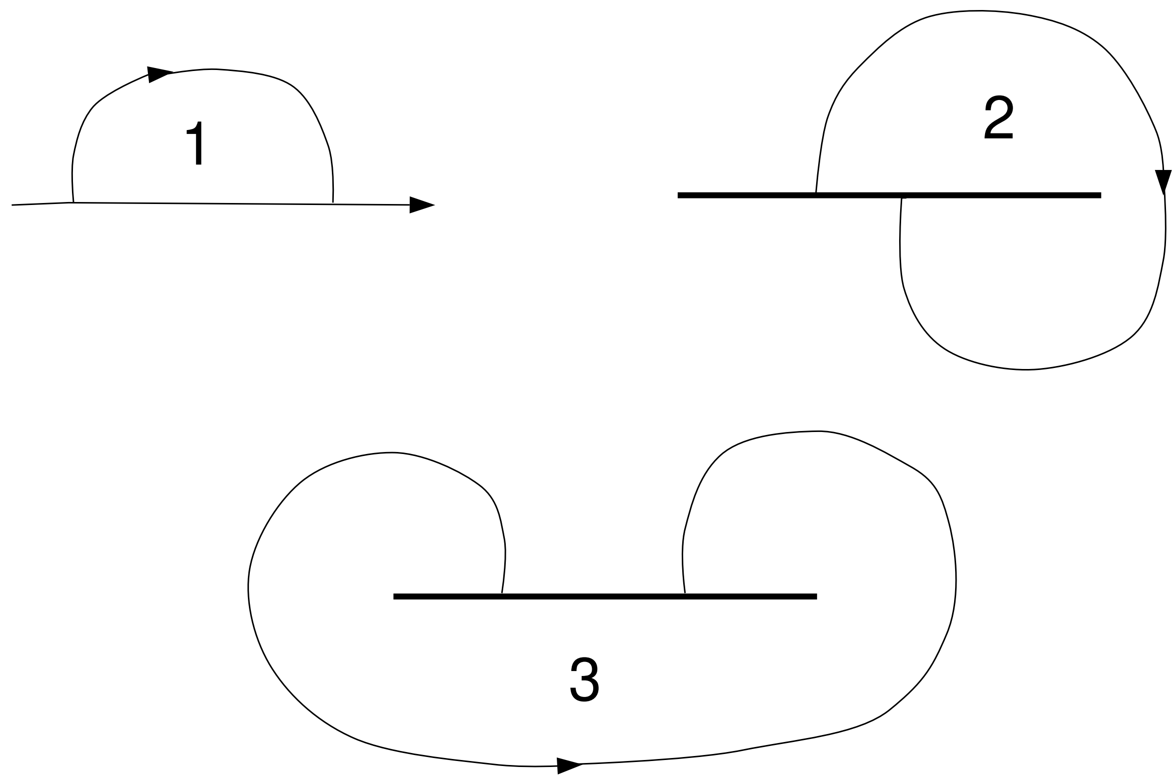
$$|2\pi[w(0,\gamma)+w(1,\gamma)]-[\arg(\gamma'(0)-\arg(\gamma'(1))]|\leq 4\pi.$$

Proof. If γ is a line segment then there is nothing to do. Otherwise, because of analyticity we may assume γ hits $[0,1]$ only finitely often. Replace γ by a homotopic smooth curve which intersects $[0,1]$ the least number of times among all curves homotopic to γ by a homotopy which is the identity in some neighborhood of 0 and 1 (thus 0 and 1 are fixed and so are the tangent direction at these points). The two quantities

$$w(0,\gamma)+w(1,\gamma), \qquad \arg(\gamma'(0)-\arg(\gamma'(1)),$$

are invariant under such homotopies (since they can only take a discrete set of values, they can not be changed under continuous deformations), so it suffices to prove the result for the new curve.

So we assume γ has the minimum number of intersections with $[0,1]$, say $\{0 = y_0, y_1, y_2, \ldots y_n = 1\}$, which map via γ^{-1} to points say $\{0 = x_0, x_1, x_2, \ldots x_n = 1\} \subset [0,1]$. Divide γ into oriented subarcs $\gamma_i = \gamma|_{[x_i, x_{i+1}]}$. Then γ_i is a Jordan arc with endpoints on $[0,1]$, but otherwise disjoint from $[0,1]$. The three possible types of arcs (up to a homeomorphism of the plane mapping $[0,1]$ to itself) are shown in the figure below. We denote the three types as 1,2 and 3.



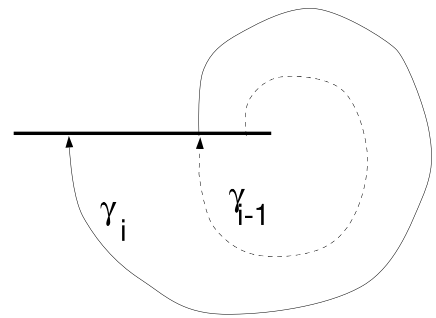
Except at the points 0 and 1 its makes sense to say that γ_i approaches it endpoints from either "above" or "below" $[0,1]$. For each x_i with $0 < i < n$ it is easy to see that γ_{i-1} and γ_i approach from different sides; otherwise there would be a smooth homotopy which removes the intersection at x_i, thus lowering the total number of intersections.

Similarly, none of these subarcs can be of type 1. Otherwise, using the fact that γ_{i-1} and γ_{i+1} approach x_i and x_{i+1} respectively from the opposite side we can homotopy γ_i across $[0,1]$ thus removing the intersections at both x_i and $x_i + 1$. Thus the subarcs of γ must be either type 2 or 3.

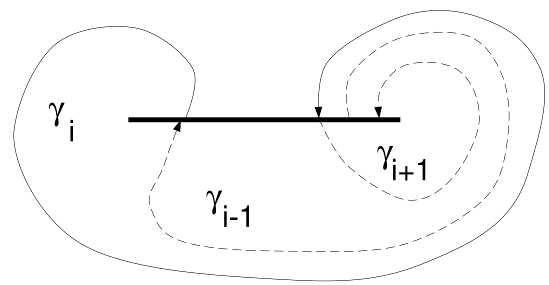
We say that γ_i is "good" if $y_{i+1} > y_i$ and is "bad" if $y_{i+1} < y_i$. We first claim that the minimality of γ implies there are no bad subarcs.

Suppose that there are bad arcs. We will consider two cases.

First suppose there is a bad arc of type 2. Then there is a bad arc γ_i of type 2 with endpoint y_i as close to 1 as possible (i.e., farthest to the right among all bad type 2 arcs). Then the preceding arc must be type 2 as well (see the figure), but this is only possible it is bad as well with a larger endpoint. This is a contradiction and implies that there are no bad arcs of type 2.

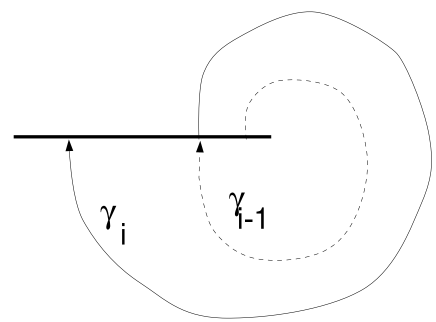


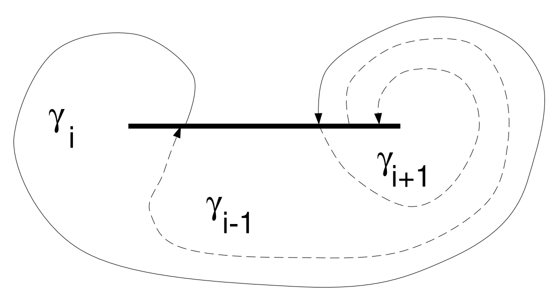
γ_i
γ_{i-1}



γ_i
γ_{i-1}
γ_{i+1}

Now suppose all the bad arcs are type 3. Choose γ_i to be the last bad arc in the ordering of γ. Then γ_{i+1} is good and must be type 2. Topologically, the only possibilities for γ_{i-1} are that it is type 1 or is a bad arc of type 2. Both are ruled out by hypothesis, so we deduce there are no bad arcs.





We can now finish the proof. Replace γ by a homotopic arc where the homotopy is the identity except in small neighborhoods of each intersection point y_i, $0 < 1 < n$ and in those neighborhoods the curve is changed so that γ' is horizontal and points to the right as γ crosses $[0, 1]$. Then for each subarc γ_i the tangents points the same direction at either endpoint. Thus the change in argument of γ' along each γ_i is a multiple of 2π. There are only a few cases and it is each of them it is trivial to check that the change in argument of γ' is 2π times $w(0, \gamma_i) + w(1, \gamma_i)$ where $w(z, \gamma)$ denotes the change in $\arg(y - z)$ for some branch of the argument function defined on γ.

By summing over i we now get that the change of argument of γ' on $[x_1, x_{n-1}$ is equal to 2π times $w(0, \gamma|_{[x_1, x_{n-1}]}) + w(1, \gamma|_{[x_1, x_{n-1}]})$. Adding in the two end intervals γ_0 and γ_{n-1} can only alter the equality by a factor of at most 2π each so we obtain the lemma. $\square$

If Ω is simply connected we say $x \in \partial\Omega$ is an *inner tangent* point of Ω if for any $\epsilon > 0$ x is the vertex of a cone in Ω with angle $\pi - \epsilon$, but is the vertex of no cone with angle $> \pi$. We say that x is a *cone point* if it is the vertex of some cone in Ω.

Lemma 11.2. *If Ω is simply connected then the set of cone points has σ-finite 1-dimensional measure and almost every cone point is an inner tangent point.*

Proof. By considering cones with rational angles and radius, we can write the set of cone points as a countable union of sets, each of each are the vertices of cones in Ω with fixed side directions and diameters. It clearly suffices to prove the claims for any such set.

Let $F \subset \partial\Omega$ be the set of points $x \in \partial\Omega$ so that

$$W_x = \{x + z : |z| < r, |\arg(-iz)| \leq \epsilon\}\} \subset \Omega.$$

By again dividing into a countable number of subsets we may assume that F is contained in the rectangle $R = \{z : |\mathrm{Im}(z)| < r/10, |\mathrm{Re}(z)| < r\epsilon/10\}$.

Let $W = \cup_{x\in F} W_x$. Then $R \cap \partial W$ is graph of a Lipschitz function (norm depending only on ϵ), and hence is rectifiable. Since it contains F, F has finite 1-dimensional measure and a.e. point of F is a tangent point of the arc.

Thus almost every point of F is an inner tangent of W and hence of Ω. $\square$

A point x is called a twist point for Ω if for any branch of $\arg(z-x)$ defined on Ω we have

$$\limsup_{z \to x, z \in \Omega} \arg(z-x) = \infty,$$

and

$$\liminf_{z \to x, z \in \Omega} \arg(z-x) = -\infty.$$

Thus to approach a twist point x through Ω we must "twist around" x arbitrarily far in both directions. It is difficult to draw a twist point on the boundary for a point with one twist), but we shall see later that such things can exist. For example, harmonic measure on the von Koch snowflake gives full measure to the twist points (see the exercises).

Theorem 11.3 (McMillan's Twist Point Theorem)**.** *If Ω is a simply connected domain then almost every point on $\partial\Omega$ (with respect to harmonic measure) is either an inner tangent point or a twist point.*

Proof. The proof is essentially Plessner's theorem. Let $\Phi : \mathbb{D} \to \Omega$ be a Riemann mapping and apply Plessner's theorem to the derivative Φ'. Plessner's theorem says that we can write $\mathbb{T} = E_0 \cup E_1 \cup E_2$ where E_0 has measure zero, Φ' has non-zero non-tangential limits at every point of E_1 and Φ' is non-tangentially dense at every point of E_2.

Clearly the set $\Phi(E_1) \cup \Phi(E_2)$ has full harmonic measure on $\partial\Omega$.

Moreover, we saw in the last section that $\Phi(E_1) \subset \partial\Omega$ consists of inner tangents. Thus if we can show that $\Phi(E_2)$ consists of twist points almost everywhere (with respect to harmonic measure) we will be done.

In fact, all we have to do is produce a sequence of points $z_n \in \Omega$ with $\arg(z_n - x) \to +\infty$ and another with arguments tending tending to $-\infty$.

To prove this, suppose it fails. Then there is a set F of positive measure on $\mathbb{T}$ on which Φ has nontangential limits, Φ' is non-tangentially dense but

$$\arg[\Phi(re^{i\theta}) - \Phi(e^{i\theta})],$$

remains bounded above as $r \to 1$. We will show this is impossible.

Since Φ' is non-tangentially dense on F, so is $\log \Phi' = \log |\Phi'| + i \arg \Phi'$. Hence $\arg \Phi'$ must be non-tangentially unbounded above and below by Plessner's theorem.

There is a M so that the images of the rays $[0, e^{i\theta})$ have length less than M except on a set of measure $|F|/2$. So be replacing F by a set of half the measure we may assume of the associated rays have bounded length.

Let $\theta_0 \in F$ be a point of density and consider a sequence $\{z_n\} \to 1$ so that $\arg \Phi(z_n) \to +\infty$.

Fix a large N and choose an n so large that $\arg \Phi'(z_n) > 4\pi(N+1)$. Let $\gamma = \Phi([0, z_n)$ be the image of the radial segment from 0 to z_n.

The change of argument of γ' from one endpoint to the other is $|\arg \Phi'(z_n)|$. By Lemma 11.1 the curve γ "winds around" one of its endpoints at least N times.

It does not wind around the origin arbitrarily often since Φ has a non-zero radial limit at x.

More precisely, we proved earlier that if the argument of γ' changes by more than $4\pi(N+1)$ between its two endpoints then

$$w(a,\gamma) + w(b,\gamma) \geq 2N,$$

By rescaling we may assume $\Phi'(0) = 1$. So by Koebe's theorem there is a disk $D_0 \subset \Omega$ of diameter similar to 1 that γ never re-enters once it leaves. Moreover, γ does not wind around 0 inside this disk.

In order to wind around $a = \Phi(0)$, γ must wind around the disk and since it has length at most M, it can wind around 0 at most $M/2\pi$ times. If N was chosen large enough, we see that most of the winding of γ must be around the point $b = \Phi(rx)$.

We would like to deduce that the curve γ also winds around the point $\Phi(e^{i\theta_0})$, but this may not be true. Instead we will show that there is another point in F near b around which the curve does wind.

Recall that θ_0 was chosen to be a point of density of F. So if $|z_n|$ is close enough to 1, more that half the interval of length $1 - |z_n|$ centered at $e^{i\theta_0}$ consists of points in F.

We can find a point x in F so that x can be connected to b in Ω by a curve of length at most $C\text{dist}(b, \partial\Omega)$. Just as we argued for the origin above, this curve cannot wind around b more than a bounded number of times. This implies that the winding of γ around b and around x can differ by at most a bounded factor. Thus the winding of γ around x must be very large. This contradicts the assumption that $x \in F$, proving the theorem. $\square$

Corollary 11.4. *Suppose Ω is simply connected and let E be a subset of the cone points on $\partial\Omega$. Then E has positive harmonic measure iff it has positive length.*

Proof. First suppose E has positive length. Then pass to a subset of positive measure contained in a rectangle R exactly as in the proof of Lemma 11.2 and let W be the union of cones constructed there. Then $W_1 = W \cap R$ is a rectifiable subdomain of Ω which hits E in positive length. By the F. and M. Riesz theorem Theorem 10.5 E has positive harmonic measure in W_1 and hence in Ω by the maximum principle.

Next suppose E has positive harmonic measure. Let Φ be a Riemann mapping of $\mathbb{D}$ to Ω. Then $F = \Phi^{-1}(E)$ has positive length and Φ' has a non-zero non-tangential limit at almost every point of F. Therefore we can find a $\alpha > 0$, $M < \infty$ and a subset $F_0 \subset F$ of positive measure so that $|\Phi'| \leq M$ on every Stolz cone of angle α with vertex in F_0.

Let W_2 be the union of these cones. Then W has rectifiable boundary, and $|\Phi'|$ is bounded on ∂W_2, so $\Phi(W_2)$ is a subdomain of Ω with rectifiable boundary. By Theorem 10.5 again, $\Phi(F_0)$ has positive length (since it has positive harmonic measure) and hence so does E. $\square$

12. Singular harmonic measures

Two measures μ and ν are called *mutually absolutely continuous* if they have the same null sets, i.e., $\mu(E) = 0$ if and only if $\nu(E) = 0$. The measures are called *mutually singular* if each is supported on a null set of the other, i.e., there is a set E with $\mu(E) = 0$ but $\nu(E^c) = 0$.

Suppose Γ is a closed Jordan curve which divides the Riemann sphere $\mathbb{C}^\infty$ into two simply connected domains Ω_1 and Ω_2. If we choose two points on the same side of Γ then the two harmonic measures will be mutually absolutely continuous with respect to each other. But what happens if we choose points from opposite sides of the curve? Can the two measures be mutually singular?

We have already see that harmonic measure for a simply connected domain Ω is mutually absolutely continuous with $\mathcal{H}^1$ on the set of inner tangents. A point of Γ is called a tangent point if it is an inner tangent for each of the two complementary domains. Thus the ω_1 and ω_2 are mutually absolutely continuous when restricted to the set tangent points of Γ. The following result says they are mutually singular on the rest of Γ.

Theorem 12.1. *Suppose $z_1 \in \Omega_1$, $z_2 \in \Omega_2$ and let ω_1, ω_2 denote the corresponding harmonic measures. Then ω_1 and ω_2 are mutually absolutely continuous on the set of tangent points of Γ and are mutually singular on the rest of Γ. In particular, $\omega_1 \perp \omega_2$ iff $\mathcal{H}^1(\textit{tangent points}) = 0$.*

This result follows from the proof of Makarov's theorem and an estimate of harmonic measure due to Beurling.

The part of the proof of Makarov's theorem we need can be summarized as

Lemma 12.2. *Suppose Ω is simply connected and let ω be harmonic measure with respect to some point in Ω. If $T \subset \partial\Omega$ denotes the set of inner tangents then there is an $F \subset \partial\Omega \setminus T$ $\omega(F) = \omega(\partial\Omega \setminus T)$ such that for any $M > 0$ there is a disjoint covering of F by disks $\{D_j\}$ with $\omega(D_j) \geq M|D_j|$.*

The estimate of Beurling we want is

Lemma 12.3. *Suppose* Γ *is a closed Jordan curve dividing the sphere into two simply connected domains* Ω_1, Ω_2*. Let* $z_i \in \Omega_i$ *satisfy* $\mathrm{dist}(z_i, \partial\Omega_1)$ *for* $i = 1, 2$*. Then there is a* $C < \infty$ *so that for any disk* D*,*

$$\omega_1(D)\omega_2(D) \le C|D|^2.$$

Proof. This follows from an estimate of harmonic measure known as the Ahlfors distortion theorem. Suppose Ω is simply connected and $x \in \partial\Omega$. For each $t > 0$, let $\theta(t)$ denote the length of the longest arc in $\Omega \cap \{|z - x| = t\}$. Then if $\mathrm{dist}(z_0, \partial\Omega) \ge 1$, the distortion theorem says

$$\omega(z_0, D(x, r), \Omega) \le C \exp(-\pi \int_r^1 \frac{dt}{\theta(t)}).$$

To apply this to our situation, let $x \in \Gamma$ and let $\theta_i(t)$ be the function corresponding to Ω_i for $i = 1, 2$. The multiplying the estimates for each domain gives

$$\omega_1(D)\omega_2(D) \leq C \exp(-\pi \int_{|D|}^{1} (\frac{1}{\theta_1(t)} + \frac{1}{\theta_2(t)})dt).$$

Since Ω_1 and Ω_2 are disjoint, $\theta_1 + \theta + 2 \leq 2\pi t$ and so a simple calculus exercise shows that $\theta_1^{-1} + \theta_2^{-1} \geq 2/\pi t$. Thus

$$\omega_1(D)\omega_2(D) \leq C \exp(-\pi \int_{|D|}^{1} \frac{2\pi t}{d} t) = C|D|^2,$$

as desired. $\square$

Proof. We can now prove the singularity of harmonic measures. Divide Γ into:

(1) Tangent points,

(2) Twist points,

(3) Inner tangents for Ω_1 which are not inner tangents for Ω_2,

(4) Inner tangents for Ω_2 which are not inner tangents for Ω_1,

(5) Everything else.

We already know that the harmonic measures are mutually absolutely continuous on (1) and that (5) has zero harmonic measure from both sides. Moreover, ω_2 gives zero mass to (3), so the measures are singular on that set. Similarly for (4). Therefore all we need to show is that the measures are singular on the twist points.

Choose a large n and by the first lemma choose disjoint disks $\{D_j^n\}$ so that

$$\omega_1(D_j^n) \geq n|D_j^n|, \quad \Rightarrow \quad |D_j^n|^2 \leq \omega_1(D_j^n)^2/n^2,$$

$$\omega(\cup_j D_j^n) = \omega_1(\text{twist points}).$$

Then if $F = \cap_n \cup_{k>n} \cup_j D_j^k$, we have

$$\omega_1(F) = \omega_1(\text{twist points}),$$

but by Beurling's estimate,

$$\omega_2(F) \ \leq \ \sum_j \frac{C|D_j^n|^2}{\omega_1(D_j^n)} \leq \frac{C}{n^2} \sum_j \omega_1(D_j^n) \leq \frac{C}{n^2} \to 0.$$

Thus the measures are singular on the twist points. $\square$