Introduction to Transcendental Dynamics

Christopher J. Bishop
Department of Mathematics
Stony Brook University

Intentions

Despite many previous opportunities and a brush with conformal dynamics in the form of Kleinian groups, I had never worked on the iteration of holomorphic functions until April 2011 when I met Misha Lyubich on the stairwell of the Stony Brook math building he told me that Alex Eremenko had a question for me. Alex was visiting for a few days and his question was the following: given any compact, connected, planar set K and an $\epsilon > 0$, is there a polynomial $p(z)$ with only two critical values, whose critical points approximate K to within ϵ in the Hausdorff metric?

It turns out that we can assume the critical values are ± 1 and then $T = p^{-1}([-1, 1])$ is a finite tree in the plane and Alex's question is equivalent to asking if such trees are dense in all compact sets. Such a tree is conformally balanced in the sense that every edge gets equal harmonic measure from ∞ and every subset of an edge gets equal harmonic measure from either side. This makes the existence of such trees seem unlikely, or at least requiring very special geometry, but I was able to show that Alex's conjecture was correct by building an approximating tree that is balanced in a quasiconformal sense and then "fixing" it with the measurable Riemann mapping theorem.

When I asked for the motivation behind the question, Alex explained that entire functions with a finite number of critical points play an important role in transcendental dynamics because they mimic certain properties of polynomials, e.g., Dennis Sullivan's "no wandering domains" theorem can be extended to such functions. He wanted to know if the geometry of polynomials of high degree but with a low number of critical points was somehow "special". At least in the case of this particular question, the answer is no.

This discussion led to an analogous question for entire functions. If f is entire and has only two critical points, say ± 1, then $T = f^{-1}([-1, 1])$ is an unbounded tree in the plane. Conversely, given an unbounded tree in the plane can we approximate it by a tree of this form? We need to impose a few conditions, but the basic answer is yes, any "reasonable" tree T can be approximated. The "reasonable" conditions include that the tree has uniformly bounded degrees, the edges are uniformly C^2 arcs, adjacent edges have comparable Euclidean lengths and one more technical looking condition. Since T is connected and unbounded, its complementary components are simply connected and can be conformally mapped to a half-plane with ∞ mapping to infinity. Pulling back Lebesgue measure on the boundary of the half-plane to T defines a conformal length for each side of each edge. We require that each side of each edge in the tree has conformal length that is uniformly bounded away from zero. Given this, we can approximate the tree in a precise sense by a tree T' of the form $f^{-1}([-1,1])$ where f is entire and has exactly two critical values.

With this construction in hand, one can create a large number of new entire functions with finite singular set $S(f)$ (critical values and finite asymptotic values). This collection is called the Speiser class. With a small variation we can also build functions in the larger Eremenko-Lyubich class (bounded singular set). One of the first applications was to build an entire function in the Eremenko-Lyubich that has a wandering domain; as noted above, this is impossible in the Speiser class, but was previously known for more general entire functions. Another application of the ideas (though not of the precise method) led to the construction of an entire function whose Julia set has Hausdorff dimension 1. Baker had proven in 1975 [**?**] that the Julia set of an entire function always contains an non-trivial continuum, so its dimension is always ≥ 1, and Stallard [**130**], [**132**] had given examples with dimension $1+\epsilon$ for any $\epsilon > 0$. Furthermore, the new example has packing dimension 1 and is the first entire function known to have packing dimension < 2. At this writing, no known examples have packing dimension strictly between 1 and 2 (but I hope this will change shortly).

In the course of thinking about my examples, I looked at many papers in the field and benefited greatly from a number of well written surveys, especially by Bergweiler and Schleicher. However, I felt that I needed to learn some of the basic theory in more detail and the standard way to do this quickly is to offer to each a graduate course on the topic, which I volunteered to due during Spring 2013, and followed by a sequel during Spring 2014. These notes are my attempt to record and organize some of the topics I hope to cover.

First, there are topics from complex analysis that are not usually covered in the standard first year graduate course, such as the hyperbolic and spherical metrics, the uniformization theorem, Koebe's $\frac{1}{4}$-theorem, normal families, the Ahlfors 5 islands theorem, Arakelian's approximation theorem (or is it Arakeljan?), Hausdorff and packing dimensions, extremal length, harmonic measure estimates. We will also need a certain amount of quasiconformal theory, especially the measurable Riemann mapping theorem, and some extensions of to more general homeomorphisms.

Second, there are the topics from transcendental dynamics that are the real goal of the course. I will interpret "transcendental dynamics" as the iteration of entire functions. One can also develop parallel theories for holomorphic functions mapping $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ to itself, for meromorphic functions on the plane, or even for maps between domains on a Riemann surface, but I will limit myself to entire functions, except in a few cases where we can obtain a more general result for no extra work or need a more general result for an application to entire functions.

If f is entire let $\{f^n\}$ denote the iterates of f. The basic objects of study are the Fatou set

$$\mathcal{F}(f) = \{z : \{f^n\} \text{ is a normal family on a neighborhood of } z\},$$

its complement the Julia set, $\mathcal{J}(f)$. The escaping set

$$I(f) = \{z : f^n(z) \to \infty\},$$

plays a crucial role in the theory; $\mathcal{J}(f) = \partial I(f)$ for all entire functions and $\mathcal{J}(f) = \overline{I(f)}$ in many cases of interest (e.g., the Eremenko-Lyubich class). $I(f)$ often has an interesting internal structure, due in part to the ability to define subsets in terms of the rate of escape to ∞. We will spend a great deal of effort trying to understand its geometry. Indeed, the first "dynamical" result we prove (after some background material) is that $I(f) \neq \emptyset$, a fundamental result of Eremenko.

Some of the other results we will cover include:

- The Julia set is non-empty, the closure of the repelling fixed points and the boundary of the escaping set. It is either the whole plane or nowhere dense.
- The Fatou set has no unbounded multiply connected components. Thus the Julia set can't be totally disconnected and must have dimension ≥ 1.

• Multiply connected components of the Fatou set must be wandering domains. Examples exists with all connectivities $2, 3, \dots, \infty$.

• Properties of various subsets of the escaping set defined in terms of rates of escape, especially the "fast escaping set".

• Special results for the Speiser and Eremenko-Lyubich classes, e.g., the Julia set contains the escaping set, the Hausdorff dimension of the Julia set is strictly bigger than 1, but the escaping set can have dimension 1. The Julia set has packing dimension 2. How to construct such functions with prescribed geometry near infinity.

• The Julia set of $\exp(z)$ is the whole plane. Examples where the Julia set has positive area, but is not the whole plane, where it has zero area but Hausdorff dimension 2, examples of dimension < 2.

• Periodic components of the Fatou set of an entire functions include all the possibilities for polynomial plus one other. There can be Baker domains, i.e., unbounded periodic domains that iterate to ∞, a sort of analog to a the petal associated to a rationally neutral fixed point. We will give examples, estimate the escape rate in such a domain and show that they force the presence of nearby singular points (although the Baker domain itself need not contain any singular points).

• Progress on Eremenko's question: is every component of the escaping set is unbounded? Among other partial results we know this is false for path components, true for the closure of the escaping set, and there is always at least one unbounded component.

• Progress on Baker's conjecture: an entire function that grows slower than $\exp(\sqrt{z})$ has only bounded Fatou components. This is known to be true under various extra conditions such as very slow growth or some regularity on growth (e.g., the maximum modulus on the circle of radius r is "nice" as a function of r).

Well, this should be enough to get started. In general, I am most interested in topics where the behavior for entire functions contrasts with that for polynomials. For example, the "simplest" examples of transcendental Julia sets have dimension 2; it requires work to build one of dimension < 2 and more work to reach the minimum value 1. For polynomials, the situation is reversed; it is easy to build Julia sets that are small Cantor sets but required sophisticated methods to construct examples with dimension 2 [**128**], [**129**] or positive area [**38**], [**39**].

The prerequisite for reading these notes is to be Lars Ahlfors, or failing this, to have read Ahlfors' book "Complex Variables", or at the very least, to believe results quoted from this book.

At present the notes are incomplete, missing many references and contain more mistakes than I care to contemplate. Use extreme caution when reading them and please let me know of the errors you find.

Chris Bishop
Stony Brook, NY
March 2014

Contents

CHAPTER 1

The hyperbolic metric

The study of conformal dynamics is made possible in great part by the use of conformal invariants, i.e., numerical values that can be associated to a certain geometric configurations and that remain unchanged (or at least change in predictable ways) under the application of conformal or holomorphic maps. In the course of these notes we shall see several examples: hyperbolic distance, harmonic measure and extremal length, to name a few. In this chapter we start with the the definition and properties of the hyperbolic metric; first on the disk, then extending it to simply connected domains via the Riemann mapping theorem and then to general planar domains (with a few exceptions) via the uniformization theorem.

For simply connected domains, the principal tool is the Koebe $\frac{1}{4}$-theorem that allows us to estimate hyperbolic distances, up to a bounded factor, by Euclidean quantities. For more general domains, similar estimates are obtained from comparisons with the hyperbolic metric for the twice puncture plane. Also of critical importance is Schwarz lemma: holomorphic maps never increase hyperbolic distances. The Schwarz lemma has a number of important implications in dynamics, e.g., it limits how quickly a point can iterate to the boundary of a Fatou component (assuming all iterates remain in the same component), and we shall end the chapter by proving that every non-trivial entire function has a point that iterates to ∞, i.e., the escaping set is non-empty.

1.1. Schwarz's lemma

The **hyperbolic metric** on $\mathbb{D}$ is given by $d\rho(z) = |dz|/(1-|z|^2)$. This means that the hyperbolic length of a rectifiable curve γ in $\mathbb{D}$ is defined as

(1) $$\ell_\rho(\gamma) = \int_\gamma \frac{|dz|}{1-|z|^2},$$

and the hyperbolic distance between two points $z, w \in \mathbb{D}$ is the infimum of the lengths of paths connecting them (we shall see shortly that there is an explicit formula for this distance in terms of z and w). In many sources, there is a "2" in the numerator of (1), but we follow [**59**], where the definition is as given in (1). For most things, this makes no difference, but the reader is warned that some of our formulas may differ by a factor of 2 from the analogous formulas in some papers and books.

We define the **hyperbolic gradient** of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ as

$$D_H^H f(z) = |f'(z)| \frac{1 - |z|^2}{1 - |f(z)|^2}.$$

More generally, given a map f between metric spaces (X, d) and (Y, ρ) we define the gradient at a point z as

$$D_d^\rho f(z) = \limsup_{x \to z} \frac{\rho(f(z), f(x))}{d(x, z)}.$$

The use of the word "gradient" is not quite correct; a gradient is usually a vector indicating both the direction and magnitude of the greatest change in a function. We use the term in a sense more like the term "upper gradient" that occurs in metric measure theory to denote a function $\rho \geq 0$ that satisfies

$$|f(b) - f(a)| \leq \int_\gamma \rho ds,$$

for any curve γ connecting a and b. I hope that the slight abuse of the term will not be confusing.

In these notes, the most common metrics we will use are the usual Euclidean metric on $\mathbb{C}$, the spherical metric

$$\frac{ds}{1 + |z|^2},$$

on the Riemann Sphere, $\widehat{\mathbb{C}}$ and the hyperbolic metric on the disk or on some other hyperbolic planar domain (these will be defined in Section 1.4). To simplify (?) notation, we use E, S and H to denote whether we are taking a gradient with respect to Euclidean, Spherical or Hyperbolic metrics. For example if $f : U \to V$, the symbol $D_H^H f$ means that we are taking a gradient from the hyperbolic metric on U to the hyperbolic metric on V (assuming the domains are clear from context; otherwise we write D_U^V or $D_{\rho_U}^{\rho_v}$ it we need to be very precise).

In this notation, the spherical derivative of a function, usually denoted

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2},$$

is written $D_E^S f(z)$ since it is a limit of quotients where the numerator is measured in the spherical metric and the denominator is measured in the Euclidean metric. Similarly D_H^S denotes a gradient measuring expansion from a hyperbolic to the spherical metric. This particular gradient will be important in Chapter 2 when we characterize conformally invariant normal families.

A **linear fractional transformation** is a map of the form

$$z \to a + bxc + dz,$$

where $a, b, c, d \in \mathbb{C}$. These exactly the 1-to-1, holomorphic maps of the Riemann sphere to itself. Such maps are also called **Möbius transformation**

EXERCISE: Show that the linear fractional transformations that map $\mathbb{D}$ 1-to-1, onto itself are exactly those of the form $z \to \lambda(z-a)/(1-\bar{a}z)$ where $|a| < 1$ and $|\lambda| = 1$.

LEMMA 1.1.1. *Möbius transformations of $\mathbb{D}$ to itself are isometries of the hyperbolic metric.*

PROOF. When f is a Möbius transformation of the disk we have

$$f(z) = \frac{z-a}{1-\bar{a}z},$$

$$f'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

Thus

$$\begin{aligned}
D_H^H f(z) &= \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|f(z)|^2} = \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|\frac{z-a}{1-\bar{a}z}|^2} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2-|z-a|^2} = \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z)(1-a\bar{z})-(z-a)(\bar{z}-\bar{a})} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z-a\bar{z}+|az|^2)-(|z|^2-a\bar{z}-z\bar{a}+|a|^2)} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1+|az|^2-|z|^2-|a|^2)} = .
\end{aligned}$$

Note that

$$\ell_\rho(f(\gamma)) \leq \int_\gamma D_H^H f(z) \frac{|dz|}{1-|z|^2}.$$

Thus Möbius transformations multiply hyperbolic length by at most one. Since the inverse also has this property, we see that Möbius transformation preserve hyperbolic length. $\square$

The segment $(-1,1)$ is clearly a geodesic for the hyperbolic metric and since isometries take geodesics to geodesics, we see that geodesics for the hyperbolic metric are circles orthogonal to the boundary.

On the disk it is convenient to define the pseudo-hyperbolic metric

$$T(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

The hyperbolic metric between two points can then be expressed as

$$\rho(w,z) = \frac{1}{2}\log\frac{1+T(w,z)}{1-T(w,z)}. \tag{2}$$

On the upper half-plane the corresponding function is

$$T(z,w) = \left|\frac{z-w}{w-\bar{z}}\right|,$$

and ρ is related as before.

EXERCISE: Show a hyperbolic ball in the disk is also a Euclidean ball, but the hyperbolic and Euclidean centers are different (unless they are both the origin). Compute the Euclidean center and radius of a hyperbolic ball of radius r centered at z in $\mathbb{D}$.

LEMMA 1.1.2 (Schwarz's Lemma). *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ then $|f'(0)| \leq 1$ with equality iff f is a rotation. Moreover, $|f(z)| \leq |z|$ for all $|z| < 1$, with equality for $z \neq 0$ iff f is a rotation.*

PROOF. Define $g(z) = f(z)/z$ for $z \neq 0$ and $g(0) = f'(0)$. This is a holomorphic function since if $f(z) = \sum a_n z^n$ then $a_0 = 0$ and so $g(z) = \sum a_n z^{n-1}$ has a convergent power series expansion. Since $\max_{|z|=r}|g(z)| \leq \frac{1}{r}\max_{|z|=r}|f| \leq \frac{1}{r}$. By the maximum principle $|g| \leq \frac{1}{r}$ on $\{|z| < r\}$. Taking $r \nearrow 1$ shows $|g| \leq 1$ on $\mathbb{D}$ and equality anywhere implies g is constant. Thus $|f(z)| \leq |z|$ and $|f'(0)| = |g(0)| \leq 1$ and equality implies f is a rotation. $\square$

In terms of the hyperbolic metric this says that

$$\rho(f(0), f(z)) = \rho(0, f(z)) \leq \mathbb{H}_r(0, z),$$

which shows the hyperbolic distance from 0 to any point is non-increasing. For an arbitrary holomorphic self-map of the disk f and any point $w \in \mathbb{D}$ we can always choose Möbius transformations τ, σ so that $\tau(0) = w$ and $\sigma(f(w)) = 0$, so that $\sigma \circ f \circ \tau(0) = 0$. Since Möbius transformations are hyperbolic isometries, this shows

COROLLARY 1.1.3. *If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic then $\rho(f(w), f(z)) \leq \rho(w, z)$.*

Another formulation is

COROLLARY 1.1.4. *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic then $D_H^H f(z) \leq 1$.*

COROLLARY 1.1.5. *If $f : U \to V$ is conformal and $z_0 = f(z_0) \in U \subset V$, then $|f'(z_0)| \geq 1$ with equality iff $U = V$.*

EXERCISE: Show that the only isometries of the hyperbolic disk are Möbius transformations and their reflections across $\mathbb{R}$.

Since $f(z) = z^2$ maps the disk to itself, it strictly contracts the hyperbolic metric; a more explicit computation shows

$$D_H^H f(z) = |2z| \frac{1 - |z|^2}{1 - |z|^4} = \frac{2|z|}{1 + |z|^2} < 1.$$

Thus $g(z) = \sqrt{z}$ is locally an expansion of the hyperbolic metric, at least on a subdomain $W \subset \mathbb{D}$ where it has a well defined branch. For $z \neq 0$,

$$D_H^H g(z) = |\frac{1}{2\sqrt{z}}| \frac{1 - |z|^2}{1 - |z|} \geq \frac{1 + |z|}{2\sqrt{z}}. \tag{3}$$

For $z = 0$

$$D_H^H g(0) = \limsup_{z \to 0} \frac{\rho(0, \sqrt{z})}{\rho(0, z)} = \infty.$$

Similarly, if $\alpha > 0$, then the map $p_\alpha(z) = z^\alpha$ sends $\mathbb{D}$ to $\mathbb{D}$, and satisfies

$$D_H^H p_\alpha(z) = \alpha |z|^{\alpha - 1} \frac{1 - |z|^2}{1 - |z|^{2\alpha}} < 1, \tag{4}$$

hence is a hyperbolic contraction if $\alpha > 1$ and an expansion if $\alpha < 1$.

EXERCISE: If $a > 1$ and $0 < r < 1$ show that

$$\alpha r^{a-1} \frac{1 - r^2}{1 - r^{2a}} < 1, \tag{5}$$

if $a > 1$ and $0 < r < 1$.

Another consequence of the Schwarz lemma that we will use later is the Borel-Carathéodory theorem: Borel and Carathéodory:

LEMMA 1.1.6. *If $g = u + iv$ is holomorphic on $\mathbb{D}$ and $g(0) = 0$ then*

$$\max_{|z|<1/2} |g(z)| \leq 2 \max_{|z|<1} u(z).$$

PROOF. Without loss of generality we may assume that $u < 1$ on $\mathbb{D}$, so that g maps the disk into the half-plane $H = \{x + iy : x < 1\}$. If $\tau(z) = z/(2 - z)$ is a Möbius transformation that takes H to $\mathbb{D}$ and fixes 0, then by the Schwarz lemma, $|\tau(g(z))| \leq |z|$. Therefore g maps $\mathbb{D}$ into $\tau^{-1}(D(0, \frac{1}{2})) \subset D(0, 2)$. $\square$

EXERCISE: Show, more generally, that

$$\sup_{|z|=r} |g(z)| \leq \frac{2r}{1-r} \sup_{|z|<1} u(z) + \frac{1+r}{1-r}|g(0)|.$$

this is the "full strength" version of the Borel-Carathéodory theorem.

1.2. Uniformization for planar domains

let $p : E \to B$ be continuous and surjective. An open set $U \subset B$ is **evenly covered** if the inverse image $p^{-1}(U)$ can be written as a disjoint union of sets V_α so that p restricted to each V_α is a homeomorphism onto U. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a **covering map**. A space X is **simply connected** if it is path connected and if its fundamental group is trivial, i.e., every closed loop in X can be homotoped to a point.

LEMMA 1.2.1 ([**102**], Lemma 8.4.1). *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Any path $f[0, 1] \to B$ beginning at b_0 has a unique lift to a path $\tilde{f}$ in E beginning at e_0.*

LEMMA 1.2.2 ([**102**], Exercise 8.4.12(a)). *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $f : Y \to B$ be continuous with $f(y_0) = b_0$. If Y is locally path connected and simply connected then f can be lifted uniquely to a continuous map $\tilde{f} : (Y, y_0), \to (E, e_0)$.*

The theory of covering spaces says that every Riemann surface has a universal covering surface that is also a Riemann surface. Koebe's uniformization theorem says that there are only three simply connected Riemann surfaces (up to conformal isomorphism): $\mathbb{D}$, $\mathbb{C}$ and $\widehat{\mathbb{C}}$. Any other Riemann surface (and there are many) is the quotient of one of these by a discrete group of Möbius transformations. An element of such group can't have a fixed point, and this implies that the sphere covers only itself and the plane covers only genus 1 tori and the once punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Every other Riemann surface is the quotient of the disk by a Fuchsian group (i.e., a discrete group of Möbius transformations acting on $\mathbb{D}$).

We will not prove the complete uniformization theorem here, although there are proofs using potential theory that take only about a dozen pages (e.g. see Don Marshall's paper [**91**]). However, we will give the proof for hyperbolic planar domains, a case that we will use throughout these notes. A planar domain Ω is called **hyperbolic** if $\mathbb{C} \setminus \Omega$ has at least two points.

THEOREM 1.2.3. *Every hyperbolic plane domain Ω is holomorphically covered by $\mathbb{D}$ (i.e., there is a locally 1-to-1, holomorphic covering map from $\mathbb{D}$ to Ω).*

We will first prove this for bounded domains, then for general simply connected domains, and then finally general hyperbolic domains. The proof will use Montel's theorem that a sequence of uniformly bounded holomorphic functions on $\mathbb{D}$ has a subsequence that converges uniformly on compact sets. This will be proved later; see Theorem 2.3.2. We will also need the following.

LEMMA 1.2.4. *If $\{f_n\}$ are holomorphic functions on a domain Ω that converge uniformly on compact sets to f and if $z_n \to z \in \Omega$, then $f_n(z_n) \to f(z)$.*

PROOF. We may assume $\{z_n\}$ are contained in some disk $D \subset \Omega$ around z. Let $E = \{z_n\} \cup \{z\}$. This is a compact set so it has a positive distance d from $\partial\Omega$. The points within distance $d/2$ of E form a compact set F on which the functions $\{f_n\}$ are uniformly bounded on E, say by M. By the Cauchy estimate the derivatives are bounded by a constant M' on E. Thus

$$|f(z) - f_n(z_n) \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_n)| \leq |f(z) - f_n(z)| + M'|z - z_n|,$$

and both terms on the right tend to zero by hypothesis. $\square$

PROOF OF UNIFORMIZATION FOR BOUNDED DOMAINS. If Ω is bounded, then by a translation and rescaling, we may assume $\Omega \subset \mathbb{D}$ and $0 \in \Omega$. We will define a sequence of domains $\{\Omega_n\}$ with $\Omega_0 = \Omega$ and covering maps $p_n : \Omega_n \to \Omega_{n-1}$ such that $p(0) = 0$. We will show that Ω_n contains hyperbolic disks centered at 0 of arbitrarily large radius and that the covering map $q_n = p_1 \circ \cdots \circ p_n : \Omega_n \to \Omega_0 = \Omega$ converges uniformly on compacta to a covering map $q : \mathbb{D} \to \Omega$.

If $\Omega_0 = \mathbb{D}$ we are done, since the identity map will work. In general assume that we have $q_n : \Omega_n \to \Omega_0$ and that there is a point $w \in \mathbb{D} \setminus \Omega_n$. Let τ and σ be Möbius transformations of the disk to itself so that $\tau(w) = 0$, choose a square root α of $\tau(0)$ and choose σ so $\sigma(\alpha) = 0$. Then $p_{n+1}(z) = \sigma(\sqrt{\tau(z)})$ and let Ω_{n+1} be the component of $U = p_{n+1}^{-1}(\Omega_n)$ that contains the origin (the set U will have one or two components; two if w is in a connected component of $\mathbb{D} \setminus \Omega_n$ that is compact in $\mathbb{D}$, and one otherwise). Since σ and τ are hyperbolic isometries and $\sqrt{z}$ expands the hyperbolic metric, we see that Ω_{n+1} contains a larger hyperbolic ball around 0 than Ω_n did. Suppose $\text{dist}(\partial\Omega_n, 0) < r < 1$ for all n. Then (3) says that

$$D_H^H p_n(0) = D_H^H \sqrt{z}(\tau(0)) > \frac{1+r}{2\sqrt{r}} > 1,$$

since $|\tau(0| = |w| > r$. Hence $D_H^H q_n(0)$ increases by this much at every step. But $D_H^H q_n(0) \le 1$, which is a contradiction. Thus $d_n \to 1$.

Thus $\{q_n\}$ is a sequence of uniformly bounded holomorphic functions on the disk. By Montel's theorem, there a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic map $q : \mathbb{D} \to \Omega$. It is non-constant since it has non-zero gradient at the origin; moreover, by Hurwitz's theorem, q' never vanishes on $\mathbb{D}$ since it is the locally uniform limit of the sequence$\{q_n\}$, and these functions never vanish since they are all covering maps. Next we show that q is a covering map $\mathbb{D} \to \Omega$.

Fix $a \in \Omega$ and let $d = \text{dist}(a, \partial\Omega)$. Since Ω is bounded, this is finite. Let $D = D(a, d) \subset \Omega$. Since q_n is a covering map, every branch of q_n^{-1} is 1-to-1 holomorphic map of D into $\mathbb{D}$ and hence each q_n is a contraction from the hyperbolic metric on D to the hyperbolic metric on $\mathbb{D}$. Thus every preimage of $\frac{1}{2}D$ has uniformly bounded hyperbolic diameter.

Now fix a point $b \in q^{-1}(a)$. Since $q_n(b) \to q(b) = a$, $q_n(b) \in \frac{1}{2}D$ for n large enough, so there is branch of q_n^{-1} that contains b. Since these branches are uniformly bounded holomorphic functions, by Montel's theorem we can pass to a subsequence so that they converge to a holomorphic function g from $\frac{1}{2}D$ into $\mathbb{D}$. Moreover,

$$q(g(z)) = \lim_n q_n(q_n^{-1}(z)) = z,$$

by Lemma 1.2.4. □

This proves the existence of a covering map for bounded domains Ω. If Ω is bounded and simply connected, then we have proved the Riemann mapping theorem for Ω. To deduce Riemann's theorem for all proper simply connected plane domains, we only need:

LEMMA 1.2.5. *Any simply connected planar domain, except for the plane itself, can be conformally mapped to a bounded domain.*

PROOF. If the domain Ω is bounded, there is nothing to do. If Ω. omits a disk $D(x, r)$ then the map $z \to 1/(z - x)$ conformal maps Ω to a bounded domain. Otherwise, translate the domain so that 0 is on the boundary and consider a continuous branch of $\sqrt{z}$. The image is a 1-1, holomorphic image of Ω, but does not contain both a point and its negative. Since the image does contain some open ball, it also omits an open ball and hence can be mapped to a bounded domain by the previous case. □

Next we want to deduce the uniformization theorem for all hyperbolic plane domains (we have only proved it for bounded domains so far). It suffices to show that any hyperbolic plane domain has a covering map from some bounded domain W, for then we can compose the covering maps $\mathbb{D} \to W$ and $W \to \Omega$.

Assume for the moment that we already have a covering map of the twice punctured plane, $q : \mathbb{D} \to \mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$. If $\{a, b\} \in \mathbb{C} \setminus \Omega$ then $h(z) = bq(z) + a$ is a covering map from $U = h^{-1}(\Omega) \subset \mathbb{D}$ to Ω. Any connected component of U shows that Ω has a covering from a bounded plane domain, finishing the proof. Thus we are reduced to proving:

THEOREM 1.2.6. *There is a holomorphic covering map from $\mathbb{D}$ to $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$*

PROOF. Let

$$\Omega = \{z = x + iy : y > 0, 0 < x < 1, |z - \frac{1}{2}| > \frac{1}{2}\} \subset \mathbb{H}.$$

This is simply connected and hence can be conformally mapped to $\mathbb{H}$ with $0, 1, \infty$ each fixed. We can then use Schwarz reflection to extend the map across the sides of Ω. Every such reflection of Ω stays in $\mathbb{H}$ maps to either the lower or upper half-planes. Continuing this forever gives a covering map from a simply connected subdomain U of $\mathbb{H}$ to W. Since U is simply connected and not the whole plane (it is a subset of $\mathbb{H}$) it is conformally equivalent to $\mathbb{D}$ and hence a covering $q : \mathbb{D} \to W$ exists. □

FIGURE OF COVERING MAP, TESSELATION OF UPPER HALF-PLANE

EXERCISE: Show that the domain U formed by repeated reflections across $\partial\Omega$ is, in fact, all of $\mathbb{H}$.

There is an explicit formula for the covering map from the upper half-plane to $\mathbb{C} \setminus \{0, 1\}$. in terms of the Weierstrass P-function. See [].

1.3. Koebe's $\frac{1}{4}$-theorem

We start by recalling Green's theorem and some useful variants. We start with the "standard version":

$$\iint_\Omega u\Delta v - v\Delta u dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} ds, \tag{6}$$

where n denotes the outward pointing normal vector of $\partial\Omega$. We will also use Green's theorem in the following form:

$$\int_{\partial\Omega} f(x, y)dx + g(x, y)dy = \iint_\Omega \frac{\partial g}{\partial x} - \frac{\partial f}{\partial g} dxdy \tag{7}$$

and its simple consequence that the area of a region Ω is given by

$$\text{area}(\Omega) = \frac{1}{2}\int_{\partial\Omega} xdy - ydx = \frac{1}{2i}\int_{\partial\Omega} \overline{z}dz. \tag{8}$$

The complex form of Green's theorem uses $dz = dx + idy$,

$$\begin{aligned}
\int_{\partial\Omega} gdz &= \int_{\partial\Omega} gdx + \int_{\partial\Omega} igdy \\
&= \iint_\Omega (i\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y})dxdy \\
&= 2i\iint_\Omega \frac{\partial g}{\partial \overline{z}} dxdy
\end{aligned}$$

Next, apply this to the domain $\Omega\setminus D(\epsilon, w)$ where $w \in \Omega$ and the function $g(z)/(z-w)$ to get

$$\begin{aligned}\int_{\partial\Omega} \frac{g(z)-g(w)}{z-w}dz - \int_{\partial D(\epsilon,w)} \frac{g(z)-g(w)}{z-w}dz &= 2i\iint_{\Omega\setminus D(\epsilon,w)} \frac{\partial}{\partial \bar{z}}(\frac{g(z)-g(w)}{z-w})dxdy \\ &= 2i\iint_{\Omega\setminus D(\epsilon,w)} \frac{\partial g}{\partial \bar{z}}(\frac{1}{z-w})dxdy.\end{aligned}$$

Since g is continuous at w, the integral over $\partial D(\epsilon, w)$ tends to 0 as $\epsilon \searrow 0$ and by the Cauchy's theorem, the integral over $\partial\Omega$ is equal to

$$\int_{\partial\Omega} \frac{g(z)}{z-w}dz - 2\pi i g(w).$$

Since $|z-w|^{-1}$ is integrable over bounded sets, the area integral above tend to the integral over all of Ω. Thus we obtain Pompeiu's formula:

$$g(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{g(z)}{z-w}dz - \frac{1}{\pi}\iint_{\Omega\setminus D(\epsilon,w)} \frac{g_{zbar}}{z-w}dxdy. \tag{9}$$

We now start towards the proof of Koebe's theorem.

THEOREM 1.3.1 (Area theorem). *Suppose $g(z) = \frac{1}{z} + b_0 + b_1 z + \dots$ is univalent in $\mathbb{D}$. Then $\sum_{n=0}^{\infty} n|b_n|^2 \le 1$. In particular, $|b_1| \le 1$.*

PROOF. For $0 < r < 1$ let $D_r = \mathbb{C} \setminus g(D(0, r))$. If $z = g(w)$ and $w = e^{i\theta}$ then $dw = iwd\theta$, so by (8),

$$\text{area}(D_r) = \iint_{D_r} dxdy = \frac{1}{2i}\int_{\partial D_r} \bar{z}dz = \frac{-1}{2i}\int_{\partial D(0,r)} \bar{g}(w)g'(w)dw.$$

To evaluate the right hand side note that

$$\begin{aligned} g(z) &= \frac{1}{z} + b_0 + b_1 z + \dots, \\ g'(z) &= \frac{1}{z^2} + 0 + b_1 + 2b_2 z + \dots, \end{aligned}$$

so that

$$\begin{aligned}\int_{|w|=r} \bar{g}(w)g'(w)dw &= i\int \bar{g}(w)g'(w)wd\theta \\ &= i\int (\frac{1}{\bar{w}} + \bar{b}_0 + \bar{b}_1\bar{w} + \dots)(-\frac{1}{w} + b_1 w + 2b_2 w + \dots)d\theta \\ &= 2\pi i(-\frac{1}{r^2} + |b_1|^2 r^2 + 2|b_2|r^4 + \dots)\end{aligned}$$

Thus,

$$0 \le \text{area}(D_r) = \pi(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}).$$

Taking $r \to 1$ gives the result. $\square$

COROLLARY 1.3.2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent on the unit disk, then $|a_2| \le 2$.*

PROOF. Let $F(z) = z\sqrt{f(z^2)/z^2}$. Then the quantity inside the square root is even and doesn't vanish in $\mathbb{D}$, so F is odd, univalent and

$$F(z) = z + \frac{a_2}{2}z + \dots.$$

Thus

$$g(z) = \frac{1}{F(z)} = \frac{1}{z} - \frac{a_2}{2}z + \dots,$$

is univalent and satisfies Theorem 1.3.1, so $|a_2| \le 2$. $\square$

THEOREM 1.3.3 (Koebe 1/4 theorem). *If f is univalent on $\mathbb{D}$, then*

$$\frac{1}{4}|f'(z)|(1-|z|^2) \le \text{dist}(f(z), \partial\Omega) \le |f'(z)|(1-|z|^2).$$

PROOF. By pre-composing with a Möbius transformation and post-composing by a linear map, we may assume $z = 0$, $f(0) = 0$ and $f'(0) = 1$. Then the right hand inequality is just Schwarz's lemma applied to f^{-1}. To prove the left hand inequality, suppose f never equals w in $\mathbb{D}$. Then

$$\begin{aligned} g(z) &= \frac{wf(z)}{w - f(z)} \\ &= w(z + a_2z^2 + \dots)\frac{1}{w}[(1 + \frac{1}{w}(z + a_2z^2 + \dots) + \frac{1}{w^2}(z + a_2z^2 + \dots)^2 + \dots)] \\ &= z + (a_2 + \frac{1}{w})z^2 + \dots, \end{aligned}$$

is univalent with $f(0) = 0$ and $f'(0) = 1$. Applying Corollary 1.3.2 to f and g gives

$$\frac{1}{|w|} \le |a_2| + |a_2 + \frac{1}{w}| \le 2 + 2 = 4.$$

Thus the omitted point w lies outside $D(0, 1/4)$, as desired. $\square$

A second proof of Koebe's theorem using extremal length is given in the Appendix, Theorem A.12.4.

The following estimate is known as the distortion theorem.

LEMMA 1.3.4. *Suppose f is univalent on $\mathbb{D}$, $f(0) = 0$ and $f'(0) = 1$. Then*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}, \tag{10}$$

PROOF. Fix a point $w \in \mathbb{D}$ and write the Koebe transform of f,

$$F(z) = \frac{f(\tau(z)) - f(w)}{(1-|w|^2)f'(w)},$$

where

$$\tau(z) = \frac{z+w}{1-\bar{w}z}.$$

This is univalent, so by Corollary 1.3.2, $|a_2(w)| \leq 2$. Differentiation and setting $z = 0$ shows

$$F'(z) = \frac{f'(\tau(z))\tau'(z)}{(1-|w|^2)f'(w)},$$

$$F''(z) = \frac{f''(\tau(z))\tau'(z)^2 + f'(\tau(z))\tau''(z)}{(1-|w|^2)f'(w)},$$

$$\tau'(0) = 1 - |w|^2, \tau''(0) = -2(1-|w|^2),$$

$$F''(0) = \frac{f''(w)}{f(w)}(1-|w|^2) - 2\bar{w}.$$

This implies that the coefficient of z^2 (as a function of w) in the power series of F is

$$a_2(w) = \frac{1}{2}((1-|w|^2)\frac{f''(w)}{f'(w)} - 2\bar{w}).$$

Using $|a_2| \leq 2$ and multiplying by $w/(1-|w|^2)$, we get

$$|\frac{wf''(w)}{f'(w)} - \frac{2|w|^2}{1-|w|^2}| \leq \frac{4|w|}{1-|w|^2}.$$

Thus

$$\frac{2|w|^2 - 4|w|}{1-|w|^2} \leq \frac{wf''(w)}{f'(w)} \leq \frac{4|w| + 2|w|^2}{1-|w|^2}.$$

Now divide by $|w|$ and use partial fractions,

$$\frac{-1}{1-|w|} + \frac{-3}{1+|w|} \leq \frac{1}{|w|}\frac{wf''(w)}{f'(w)} \leq \frac{3}{1-|w|} + \frac{1}{1+|w|}$$

Note that

$$\begin{aligned}
\frac{\partial}{\partial r}\log|f'(re^{i\theta})| &= \frac{\partial}{\partial r}\mathrm{Re}\log f'(z) \\
&= \mathrm{Re}\frac{z}{|z|}\frac{\partial}{\partial z}\log f'(z) \\
&= \frac{1}{|z|}\mathrm{Re}(\frac{zf''(z)}{f'(z)})
\end{aligned}$$

Since $w = re^{i\theta}$ and $f'(0) = 1$, we can integrate to get

$$\log(1-r) - 3\log(1+r) \leq \log|f'(re^{i\theta})| \leq -3\log(1-r) + \log(1+r).$$

Exponentiating gives the result. $\square$

EXERCISE: Integrate the distortion theorem to obtain the growth estimate:

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

If $f : \omega \to \mathbb{D}$ is conformal, then clearly $\int_\Omega |f'(z)|^2 dxdy = \mathrm{area}(\mathbb{D}) = \pi$. The distortion theorem implies a better estimate is true.

COROLLARY 1.3.5. *If $f : \Omega \to \mathbb{D}$ is conformal then $\int_\Omega |f'(z)|^p dxdy < \infty$ for all $2 \leq p < 3$.*

PROOF. Let $g = f^{-1}$. Then

$$\int_\Omega |f'(z)|^p dxdy = \int_{\mathbb{D}} |g(z)|^{2-p} dxdy,$$

and the right-hand side is finite for $2 \leq p < 3$ by the left-hand side of (10). $\square$

The famous Brennan conjecture [**37**] states that this is true for all $\frac{4}{3} < p < 3$. For $p \leq \frac{4}{3}$ or $p \geq 4$ the integral diverges for $f^{-1}(z) = g(z) = z/(1-z)^2$, where f maps a slit plane to the disk. Pommerenke [**108**] proved the integral converges for $p < 3.39$, and this has been improved by Bertilsson [**22**], Shimorin [**127**] and Hedenmalm and Shimorin [**66**].

1.4. The hyperbolic metric in simply connected domains

If $\Omega \subset \mathbb{C}$ has at least two boundary points we want to use the covering map $p : \mathbb{D} \to \Omega$ to define a metric $\rho_\Omega(z)ds$ on Ω. ρ should be defined so that p is locally an isometry, i.e., for $w \in \mathbb{D}$, $z = p(w)$,

$$\begin{aligned} 1 &= D^{\rho_\Omega}_{\rho_\mathbb{D}} p(w) \\ &= D^{E}_{\rho_\mathbb{D}} \mathrm{Id}(w) \cdot D^{E}_{E} p(w) \cdot D^{\rho_\Omega}_{E} \mathrm{Id}(p(w)) \\ &= \frac{1}{\rho_\mathbb{D}(w)} \cdot |p'(w)| \cdot \rho_\Omega(z) \end{aligned}$$

and so we take

$$\rho_\Omega(z) = \frac{|p'(w)|}{1-|w|^2} = |p'(w)|\rho_\mathbb{D}(w)$$

where $p(w) = z$. Different choices of p and w give the same value for $\rho_\Omega(z)$ since they differ by an isometry of $\mathbb{D}$. Thus every hyperbolic planar domain has a hyperbolic metric.

For parabolic planar domains (i.e., $\mathbb{C}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$) it is convenient to define the "hyperbolic metric" to be $\rho(z) = 0$. This reflects the fact that there are only constant holomorphic maps from a parabolic domain to a hyperbolic domain (Picard's little theorem, Theorem 2.3.1) and no restrictions on a holomorphic map from a hyperbolic to a parabolic domain.

We want to give some useful estimates for ρ_Ω in terms of more geometric quantities, such as the quasi-hyperbolic metric, defined as

$$\tilde{\rho}_\Omega(z)ds = \frac{ds}{\mathrm{dist}(z, \partial\Omega)}.$$

For simply connected domains, ρ and $\tilde{\rho}$ are boundedly equivalent; for more general domains this can fail, but some useful estimates are still available.

The first observation is that if $f : U \to V$ is conformal and $\rho_U(z)ds$ and $\rho_V(z)ds$ are the densities of the hyperbolic metrics on U and V then

$$\rho_V(f(z)) = \rho_U(z)/|f'(z)|.$$

Applying this to the map $\tau(z) = (z+1)/(z-1)$ that maps the right half-plane $\mathbb{H}_r = \{x + iy : x > 0\}$ to the unit disk $\mathbb{D}$, we see that the hyperbolic density for the half-plane is

$$\rho_{\mathbb{H}_r}(z) = |\tau'(z)|\rho_{\mathbb{D}}(\tau(z)) = \frac{2}{|z-1|^2}\frac{1}{1-|\tau(z)|^2} = \frac{1}{2x} = \frac{1}{2\mathrm{dist}(z,\partial\mathbb{H}_r)}.$$

Thus the hyperbolic density on a half-plane is approximately the same as the quasi-hyperbolic metric. Using Koebe's theorem we can deduce that that this is true for any simply connected domain.

LEMMA 1.4.1. *For simply connected domains, the hyperbolic and quasi-hyperbolic metrics are bi-Lipschitz equivalent, i.e.,*

$$d\rho_\Omega \leq d\tilde{\rho}_\Omega \leq 4d\rho_\Omega. \tag{11}$$

PROOF. Using Koebe's theorem,

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \leq \rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\mathrm{dist}(f(z),\partial\Omega} = \frac{1}{\mathrm{dist}(f(z),\partial\Omega} = \tilde{\rho}(f(z)),$$

which is one half of the result. The other half is similar:

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \geq \frac{1}{4}\rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\mathrm{dist}(f(z),\partial\Omega)} = \frac{1}{4}\tilde{\rho}(f(z)).$$

□

COROLLARY 1.4.2. *Suppose Ω is simply connected, $z, w \in \Omega$. Then*

$$\rho(z,w) \geq \frac{1}{4}|\log\frac{\mathrm{dist}(z,\partial\Omega)}{\mathrm{dist}(w,\partial\Omega)}|.$$

PROOF. Suppose γ is a curve in Ω connecting the two points. Then the quasi-hyperbolic length of γ is at least

$$|\int_{\mathrm{dist}(z,\partial\Omega)}^{\mathrm{dist}(w,\partial\Omega)}\frac{dt}{t}| = |\log\frac{\mathrm{dist}(z,\partial\Omega}{\mathrm{dist}(w,\partial\Omega)}|.$$

By our previous remarks, the hyperbolic distance is at least $\frac{1}{4}$ of this. □

COROLLARY 1.4.3. *If $f:\Omega\to\Omega'$ is conformal, then*

$$\frac{\mathrm{dist}(f(z),\partial\Omega')}{4\mathrm{dist}(z,\partial\Omega)} \leq |f'(z)| \leq \frac{4\mathrm{dist}(f(z),\partial\Omega')}{\mathrm{dist}(z,\partial\Omega)}.$$

PROOF. Write $f = g\circ h^{-1}$ where $g:\mathbb{D}\to\Omega'$ and $h:\mathbb{D}\to\Omega$ and use the chain rule and Koebe's theorem. □

The following is a consequence we will use repeatedly

COROLLARY 1.4.4. *Suppose Ω is simply connected and $a \notin \Omega$. Then if $|z| > 2|a|$ and $w \in \Omega$,*

$$|w| \leq |z| \exp(8\rho(z, w)).$$

PROOF. If $|w| \leq |z|$, there is nothing to do, so assume the reverse is true. Connect z and w by a curve in Ω and let γ be a subarc that connects the circles of radius $|z|$ and $|w|$ without leaving the annulus between these circles. For $y \in \gamma$ with $|y| = t$, the pseudo-hyperbolic metric is bounded below by

$$\frac{1}{\mathrm{dist}(y, \partial\Omega} \geq \frac{1}{|y-a|} \geq \frac{1}{|y|+|a|} \geq \frac{1}{2|y|},$$

so the length of γ in this metric is at least $\frac{1}{2} \log |w|/|z|$. Thus

$$|w| \leq |z| \exp(2\tilde{\rho}(z, w)) \leq |z| \exp 8\rho(z, w)).$$

□

If we apply the previous result and the Schwarz inequality we get:

COROLLARY 1.4.5. *Suppose Ω is simply connected, $0 \notin \Omega$ and $f : \Omega \to \Omega$ is holomorphic. If $z, w \in \Omega$, then*

$$|f(w)| \leq |f(z)| \exp(8\rho(z, w)).$$

This will be useful later when we consider iterations of f, e.g., if Ω is unbounded, proper, simply connected domain and $f : \Omega \to \Omega$ then the iterates of z cannot grow faster than C^n where $C = \exp(8\rho(z, f(z))$. As we shall see later, this is rather slow escape to ∞; a transcendental entire function always has points that iterate to ∞ much faster than this (see Chapter 6).

COROLLARY 1.4.6. *If Ω is simply connected and not the whole plane, $f : \Omega \to \Omega$ is holomorphic, and $z \in \Omega$, then there is a $C < \infty$ so that $|f^n(z)| \leq C^n$*

1.5. Hyperbolic metric multiply connected domains

The following is immediate from Schwarz's lemma.

COROLLARY 1.5.1. *If $U \subset V$ are both hyperbolic, then $\rho_U \geq \rho_V$.*

PROOF. If $\Pi_U : \mathbb{D} \to U$ and $\Pi_V : \mathbb{D} \to V$ are the covering maps then the inclusion map $U \to V$ can be lifted to conformal map $\mathbb{D} \to \Pi_V^{-1}(U) \subset \mathbb{D}$. Applying Schwarz's lemma to this map (and using the fact that the projections are local isometries) gives the result. $\square$

We will also need estimates for hyperbolic metrics in domains that are not simply connected. The most important (and useful) cases are the punctured disk $\mathbb{C}^* = \mathbb{D} \setminus \{0\}$, an annulus, and the twice punctured plane $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$.

For the punctured disk, $z \to \exp(z)$ is a covering map from the left half-plane to $\mathbb{C}^*$. Transferring the hyperbolic metric on the half-plane gives

$$\rho_{\mathbb{D}*}(z) = \frac{1}{2|z| \log(1/|z|)}. \tag{12}$$

Note that in this case, the hyperbolic and quasi-hyperbolic metrics are not boundedly equivalent: the circle of radius r around the origin has length 2π for the quasi-hyperbolic metric but only has length $2\pi / \log 1/r$.

The exponential map sends the strip $S = \{x + iy : |y| < \pi\}$ to the right half-plane and this can be used to show that the hyperbolic metric density on S is

$$\rho_S(z) = \frac{1}{\cos y}.$$

The vertical strip $V = \{x + iy : \log a < x < \log b\}$ is mapped by the exponential to the annulus $A = \{z : a < |z| < b\}$. If we simplify slightly to the case $0 < a < 1$ and $b = 1/a$, then we get

$$\rho_A(z) = \frac{\pi}{|\log a| \cos(\frac{\pi}{|\log a|} \log r)}. \tag{13}$$

The next case we consider is $\mathbb{C}^{**} = \mathbb{C} \setminus \{-1, 1\}$.

LEMMA 1.5.2. *The hyperbolic density ρ for $\mathbb{C}^{**} = \mathbb{C} \setminus \{-1, 1\}$ satisfies*

$$\rho(z) \simeq \frac{1}{|z| \log |z|}, \quad |z| > 2,$$

$$\rho(z) \simeq \frac{1}{|z-1| \log 1/|z-1|}, \quad |z-1| < 1/2,$$

$$\rho(z) \simeq \frac{1}{|z+1| \log 1/|z+1|}, \quad |z+1| < 1/2,$$

and $\rho \simeq 1$ elsewhere.

PROOF. The upper bounds are immediate by Lemma 1.5.1 and comparing to a punctured disk; the lower bounds require some more work.

We saw earlier that the covering map $F : \mathbb{H} \to \mathbb{C}^{**}$ was obtained by conformally mapping

$$W = \{x + iy : |x| < 1, |z| > 1\}$$

to the upper half-plane $\mathbb{H}$ with $-1, 1, \infty$ each fixed and then extending this to the rest of $\mathbb{H}$ by repeated reflections. Thus F restricted to W can be written as a composition of three conformal maps:

$$f : W \to S = \{x + iy : y > 0, |x| < 1\},$$

$$\exp : S \to U = \{x + iy : y > 0, |z| > 1\}.$$

$$J : U \to \mathbb{H}.$$

Here $J(z) = \frac{1}{2}(z+\frac{1}{z})$ the Joukowsky map. Let V be the union of W and its horizontal translates by $2\mathbb{Z}$. By reflection, f extends to be a conformal map of V to $\mathbb{H}$ and since $0 \leq \Im(f(z)) \leq 1$ on ∂V, one can prove that $\Im(z) - 1 \leq \Im(f(z)) \leq \Im(z)$. By the Koebe theorem,

$$|f'(x + iy)| \simeq \frac{\Im(f(z))}{\Im(z)} \simeq 1,$$

for $y > 2$. Similarly,

$$J'(z) = \frac{1}{2}(1 - \frac{1}{z^2}),$$

so $\frac{3}{8} \leq |J'| \leq \frac{5}{8}$ for $|z| > 2$. Thus if $z = F(w)$,

$$\rho_{\mathbb{C}^{**}}(z) = \frac{\rho_{\mathbb{H}}(w)}{|F'(w)|} \simeq \frac{1}{|\exp(w)|\Im(w)} = \frac{1}{|z| \log |z|},$$

for $|z| \geq 2$. The estimates in $D(-1, 1)$ and $D(1, 1)$ follow by applying Möbius transformations that permute $-1, 1, \infty$. The remaining region of the plane is compact and so ρ is bounded above and below there. □

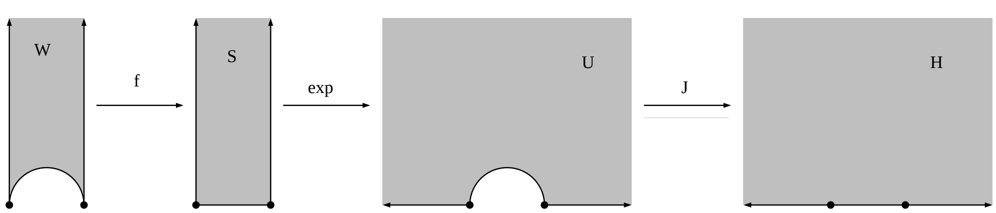


FIGURE 1. The sequence of maps in the proof of Lemma 1.5.2.

The twice punctured plane is one of the few domains for which there is an explicit expression for the hyperbolic metric. Assuming the punctures are $0, 1$, the hyperbolic density is given by

$$\frac{1}{\rho(w)} = \frac{1}{\pi}\iint_{\mathbb{C}} \left|\frac{w(w-1)}{z(z-1)(z-w)}\right| dxdy.$$

See [**1**], [**85**]. For $\widehat{\mathbb{C}} \setminus \{a, b, c\}$ it has the more symmetric looking form

$$\frac{1}{\rho(w)} = \frac{1}{\pi}\iint_{\mathbb{C}} \left|\frac{(w-a))(w-b)(w-c)}{(z-a)(z-b)(z-c)(z-w)}\right| dxdy.$$

We will not prove these formulas since the simpler estimate in Lemma 1.5.2, will be enough for our applications.

We can now give a geometric estimate of the hyperbolic density that is valid in all planar hyperbolic domains. We normalize by supposing $1 \in \Omega$ and that $0 \in \partial\Omega$ is a closest point of $\partial\Omega$ to 1 (thus $D(1, 1) \subset \Omega$). Let R be the maximum value such that $A = \{\frac{1}{R} < |z| < R\} \subset \Omega$. R must be finite, for otherwise Ω is the once punctured plane and hence a parabolic, not hyperbolic, domain.

LEMMA 1.5.3. *With notation as above, $\rho_\Omega(1) \simeq 1/(1 + \log R)$.*

PROOF. The inclusion $A_R \subset \Omega$ gives $\rho_\Omega(1) \lesssim 1/(1 + \log R)$. By the maximality of R there is a point $w \in \partial\Omega$ on either $\{|z| = R\}$ or $\{|z| = 1/R\}$. The inclusion $\Omega \subset \mathbb{C} \setminus \{0, w\}$ gives the other direction. $\square$

COROLLARY 1.5.4. *For $a, b \in \mathbb{C}$ let $\rho_{a,b}(z)$ be the hyperbolic density for the $\mathbb{C} \setminus \{a, b\}$ and let*

$$\Upsilon_\Omega(z) = \sup_{a,b} \rho_{a,b}(z).$$

There is a $C < \infty$ so that for any hyperbolic planar domain Ω $\Upsilon_\Omega(z) \leq \rho_\Omega(z) \leq C\Upsilon_\Omega(z)$.

PROOF. The first inequality is clear by Lemma 1.5.1. The second is given by the proof of Lemma 1.5.3. $\square$

This result is due to Gardiner and Lakic [**57**], and explicit estimates for C are given in [**23**], [**5**] and [**134**].

A special case that we will need is:

Corollary 1.5.5. *There is a constant $c > 0$ so that the following holds. Suppose $w \in \Omega$, but there are points $a, b \notin \Omega$ with $|a| = |w|$ and $|b| = 2|w|$. Then $\rho(w) \geq c/|w|$.*

Lemma 1.5.6 (Bohr's lemma). *If $0 < r < 1$ let $\rho = \rho(r)$ be the hyperbolic distance from 0 o0 r in $\mathbb{D}$, let c be the constant from Corollary 1.5.5 and let $\beta = \beta(r) =$* $\exp(r/c) > 0$*. If f is holomorphic on $\mathbb{D}$, $|f(0)| \leq \beta$, $|f| < 1$ on $\mathbb{D}_r$ and $|f| = 1$ somewhere on $\partial\mathbb{D}_r$, then $\partial\mathbb{D}_t \subset f(\mathbb{D})$ for some $t \geq \beta$.*

Proof. Let $R = \sup\{t : \{|z| = t\} \subset disk\}$. Choose $|w| = r$ so that $|f(w)| = 1$. We will estimate the hyperbolic distance between $f(0)$ and $f(w)$ in $f(\mathbb{D})$ in two different ways, that give a contradiction if $R < \beta$. This implies the lemma.

So suppose $R < \beta$. Let γ be the radial segment joining 0 and w; by definition it has hyperbolic length ρ. Since $|f(0)| \leq \beta$, $f(\gamma)$ contains a subarc σ that connects $\{|z| = \beta\}$ and $\{|z| = 1\}$ and is contained in the annulus between these circles. At each point $z \in \sigma$, Corollary 1.5.5 says the hyperbolic metric is bounded below by $c/|z|$ and hence

$$\rho_{f(\mathbb{D})}(w, f(0)) > \rho_{f(\mathbb{D})}(\sigma) \geq c \int_\beta^1 \frac{dt}{t} = c \log \frac{1}{\beta}.$$

Thus by Schwarz's lemma

$$\rho > c \log \frac{1}{\beta},$$

or

$$\beta < \exp(r/c).$$

This contradicts the definition of $\beta = \exp(r/c)$, and hence $R \geq \beta$. Thus $f(\mathbb{D})$ contains $\partial\mathbb{D}_t$ for some $t \geq \beta$. □

There is a beautiful estimate of Weitsman [**141**] that says for any hyperbolic domain W,

$$\min\{\rho_D(z) : z \in D, |z| = r\} \geq \rho_{D^*}(r),$$

where D^* is the circular symmetrization of D (if $\{|z| = t\} \subset D$, then it is also in D^*, otherwise $-t \notin D^* \cap \{|z| = t\}$ and $D^* \cap \{|z| = t\}$ is an open arc centered at t whose total length equals that of $D \cap \{|z| = t\}$.) In the proof of Bohr's lemma, this allows us to estimate the hyperbolic density in U by comparing it to the density for

$\mathbb{C} \setminus [-R, -\infty)$ and using Koebe's theorem gives the estimate

$$\rho_U(w, f(0)) \geq \frac{1}{4} \log(1 + \frac{w}{R}),$$

which leads to $R \geq q(e^{4\rho} - 1)^{-1}$. Taking $r = 1/2$ gives $\rho = \frac{1}{2} \log 3$ and $s = (e^{4\rho} - 1 = (e^{\log 9} - 1)^{-1} = 1/8$. For this particular application, the symmetrized domain D^* is simply connected, and in this case the necessary estimate had been proven earlier by Hayman [**63**]. See also [**69**]

Next we record a result that will be used later in the book when we discuss fast escaping points (see Theorem 6.8.4).

LEMMA 1.5.7 (Schottky's Lemma). *Suppose f is a holomorphic function on a hyperbolic domain Ω and that f omits the values $0, 1$. If $K \subset \Omega$ is compact, then there is are constants B, C, depending only on K, so that*

$$\max_K |f(z)| \leq B(1 + \min_K |f(z)|)^C.$$

PROOF. f is a holomorphic map from Ω to $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$ and hence is a contraction of the hyperbolic metrics. Since $K \subset \Omega$ is compact, it has finite hyperbolic diameter in Ω and hence $f(K)$ has finite hyperbolic diameter d in $\mathbb{C}^{**}$. We may also assume $f(K)$ is connected (if not, replace it by a covering hyperbolic ball of at most twice the diameter).

Let $r_0 = 1$, $r_1 = 2$ and, in general, $r_n = (r_{n-1})^2 = 2^{2^n}$. By Corollary 1.5.4 the hyperbolic distance between these circles is at least

$$\int_{2^{2^n}}^{2^{2^{n+1}}} \frac{dr}{r \log r} \geq \log \log 2^{2^{n+1}} - \log \log 2^{2^n} = \log 2.$$

Thus the circles are a uniformly positive hyperbolic distance apart in $\mathbb{C}^{**}$ and hence $f(K)$ hits at most a bounded number of them and hence is trapped between r_k and r_{k+M} some k (depending on f) and some M (depending only on d). This proves the result with $B = 2 = C = 2^{M+1}$. □

1.6. Maximum modulus

If f is entire we define

$$m(r, f) = \min\{|f(z)| : |z| = r\},$$

$$M(r, f) = \max\{|f(z)| : |z| = r\}.$$

The iterates of M are defined iteratively as

$$M^n(r,f) = M(M^{n-1}(r,f),f),$$

where $M^1(r,f) = M(r,f)$. Note that

$$M(r,f^n) \le M^n(r,f),$$

and strict inequality is possible, but we shall see later (Theorem 6.5.2) that the two quantities are always close in a certain sense. Also note that if $|z| \le M^n(r,f)$ then $|f^k(z)| \le M^{n+k}(r,f)$. The most important properties of $M(r,f)$ are described in the following results.

Lemma 1.6.1. *For $r > 0$, $M(r,f)$ is an increasing, convex function of* $\log r$.

Proof. Clearly $M(r,f)$ increases with r by the maximum principle. Since $g(z) = f(e^z)$ is entire, $\log|g(z)|$ is subharmonic on the plane. Hence

$$M(e^x,f) = \sup\{\log|g(x+iy))| : y \in \mathbb{R}\},$$

is a subharmonic function of x alone and hence convex (a supremum of subharmonic functions is subharmonic). $\square$

As a consequence of convexity we see that

$$\frac{\log M(r,f) - \log M(1,f)}{\log r}$$

increases with $r \ge 0$ and hence has a limit in $[0,\infty]$ as $r \to \infty$. If f is a polynomial of degree d then it is easy to check the limit is d. Conversely, if the limit is $\le d$, then for $r = |z|$ large we have

$$\log M(r,f) \le \log M(1,f) + d\log r,$$

or

$$|f(z)| \le M(1,f)|z|^d,$$

which implies f is a polynomial of degree $\le d$ by a well known argument (use the Cauchy estimates to show all derivatives of order $> d$ vanish at the origin). Thus

Lemma 1.6.2. *If f is a transcendental entire function,*

$$\lim_{r\to\infty} \frac{\log M(r,f)}{\log r} = \infty.$$

LEMMA 1.6.3. *If f is a transcendental entire function and $\lambda > 1$, then*

$$\lim_{r\to\infty} \frac{M(\lambda r, f)}{M(r, f)} = \infty.$$

PROOF. Since $\log M(r,f)$ is an increasing convex function of $\log r$, $\log(M(\lambda r, f) - \log M(r,f)$ is increasing in r if $\lambda > 1$ is fixed. If this difference is bounded above by $A < \infty$ for all $r \geq 0$, then setting $r_n = \lambda^n$ and summing a telescoping series shows

$$M(r_n, f) \leq (n+1)A = (1 + \frac{1}{n}) \frac{A}{\log \lambda} \log r_n,$$

which implies f is a polynomial of degree $d \leq A \log \lambda$ by Lemma 1.6.2. □

LEMMA 1.6.4. *If f is a transcendental entire function and $\lambda > 1$, then there is an $R > 0$ so that for all $r \geq R$ and all $c \geq 1$,*

$$\log M(r^c, f) \geq \lambda(c-1) \log M(r, f).$$

PROOF. Since $\log M(r,f)$ is an increasing, convex function of $\log r$ it is the integral of some positive, increasing function, say

$$\log M(r, f) = \int_{-\infty}^{\log r} a(t) dt.$$

Choose R so that $a(t) \geq \lambda$ if $t \geq \log R$. Then if $r \geq R$,

$$\log M(r^c, f) - \log M(r, f) = \int_{\log r}^{c \log r} a(t) dt \geq \lambda(c-1) \log r.$$

□

EXERCISE: For any increasing function $\phi : [0, \infty) \to [0, \infty)$ there is an entire function f so that $M(r, f) > \phi(r)$ for all sufficiently large r. (Hint: apply Arakelian's theorem (Theorem A.5.4) to ϕ on $[0, \infty)$.)

EXERCISE: [Polya's lemma [**106**]] Suppose g, h are entire and define an entire function $f = g \circ h$. Then

$$M(r, f) \geq M(\beta(\frac{1}{2}) M(\frac{1}{2} r, h), g),$$

where β is the constant from Bohr's lemma.

EXERCISE: Use Polya's lemma to prove: if g and h are entire functions such that $f = g \circ h$ has finite order then either

(1) h is polynomial and g has finite order, or
(2) h transcendental of finite order and g is zero order.

1.7. The escaping set is non-empty

If f is holomorphic on $\mathbb{C}$, we define the escaping set as

$$I(f) = \{z : f^n(z) \to \infty\}.$$

Our first 'dynamical" result is to show this set is non-empty. This important result is due to Eremenko [**47**], but here we give a later proof by Dominguez [**42**] that uses Bohr's lemma. (Dominguez's proof applies to meromorphic functions with finitely many poles, but we only apply in the case of entire functions.) Eremenko's proof uses delicate estimates of Wiman and Valiron about the behavior of an entire function f near a point where $|f(z)| = M(|z|, f)$, and we will give his proof in Chapter 6 when we study the rate of escape of iterates of f.

THEOREM 1.7.1. *If f is a transcendental entire function then $I(f) \neq \emptyset$. In fact, $I(f)$ intersects every circle $\{|z| = r\}$ for all sufficiently large r.*

PROOF. Let $\beta = \beta(\frac{1}{2})$ be the constant from Bohr's lemma (Lemma 1.5.6). Choose r so large that $M(\frac{1}{2}s, f) \geq \frac{2}{\beta}s$ for $s \geq r$. It then follows from Bohr's lemma (Lemma 1.5.6) that $f(\mathbb{D}_r)$ contains a circle $\partial\mathbb{D}_s$ for some

$$s \geq \frac{1}{8}M(\frac{1}{2}r, f) \geq 2r.$$

Let $\gamma_1 = \partial\mathbb{D}_r$. Let γ_2 be the boundary of the unbounded component of $\mathbb{C} \setminus f(\mathbb{D}_r)$. We call this the outer boundary of $f(\mathbb{D}_r)$. It is a subset of $f(\partial\mathbb{D}_r)$, so $K_2 = f^{-1}(\gamma_2) \cap \gamma_1$ is a non-empty, compact subset of γ_1 and $|f| > 2r$ on K_2.

Now repeat the argument with $\mathbb{D}_r$ replaced by $\mathbb{D}_{2r}$. Bohr's lemma and our assumptions imply $f(\mathbb{D}_{2r})$ contains a circle $\mathbb{D}_s$ for some

$$s \geq \beta M(\frac{1}{2}(2r), f) \geq \beta \cdot \frac{2}{\beta} \cdot 2r = 4r.$$

Since $\mathbb{D}_{2r}$ is contained in the region D_2 bounded by γ_2, $f(D_2)$ also contains this circle. Let γ_3 be the boundary of the unbounded component of $\mathbb{C} \setminus f(D_2)$ and let $K_3 = f^{-2}(\gamma_3) \cap \gamma_1$. Thus $K_3 \subset K_2 \subset \gamma_1$ and $|f^2| \geq 4r$ on K_3.

In general, suppose γ_n bounds a region D_n that contains $\mathbb{D}_{2^{n-1}r}$. By Bohr's lemma, $f(\mathbb{D}_{2^{n-1}r})$ contains a circle of radius $s \geq 2^1 r$ and hence the same is true of the larger region $f(D_n)$. Thus the outer boundary γ_{n+1} of $f(D_n)$ bounds a region D_{n+1} that contains the disk $\mathbb{D}_{2^n r}$. Thus $f^{-1}(\gamma_{n+1}) \cap \gamma_n$ is a non-empty, compact subset of γ_n and $|f| > 2^n$ on this subset. Let $K_{n+1} \subset K_n$ be the f^n-preimage of γ_{n+1} in γ_1.

Continuing in this way we build a sequence of nested sets $\gamma_1 \supset K_2 \supset K_3 \supset \dots$ so that $|f^n| \geq 2^{n-1}$ on K_n. Clearly $\cap_n K_n$ is non-empty and contained in $I(f)$. $\square$

Later we will want to use a variation of this argument to show that the Julia set of a transcendental entire function always contains an escaping point (see Theorem 2.6.10). The lower bound on the rate of escape we have given is rather slow. Eremenko's proof, based on Wiman-Valiron theory, constructs points that escape as fast as possible, essentially as fast as $M^n(R, f)$ for some R. In particular, it shows show that every transcendental entire function has points whose iterates satisfy

$$\liminf_{n\to\infty} \frac{1}{n} \log\log|f^n(z)| = \infty.$$

Thus $|f^n(z)|$ is eventually larger than $\exp(\exp(Cn))$ for any finite C See Lemma 6.8.1.

The fact that $I(f)$ hits $\{|z| = r\}$ for all sufficiently large r implies that $I(f)$ always has Hausdorff dimension ≥ 1; proving that this is also true for Julia sets is one of the main goals of the next chapter. This fact also suggests that

- $I(f)$ may contain unbounded connected components (known true, Theorem 6.7.12),
- $I(f)$ contains paths to ∞ (known false, Theorem 9.3.1), or
- every connected component of $I(f)$ is unbounded.

The latter is still open and known as Eremenko's question; a great deal of research effort has been focused on answering his question and we will return to it several times in these notes.

CHAPTER 2

Normal families

Normal families refer to pre-compact families of holomorphic functions and thus many arguments using normal families are essentially compactness arguments. Such techniques are ubiquitous in conformal dynamics, indeed, the standard definitions of the Fatou and Julia sets are simply the sets where the sequence of iterates either form a normal family or do not.

We start with a review of the basic definitions and properties of normal families and introduce the idea of a conformally invariant normal family. These can be characterized by certain boundedness conditions on the derivatives of members of the family. We shall also discuss Zalcman's lemma, a remarkable result that turns the failure of normality into a very useful property: we can always extract a sequence from a non-normal family that, after normalization, converges to limiting function with certain nice properties.

We then turn to the definitions and basic properties of the Fatou and Julia sets, showing, for example, that the Julia set is the boundary of the escaping set and that it is the closure of the pre-periodic points that it contains (in the next chapter we will show that it is the closure of the repelling fixed points). We then discuss multiply connected Fatou components and prove Baker's theorem that any such component must be bounded and must be wandering (all of its iterates are disjoint); we also give examples to show that this can actually occur. This fact is one of the main distinctions between the dynamics of polynomials and the dynamics of transcendental entire functions.

2.1. Zalcman's lemma

A family $\mathcal{F}$ of functions from one metric space (X, d) to another (Y, ρ) is called **equicontinuous** if for each $\epsilon >$ there is a $\delta > 0$ so that $d(x, y) < \delta \Rightarrow \rho(f(x)-f(y)) < \epsilon$ for every $f \in \mathcal{F}$. This is the same as the definition of continuity at a point, except that δ can be chosen independent of the point and of the function f. and it consists of all transcendental entire functions have have a bounded singular set (defined below). Such functions have nice behavior when $|f|$ is large; in particular, they have a strong expansion property near infinity that is extremely useful. For example, we shall use this expansion property to show that the Julia set of an Eremenko-Lyubich function is the closure of the escaping set; for general entire functions the Julia set is the boundary of the escaping set (). A family $\mathcal{F}$ of meromorphic functions on a planar domain Ω is a **normal family** if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on every compact set or converges uniformly to ∞ on every compact set.

LEMMA 2.1.1. *If a sequence of meromorphic functions converges uniformly on compact sets in the sense of spherical distance, then the limit is meromorphic or identically ∞. If a sequence of homomorphic functions converges in the same sense, then the limit is either holomorphic or identically ∞.*

PROOF. □

If a sequence of homomorphic functions converges uniformly on compacta to a holomorphic limit, then the derivatives also converge uniformly on compacta.

EXERCISE: Show that if $\mathcal{F}$ is a normal family, $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ need not be normal. (Hint: consider $f_n(z) = n(z^2 - n)$.)

THEOREM 2.1.2 (Arzela-Ascoli). *A family $\mathcal{F}$ of continuous functions from a planar domain Ω to a metric space (X, d) is normal if and only if*

(1) *$\mathcal{F}$ is equicontinuous on every compact $E \subset \Omega$.*
(2) *For any $z \in \Omega$, $\{f(z) : f \in \mathcal{F}\}$ is pre-compact (lies in a compact subset).*

The following is taken from Ahlfors; book [**2**], but since it plays such a crucial role in what follows, we repeat the proof here.

THEOREM 2.1.3 (Marty's theorem). *A family $\mathcal{F}$ of meromorphic functions on a hyperbolic planar domain Ω is normal iff*

$$\sup_{f \in \mathcal{F}} \sup_{z \in K} D_E^S f(z) < \infty,$$

for every compact $K \subset \Omega$.

PROOF. One direction is easy; if the spherical gradient is bounded at each point then $\mathcal{F}$ is equicontinuous, hence normal. Conversely, suppose $\mathcal{F}$ is normal but there is compact set K, a sequence $\{z_n\} \subset K$ and a sequence $\{f_n\} \subset \mathcal{F}$ such that $D_E^S f_n(z_n) \to \infty$. By passing to a subsequence if necessary, we may assume f_n converges uniformly to a meromorphic function f on an open disk around z ($f \equiv \infty$ is allowed).

First assume the limit function f is finite at z. Then f is bounded on some disk around z and hence f' is bounded on a smaller disk by the Cauchy estimate

$$|f'(w)| \leq \frac{1}{r} \sup_{\partial D(w,r)} |f(z) - f(w)|.$$

Since $f_n' \to f'$ on this disk, f_n' is uniformly bounded. Since $D_E^S f \leq |f'|$, the spherical gradient is uniformly bounded on a neighborhood of z.

On the other hand, if $f(z) = \infty$, then consider $1/f$. Since $z \to 1/z$ corresponds to a rotation of the Riemann sphere by 180 degrees, its spherical gradient is 1 everywhere and hence

$$D_E^S \frac{1}{f}(z) = D_S^S \frac{1}{z} \cdot D_E^S f(z) = 1 \cdot D_E^S f(z).$$

Thus the argument above, applied to $1/f$, again shows that the spherical gradient of $\{f_n\}$ is uniformly bounded on some neighborhood of K.

Since K is compact, it can be covered by a finite number of these neighborhoods where f has bounded spherical gradient, and we deduce that $D_E^S f$ is uniformly bounded on K. $\square$

The following lemma, due to Zalcman, is extremely helpful. It turns the failure of normality into a useful property.

LEMMA 2.1.4 (Zalcman's lemma). *Suppose Ω is a planar domain and $\mathcal{F}$ is a family of meromorphic functions on Ω. If $\mathcal{F}$ is not normal, then there is a sequence of points $\{z_k\}$ in Ω converging to a point $z_0 \in \Omega$, a sequence $\{\rho_k\} \subset (0,1)$ converging to 0 and a sequence $\{f_k\} \subset \mathcal{F}$ so that $f_k(z_k + \rho_k z)$ converges uniformly on compact sets to a meromorphic function f on $\mathbb{C}$. Moreover,*

$$\sup_{z \in \mathbb{C}} D_E^S f(z) \leq 1 = D_E^S f(0).$$

PROOF. By Marty's theorem, there is a sequence $\{f_n\} \in \mathcal{F}$ and $\{w_n\} \in \Omega$ so that $w_n \to w_0 \in \Omega$ and $D_E^S f_n(w_n) \nearrow \infty$. Without loss of generality, we assume $w_0 = 0$ and $\overline{\mathbb{D}} \subset \Omega$. Since $(1 - |z|) D_E^S f_n(z)$ is continuous on $\overline{\mathbb{D}}$ and vanishes on its boundary, it attains a maximum M_n at some point $z_n \in \mathbb{D}$. Note that $M_n \to \infty$ since $M_n \geq (1 - |w_n|)D_E^S f_n(w_n) \to \infty$ and since $w_n \to 0$. Define

$$\rho_n = \frac{1}{D_E^S f_n(z_n)} = \frac{1 - |z_n|}{M_n},$$

and

$$g_n(z) = f_n(z_n + \rho_n z).$$

Note that

$$D_E^S g_n(0) = \rho_n D_E^S f_n(z_n) = 1.$$

Also note that $0 < \rho_n \leq 1/M_n \to 0$ and $|z_n + \rho_n z| < 1$ if $|z| < (1 - |z_n|)/\rho_n = M_n$. Thus g_n is defined on $\mathbb{D}_{M_n}$, and for $z \in \mathbb{D}_{M_n}$,

$$\begin{aligned}
D_E^S g_n(z) &= \rho_n D_E^S f_n(z_n + \rho_n z) \\
&= \frac{1 - |z_n|}{M_n} D_E^S f_n(z_n + \rho_n z) \\
&= \frac{1 - |z_n|}{1 - |z_n + \rho_n z|} \cdot \frac{(l1 - |z_n + \rho_n z|)D_E^S f_n(z_n + \rho_n z)}{M_n} \\
&\leq q \ \frac{1 - |z_n|}{1 - |z_n + \rho_n z|} \cdot 1 \\
&\leq \frac{1 - |z_n|}{1 - |z_n| - |\rho_n z|} \\
&= \frac{1}{1 - \frac{\rho_n |z|}{1 - |z_n|}} \\
&= \frac{1}{1 - |z/M_n|} \\
&\leq \frac{1}{1 - |z/M_n|}.
\end{aligned}$$

This tends to 1 for z fixed and $n \to \infty$. By Marty's theorem, $\{g_n\}$ has a subsequence that converges uniformly on compact subsets of $\mathbb{C}$ to a function f with $D_E^S f \leq 1$. Passing to another subsequence, if necessary, we may assume z_k converges to some point $z_0 \in \overline{\mathbb{D}} \subset \Omega$. Since $D_E^S g_n(0) = 1$ for all n, $D_E^S f(0) = 1$, hence f is non constant and $D_E^S f \leq 1$ everywhere on $\mathbb{C}$. $\square$

2.2. Normal functions

Let $\mathcal{A}(\mathbb{D})$ denote the collection of Möbius transformations of the unit disk to itself (i.e., its conformal automorphisms). A meromorphic function f on $\mathbb{D}$ is called a **normal function** if the family

$$\mathcal{F} = \{f \circ \sigma : \sigma \in \mathcal{A}(\mathbb{D}), \},$$

is a normal family. A family of functions $\mathcal{F}$ on $\mathbb{D}$ is called a **conformally invariant normal family** if

$$\mathcal{G} = \{f \circ \sigma : f \in \mathcal{F}, \sigma \in \mathcal{A}(\mathbb{D})\},$$

is a normal family.

THEOREM 2.2.1. *A meromorphic function f is normal on $\mathbb{D}$ iff $D_H^S f$ is bounded on $\mathbb{D}$.*

PROOF. Let

$$\sigma(z) = \frac{z+a}{1+\bar{a}z}.$$

This is a Möbius transformation of $\mathbb{D}$ to itself that maps 0 to a. Then

$$\begin{aligned} & D_H^S(f \circ \sigma)(0) \\ &\quad = D_H^S f(\sigma(0)) \cdot D_H^H \sigma(a) \cdot D_E^H \mathrm{Id}(0) \\ &\quad = D_H^S f(0) \cdot 1 \cdot 1. \end{aligned}$$

Thus for any compact set $K \subset \mathbb{D}$,

$$\sup_\sigma \sup_{a \in K} D_H^S(f \circ \sigma)(a) = \sup_{a \in \mathbb{D}} D_H^S f(a).$$

By Marty's theorem we see that f is a normal function if and only if $D_S^H f$ is uniformly bounded on the disk. $\square$

The following is given by essentially the same argument.

COROLLARY 2.2.2. *$\mathcal{F}$ is a conformally invariant normal family on $\mathbb{D}$ iff $D_H^S f$ is uniformly bounded over $\mathbb{D}$ and $\mathcal{F}$.*

Next we generalize from the disk to more general domains. A function f on a hyperbolic plane domain is called **normal** if $f \circ p$ is normal on the disk, where $p : \mathbb{D} \to \Omega$ is the covering map. A family of meromorphic functions on Ω is called a **conformally invariant normal family** on Ω if $\mathcal{G} = \{f \circ p : f \in \mathcal{F}\}$ is conformally invariant normal family on the disk. The following is immediate from these definitions.

COROLLARY 2.2.3. *Suppose Ω is a hyperbolic planar domain. Then f is normal on Ω iff $D^S_H f$ is bounded on Ω. $\mathcal{F}$ is a conformally invariant normal family on Ω iff $D^S_H f$ is uniformly bounded over Ω and $\mathcal{F}$.*

Our next goal is to prove a result of Lehto that says normal functions can't have essential singularities at isolated boundary points. The proof uses the following fact.

LEMMA 2.2.4. *If f is holomorphic on $\overline{\mathbb{D}}^c_r = \{z : |z| > r\}$ and has an essential singularity at ∞ then $F_\lambda(z) = f(\lambda z)\overline{f(\overline{z})}$ also has an essential singularity at ∞ for some λ with $|\lambda| = 1$ (in fact, this holds for all but countably many λ on the unit circle).*

PROOF. By scaling, we can assume $r < 1$, say $r = 1/2$. f has a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ that converges in $\overline{\mathbb{D}}^c_r$. This implies $|a_n| = O(r^{|n|})$, and this in turn implies that if $b_{n,k} = a_k \overline{a_{n-k}}$, then

$$|b_{n,k}| = O(r^{|k|}). \tag{14}$$

Thus the Laurent series for F satisfies ($n = m + k$),

$$\begin{aligned} F(z) &= f(\lambda z)\overline{f(\overline{z})} \\ &= (\sum_{k=-\infty}^{\infty} a_k(\lambda z)^k)(\sum_{m=-\infty}^{\infty} \overline{a_m} z^m) \\ &= \sum_{n=-\infty}^{\infty} (\sum_{k=-\infty}^{\infty} a_k \overline{a_{n-k}} \lambda^k) z^n \\ &= \sum_{n=-\infty}^{\infty} A_n(\lambda) z^n \end{aligned}$$

and $A_n(\lambda)$ is a holomorphic function of λ on some neighborhood of $\{|\lambda| = 1\}$ (since (14) implies that for each n, the $\{b_{n,k}\}_{k=-\infty}^{\infty}$ define a Laurent series that converges in $r < |\lambda| < 1/r$).

Fix some k so that $a_k \neq 0$. If $a_{n-k} \neq 0$ as well, then $a_k \overline{a_{n-k}} \neq 0$. By assumption, F has an essential singularity at ∞, so a_{n-k} is non-zero for arbitrarily large positive

n, hence the Laurent series for A_n has some non-zero coefficients for arbitrarily large n. For such n's, $A_n(\lambda)$ is not the constant zero function, and hence vanishes at most finitely often on the circle $\{|\lambda| = 1\}$. Thus, except for a countable set of λ's, $A_n(\lambda) \neq 0$ for an infinite set of positive n's. For these λ, $F_\lambda(z)$ has an essential singularity at ∞. $\square$

LEMMA 2.2.5 (Lehto [**86**]). *If f is meromorphic in $\overline{\mathbb{D}}_r^c$ and has an essential singularity at ∞, then*

$$\limsup_{|z|\to\infty} |z| \cdot D_E^S f(z) \geq 1/2.$$

PROOF. Recall that points w, z are antipodal on $\widehat{\mathbb{C}}$ iff $z\overline{w} = -1$. For any $|\lambda| = 1$,

$$F(z) = f(\lambda z)\overline{f(\overline{z})} = (\sum_{n=-\infty}^{\infty} a_n(\lambda z)^n)(\sum_{n=-\infty}^{\infty} \overline{a_n} z^n)$$

is meromorphic in $\overline{\mathbb{D}}_r^c$, and for some choice of λ, F has an essential singularity at ∞ by Lemma 2.2.4.

If $F(z) = -1$ let $r = |z|$ and note that f maps $\overline{z}, \lambda z \in \partial\mathbb{D}_r$ to antipodal points on $\widehat{\mathbb{C}}$. Hence $\gamma = f(\partial\mathbb{D}_r)$ has spherical length at least π. This means that $D_E^S f \geq 1/2r$ somewhere on $\partial\mathbb{D}_r$. Similarly, if $|F(z) + 1| < \epsilon$, then we can deduce $D_E^S f \geq \frac{1}{(2+\delta)r}$ somewhere on $\partial\mathbb{D}_r$, for some δ that tends to zero with ϵ.

To finish the proof, note that either there is a sequence $z_n \to \infty$ so that $F(z_n) \to -1$ or there is not. If there is, then the argument above shows $|z| \cdot D_E^S f \to \frac{1}{2}$ along some sequence tending to ∞. If not, then $1/(1+F)$ is bounded in a neighborhood of ∞, which implies it has a removable singularity there and hence F is meromorphic at ∞, which is a contradiction. $\square$

This immediately gives its contrapositive:

COROLLARY 2.2.6. *If f is meromorphic in $\mathbb{D}_r^*$ and*

$$\limsup_{|z|\to\infty} |z| \cdot D_E^S f(z) < 1/2,$$

then f is meromorphic at ∞.

THEOREM 2.2.7. *Suppose f is a normal function on a hyperbolic planar domain Ω. Then f has a meromorphic extension to any isolated boundary point of Ω.*

PROOF. Without loss of generality, we may assume the boundary point is ∞. Since f is normal

$$\begin{aligned} D_E^S f(z) &= D_E^H \mathrm{Id}(z) \cdot D_H^S f(z) \\ &= O(\frac{1}{|z|\log|z|}) \cdot O(1) \\ &= o(\frac{1}{|z|}), \end{aligned}$$

so by Corollary 2.2.6, f has a meromorphic extension at ∞. $\square$

2.3. Picard's theorems

The following could have been proven in the previous chapter as a corollary of Theorem 1.2.6.

THEOREM 2.3.1 (Picard's little theorem). *If f is a non-constant entire function, then $E = \mathbb{C} \setminus f(\mathbb{C})$ contains at most one point.*

PROOF. If E contains two points $\{a, b\}$, then using the covering map $p : \mathbb{D} \to \mathbb{C} \setminus \{a, b\}$, f can be lifted to a holomorphic map $f : \mathbb{C} \to \mathbb{D}$. By Liouville's theorem, the lift is constant and hence so must f. $\square$

THEOREM 2.3.2 (Montel's theorem). *If $\mathcal{F}$ is a family of holomorphic functions on a planar domain Ω all taking values in $W = \mathbb{C} \setminus \{a, b\}$ for some $a \neq b$, then $\mathcal{F}$ is a normal family.*

We offer three proofs of this fundamental result.

FIRST PROOF. $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$ is covered by a map p from the unit disk, so each map $f : \Omega \to \mathbb{C}^{**}$ can be lifted to a map $F : \Omega \to \mathbb{D}$. The family of lifted maps is normal by the first version of Montel's theorem. Thus any sequence $\{f_n\}$ in $\mathcal{F}$ can be lifted to a sequence $\{F_n\}$ that has a convergent subsequence $\{F_{n_k}\}$ and $\{f_{n_k}\} = \{p \circ F_{n_k}\}$ is convergent in the original family. $\square$

SECOND PROOF. If not, then by Zalcman's lemma we can form a sequence

$$g_n(z) = f_n(\rho_n + z_n)$$

so that $f_n \in \mathcal{F}$, $\rho_n \searrow 0$ and $z_n \to z \in \Omega$ and g_n converges uniformly on compact sets to a non-constant entire function g. If g ever took the value a, then so would f_n for

n sufficiently large (apply Rouche's theorem to a small disk around some g-preimage of a). Thus g omits a, b and is constant, a contradiction. $\square$

THIRD PROOF. Use the covering map $p : \mathbb{D} \to W$ to lift the family $\mathcal{F}$ to a family $\mathcal{G}$ of holomorphic functions mapping Ω into $\mathbb{D}$. Then

$$D_E^S f(z) = D_H^S f(z) = D_H^H \tilde{f}(z) D_H^H p(z) D_H^S \mathrm{Id}(z) D_E^H \mathrm{Id}(z) \leq 1 \cdot 1 \cdot O(1) \cdot D_E^H \mathrm{Id}(z),$$

where the first three bounds hold by the Schwarz lemma, the definition of the hyperbolic metric on Ω, and because the spherical distance between points in W is smaller that a fixed constant times the hyperbolic distance (the constant depends on a, b). The final term, $D_E^H \mathrm{Id}$ is bounded on compact sets of Ω, so this implies $D_E^S f$ is locally bounded, which implies the normality of $\mathcal{F}$ by Marty's theorem. $\square$

Thus omitting two values has two consequences: it implies normality when applied to functions on a hyperbolic domain and it implies constancy when applied to functions on $\mathbb{C}$. This is a common phenomenon and **Bloch's principle** says that a property P implies one of these conclusions iff it implies the other. This is not always true, but it does hold for a number of interesting cases and can be made into a precise mathematical statement. See Bergweiler's paper [**16**].

THEOREM 2.3.3 (Picard's great theorem). *If f is meromorphic on $A_R = \{R < |z| < \infty\}$ and has an essential singularity at ∞, then for every $r \geq R$, $E = \mathbb{C} \setminus f(A_r)$ contains at most one point.*

PROOF. Suppose for r sufficiently large, $f(\overline{\mathbb{D}}_r^c)$ omits two points. Then f is a normal function on $\overline{\mathbb{D}}_r^c$ and hence has a meromorphic extension to ∞ by Lemma 2.2.7, a contradiction. $\square$

Omitted points are values which have no preimages under f. In the next chapter we will generalize Picard's theorem by considering points that have only a finite number of preimages. See Theorem 3.2.1.

2.4. The Julia and Fatou sets

The **Fatou set**, $\mathcal{F}(f)$, of an entire function f is the union of open disks on which $\{f^n\}$ forms an open family. It is also clear that $f(\mathcal{F}(f)) \subset \mathcal{F}(f)$ (forward invariance), but equality need not hold if f has an omitted value. For example, $\frac{1}{10}e^z$ has a Fatou component that contains 0, but $0 \notin f(\mathcal{F}(f))$. It turns out that if U is a Fatou component that is mapped into a component V then $V \setminus U$ can have at most one point (see Theorem 5.6.3) and $U = V$ if U is bounded (see Lemma 2.6.2). Similarly, $f^{-1}(\mathcal{F}(f)) \subset \mathcal{F}(f)$ (backwards invariance. A set that is both forward and backwards invariant is called **totally invariant**.

LEMMA 2.4.1. *For any $n \geq 1$, $\mathcal{F}(f) = \mathcal{F}(f^n)$.*

PROOF. Since $\{f^{nk}\}_{k=1}^{\infty}$ is a subset of $\{f^k\}_{k=1}^{\infty}$ the first collection is normal wherever the second one is. Thus $\mathcal{F}(f) \subset \mathcal{F}(f^n)$. On the other hand, any iterate can be written as $f^k = f^{nj+r}$ where $0 \leq r < n$, so if K is a compact disk in $\mathcal{F}(f)$, then

$$D_E^S f^k(z) = D_E^E f^r(z) \cdot D_E^S f^k(f^r(z)) \leq \sup_{z \in K} D_E^E f^r(z) \cdot \sup_{w \in f^r(K)} D_E^S f^k(w).$$

Since $f^r(K)$ is a compact subset of $\mathcal{F}(f)$, Marty's theorem (Theorem 2.1.3) proves that $\{f^k\}_k$ is normal wherever $\{f^{nk}\}_k$ is, showing Thus $\mathcal{F}(f) \supset \mathcal{F}(f^n)$. $\square$

The complement $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$ is called the **Julia set** of f and is clearly a closed, totally invariant set and satisfies $\mathcal{J}(f) = \mathcal{J}(f^n)$ for every $n \in \mathbb{N}$.

Our next goal is to prove that $\mathcal{J}(f)$ is non-empty by proving that $\mathcal{J}(f) = \partial I(f)$. We already know that $I(f) \neq \emptyset$ (Theorem 1.7.1) but we still must prove $I(f) \neq \mathbb{C}$. We will do this by proving that f must have pre-periodic points (which obviously can't escape). We start with:

LEMMA 2.4.2 (Rosenbloom, 1952). *If g is entire and $h(z) = (g(g(z)) - z)/(g(z) - z)$ is constant then g is constant or linear.*

PROOF. If $h \equiv 0$, then $g(g((z)) = z$ implying g is 1-to-1, hence linear. If $h \equiv 1$, then $g \circ g = g$ so g is constant or $g(z) = z$. So assume h is a constant $c \neq 0, 1$, i.e,

$$g^2(z) - z = c(g(z) - z),$$

and differentiate to get

$$g'(g(z))g'(z) - 1 = c(g'(z) - 1),$$

or

$$g'(z)(g'(g(z)) - c) = 1 - c.$$

Since $c \neq 1$, the left side is never zero, hence both factors are never zero. Thus g' omits 0. It also omits c, for if g covers the whole plane this is obvious; if $g' = c$ only at the single possible omitted value of g, then g' takes the values $0, c$ only finitely often; by the great Picard theorem g' is a polynomial omitting 0, hence constant. Thus g is linear. $\square$

EXERCISE: Show that if h is rational, then g must be rational too.

A **periodic point** z for f is a point such that $f^n(z) = z$ for some $n \geq 1$. A point is **pre-periodic** if some iterate of it is periodic. Note that periodic points are automatically pre-periodic; we will use the term **strictly pre-periodic** to mean a pre-periodic point that is not itself periodic.

THEOREM 2.4.3. *If g is entire and not constant or linear then it has at least two pre-periodic points.*

PROOF. Consider the function

$$h(z) = (g(g(z)) - z)/(g(z) - z),$$

as in Lemma 2.4.2. Our assumption implies that h is a non-constant meromorphic function. If $h(z) = \infty$ then $g(z) = z$, so every such point is a fixed point of g. If $h(z) = 0$ then $g^2(z) = z$ so every such point is periodic or period 2. If $h(z) = 1$, then $g^2(z) = g(z)$ so $g(z)$ is a fixed point of g and hence z is pre-periodic.

If h is a rational of degree $d \geq 1$, then each of $\{0, 1\}$ has at least one preimage and hence g has at least two pre-periodic points. Otherwise h has an essential singularity at ∞ and then Picard's great theorem says that it takes on at least one of the values $\{0, 1, \infty\}$ infinitely often. Hence g has infinitely many pre-periodic points. $\square$

COROLLARY 2.4.4. *If g is entire and not constant or linear, then $\mathbb{C} \setminus I(g)$ (the non-escaping set) contains at least two points. If g is transcendental, it contains infinitely many points.*

PROOF. The first part was proven above. If f is transcendental, then by the great Picard theorem, one of the two pre-periodic points has infinitely many preimages (which are obviously not escaping). $\square$

THEOREM 2.4.5. *If f is entire, $\mathcal{J}(f) = \partial I(f) \neq \emptyset$.*

PROOF. We will show that $\mathcal{J}(f) \subset \overline{I(f)}$, $\mathrm{int}(I(f)g \cap \mathcal{J}(f) = \emptyset$ and $\partial I(f) \subset \mathcal{J}(f)$. Together, these imply $\mathcal{J}(f) = \partial I(f)$.

First, suppose $z \in \mathcal{J}(f)$ and V is a neighborhood of z. Then $\{f^n\}$ is not normal on V, so takes every complex value except possibly one (Theorem 2.3.2). For any $w \in I(f)$ we have $f(w) \neq w$ (since fixed points don't escape), so $f^n(V)$ eventually contains either w or $f(w)$. Hence V contains escaping points. Thus the Julia set is contained in the closure of the escaping set.

Second, if D is a disk in $I(f)$, then $f^n(D)$ never hits a non-escaping point. We know that there are at least two pre-periodic points, so $\{f^n\}$ is normal on D, hence $D \subset \mathcal{F}(f)$. Thus the Julia set is disjoint from the interior of $I(f)$.

Finally, if $D \subset \mathcal{F}(f)$ is a disk that contains an escaping point, then the whole disk escapes by normality. Hence boundary points of $I(f)$ can't lie in the Fatou set and so must be in the Julia set. This completes the proof that $\mathcal{J}(f) = \partial I(f)$. Since both $I(f)$ and its complement are non-empty, it is is easy to see that the boundary of $I(f)$ is non-empty. □

Some important corollaries of this proof are:

COROLLARY 2.4.6. *If Ω is a Fatou component that contains an escaping point, then all of Ω escapes.*

COROLLARY 2.4.7. *If f is entire and V is any neighborhood of any point $z \in \mathcal{J}(f)$ then $\cup_n f^n(V)$ covers the whole plane with at most one exception.*

Later we will prove another version of this that does not require taking a union; see Lemma 3.5.2.

COROLLARY 2.4.8. *The Julia set is either nowhere dense or is the whole plane.*

PROOF. If $\mathcal{J}(f)$ contains the open set U it also contains all iterates of U. Since these cover the plane, except possibly for one point, and since $\mathcal{J}(f)$ is closed, we see that $\mathcal{J}(f) = \mathbb{C}$ in this case. □

LEMMA 2.4.9. *The Julia set is contained in the accumulation set of the backwards orbits $\cup_n f^{-n}(z)$, except possibly for one exceptional point z.*

PROOF. Take z, V as in Theorem 2.4.7 and suppose w is not the exceptional point. Then V contains some preimage of w, so $\mathcal{J}(f)$ is contained in the accumulation set of the preimages. $\square$

In general, the Julia set need not be the whole accumulation set of a backwards orbit. For example, there can be simply connected Fatou components where f is conjugate to an irrational rotation (Siegel disks) and the accumulation set of a point in such a component contains a closed curve inside the Fatou component.

COROLLARY 2.4.10. *The Julia set is the minimal set among closed backwards invariant sets with at least two elements. Thus it is contained in the closure of any backwards invariant set with at least two elements.*

PROOF. Suppose A is a closed, completely invariant set and $z \in A$ has infinitely many preimages under f (this choice is possible if A has at least two elements). Then $\mathcal{J}(f)$ is contained in the accumulation set of the backwards orbit of z and that set is contained in A since A is closed and invariant. Thus $\mathcal{J}(f) \subset A$, as claimed. The final claim follows from the following exercise. $\square$

EXERCISE: the closure of a backwards invariant set is also backwards invariant.

For example, the closure of the escaping set is a closed, completely invariant set, and hence contains the Julia set (we already knew this since $\mathcal{J}(f) = \partial I(f)$. Since we know that there are at least two pre-periodic points in the Julia set and since preimages of pre-periodic points are pre-periodic, we get:

COROLLARY 2.4.11. *If f is entire, $\mathcal{J}(f)$ is the closure of the pre-periodic points it contains.*

Of course, there can be pre-periodic points in the Fatou set as well, e.g., an attracting fixed point. A stronger result, that the Julia set is the closure of repelling fixed points will be proved later, Theorem 3.4.2.

COROLLARY 2.4.12. *The Julia set of a transcendental entire function is unbounded.*

PROOF. We have already proved that $I(f)$ and its complement each contain at least two points and hence $\mathcal{J}(g) = \partial I(g)$ has at least two points. By Picard's great theorem, one of these points has infinitely many preimages converging to ∞, and all of these are in $\mathcal{J}(g)$ since the Julia set is totally invariant. □

2.5. Multiply connected Fatou components escape

LEMMA 2.5.1. *If Ω is multiply connected Fatou component of an entire function f, then $f^n \nearrow \infty$ uniformly on compact subsets of Ω.*

PROOF. Suppose γ is closed curve in Ω. By normality, any subsequence of $\{f^n\}$ contains a subsequence that converges uniformly, either to ∞ or to a bounded function. In the latter case, the maximum principle implies it converges uniformly to a bounded function on the whole interior of γ (i.e., the bounded complementary component of γ), and therefore the interior of γ is in the Fatou set. Thus if the Fatou set contains a curve γ surrounding a point in the Julia set, every subsequence of f^n must converge uniformly to ∞ on γ. Normality then implies it converges to ∞ on every compact subset of component containing γ. □

Herman rings are a type of multiply connected Fatou component that does not iterate to ∞, but these can only occur for rational or meromorphic functions; they do not occur for polynomials or entire functions. There do exist entire functions that have Fatou components that are topological annuli; this is due to Kisaka and Shishikura [**80**].

FIGURE OF HOW A SEQUENCE OF CLOSED CURVES CAN TEND TO INFINITY

LEMMA 2.5.2. *If Ω is multiply connected Fatou component of an entire function f, and $\gamma \subset \Omega$ surrounds a point of the Julia set, then $f^n(\gamma)$ has positive index with respect to 0 for all sufficiently large n.*

PROOF. Suppose the index is zero for an infinite subsequence of γ, and tends to ∞ uniformly on γ. The minimum principle implies f^{n_k} tends to ∞ inside γ. This contradicts the fact that pre-periodic points are dense in the Julia set, Corollary 2.4.11. □

We will say that curve with this property **eventually surrounds every point**, i.e., the iterates of γ under f eventually surround arbitrarily large disks centered at the origin.

LEMMA 2.5.3. *If Ω is multiply connected Fatou component of an entire function f, and $\gamma \subset \Omega$ surrounds a point of the Julia set, then the winding number of $f^n(\gamma)$ around zero tends to ∞ as $n \to \infty$*

PROOF. Lemma 2.5.2 implies that γ eventually has winding number ≥ 1 around zero and lies outside any disk $D(0, r)$ we choose. Thus the winding number around any point of $D(0, r)$ will be the same. By Picard's great theorem (Theorem 2.3.3), either $f = 0$ or $f = 1$ infinitely often in $\mathbb{C}$ and hence the number of solutions in $D(0, r)$ for one of these increases to ∞ with r. If γ_n does not hit $\overline{\mathbb{D}}$, then the winding of f around 0 or 1 is the same. By the argument principle and our previous observation, at least one of these numbers (hence both) tends to infinity with n. $\square$

Next we want to prove that iterates of Ω contain, not just a curve surrounding zero, but a "fat" circular annulus surrounding zero. First we need another fact about the hyperbolic metric.

LEMMA 2.5.4. *There is a $C > 0$ and $\epsilon_0 > 0$ so that the following holds. Suppose γ is a closed Jordan curve that has hyperbolic length $\epsilon < \epsilon_0$ in a planar domain Ω. Suppose further that $0 \notin \Omega$ and γ winds at least once around 0. Then*

$$\gamma \subset A = \{r < |z| < R\} \subset \Omega,$$

where $R/r \geq \exp(C/\epsilon)$.

PROOF. The Jordan curve γ separates the boundary of Ω into two sets E_0, E_∞ that are on the same side of γ as 0 and ∞ respectively. By dilating and rotating we may assume $1 \in \gamma$, and this is the point of γ farthest from the origin. Note that $\gamma' = \gamma \cap D(1, 1/2)$ has Euclidean length at least 1. Choose $z_0 \in E_0$ so that $|z_0| = r$ is maximized and and $z_\infty \in E_\infty$ so that $|z_\infty| = R$ is minimized. Since z_0 is inside γ, $r < 1$.

Let ϵ_0 be the shortest possible hyperbolic length of γ in $\mathbb{C} \setminus \{z_0, z_\infty\}$ for any $z_0 \in \mathbb{D}$ for any $z_\infty \neq z_0$ with $|z_\infty| \leq 2$. This is strictly positive since the length blows up as $|z_0| \to 1$ or $\min(|z_\infty|, |z_0 - z_\infty|) \to 0$.

Thus we may assume $R > 2$, in which case $\rho(\gamma) \gtrsim 1/\log R$. Hence $R/r \geq R \geq \exp(c/\epsilon)$. $\square$

LEMMA 2.5.5 (Zheng [**143**]). *If Ω is multiply connected Fatou component of an entire function f, then for large n, $f^n(\Omega)$ contains an annulus $A = \{r_n < |z| < R_n\}$ with $R_n/r_n \to \infty$.*

PROOF. Suppose $\gamma \subset \Omega$ surrounds a point of the Julia set and has hyperbolic length $L < \infty$ in Ω. Without loss of generality γ is a closed hyperbolic geodesic and hence is smooth and does not self-intersect. i By our previous results we can choose n large so that $f^n(\Omega)$ surrounds 0, and $\gamma_n = f^n(\gamma)$ winds around zero N times, where N is as large as we wish.

We claim γ_n can be written as a union of N piecewise smooth curves, each of which winds once around zero. To see this, consider $\mathbb{C} \setminus \gamma_n$. The boundary of the unbounded component is one such curve. Removing this curve gives a larger unbounded components whose boundary is the next curve, and so on. This can be repeated N times since the winding number around a point only goes down by one each time we cross γ_n (except at the finitely many self-intersections).

By the Schwarz lemma the total hyperbolic length of γ_n is $\leq L$, so one of the sub-curves, call it σ, has hyperbolic length $\leq L/N$. The lemma then follows from Lemma 2.5.4. □

This result has been considerable strengthened by Bergweiler, Rippon and Stallard [**18**]. They show that $f^n(\Omega)$ contains an annulus $\{r_n < |z| < R_n\}$ with

$$\liminf_{n\to\infty} \frac{\log R_n}{\log r_n} > 1,$$

(but ratios close to 1 are possible) and provide detailed information about the dynamics inside the Fatou component. Some of their results will be considered in greater detail in Section **??**.

LEMMA 2.5.6. *The Julia set has no isolated points.*

PROOF. Suppose $D = D(z, \epsilon)$ is a disk such that $\overline{D} \cap \mathcal{J}(f) = \{z\}$. Then Lemma 2.5.2 implies that $f^n(\partial D)$ eventually surrounds every point, and hence surrounds at least two points of the Julia set. At most one can have n-th preimage equal to z, so the nth-preimage of the other is a distinct point of $\mathcal{J}(f) \cap D$, contrary to assumption. □

A closed set is called **perfect** if it has no isolated points. It is a well known fact from analysis that any perfect set with more than one point is uncountable, in fact, it is **locally uncountable**; this means that any neighborhood of a point in the set contains uncountably many points of the set.

COROLLARY 2.5.7. *If f is entire and not linear or constant, then the Julia set is locally uncountable.*

A closed set K is called **uniformly perfect** if there is a finite constant C so that for every $r \in (0, \operatorname{diam}(K))$ and every $x \in K$ there is $y \in K$ such that

$$\frac{1}{C} \leq \frac{|x-y|}{r} \leq C.$$

Connected sets are uniformly perfect with $C = 1$. It turns out that a Julia set of a transcendental entire functions a is either connected (and hence uniformly connected with $C = 1$) or it fails to be uniformly connected with any constant. This happens because if the Julia set is not connected, then there is a multiply connected Fatou component and we shall see later that the iterates of this component contain round annuli with arbitrarily large moduli (Theorem 2.5.5), and this contradicts the uniform perfectness condition. These remarks prove:

COROLLARY 2.5.8. *The Julia set of a transcendental entire function is uniformly perfect if and only if it is connected.*

All Julia sets of polynomials are uniformly perfect and hence this is another difference between the polynomial and transcendental cases.

2.6. Multiply connected components are bounded

In this section we prove Noel Baker's result that any multiply connected Fatou component must be bounded and a number of interesting consequences, such as the fact that the Julia set of a transcendental entire function must contain a non-trivial continuum. We start by showing that there can be at most one unbounded, multiply connected component.

LEMMA 2.6.1 (Töpfer). *If the Fatou set of a transcendental entire function has an unbounded component, then all other components are simply connected.*

Proof. Suppose W is the unbounded component and Ω is some multiply connected component. Let $\gamma_n = f^n(\gamma) \subset \mathcal{F}(f)$ be the curve given by Lemma 2.5.2 that eventually surrounds every point. Then for large enough n, γ_n must hit W and hence is contained in W. We claim this implies $\Omega = W$. If Ω were bounded then $f^n(\Omega) \subset W$ is bounded and hence there would be a point $w \in W \cap \partial f^n(\Omega)$; thus there are points $\{z_k\} \subset \Omega$ so that $f^n(z_k) \to w$. If Ω were bounded we could pass to a subsequence so that $z_k \to z \in \overline{\Omega}$. If $z \in \partial\Omega \subset \mathcal{J}(f)$, then $w = f^n(z) \in \mathcal{J}(f)$, a contradiction. If $z \in \Omega$, then $w = f^n(z) \in f^n(\Omega)$, also a contradiction. Hence Ω can't be bounded. Thus $f^n(\gamma)$ is eventually in Ω, for the same reasons as above and hence $\Omega = W$. □

Part of the previous proof will be useful later, so we state it as a lemma:

Lemma 2.6.2. *If Ω is a bounded Fatou component of f, then $f(\Omega)$ is contained in a bounded Fatou component and equals the whole component. The map is a branched covering.*

In fact, if U, V are Fatou components of f so that $f(U) \subset V$, then $V \setminus U$ can contain at most one point. This is due to M. Herring and independently Bergweiler and Rohde. See Theorem 5.6.3.

Theorem 2.6.3 (Baker []). *If f is a transcendental entire function, then every multiply connected component of the Fatou set is bounded.*

Proof. Suppose not, i.e., suppose Ω is an unbounded multiply connected Fatou component and let $\gamma \subset \Omega$ is a closed curve surrounding a point of the Julia set. Then by Lemma 2.5.2 the iterates of $\gamma_n = f^n(\gamma)$ hit Ω (and hence are contained in Ω for all large enough n. Thus Ω is forwards invariant.

Choose a compact, connected set $K \subset \Omega$ that contains both γ and $f(\gamma)$ and choose a domain V so that $K \subset V \subset \overline{V} \subset \Omega$. Since $|f^n| \to \infty$ uniformly on $\overline{V}$, $\log|f^n|$ is a sequence of well defined, positive harmonic functions on V and so by Harnack's inequality there is a constant $C = C(K)$ so that

$$\log|f^n(w)| \leq C \log|f^n(z)|,$$

for all $z, w \in K$, independent of n. Thus

$$|f^n(w)| \leq |f^n(z)|^C.$$

Since $\gamma_{n-1} \cup \gamma_n \supset f^{n-1}(K)$, we have

$$\sup_{\gamma_n} |f(z)| \leq \inf_{\gamma_{n-1}} |f(z)|^C = \inf_{\gamma_n} |z|^C.$$

In particular, $|f(z)| \leq |z|^C$ for every $z \in \gamma_n$. Since the curves $\{\gamma_n\}$ eventually surround every point and from this we can easily deduce f is a polynomial. This contradiction proves the theorem. $\square$

COROLLARY 2.6.4. *If f is a transcendental entire function then every multiply connected component of the Fatou set is a wandering domain.*

PROOF. We already know that multiply connected components are bounded and iterate to infinity uniformly on compact sets, so they can't be periodic. If they were pre-periodic they would have to land on a periodic domain that iterates to infinity, i.e., a Baker domain. However such a domain is unbounded, whereas $f(U)$ must be bounded, contradicting Lemma 2.6.2. Thus there are no pre-periodic, multiply connected Fatou components. $\square$

COROLLARY 2.6.5. *If the Fatou set of a transcendental entire function has an unbounded component, then all components are simply connected.*

PROOF. Clear from Lemma 2.6.1 and Theorem 2.6.3. $\square$

COROLLARY 2.6.6. *The Julia set of a transcendental entire function contains a non-trivial continuum.*

PROOF. If the Julia set is connected, this is obvious since the Julia set contains at least two points. If the Julia set is not connected, there is a multiply connected Fatou component. By Baker's theorem it is bounded and so the boundary of its unbounded complementary component is a non-trivial continuum in the Julia set. $\square$

Although we have not discussed Hausdorff dimension in detail yet, let us note that every non-trivial connected set has Hausdorff dimension ≥ 1 and hence:

COROLLARY 2.6.7. *The Julia set of a transcendental entire function has Hausdorff dimension at least* 1.

In fact, many examples of transcendental Julia sets have Hausdorff dimension 2 (e.g., [**95**]), but it is somewhat harder to construct examples with dimensions between

1 and 2 [**130**], [**132**]. Constructing an example with dimension equal giving to 1 was only accomplished recently. Some of these examples will be discussed later in the book (see Chapter **??**.

We can say slightly more than Corollary 2.6.7. Since there are no unbounded multiply connected components, $\mathcal{J}(f)$ either contains an unbounded continuum or contains a sequence of continua with diameters growing to ∞. In either case the 1-dimensional Hausdorff measure of $\mathcal{J}(f)$ is infinite. On the other hand, examples exist where the 1-dimensional measure of any bounded subset of $\mathcal{J}(f)$ is finite [**27**].

COROLLARY 2.6.8. *If f is a transcendental entire function that is bounded along a curve σ tending to ∞, then all Fatou components are simply connected. In particular, this happens if f has a finite asymptotic value.*

PROOF. If U is a multiply connected component, then by Lemma 2.5.2, it contains a curve γ whose iterates $f^n(\gamma)$ intersect σ for all sufficiently large n. This contradicts the assumption that f is bounded on σ since $f(f^n(U)\cap\sigma) \subset f^{n+1}(U)$ is as far from the origin as we wish. Thus f can't have any multiply connected Fatou components. □

COROLLARY 2.6.9. *If f is transcendental, any completely invariant Fatou component is simply connected.*

PROOF. A completely invariant Fatou component Ω must be unbounded (since by Picard's great theorem $f^{-1}(\Omega)$ contains unbounded sets) and hence is simply connected by Baker's theorem (Theorem 2.6.3). □

THEOREM 2.6.10. *If f is a transcendental entire function then $\mathcal{J}(f) \cap I(f) \neq \emptyset$.*

PROOF. There are two cases depending on whether there are multiply connected Fatou components or not.

If there is a multiply connected component Ω, then by Lemma 2.5.2 there is a closed curve γ in Ω that eventually surrounds every point in the plane. Since Ω wanders and iterates to ∞, $\Omega_m = f^m(\Omega)$ is eventually outside $\gamma_n = f^n(\gamma)$ and hence $\partial\Omega_m$ is outside γ_n. Thus $\partial\Omega$ consists of escaping points in the Julia set.

If there are no multiply connected Fatou components, then the Julia set contains an unbounded continuum. So if we consider the curves γ_n constructed in the proof of Theorem 1.7.1, we see that each of them must hit $\mathcal{J} = \mathcal{J}(f)$ (if r is chosen sufficiently large). Thus $J_n = f^{-n}(\mathcal{J} \cap \gamma_n) \cap \gamma_1$ is a non-empty compact subset of γ and $J_{n+1} \subset J_n$ since $f^{-1}(\mathcal{J} \cap \gamma_{n+1}) \subset \mathcal{J} \cap \gamma_n$. Thus $\cap_n J_n$ is non-empty and contains only escaping points of the Julia set. $\square$

THEOREM 2.6.11 (Eremenko, []). *The closure of the escaping set has no bounded components.*

PROOF. Since $\mathcal{J}(f) \subset \overline{I(f)}$, if there is a bounded component of $\overline{I(f)}$, it is surrounded by a curve in a non-escaping component of the Fatou set. But this component is then multiply connected component, hence it does escape. $\square$

Eremenko conjectured that all the components of $I(f)$ are unbounded. This is still open, but the so called "Strong Eremenko Conjecture" that all path components of $I(f)$ are unbounded has been disproven by Rottenfusser, Rückert, Rempe and Schleicher; see Theorem 9.3.1. Partial progress towards Eremenko's conjecture has been given by Rippon and Stallard who showed that $I(f)$ always contains at least one unbounded component. See Theorem 6.7.12.

COROLLARY 2.6.12. *If Ω is a multiply connected Fatou component of f, then $f(\Omega)$ is contained in a bounded Fatou component and equals the whole component. The map is a branched covering.*

PROOF. Clear from Lemma 2.6.2 and Theorem 2.6.3. $\square$

In particular, the connectivity of V and $f(U)$ are the same. For a multiply connected Fatou component U, how do the connectivities of $\{f^n(U\}$ behave?

THEOREM 2.6.13 (Kisaka-Shishikura). *For a bounded wandering domain Ω of a transcendental function f, the connectivity of $\Omega_n = f^n(\Omega)$ is non-increasing and eventually either 1, 2 or ∞. If it is eventually 1, then Ω is itself simply connected. If the eventual connectivity is 2 then $f : \Omega_n \to \Omega_{n+1}$ is a covering of annuli for all large enough n.*

PROOF. Since Ω is bounded, $f : \Omega \to f(\Omega)$ is a finitely branched covering map. Thus if Ω is simply connected, any closed curve in $f(\Omega)$ pulls back to a closed curve in Ω, hence is deformable to point in Ω. The image of this deformation under f shows that $f(\Omega)$ is also simply connected.

If $f(\Omega)$ is finitely connected, then so is any finite branched covering. Thus if Ω is infinitely connected, so is $f(\Omega)$.

Finally, assume Ω and $f(\Omega)$ have finite connectivity, both ≥ 2. Let $c(\Omega), c(f(\Omega))$ be the connectivities of Ω and $f(\Omega)$, d the degree of f as a map from Ω to $f(\Omega)$ and N the number of critical points of f in Ω, counted according to multiplicity. By the Riemann-Hurwitz theorem

$$c(\Omega) - 2 = d(c(f(\Omega) - 2) - N.$$

Since $d \geq 1$ and $N \geq 0$, we see that $c(\Omega) \geq c(f(\Omega))$, hence $c(f^n(\Omega))$ is eventually constant. Suppose p is this eventual value. If $p > 2$, then we must eventually have $d = 1$ and $N = 0$. By the argument principle, f must also be 1-to-1 on the bounded complementary components of Ω_n and hence on the whole plane (since Ω_n eventually surrounds every point). This is impossible for a transcendental entire function, so $p = 2$. We still must have $N = 0$, but d can be any value $d \geq 1$, i.e., $f : \Omega_n \to \Omega_{n+1}$ can be a d-to-1 covering map between annuli. □

Corollary 2.6.14. *If U, V are Fatou components, $f(U) \subset V$ and U is simply connected, then V is simply connected too.*

Proof. If either U or V is unbounded then V is simply connected by Lemma 2.6.5. If they are both bounded then V is simply connected by Theorem 2.6.13. □

Corollary 2.6.15 (Kisaka and Shishikura). *The eventual connectivity of a multiply connected Fatou component is 2 or ∞.*

The argument that eliminated $p > 2$ as a possibly eventual connectivity shows that $f : \Omega_n \to \Omega_{n+1}$ cannot have degree 1 infinitely often, since otherwise f would be injective on the whole plane. Similarly, the degree can't be bounded by a finite value d infinitely often, or f would be finite-to-1 on the whole plane and hence a polynomial.

Thus in the case when Ω is finitely connected, the $f : \Omega_n \to \Omega_{n+1}$ is eventually a covering map with degree tending to ∞ and hence the modulus of the the topological annuli Ω_n also tends to ∞. A stronger result is:

Theorem 2.6.16 (Bergweiler, Rippon and Stallard [**18**]). *If f is a transcendental entire function with multiply connected Fatou component U.*

(1) $c(U_n) = 2$ *iff* $\cup_{m=n}^{\infty} U_n$ *contains no critical point of* f.
(2) $2 < c(U_n) < \infty$ *iff* $\cup_{m=n}^{\infty} U_n$ *contains a finite number of critical points.*
(3) *If* $c(U_n) = \infty$ *iff* $\cup_{m=n}^{\infty} U_n$ *contains infinitely many critical points.*

OPEN PROBLEM: Is there a bounded wandering domain whose iterates are all uniformly bounded? Such a domain would have to be simply connected and perhaps could "orbit" around an irrational neutral fixed point (e.g. a Cremer point).

2.7. Some wandering domains

THEOREM 2.7.1 (Baker). *There exists an entire function with a multiply connected Fatou component, hence with a wandering domain.*

PROOF. The function will be

$$f(z) = z^2 \prod_{k=1}^{\infty} (1 + \frac{z}{R_k}),$$

where $R_k \nearrow \infty$ is a sequence of positive real numbers that we define inductively. Suppose $R_0 > 0$ is large and $S \subset \mathbb{N} = \{1, 2, 3, \dots\}$ (one can take $S = \mathbb{N}$ at a first reading) and set $f_0(z) = F_0(z) = z^2$.

In general, let

$$R_n = \max_{|z|=R_{n-1}} |f_{n-1}(z)|.$$

If $n \in S$, let

$$F_n(z) = (1 + \frac{z}{R_n}),$$

and if $z \notin S$ let $F_n(z) \equiv 1$. Set

$$f_n(z) = \prod_{k=0}^{n} F_k(z),$$

and

$$f(z) = \lim_{n\to\infty} f_n(z) = z^2 \prod_{k\in S} (1 + \frac{z}{R_k}).$$

The first step is to check that the product defining f converges and for this we need to know that $R_k \nearrow \infty$ fast enough. However, each F_k (and hence each f_k) takes its maximum modulus on $\{|z| = r\}$ where this circle intersects the positive real axis, so

$$\begin{aligned} R_n &= \max_{|z|=R_{n-1}} |f_{n-1}(z)| \\ &\geq {R_{n-1}}^2 \prod_{k\in S, k<n} (1 + \frac{R_{n-1}}{R_k}) \\ &\geq {R_{n-1}}^2, \end{aligned}$$

since every term in the product is ≥ 1. Thus $R_n \geq R_0^{2^n}$ and, more generally, $R_n \geq R_k^{2^{n-k}}$ for $1 \leq k \leq n$. From this it easily follows that the product defining f converges uniformly on compact sets.

Next, for $n \in \mathbb{N}$, define the annulus

$$A_n = \{z : \frac{1}{4}R_n \leq |z| \leq 4R_n\},$$

and let B_n be the annulus separating A_n and A_{n+1}, i.e.,

$$B_n = \{z : 4R_n < |z| \leq \frac{1}{4}R_{n+1}\}.$$

We claim that $f(B_n) \subset B_{n+1}$. If this is true, then the iterates of B_n clearly converge uniformly to ∞, so that $B_n \subset \mathcal{F}(f)$. On the other hand, if $n \in S$, then A_n contains a zero of f and 0 is a super-attracting fixed point of f. Thus A_n contains a Fatou component that does not iterate to ∞ and hence must contain some point of the Julia set (in fact a continuum of such points). Thus B_n surrounds a point of $\mathcal{J}(f)$ and the Fatou component containing it must be multiply connected.

FIGURE OF ANNULI

Thus we must prove $f(B_n) \subset B_{n+1}$. The idea is that A_n is bounded by two circles and that after applying f these two circles are further apart; enough so that the region between them contains A_{n+1}.

We break the product for f into three pieces

$$f(z) = (z^2 \prod_{k\in S, k<n} (1 + \frac{z}{R_k})) \cdot F_n(z) \cdot (\prod_{k\in S, k>n} (1 + \frac{z}{R_k})) = I(z) \cdot II(z) \cdot III(z) \tag{15}$$

For $z \in A_n$, the third term is bounded between

$$\prod_{k\in S, k>n} (1 + \frac{-R_n}{R_k})) \leq III \leq \prod_{k\in S, k>n} (1 + \frac{R_n}{R_k}))$$

Now use the estimate $R_k \geq R_n^{2^{k-n}}$ for $k > n$,

$$\prod_{k\in S, k>n} (1 - {R_n}^{1-2^{n-k}}) \leq III \leq \prod_{k\in S, k>n} (1 + {R_n}^{1-2^{k-n}})$$

$$1 - O(R_n^{-1}) \leq III \leq 1 + O(R_n^{-1})$$

and this gives

$$\frac{9}{10} \leq III \leq \frac{10}{9},$$

if R_0 is large enough.

The second term in (15) satisfies $II = 1$ if $n \notin S$, and if $n \in S$, then

$$|II(z)| \leq 3, \quad |z| = 2R_n,$$

$$|II(z)| \geq \frac{1}{2}, \quad |z| = R_n/2,$$

and

$$|II(z)| \leq 2, \quad |z| = R_n.$$

Let $s_n = |S \cap [1,k]|$ be the number of elements in S less than or equal to k and let

$$C_n = \prod_{k\in S, k<n} R_k^{-1}.$$

Then the first term in (15) satisfies

$$\begin{aligned} z^2 \prod_{k\in S, k<n} (1 + \frac{z}{R_k}) &= z^2 \prod_{k\in S, k<n} \frac{z}{R_k}(1 + \frac{R_k}{z}) \\ &= C_n z^{2+s_n} \prod_{k\in S, k<n} (1 + \frac{R_k}{z}) \\ &= C_n z^{2+s_n} \prod_{k\in S, k<n} (1 + O(\frac{R_k}{R_n}) \\ &= C_n z^{2+s_n} (1 + O({R_n}^{-1/2})). \end{aligned}$$

Thus if R_0 is large enough,

$$I = (1 + o(1)) C_n z^{2+s_n}.$$

Thus we can deduce that

$$R_{n+1} = (1 + o(1)) 2 C_n R_n^{2+s_n}$$

$$|f(z)| \leq 2(1 + o(1)) 2^{-2-s_n} < \frac{1}{4} R_{n+1}, \quad |z| = R_n/2,$$

$$|f(z)| \geq 2(1+o(1))2^{2+s_n} > 4R_{n+1}, \quad |z| = 2R_n.$$

Thus the two boundaries of B_n both land inside B_{n+1}, and since, by construction, f has no zeros in B_n (they all lie in the A_n's) the minimum and maximum principles imply $f(B_n) \subset B_{n+1}$. □

By taking the set S very sparse, we can make f grow as slowly as we wish, i.e.,

COROLLARY 2.7.2. *For any $\varphi(t)$ that grows faster than any polynomial, there is an entire function with a wandering domain so that $|f(t)| \leq \varphi(t)$ for t sufficiently large.*

It is not immediately clear whether the wandering domains constructed above are finitely or infinitely connected, but it is easy to make a small change which forces infinite connectivity. With the same inductive definition of $\{R_n\}$, place the zeros slightly outside the circles of radius R_n, i.e.,

$$f(z) = \prod_{k \in S}(1 - \frac{z}{3R_k}).$$

Everything goes through as above to show that $f(B_n) \subset B_{n+1}$ and hence f has multiply connected wandering domains, but now we also can show that for $z \in \gamma$, $\gamma = \{z : |z + 3R_n| = R_n\}$,

$$|f(z)| \geq 4R_{n+1},$$

Hence γ iterates into B_{n+1}, so is in the Fatou set. Moreover, γ is clearly not homotopic to $\{|z| = 4R_n\}$ in the Fatou set since $\mathbb{D}_{R_n}$ contains points of the Julia set. Thus the Fatou component containing B_n always has connectivity at least 3 and hence by Corollary 2.6.15 has infinite connectivity.

THEOREM 2.7.3 (Herman). *$f(z) = z - 1 + e^{-z} + 2\pi i$ has a wandering domain.*

PROOF. This is due to Michel Herman around 1985 (see [**125**]). The map $N(z) = z - 1 + e^{-z}$ is the Newton's method map for $g(z) = e^z - 1$. The basin of attraction for $z = 0$ is invariant under N and the basins for $z = 2\pi i n$ are each translates of this basin (and are disjoint since they iterate to different points). Note that $N(z + 2\pi i) = N(z) + 2\pi i$ so the Julia and Fatou sets of N are $2\pi i$ periodic. Since $f(z) = N(z) + 2\pi i = N(z + 2\pi i)$, if z is a repelling fixed point of N of period k and multiplier λ, then $f^{nk}(z) = z + 2\pi i n k$ and

$$D_E^S f^{nk}(z) \geq \frac{|\lambda|^n}{1 + (|z| + |2\pi n k|)^2} \to \infty,$$

so $\{f^n\}$ is not normal at z by Marty's theorem. Since the repelling fixed points of N are dense in $\mathcal{J}(N)$ (this will be proved later, Theorem 3.4.2), we deduce $\mathcal{J}(N) \subset \mathcal{J}(f)$. On the other hand, since f preserves the Fatou components of N, it is normal on these components, hence $\mathcal{F}(N) \subset \mathcal{F}(f)$. Thus equality holds. Thus each basin for $2\pi i n$ moves by up by $2\pi i$ under f and hence are wandering domains. $\square$

THEOREM 2.7.4 (Baker, [**7**]). *$f(z) = z+\sin z+2\pi$ has a bounded, simply connected wandering domain.*

PROOF. For $g(z) = z+\sin z$, all points $(2n+1)\pi$ are super-attracting fixed points, hence in different Fatou components. Since $g(z + 2\pi) = g(z) + 2\pi$, the Julia set is 2π-periodic and arguing as in the previous proof, $\mathcal{J}(f) = \mathcal{J}(g)$. Thus f maps the g-basin for $(2n+1)\pi$ to the g-basin for $(2n+3)\pi$ and so these are wandering domains for f. All the critical points of g are super-attracting fixed points, so their basins of attraction are simply connected (otherwise they are in the escaping set, but a fixed point can't be).

To see that these components are bounded, note that the imaginary axis is preserved by g, as are its translates by $2\pi\mathbb{Z}$ and that all points on these lines iterate to ∞, except for those on the real line. Thus these vertical lines cannot be in the basins of attraction of $\{(2n + 1)\pi\}$, so these basins are separated by these lines.

We claim these basins are bounded. Suppose Ω_n is the basin of attraction of $(2n + 1)\pi$. We know it is trapped between the vertical lines $L_0 = \{x = 2n\pi\}$ and $L_1 = \{x = (2n + 2)\pi\}$. Suppose Ω intersects the horizontal segment $S = \{2n\pi < x < (2n+2)\pi, y = \frac{\pi}{2}\}$ and let γ be the shortest hyperbolic curve connecting the fixed point $(2n+1)\pi$ to S. Suppose the endpoint on S is $x+iy$ ($y = \pi/2$). By the Schwarz lemma, $f(\gamma)$ has at most the hyperbolic length of γ. Since $y = \pi/2$,

$$\Im g(x+iy) = y+\Im\sin(x+iy) = y+\frac{1}{2}(e^x\sin(y)-e^{-x}\sin(-y)) = y+\frac{1}{2}(e^x+e^{-x}) > y,$$

so $g(\gamma)$ connects the fixed point to a point above S. By the Schwarz lemma the hyperbolic length of $g(\gamma)$ is less or equal the hyperbolic length of γ. Thus a subset of $g(\gamma)$ connects the fixed point to S and has strictly shorter hyperbolic length than

γ, a contradiction. Thus the attracting basins do not intersect the lines $|y| = \pi/2$, and hence the basins are bounded sets. $\square$

CHAPTER 3

Ahlfors islands and repelling fixed points

One of the most important results of the previous chapter was that a family of holomorphic functions that omits two values must be normal. In other words, if there are two complex values whose preimages are empty under every member of the family, then the family is normal. This result is generalized in this chapter to a normality criterion that involves the points with preimage sets of finite size. This in turn gives the famous Ahlfors' Islands theorem that says (rather imprecisely) that a family of holomorphic functions that all fail to be coverings maps in a certain way form a normal family. The power of the result comes from the converse: if the iterates of f fail to be normal (e.g., consider them in a disk centered on the Julia set), then at least one of the iterates must be a covering map in a certain way. In particular, this allows us to show that every disk centered on the Julia contains a connected component of some preimage of itself, and this implies the disk contains a repelling fixed point. Thus repelling fixed points are dense in the Julia set.

3.1. Schwarz lemma for functions with multiple zeros

It is a useful curiosity that the hyperbolic gradient of f can be well defined even if f itself is not well defined. Consider $f(z) = z^{1/2}$ on $\mathbb{D}$. There is a not a well defined branch of f on the whole unit dist, but if $z \neq 0$, and we take a small disk D around z, then $f(z)$ does have two well defined branches in this disk. If we use either branch to compute the hyperbolic gradient at z we get the same answer since we take absolute values in the definition and the branches differ by ± 1.

LEMMA 3.1.1 (Product Rule). *If F, G are multi-valued maps $\mathbb{D} \to \mathbb{D}$, but $|F|, |G|$ are single valued, then*

$$D_H^H FG(z) \leq \max(D_H^H F(z), D_H^H G(z)) \cdot \frac{|F(z)| + |G(z)|}{1 + |F(z)G(z)|} \tag{16}$$

$$\leq \max(D_H^H F(z), D_H^H G(z)). \tag{17}$$

If both F and G are holomorphic, then

$$D_H^H FG(z) \leq \frac{|F(z)|+|G(z)|}{1+|F(z)G(z)|} = T(|F(z)|, -|G(z)|) < 1. \tag{18}$$

PROOF. We copy the usual proof of the product rule to get

$$D_H^H FG(z) \quad = \quad D_H^H F(z) \cdot \frac{|G(z)|(1-|z|^2)}{1-|F(z)G(z)|^2} + D_H^H G(z) \cdot \frac{|F(z)|(1-|z|^2)}{1-|F(z)G(z)|^2}$$

The first claim follows from the equality

$$x\frac{1-y^2}{1-x^2y^2} + y\frac{1-x^2}{1-x^2y^2} = \frac{x+y}{1+xy},$$

which can be proved by simple algebra. The second claim is then immediate from Schwarz's lemma. □

LEMMA 3.1.2. *If $F : \mathbb{D} \to \mathbb{D}$ is holomorphic and every zero has order at least m, then $D_H^H F^{1/m}(z) \leq 1$.*

PROOF. This is simple if every zero has order divisible by m, for then $F^{1/m}$ is a well defined holomorphic function. In general, we may assume that F is holomorphic on a neighborhood of $\overline{\mathbb{D}}$; otherwise we replace $F(z)$ by $F(rz)$ and let $r \nearrow 1$ at the end of the proof. We can write $F = GB$ where $G = H^m$ has a holomorphic m-root and B is a finite Blaschke product

$$B(z) = \prod_k B_k(z) = \prod_k (\frac{z - z_n}{1 - \overline{z_k} z})^{a_k}$$

where $a_k \geq m$ for all k. Then by (16)

$$D_H^H F^{1/m}(z) \quad \leq \quad D_H^H H B^{1/m}(z) \leq D_H^H B^{1/m}(z).$$

Using (18) and induction gives

$$D_H^H F^{1/m}(z) \quad \leq \quad \max_k D_H^H B_k^{1/m}(z).$$

However, B_k is the composition of two maps: $z \to \frac{z-z-k}{1-\overline{z_k}z}$, and $z \to z^{a_k/m}$. The first is an isometry and second is a contraction by (4). □

COROLLARY 3.1.3 (Schwarz's lemma for multiple roots). *If $F : \mathbb{D} \to \mathbb{D}$ is holomorphic and every zero of F has order at least m, then*

$$|F'(0)|^m \leq m^m |F(0)|^{m-1}.$$

PROOF. The stated inequality follows by noting

$$D_H^H F^{1/m}(z) = \frac{|F(z)|^{\frac{1}{m}-1}|F'(z)|(1-|z|^2)}{m(1-|F(z)|^{2/m})} \leq 1.$$

Thus setting $z = 0$ we get

$$|F(0)|^{\frac{1}{m}-1}|F'(0)| \leq m(1-|F(z)|^{2/m}).$$

Finally, raise both sides to the mth power and use the fact that $1-|F(z)|^{2/m} \leq 1$. $\square$

3.2. Perfectly branched points

Our next goal is a result that contain both Picard theorems and Montel's theorem as special cases, as well as various results of Ahlfors, Nevanlinna and others.

Recall from Corollary 2.2.3 that a family is conformally invariant normal if and only if

$$\sup_{z\in\Omega} \sup_{f\in\mathcal{F}} D_H^S f(z) < \infty.$$

and that if Ω is not hyperbolic, then $\mathcal{F}$ contains only constants. Also recall that every function in an conformally invariant normal family is normal and has a meromorphic extension to any isolated boundary point of Ω.

THEOREM 3.2.1. *Suppose $\mathcal{E} = \{a_1, \dots, a_q\}$ are distinct points and suppose $\mathbf{m} = \{m_1, \dots, m_q\} \subset \mathbb{N}$ satisfies*

$$m \equiv \sum_{k=1}^{q}(1-\frac{1}{m_k}) > 2. \tag{19}$$

If $\Omega \subset \widehat{\mathbb{C}}$ is a hyperbolic domain let $\mathcal{F} = \mathcal{F}(\mathcal{E}, \mathbf{m})$ be the collection of meromorphic functions f on Ω so that each preimage $f^{-1}(a_k)$ has degree at least m_k for $k = 1, \dots, q$. Then $\mathcal{F}$ is a conformally invariant normal family.

If a is an omitted value of f then $f^{-1}(a)$ has no pre-images, so trivially every pre-image has degree m for any m. Thus the hypothesis is trivially fulfilled for omitted points with any m_j. In this case we may take $m_j = \infty$ in (19). Thus a meromorphic function on a hyperbolic domain Ω with 3 omitted values is normal, and if Ω is not hyperbolic, then the function is constant. Similarly, holomorphic function on a hyperbolic domain with two omitted values is normal, and an entire functions with two omitted values is constant. Thus this result contains Picard's little theorem. It also contains Picard's great theorem, for if an entire function f omits two values on some neighborhood $\overline{\mathbb{D}}_r^c = \{|z| > r\}$ of ∞, then it has a meromorphic extension to ∞ and hence is a polynomial.

An a-point of f is any point $z \in f^{-1}(a)$. It is called **simple** if $f'(z) \neq 0$ and a is called **perfectly branched** if there are no simple a points (hence $f(z) = a \Rightarrow f'(z) = 0$). The result above implies that a meromorphic function can have at most 4 perfectly branched values and an entire function can have at most 2 such values ($\sin z$ shows this is sharp).

Proof of Theorem 3.2.1. It is easy to see that $\mathcal{F}$ is conformally invariant; if we replace $f : \Omega \to \widehat{\mathbb{C}}$ by pre-composing with a covering map $\varphi : \mathbb{D} \to \Omega$ then a component W $(f \circ \varphi)^{-1}(\{E_j\})$ is compact iff $\varphi(W)$ is compact and the critical points of f in W correspond 1-to-1 with the critical points of $f \circ \varphi$ in $\varphi(W)$ and have the same degrees.

Assume that Ω is hyperbolic (i.e., has at least three boundary points on $\mathbb{S}$); we will deal with the other cases at the end of the proof.

If $\mathcal{F}$ is not a normal family, then we can use Zalcman's lemma (Lemma 2.1.4) to build a sequence

$$f_n(\rho_n z + w_n),$$

that converges to a non-constant, meromorphic f on the whole plane that has bounded spherical gradient

$$D_E^S f(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \leq 1 = D_E^S f(0),$$

and still satisfies (19). (VERIFY?)

If we can prove any such f must be constant, we get a contradiction and can deduce $\mathcal{F}$ is, indeed, a normal family. Our hypothesis says that each $a_j \in \mathcal{E}$ is either an omitted value, (in which case we may take m_j in (19) as large as we wish) or every preimage of a_j is a critical point of degree at least m_j.

Let M be the least common multiple of $\{m_1, \dots, m_q\}$ and define

$$g(z) = \frac{(f'(z))^M}{\prod_{j=1}^q (f(z) - a_j)^{(m_j - 1)M/m_j}}.$$

We claim that g is an entire function. The only way this might fail is at points where the denominator is zero or where the numerator is infinite. To deal with the first case, consider a preimage w of a_j. By definition f has degree $p \geq m_j$ at w, so for z near w we have

$$|f(z) - a_j| \simeq |z - w|^p,$$

and

$$|f'(z)| \simeq |z - w|^{p-1}.$$

Hence the denominator of g satisfies

$$|f(z) - a_j|^{(m_j-1)M/m_j} \sim |z - w|^{(p-p/m_j)M},$$

and its numerator satisfies

$$|f'(z)| \simeq |z - w|^{(p-1)M}.$$

Since $p - p/m_j \leq p - 1$, g is bounded near such points (and is zero at w if $p > m_j$).

In the second case, the numerator is infinite, so f and f' have a pole at w. Recall

$$m = \sum_{k=1}^{q}(1 - \frac{1}{m_k}),$$

and note that

$$|g(z)|^{1/M} \leq \frac{|f'(z)|}{|f(z)|^m}.$$

If f has a pole w, then it tends to ∞ at w and so is bigger than $\max(1, 2A)$ where $A = \max_j |a_j|$ in some neighborhood W of w. If $|f(z)| \geq 2A$, then

$$|f(z) - a_j| \geq |f(z)|/2,$$

and because of the spherical derivative bound,

$$|f'(z)| \leq 1 + |f(z)|^2 \leq 2|f(z)|^2,$$

if $|f(z)| \geq 1$. Thus for $z \in U$,

$$|g(z)| \leq \frac{2|f(z)|^{2M}}{2^{-mM}|f(z)|^{mM}},$$

or

$$|g(z)|^{1/M} \leq \frac{2|f(z)|^2}{2^{-m}|f(z)|^m}.$$

Since $m > 2$, we deduce that $|g| \to 0$ where $|f| \to \infty$, so g is bounded near any pole of f'. Thus g is entire.

Next we claim that g is non-constant. The function f is non-constant by construction (it has spherical derivative 1 at some point) and hence it tends to ∞ along some sequence. By our remarks above g tends to zero along this same sequence, so if g were constant, it would have to be the constant zero. However, this would imply f' is constant zero, and hence f is constant, a contradiction. Thus g is a non-constant entire function.

Since g is entire and not constant, it is unbounded along some sequence $\{z_n\}$. Because f has bounded spherical gradient, Marty's theorem says we can find a convergent subsequence of $h_k(z) = f(z + z_k)$ that converges to a meromorphic h.

We claim h must be constant and equal to one of the a_j. Otherwise, there would be a circle around 0 on which h is never equal to any of the a_j's and h' is never equal to ∞. This implies

$$g(z + z_k) \to \frac{(h'(z))^M}{\prod_{j=1}^{q}(h(z) - a_j)^{(m_j-1)M/m_j}},$$

is bounded on the circle. By the maximum principle, $g(z+z_k)$ would have a bounded limit at $z = 0$, contradicting the fact that we chose $\{z_n\}$ so that $g(z_n) \to \infty$.

So h is one of the a_k's. Suppose $h = a_k$. The function $F(z) = f(z+z_n) - a_k$ only has zeros of order at least m_k so we can apply the Schwarz Lemma for m_kth roots (Corollary 3.1.3). This gives

$$|f'(z_n)|^{m_k} = |F'(0)|^{m_k} \leq m_k^{m_k}|F(0)|^{m_k-1} = m_k^{m_k}|f(z_n) - a_k|^{m_k-1},$$

or, raising to the M/m_k power,

$$|f'(z_n)|^M \leq m_k^M|f(z_n) - a_k|^{(m_k-1)M/m_k}.$$

Thus

$$|g(z)| \leq \frac{m_k^M}{\prod_{j\neq k}(f(z) - a_j)^{(m_j-1)M/m_j}}.$$

However, this is impossible since $f(z_n) \to h(0) = a_k$ and $g(z_n) \to \infty$. The contradiction proves that $\mathcal{F}$ is a normal family. $\square$

COROLLARY 3.2.2. *If the hypotheses of Theorem 3.2.1 hold, but $\Omega \subset \mathbb{S}$ is not hyperbolic, then $\mathcal{F}$ consists of constant functions.*

PROOF. If Ω is not hyperbolic we can assume its boundary points are contained in $\{0, \infty\}$ and consider the family $\mathcal{F}$ restricted to the annulus $A_\epsilon = \{z : \epsilon < |z| < 1/\epsilon\}$. The annulus is a hyperbolic domain, so the previous argument implies that the restricted family is normal, and hence

$$D^S_{\rho(\epsilon)} f \leq C_\epsilon,$$

on A_ϵ where $\rho(\epsilon)$ is the hyperbolic metric on A_ϵ. For $\epsilon \searrow 0$, $D^{\rho(1/2)}_{\rho(\epsilon)}\mathrm{Id} \to \infty$. Thus

$$C_\epsilon \geq D^S_{\rho(\epsilon)} f = D^{\rho(1/2)}_{\rho(\epsilon)}\mathrm{Id} \cdot D^S_{\rho(1/2)} f(z),$$

implies $D^S_{\rho(1/2)} f = 0$, i.e., f is constant on the open set $A_{1/2}$ and hence on all of Ω. $\square$

3.3. The Ahlfors islands theorem

A map $f : \Omega \to \widehat{\mathbb{C}}$ is said to have a **simple island** over an Jordan domain D if there is a component W of $f^{-1}(D)$ that has compact closure in Ω and so that $f : W \to D$ is conformal (1-to-1). If $\overline{W}$ is compact in Ω, then this occurs iff f has no critical points in W.

In some sources, the requirement that an island have compact closure is dropped. Saying there are no islands over D then becomes a stronger conditions, since more components of $f^{-1}(D)$ have to be checked for critical points. However, the weaker version that checks only pre-compact components is sufficient to prove normality.

For our dynamical applications of Theorem 3.2.1, we will only need to consider islands over small disks, and a self-contained proof of this case is given below. The more general case of Jordan domains requires the use of quasiconformal methods; we will state and sketch the result below, but its proof uses the measurable Riemann mapping theorem that will not be discussed until later.

Theorem 3.3.1 (Ahlfors' Island theorem for small disks). *There is an $\epsilon_0 > 0$ so that that the following holds. Suppose $\mathcal{E} = \{D_1, \dots, D_q\}$ are disjoint disks $D_k(a_k, r_k)$ with $r_k \leq \epsilon_0$ and and suppose $\mathbf{m} = \{m_1, \dots, m_q\}$ are positive integers that satisfy*

$$m = \sum_{k=1}^{q} (1 - \frac{1}{m_k}) > 2. \tag{20}$$

If $\Omega \subset \widehat{\mathbb{C}}$ is a domain let $\mathcal{F} = \mathcal{F}(\Omega, \mathcal{E}, \mathbf{m})$ be the collection of meromorphic functions f on Ω so that every connected component of $f^{-1}(D_k)$ that is compact in Ω contains critical points whose degrees sum to at least m_k, for each $k = 1, \dots q$. If Ω is hyperbolic, then $\mathcal{F}$ is a conformally invariant normal family and otherwise $\mathcal{F}$ consists of *constants. If a_k is an omitted value of f then $f^{-1}(D_k)$ has no compact components, so trivially every such component contains critical points of arbitrarily high degree, i.e., we can take $m_k = \infty$ in the definition of m.*

PROOF. As before, we start by checking that $\mathcal{F}$ is conformally invariant; if we replace $f : \Omega \to \widehat{\mathbb{C}}$ by pre-composing with a covering map $\varphi : \mathbb{D} \to \Omega$ then a connected component W of the set $(f \circ \varphi)^{-1}(\{D_j\})$ is compact iff $\varphi(W)$ is compact. Moreover, the critical points of f in W correspond 1-to-1 with the critical points of $f \circ \varphi$ in $\varphi^{-1}(W)$ and have the same degrees. Thus $\mathcal{F}$ is a conformally invariant family of functions.

First suppose Ω is hyperbolic. To show $\mathcal{F}$ is a normal family, choose any q distinct points $\{a_1, \dots a_q\}$ and fix any sequence of $\epsilon_k \searrow 0$. We claim that for ϵ small enough, the family $\mathcal{F}_\epsilon = \mathcal{F}(\{D(a_j, \epsilon)\}_1^q, \mathbf{m})$ is normal. If not, then by Zalcman's lemma we can form a sequence $\epsilon_k \searrow 0$ and $f_k \in \mathcal{F}_{\epsilon_k}$ that converges uniformly on compact sets to a non-constant, meromorphic function $f \in \mathcal{F}(\{a_1, \dots a_q\}, \mathbf{m})$. By Corollary 3.2.2, such a f must be constant, and the contradiction shows that $\mathcal{F}_\epsilon$ must be normal for ϵ small enough. When Ω is not hyperbolic, the argument is exactly as in the proof of Corollary 3.2.2. $\square$

Taking the case $m_1 = \dots m_5 = 1$ in the theorem above gives:

COROLLARY 3.3.2. *Suppose Ω is a hyperbolic domain and $\{a_1, \dots, a_5\}$ are distinct points on the sphere. Suppose $D_1, \dots, D_5$ are sufficiently small disks around these points and $\{\mathcal{F}\}$ is a family of meromorphic functions on Ω so that no element of $\mathcal{F}$ has a simple island over any of the five disks. Then $\mathcal{F}$ is a normal family.*

Taking $m_1 = m_2 = m_3 = 1$ and $m_4 = \infty$ (for the omitted value ∞) gives:

COROLLARY 3.3.3. *Suppose Ω is a hyperbolic domain and $\{a_1, a_2, a_3, \}$ are distinct points on the sphere. Suppose D_1, D_2, D_3 are sufficiently small disks around these points and $\{\mathcal{F}\}$ is a family of holomorphic functions on Ω so that no element of $\mathcal{F}$ has a simple island over any of the three disks. Then $\mathcal{F}$ is a normal family.*

The final version replaces disks by Jordan domains. As noted earlier, we do not use this version in our applications to dynamics, but include it for completeness.

THEOREM 3.3.4. *Suppose* $\mathcal{E} = \{E_1, \dots, E_q\}$ *are disjoint smooth Jordan domains that have pairwise disjoint closures. Suppose* $\mathbf{m} = \{m_1, \dots, m_q\} \subset \mathbb{N}$ *satisfy*

$$\sum_{k=1}^{q}(1 - \frac{1}{m_k}) > 2. \tag{21}$$

If $\Omega \subset \widehat{\mathbb{C}}$ *is a hyperbolic domain let* $\mathcal{F} = \mathcal{F}(\Omega, \mathcal{E}, \mathbf{m})$ *be the collection of meromorphic functions* f *on* Ω *so that every compact connected component of* $f^{-1}(E_k)$ *contains critical points whose degrees sum to at least* m_k*, for each* $k = 1, \dots q$*. If* Ω *is hyperbolic, then* $\mathcal{F}$ *is a conformally invariant normal family and otherwise* $\mathcal{F}$ *consists of constants.*

PROOF. We sketch the proof here, but use several facts about quasiconformal maps that will be discussed in more detail later. The first fact is that a smooth diffeomorphism of the plane that is the identity off a compact set must be K-quasiconformal for some K. The next fact we need is the measurable Riemann mapping theorem. This theorem implies that if f is a meromorphic function and ψ is K-quasiconformal, then there is a K-quasiconformal map ϕ so that

$$g = \psi \circ f \circ \phi$$

is meromorphic.

Assuming these facts, we can finish the proof of the theorem. Again by Zalcman's lemma it suffices to show that $\mathcal{F} = \mathcal{F}(\mathbb{C}, \{E_j\}_1^q)$ consists of constant functions. Suppose $\mathcal{F}$ contains a non-constant function f. Choose any five distinct points $\{a_1, \dots a_5\}$ and choose ϵ so small that the disks $D_j = D(a_j, \epsilon)$ have pairwise disjoint closures and the family $\mathcal{F}_\epsilon$ defined above is normal. There is a diffeomorphism ψ defined on $\mathbb{C}$ that maps each E_j into D_j for $j = 1, \dots, q$. By the measurable Riemann mapping theorem (Theorem 12.4.1), there is another quasiconformal map ϕ so that

$$g = \psi \circ f \circ \phi,$$

is meromorphic and in $\mathcal{F}_\epsilon$. But then g is constant and therefore f is constant as well. $\square$

Some special cases of the results above include:

COROLLARY 3.3.5. *Suppose* $\Omega \subset \widehat{\mathbb{C}}$ *and* $\mathcal{F}$ *is one of the following families:*

H1: *Meromorphic functions on Ω with no simple islands over five Jordan domains with disjoint closures.*

H2: *Meromorphic f with five perfectly branched values.*

H3: *Meromorphic functions with three omitted values.*

H4: *Holomorphic functions on Ω with no simple islands over three Jordan domains with disjoint closures.*

H4: *Holomorphic functions with three perfectly branched values.*

H6: *Holomorphic functions with two omitted values.*

Then all the following hold

C1: *$\mathcal{F}$ is a conformally invariant normal family if Ω is hyperbolic.*

C2: *$\mathcal{F}$ contains only constants if Ω is not hyperbolic.*

C3: *Each $f \in \mathcal{F}$ is meromorphic at any isolated boundary point of Ω.*

The implication "H6 $\Rightarrow$ C1" is Montel's theorem, "H6 $\Rightarrow$ C2" is Picard's little theorem, and "H6 $\Rightarrow$ C3" is Picard's great theorem. The case "H1 $\Rightarrow$ C1" is called the Ahlfors 5-Island theorem. "H2 $\Rightarrow$ C2" is due to Nevanlinna page 102 0f [**105**], Section X.3 of [**104**], while "H2 $\Rightarrow$ C1" appears in the work of Bloch, Theorem XILV, [**32**] and Valiron Theorem XXVI, [**140**].

3.4. Repelling points are dense

LEMMA 3.4.1. *There is an $\epsilon > 0$ so that for any meromorphic f and any five disjoint disks $D_1, \dots D_5$, each of radius $\leq \epsilon$ hitting $\mathcal{J}(f)$, there is a $j \in \{1, \dots, 5\}$ and a simple island for some iterate of f over D_j contained inside D_j. For entire functions we can replace "5" by "3".*

PROOF. For a disk hitting the Julia set of f, the iterates of f form a non-normal family and the Ahlfors island theorem may be applied. Inside each of the five disks D_k, $k = 1, \dots 5$, choose 5 disks $\{D_{kj}\}_{j=1}^5$ centered on $\mathcal{J}$ and with disjoint closures (we can do this because the Julia set has no isolated points, Lemma 2.5.6). By 25 applications of Theorem 3.3.1, each of these 25 disks contains a simple island for some iterate of f over one of the D_j's. Thus one of the D_j's occurs at least 5 times, i.e., there is a D_j that has a simple island in D_{kn} for at least five distinct disks. Choose such a D_j and choose five of the smaller disks containing an island over D_j. Apply the

Ahlfors islands theorem again with domain D_j to deduce that D_j contains a simple island over one of these disks, say D_{ki}. Thus there are subdomains $W_1 \subset D_j$ and $W_2 \subset D_{ki}$ and iterates of f so that $f^n : W \to D_{ki}$ is conformal and $f^m : W_2 \to D_j$ is conformal. Thus if $W_3 = f^{-n}(W_2) \subset W_1 \subset D_j$, then $f^{n+m} : W_3 \to D_j$ is simple island over D_j that is contained in D_j. For entire functions, the same argument works with three in place of five. □

FIGURE OF 25 ISLANDS

Recall that a point z is periodic for f if there is a positive integer n so that $f^n(z) = z$. The multiplier at z for f^n is $\lambda = (f^n)'(z)$. The periodic point is called attracting if $|\lambda| < 0$; such points are clearly in the Fatou set. The point is called neutral if $|\lambda| = 1$. This case is split into two sub-cases, rational and irrational, depending on whether $\lambda = e^{i\theta}$ with θ a rational multiple of π. We shall see later that rational neutral points are always in the Julia set; irrational neutral points may be in either the Fatou set (Siegel disk) or the Julia set (Cremer points). The final possibility is when $|\lambda| > 1$; these are called repelling points are are clearly always in the Julia set.

We already know that the Julia set is the closure of the pre-periodic points that it contains (Corollary 2.4.11). Here we prove a refinement of this:

THEOREM 3.4.2. *Suppose f is an entire function and $\mathcal{J}(f)$ contains at least three points. Then $\mathcal{J}(f)$ is the closure of its repelling periodic points.*

PROOF. Repelling periodic points are clearly in $\mathcal{J}$ so we only need show every disk D hitting $\mathcal{J}$ contains such a point. Choose $\epsilon > 0$ so small that we can choose three disks $D_j = D(a_j, \epsilon)$ centered on at points of the Julia set that have disjoint closures and so that Lemma 3.4.1 applies. Thus there is a domain U properly contained in one of the D_j that is mapped univalently to D_j by some f^n. The inverse branch is a mapping of D_j into itself and must have an attracting fixed point This fixed point is clearly a repelling periodic point for f. □

The hypothesis holds whenever f is entire and not constant or linear, since in this case $\mathcal{J}(f)$ is a perfect set, and hence uncountable.

Let $K(f)$ denote the set of points such that $\{f^n(z)\}$ is a bounded set; this generalizes the "filled Julia set" of a polynomial.

THEOREM 3.4.3. *If f is entire but not linear, then* $\dim(K(f) \cap \mathcal{J}(f)) > 0$.

PROOF. A minor variation of the proof of Lemma 3.4.1 shows that given distinct a_1, a_2, a_3 in the Julia set there is a disk D around one of them that contains two islands U_1, U_1 over itself with disjoint closures. Then $f^n : U_1 \cup U_2 \to D$ is an iterated function system that has a Cantor repeller of positive Hausdorff dimension, and all points in the Cantor set have orbits that return to D every n steps, and hence are bounded. $\square$

It is easy to construct polynomials whose Julia set is a Cantor set with dimension close to zero. However, we earlier (Theorem 2.6.7) that a transcendental Julia set always has dimension at least 1. The result above is still sharp since it it possible to construct examples where the set of points with bounded orbits has dimension close to zero. This is due to Bergweiler [].

3.5. The blowing-up property

Recall that $\mathcal{J}^*(f)$ denotes the Julia set of f minus the exceptional point (if one exists; an exceptional point is one whose preimages do not accumulate on the entire Julia set).

LEMMA 3.5.1. *A point z is exceptional for a transcendental f if and only if it is exceptional for every iterate of f.*

PROOF. Note that if z is exceptional for f, then $f^{-1}(z) \subset \{z\}$, for every preimage would have to be exceptional too, and there is at most one exceptional point. Thus exceptional points for f are obviously exceptional for every iterate of f. On the other hand, if f has no exceptional point, then $f^{-1}(z)$ contains infinitely many points and thus f^{-n} contains infinitely many points for every ≥ 1. Thus no iterate of f has an exceptional point. $\square$

The following is the "blowing-up property" of the Julia set. We had proved a weaker version of this earlier (see Lemma 2.4.7). Recall that $\mathcal{J}^*(f)$ is the Julia set of f with the exceptional point for f removed (if there is an exceptional point).

LEMMA 3.5.2. *If U is an open set that intersects the Julia set and $K \subset \mathcal{J}^*(f)$ is compact, then $K \subset f^n(U)$ for all sufficiently large n.*

PROOF. Choose a repelling periodic point $z_0 \in U$, let p denote its period and let $g = f^p$. Without loss of generality we may assume $p \geq 2$, and this case no point in the orbit of z is the exceptional point of f (for then they would all be exceptional, contradicting the fact there is at most one exceptional point).

Then z is a repelling fixed point of g. Choose $\epsilon > 0$ so small that $D = D(z_0, \epsilon) \subset g(D)$. Let $V = g(D)$ Clearly $V \subset g(V) \subset g^2(V) \cdots$. Since the sequence of iterates of g is not normal on D, the images of D under these iterates cannot omit two points by Montel's theorem. Thus the union of images covers the whole plane with at most one exception. Thus a finite union of these sets covers any compact set K disjoint from this exceptional point. Since the sets are nested, one them covers K and every following iterate also covers K.

Take K_0 to be the closure of the first p iterates of D under f. Note that $K_0 \subset f(K_0) \subset f^2(K_0) \subset \ldots$. If D is small enough this does not contain the exceptional point for g. Thus there is a q so that $K_0 \subset g^q(D)$. Any integer $n \geq qp$ can be written as $n = qp + kp + r$ for some integers $k \geq 0$, $0 \leq r < p$, so for such an n,

$$f^n(D) = f^{qp+kp+r}(D) \supset g^k(f^r(g^q(D)) \supset g^k(f^r(K_0)) \supset g^k(K_0) \supset g^k(D).$$

Thus for all sufficiently large n, $f^n(D)$ covers any compact set that $g^k(D)$ can cover. Since $g^k(V) = f^{pk}(V)$, this proves the theorem.

Note that □

Another useful formulation is:

COROLLARY 3.5.3. *Suppose U is a bounded open set that hits the Julia set. Then the outer boundary of $f^n(U)$ eventually surrounds every point of the plane.*

PROOF. The previous lemma implies that for each $r > 0$, either $\{|z| = r\}$ or $\{|z| = r + 1\}$ is covered by some iterate of U (only one of these can contain the exceptional value). In either case, the outer boundary of $f^n(U)$ separates $\{|z| = r\}$ from ∞. □

COROLLARY 3.5.4. *Suppose U is a bounded open set that hits the Julia set. Then* $\mathrm{diam}(f^n(U)) \to \infty$.

PROOF. Obvious from the previous result. □

Corollary 3.5.5. *If f is a transcendental entire function and U is an open set that intersects the Julia set, then no subsequence of iterates of f converges uniformly on compact subsets of U.*

Proof. If not, there would be a compact disk D in U that hits $J(f)$ and a sequence of iterates of f that converged to a bounded function of f. However, in the previous result we can choose K to have arbitrarily large diameter, so iterates of D must have diameters tending to ∞. □

3.6. Neutral periodic points

Let $\mathcal{P}S(f) = \cup_{k=0} f^k(\mathcal{S}(f))$ denote the union of the singular set of f and all of its iterates under f. As usual, $\overline{\mathcal{P}S(f)}$ denotes its closure. Given a disk D that does not intersect $\overline{\mathcal{P}S(f)}$, we can define branches of f^{-n} and each branch is holomorphic, 1-to-1 and does not hit $\overline{\mathcal{P}S(f)}$. If $\overline{\mathcal{P}S(f)}$ has only one point then f is a self-mapping of a punctured plane. Otherwise, $\overline{\mathcal{P}S(f)}$ has at least two points and by Montel's theorem the inverse branches of f^{-n} on D form a normal family.

A point z is **periodic** for f if $f^p(z) = z$ and the minimal $p \geq 1$ for which this holds is called the **period** of z. Thus z has period p iff z is a fixed point of f^p. The power series for f^p has the form

$$f^p(z) = z_0 + \lambda(z - z_0) + a(z - z_0)^m + \dots .$$

Here λ is the called the **multiplier** of z_0 and m is called the **multiplicity**. If $|\lambda| < 1$ then z_0 is an attracting fixed point and is in the Fatou set. If $|\lambda| > 1$ then z_0 is a repelling fixed point and is in the Julia set. If $\lambda = \exp(i2\pi\frac{p}{q})$ for $p, q \in \mathbb{N}$, $q \neq 0$, then z_0 is called a rationally neutral and is in the Julia set. Later we will need to know that "most" points near a rationally neutral fixed point are attracted to it:

Lemma 3.6.1. *If z_0 is on a rationally neutral cycle for f then*

$$X = \{z : |z - z_0| < \delta \arg(z^m)| \leq \epsilon\} \subset \mathcal{F}(f),$$

for any $\epsilon > 0$ and some choice of m and δ. In particular, every point of X converges to z_0.

Proof. The proof is a computation that shows that X maps into itself. □

If $\lambda = \exp(i2\pi\alpha)$ for an irrational real number α, then z_0 is called an **irrationally neutral** foxed point. Such a point can either be in the Julia set or in the Fatou set. In the first case, z_0 is called a Cremer point and in the second case the component of $\mathcal{F}(f)$ is called a Siegel disk.

LEMMA 3.6.2. *If z_0 is a irrationally neutral periodic point in the Julia set, then $z_0 \in \overline{\mathcal{P}S(f) \setminus \{z_0\}}$.*

PROOF. If not, then we can choose a punctured disk U around z_0 that misses the post-singular set and hence find holomorphic branches of f^{-1} on U that fix z_0. These form a normal family since the images miss any periodic orbit disjoint from U and hence there is a subsequence converging a holomorphic limit h so that $|h'(z_0)| = 1$. Then $V = h(U)$ is a neighborhood of z_0 and there is a sub-sequence of $\{f^n\}$ that converges to h^{-1} on V. This contradicts Corollary 3.5.5, proving the result. □

LEMMA 3.6.3. *If z_0 is a irrationally neutral periodic point in a component Ω of the Fatou set, then $\partial\Omega \subset \overline{\mathcal{P}S(f)}$.*

PROOF. If Ω were multiply connected, then it would be escaping (Theorem 2.5.1) and hence could not contain a periodic point. Thus Ω must be simply connected (and not the whole plane since the Julia set must be non-empty, Theorem 2.4.5). Using the Riemann mapping theorem, we see that $f : \Omega \to \Omega$ lifts to a map $F : \mathbb{D} \to \mathbb{D}$ that also has an irrationally neutral fixed point. Thus there is a sequence of iterates if F that tend to the identity and the same sequence of iterates of f tends to the identity uniformly on compact subsets of Ω.

Suppose U is an open set that hits $\partial\Omega$ and suppose $U \cap \overline{\mathcal{P}S(f)} = \emptyset$. We can find holomorphic branches of f^{-n} on U and these avoid values on a neighborhood of z_0, so they form a normal family. Choose a compact disk $D \subset U \cap \Omega$ and choose a subsequence that converges to the identity on D and a further subsequence that converges uniformly on all compact subsets of U. The limit function is necessarily the identity on all of U since it is the identity on D. For the same subsequence, f^n converges to the identity on U, contrary to Corollary 3.5.5. □

LEMMA 3.6.4. *A rationally neutral fixed point only attracts points in the Fatou set.*

Proof. Standard result as in Corollary 7.4 of Milnor's book. Must show that the attracting petal is in Fatou set, so the Julia set near 0 must

∎

CHAPTER 4

The Eremenko-Lyubich class

In this chapter we consider a class of transcendental entire functions that share some properties with polynomials and that make them easier to study than general entire functions. The class is named for Alexandre Eremenko and Misha Lyubich defined and studied this class in [**49**] and it consists of all transcendental entire functions have have a bounded singular set (defined below). Such functions have nice behavior when $|f|$ is large; in particular, they have a strong expansion property near infinity that is extremely useful. For example, we shall use this expansion property to show that the Julia set of an Eremenko-Lyubich function is the closure of the escaping set; for general entire functions the Julia set is the boundary of the escaping set (Theorem 2.4.5).

4.1. The singular set

Suppose f is a transcendental entire function. A critical point of f is a zero of f' and a critical value is $f(z)$ where z is a critical point. A asymptotic value is a $w \in \widehat{\mathbb{C}}$ so that $\lim f(z) = w$ along a curve $\gamma : [0, \infty)$ that tends to ∞.

The assumption that γ tends to ∞ is stronger than we need. If f has a limit along a path γ that is unbounded, but returns to some compact set infinitely often, then there is some annulus $A = \{z : r < |z| < 2r\}$ that γ crosses infinitely often and hence subarcs $\gamma_n \subset A$ with endpoints on different boundary components such that $|f - a| < 1/n$ on γ_n. This implies $f = a$ at some point on every circle $\{|z| = t\}$ for $r < t < 2t$ and hence f is constant. Thus to check that a is an asymptotic value of f, we only need verity f has limit a along some unbounded path. We will use this observation below.

The singular values of f, denoted $S(f)$, is defined as the closure or the union of critical values and finite asymptotic values. This is not completely standard; some authors do not take the closure in this definition, or take the following lemma as the definition.

Using the notation in the book of Ahlfors and Sario, a map $f : X \to Y$ is a **smooth covering map** if every point $x \in X$ has a neighborhood V that is mapped homeomorphically by f onto a neighborhood of $f(x)$.

LEMMA 4.1.1. *Suppose f is entire and U contain no critical values. Then f is a smooth covering map from $V = f^{-1}(\Omega)$ to Ω.*

PROOF. If $z \in V$ then $f(p) \neq S(f)$, so $f'(z)$ exists and is non-zero. Thus a small enough disk around z maps homeomorphically to a neighborhood of $f(z)$. $\square$

The map is called a **regular covering map** if given any $y \in Y$ and any $x \in X$ any such that $f(x) = y$, then any arc in Y starting at y can be lifted to an arc in X starting at x. It is a standard result (e.g., Theorem 14C of [**4**]) that any two liftings of the same arc with the same initial point must agree, but the existence of a lifting is not always true. The monodromy theorem say that if two arcs in Y have the same endpoints and are homotopic by a homotopy that keeps the endpoints fixed , then any lifts of these arcs that have the same initial point, must also have the same terminal point. This is proved by noting that the homotopy lifts to a homotopy whose terminal point must always lie in $f^{-1}(b)$; since this is a discrete set, any continuous motion within it must be constant.

LEMMA 4.1.2. *Suppose f is entire and U contain no singular points. Then f is a regular covering map from $V = f^{-1}(\Omega)$*

PROOF. From the previous lemma we know f is smooth covering map on V. Choose points $z \in V$, $w \in U$ such that $f(z) = w$ and let $D = D(w, \epsilon)$ be so small that $D \cap S(f) = \emptyset$. Define a branch g of f^{-1} so that $g(w) = z$ and extend it along a radius of D as far as possible. Because f is a smooth covering map, this extension is possible along some maximal open interval $[0, t)$. If $t < \epsilon$, consider the lifted arc corresponding to this radial segment. We claim it leaves every compact set, for if it stayed within some compact set then we could take a sequence of points on the lifted path that converged to a point that, by continuity of f, must map to $w + te^{i\theta}$. This contradicts the maximality of t. Thus the lift leaves every compact set, but f has a limit along the lift, showing f has an asymptotic value in D, a contradiction. Thus g can be defined on all of D. Thus the connected component W of $f^{-1}(D)$ containing z is mapped onto D by f. If two points of this component map to the same point of D, then an arc connecting these points maps to a closed loop in D. Since D is simply connected, this loop i s homotopic to constant path, hence the arc in W must have been constant, hence the two points were actually a single point. Thus f is a bijection from W to D. This proves that f is a regular covering map over U. $\square$

Corollary 4.1.3. *Suppose f is entire and $S(f) \subset \mathbb{D}_R = \{z : |a| < R\}$. Then f is covering map from $\Omega = f^{(}\overline{\mathbb{D}}_R^c) = \{z : |f(z)| > R\}$ to $\overline{\mathbb{D}}_R^c = \{z : |z| > R\}$. Each connected component of Ω (called a tract of f) is an unbounded, simply connected domain whose boundary is an analytic Jordan curve that tends to ∞ in both directions.*

Singular values are further classified by the behavior of f. For $a \in \widehat{\mathbb{C}}$ and $r > 0$, let D_r be a disk of radius r around a in the spherical metric. Let $\{U_r\}_{r>0}$ be a family of connected components of $f^{-1}(D_r)$ chosen so that $U_r \subset U_s$ if $r < s$. Then $E = \cap_{r>0} U(r)$ is a connected set and $f = a$ on E, so either $E = \emptyset$ or E is a single point (otherwise f would be constant on a non-trivial connected set, hence constant everywhere). If E is a single point z and $f'(z) \neq 0$ then z is called a regular a-point. If $E = \{z\}$ and $f'(z) = 0$ then z is a critical a-point. If $E = \emptyset$ then a is an asymptotic value of f (argue as in Lemma 4.1.2). In this case, the tract $\{U_r\}$ is called an asymptotic tract of f.

Suppose f is a transcendental entire function. If $S(f)$ is finite, we say f is finite type or in the Speiser class, denoted $\mathcal{S}$. If $S(f)$ is bounded, we say f is bounded type or in the Eremenko-Lyubich class, denoted $\mathcal{B}$. A little care needs to be taken with the terms "finite type" and "bounded type" since these are also used to mean something different in Nevanlinna theory.

Lemma 4.1.4. *If $f \in \mathcal{B}$, then every component of $\mathcal{F}(f)$ is simply connected.*

Proof. Suppose $f \in \mathcal{B}$ and choose $R > 0$ so that $S(f) \subset \mathbb{D}_R$. Let $\Omega = f^{-1}(\mathbb{D}_R^*)$. By Lemma 4.1.2, f is a regular covering map from each component of Ω to $\mathbb{D}_R^*$. Since $\mathbb{D}_R^*$ is unbounded, each component of Ω is unbounded, but $|f| = R$ on the boundary. Thus by Corollary 2.6.8 we get $\square$

4.2. Logarithmic coordinates and expansion near infinity

The right half-plane $\mathbb{H}_r = \{x + iy : x > 0\}$ is a simply connected covering space of $\mathbb{D}_R^*$ with the map $z \to R\exp(z) = \exp(z + \log R)$. Thus if W is a component of Ω we can lift $f : W \to \mathbb{D}_R^*$ to a conformal map $\tau : W \to \mathbb{H}_r$, so that $f(z) = R\exp(\tau(z))$ on W.

Choose a point $a \notin W$ and let $U = \log(W - a)$. Since W is simply connected, this is well defined and single valued and the domain U intersects each vertical line in length $\leq 2\Pi$. In particular, no point of U is more than distance π from some boundary point. If $R > |f(0)|$ then we can take $a = 0$; we shall make this assumption from here on.

Define $F : U \to \mathbb{H}_r$ by $F(z) = \tau(a + \exp(z))$. Then F is conjugate to f in the sense that

$$\exp(F(z)) = f(a + \exp(z)),$$

and

$$\exp(F^n(z)) = f^n(\exp(z)),$$

if the orbit of z stays inside Ω. We refer to F as "f in logarithmic coordinates".

LEMMA 4.2.1. *With notation as above,* $|F'(z)| \geq \frac{1}{4\pi}(\Re(F(z)) - \log R)$.

PROOF. If we apply the Koebe $\frac{1}{4}$-theorem to f at z, we get

$$\pi|F'(z)| \geq \operatorname{dist}(z, \partial U)|F'(z)| \geq \frac{1}{4}\operatorname{dist}(F(z), \partial\mathbb{H}_r) = \frac{1}{4}\Re F(z).$$

□

THEOREM 4.2.2. *If* $f \in \mathcal{B}$ *then* $I(f) \subset \mathcal{J}(f)$.

PROOF. Suppose not, i.e., suppose there is a point $z \in \mathcal{F}(f) \cap I(f)$ and let D be a closed disk centered at z and inside $\mathcal{F}(f)$. Then f^n converges uniformly to ∞ on D and hence $f^{n+1}(D) \subset \mathbb{D}_R^*$ for n large enough. This means $f^n(D) \subset \Omega = f^{-1}(\mathbb{D}_R^*)$ for all $n \geq m$, for some m. By replacing D by a disk centered at z_m, we may assume $m = 0$.

Let $w = \log(z) \in U$ and let $V = \log D \subset U$. Then the iterates of V under F stay in U forever, Then $w_n = F^n(w) = \log(f^n(z))$. Since $|f^n(z)| \to \infty$, we have

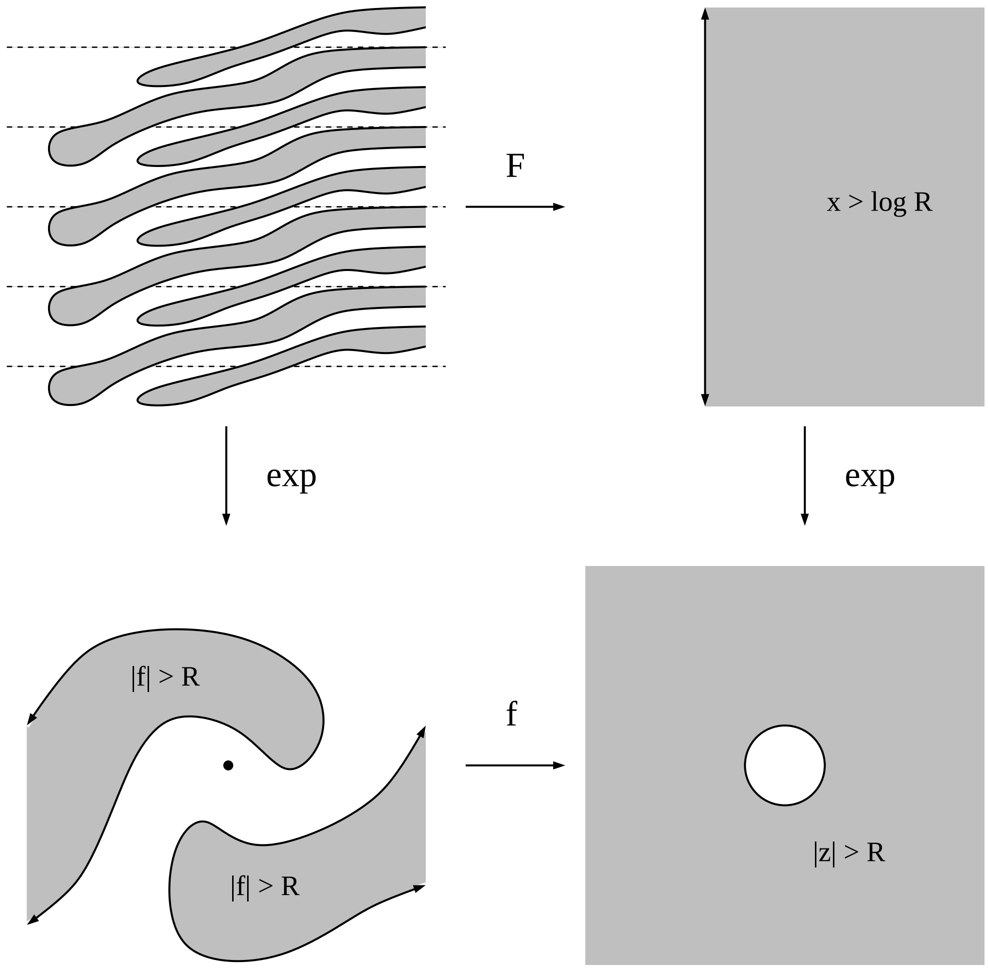


FIGURE 1. Logarithmic coordinates. On the lower left are the tracts of f (two in this case). At upper right are the logarithmic tracts; there are countable many log-tracts for each tract of f, and each is a vertical translate of any other by an integer multiple of 2π.

$\Re F(w_n) \to +\infty$, and hence

$$(F^n(w))' = \prod_{k=1}^{n} F'(w_k) \geq \prod_{k=1}^{n} \frac{\Re(w_{k+1})}{4\pi}.$$

Since w_k is never a critical point of F and since $\Re w_k \to \infty$, the product on the right tends to ∞. However, by Koebe's $\frac{1}{4}$-theorem

$$|(F^n(w))'| \leq 4\mathrm{dist}(w_n, U)/\mathrm{dist}(w, V) \leq 4\pi/\mathrm{dist}(w, V),$$

is bounded independent of n. $\square$

Note that although Fatou points don't escape, they can have unbounded orbits (but these orbits return to some compact set infinitely often).

Corollary 4.2.3. *If* $f \in \mathcal{B}$ *then* $\mathcal{J}(f) = \overline{I(f)}$.

Proof. For any transcendental entire function we have $J(f) = \partial I(f) \subset \overline{I(f)}$. We have just proved $I(f) \subset \mathcal{J}(f)$, hence $\overline{I(f)} \subset \mathcal{J}(f)$. Thus equality holds. □

The proof of Theorem 4.2.2 only uses that the sequence $\{w_k\}$ has sufficiently large real parts, not that it tends to ∞. Thus the proof actually shows

Corollary 4.2.4. *If* $f \in \mathcal{B}$, *then there is* $R > 0$ *so that* $\{z : |f^n(z)| \geq R \forall n\} \subset \mathcal{J}(f)$.

Since no Fatou point can escape, we see immediately that

Corollary 4.2.5. *If* $f \in \mathcal{B}$ *has a wandering domain, then it either has a bounded orbit or the orbit hits some compact set infinitely often.*

The latter case is known to occur. As of this writing, it is still an open problem whether a function in $\mathcal{B}$ (or any entire function) can have a wandering domain with bounded orbits.

4.3. The Julia set of e^z is the whole plane

The following is due to by Misiurewicz [**99**], but we will give a proof due to Rempe and Mihaljević-Brandt [**97**]

Theorem 4.3.1. $\mathcal{J}(e^z) = \mathbb{C}$.

Proof. Since $f(z) = e^z$ has no critical points and one finite asymptotic value, it is in the Speiser class and hence the Julia set contains the escaping set (Corollary 4.2.3). Moreover, it is easy to check that $\mathbb{R} \subset I(f)$, so $\mathbb{R} \subset \mathcal{J}(f)$. Similarly, every preimage of $\mathbb{R}$ must be in the Julia sets and this consists of all the horizontal lines

$$\{z = x + iy : y = \pi n, n \in \mathbb{Z}\}.$$

These lines cut the plane into strips

$$S_n = \{x + iy : n\pi < y < (n+1)\pi\},$$

and f is a univalent map from each strip onto either the upper half-plane, $\mathbb{H}$, or the lower half-plane, $\mathbb{L}$. Thus

$$D^U_{S_n} f \equiv 1, \tag{22}$$

where $U = \mathbb{H} \cup \mathbb{L} = \{x + iy : |y| > 0\}$.

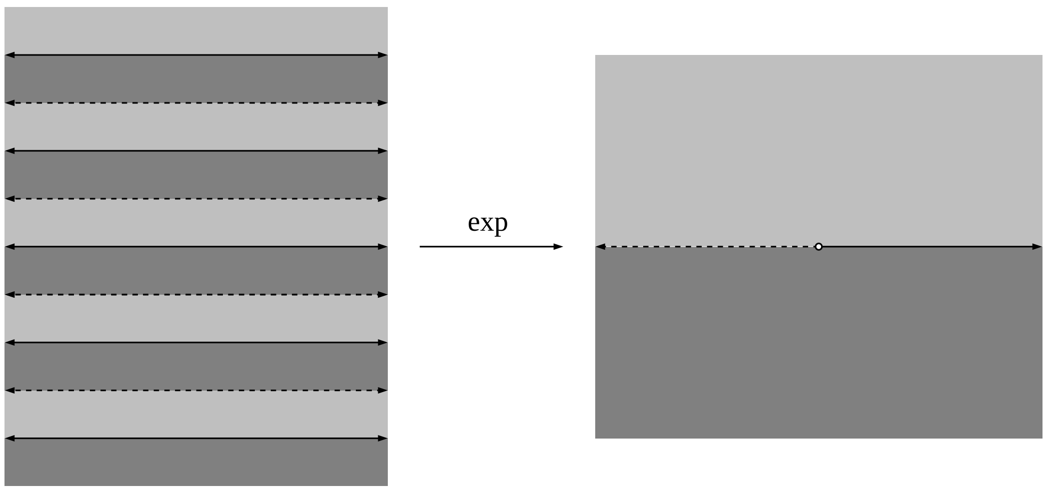


FIGURE 2. The action of the exponential map. Each horizontal strip of width π is mapped conformally to either the upper or lower half-plane.

Suppose $\mathcal{J}(f) \neq \mathbb{C}$ and let W be a component of the Fatou set. Then W lies in some S_n, and hence f restricted to W is 1-to-1. In particular it is an isometry from the hyperbolic metric on W to the hyperbolic metric on $f(W)$. Thus

$$D^{f(W)}_W f \equiv 1. \tag{23}$$

The inclusion map S_n into its corresponding half-plane is a contraction of hyperbolic metrics; a strict contraction if $|y| > \epsilon$. Thus

$$D^U_U f(x+iy) = D^{S_n}_U f(x+iy) D^U_{S_n} \mathrm{Id} * x + iy) \geq 1 \cdot D^U_{S_n} \mathrm{Id} * x + iy) \geq 1 + \eta(|y|) > 1,$$

where η is strictly increasing and $\eta(0) = 1$.

Now suppose $z_0 \in W$ and $z_k = f^k(z_0)$ is its orbit under f. Then since $f^n : W \to f^n(W)$ is conformal,

$$\begin{aligned} 1 = D_W^{f^n(W)} f^n(z) &= D_W^U \mathrm{Id}(z) \cdot D_U^U f^n(z) \cdot D_U^{f^n(W)} \mathrm{Id}(z_n) \\ &\geq D_W^U \mathrm{Id}(z) \cdot (\prod_{k=0}^{n-1} D_U^U f(z_k)) \cdot 1 \\ &\geq D_W^U \mathrm{Id}(z) \cdot \prod_{k=0}^{n-1} (1 + \eta(y_k)). \end{aligned}$$

The first term in the last line is a positive constant that is independent of n, and the product in the last term tends to ∞ unless $\eta(y_k) \to 0$, which happens if and only if $y_k \to 0$. Thus $\lim_k y_k = 0$.

However, $|\Im(z)| \leq \pi/4$ implies

$$\Re(f(z)) = e^x \cos y \geq e^x/\sqrt{x} \geq x + (1 - \ln 2) > x + \frac{3}{10},$$

(use calculus to minimize $(e^x/\sqrt{2}) - 1$). Thus if $|y_k| < \pi/4$ for all large k we must have $x_k = \Re f^k(z) \nearrow +\infty$. This implies $z_0 \in I(f) \subset \mathcal{J}(f)$, contrary to our assumption that $z \in \mathcal{F}(f)$. Thus $\mathcal{F}(f) = \emptyset$ and $\mathcal{J}(f) = \mathbb{C}$. $\square$

4.4. Julia sets with zero area

Next we consider some examples of slightly smaller Julia sets:

THEOREM 4.4.1. area$(\mathcal{J}(\lambda e^z)) = 0$ *if* $0 < \lambda < 1/e$.

We will prove this by way of a more general result. First we will give a condition that implies the escaping set, $I(f)$ has zero area and then another condition that implies $\mathcal{J}(f) \setminus I(f)$ has zero area. Both conditions will apply to

The Julia set of the e^z is the whole plane, but the following result implies its escaping set has zero area.

The following is Theorem 7 from [**49**].

THEOREM 4.4.2. *Suppose* $f \in \mathcal{B}$ *satisfies*

$$\liminf_{r \to \infty} |\{\theta : |f(re^{i\theta})| < R\}| > 0 \tag{24}$$

for some $R < \infty$. Then $\text{area}(I(f)) = 0$*. Moreover, there is a $M < \infty$ so that for a.e. $z \in \mathbb{C}$,*

$$\liminf_{n\to\infty} |f^n(z)| < M.$$

PROOF. If (24) holds for some R, it also holds for all larger values, so we assume that R is so large that (24) holds, $|f(0)| < R$ and $S(f) \subset D(0, R/2)$. Estimate (24) implies that if U is the logarithmic tract of f, then

$$\text{area}(D(x+iy, \frac{x}{4}) \cap U) \leq (1-\epsilon)\text{area}(D(x+iy, \frac{x}{4})), \tag{25}$$

for some uniform $\epsilon > 0$, as long as x is sufficiently large. For the proof of this, see Figure 3.

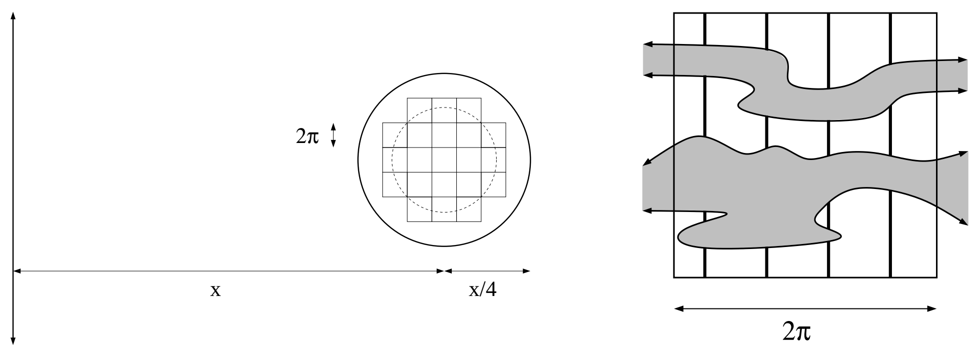


FIGURE 3. In a disk D of radius $r = x/4$ the concentric disk of radius $r - 2\pi\sqrt{2}$ is covered by squares of side length 2π. In each such square, a fixed fraction of the length of each vertical line lies outside the logarithmic tracts by assumption, hence a fixed fraction of the area of the square outside the logarithmic tracts. Thus if $x > 8r = 16\pi\sqrt{2}$, then a fixed fraction of the area of D lies outside these tracts.

Fix M large enough so that $z = x + iy \in U$ and $x \geq M$ imply $|F'(x+iy)| \geq 8$. This is possible by Lemma 4.2.1. Define

$$Y = \{z : \Re(F^n(z)) > M \quad \forall n \in \mathbb{N}\}.$$

We claim that $\text{area}(Y) = 0$ if M is large enough. To prove this, suppose $z_0 \in Y$. We will construct a sequence of disks $D_n = D(z_0, r_n)$ that shrink down to z_0 with the property

$$\limsup_n \frac{\text{area}(D_n \cap Y)}{\text{area}(D_n)} < 1 - \delta < 0.$$

The Lebesgue density theorem then implies that Y has zero area.

Let $z_n = F^n(z_0)$ and for $k = 1, \dots, n$ let F_k^{-1} denote the branch of F^{-1} taking z_k to z_{k-1}. A direct computation shows that the Euclidean disk $D(z_n, x_n/4)$ is contained in a hyperbolic ball (for the right half-plane) ball centered at x of radius

$$\int_{3x_n/4}^{x_n} \frac{dt}{t} = \log \frac{4}{3} \approx 0.287682.$$

Thus $F^{-1}(D)$ is contained in a hyperbolic ball of the same radius centered at z_{n-1}. We claim this hyperbolic ball is contained in the Euclidean $D' = D(z_{n-1}, 4\pi)$. To prove this we use the comparison between the hyperbolic and quasi-hyperbolic metrics, Lemma 1.4.1,

$$d\rho_\Omega \le d\tilde{\rho}_\Omega \le 4 d\rho_\Omega.$$

If w is a point outside D', then any path γ connecting it and z_{n-1} has Euclidean length at least 4π, so its hyperbolic length is at least

$$\rho(\gamma) \ge \frac{1}{4}\tilde{\rho}(\gamma) = \frac{1}{4}\int_{z\in\gamma} \frac{1}{\mathrm{dist}(z, \partial U)} ds \ge \frac{1}{4} \cdot \ell(\gamma)\frac{1}{\pi} = 1 > \log\frac{4}{3}.$$

This proves the claim.

Similarly, the Euclidean disk $D(z_k, r)$ is contained in a hyperbolic disk centered at z_k of radius at most $-log(1 - \frac{r}{x})$ and this is $\le 2r/x$ if $r \le x/2$. Thus its preimage under F is contained in a hyperbolic disk of the same radius centered at z_{k-1} and this, in turn, in contained in a Euclidean disk of radius R where

$$\frac{1}{4} \cdot R \cdot \frac{1}{\pi} = 2r/x,$$

or

$$R = \frac{8\pi r}{x}.$$

If $x \ge M \ge 16\pi$ we can take $R = r/2$, i.e.,

$$F_k^{-1}(D(z_k, r)) \subset D(z_{k-1}, r/2), k = 1, \dots, n-1.$$

Using induction we see that $D(z_0, 2^{3-n}\pi)$ contains a neighborhood W of z_0 that is mapped univalently by F^m to $D(z_n, x_n/4)$. Thus there is subdomain $W' \subset W$ that is mapped univalently to $D(z_n, z_x/8)$ and this W' is a bounded distortion of a disk centered at z_0. Thus there is a disk D centered at z_0 that contains W' and such that

$$\mathrm{area}(D) \simeq \mathrm{area}(W').$$

Moreover by the distortion theorem for univalent maps (Theorem 1.3.4),

$$\frac{\text{area}(W' \setminus Y)}{\text{area}(W')} \simeq \frac{\text{area}(D(z_n, x_n/8) \setminus Y)}{\text{area}(D(z_n, x_n/8))} \geq \frac{\text{area}(D(z_n, x_n/8) \setminus U)}{\text{area}(D(z_n, x_n/8))} > \epsilon > 0.$$

Putting these together gives

$$\frac{\text{area}(D \cap Y)}{\text{area}(D)} < 1,$$

uniformly on a sequence of disks shrinking down to z_0, proving that z_0 is not a point of density of Y. Since z_0 was any point of Y, the Lebesgue density theorem implies Y has zero area. $\square$

We shall see later that the escaping set of a function in $\mathcal{B}$ can have dimension equal 1 (it is at least dimension one by Theorem 1.7.1), but that the Julia set of such a function is always strictly larger than 1.

THEOREM 4.4.3. *Assume that $f \in \mathcal{S}$ (finitely many singular values) and that the orbit of every singular value converges to an attractive fixed point. Then either $\mathcal{J}(f) = \mathbb{C}$ or* $\text{area}(\mathcal{J}(f) \setminus I(f)) = 0$.

PROOF. Assume that $\mathcal{J}(f) \neq \mathbb{C}$ and let

$$d(z) = \text{dist}(z, \overline{\mathcal{P}S(f)}).$$

Then $\mathcal{F}(f)$ is an open dense set, so

$$a(z) = \frac{\text{area}(D(z, d(z)) \cap \mathcal{F}(f))}{\pi d^2(z)},$$

is a positive continuous function on $U = \mathbb{C} \setminus \overline{\mathcal{P}S(f)}$.

There is a $M < \infty$ so that for almost every z in the Julia set, $|f^n(z)| \leq M$ infinitely often. Choose a n when this occurs and pull back the the disk $D(f^n(z), d(f^n(z)))$ under a univalent branch of f^{-n} to a neighborhood U of z. By the distortion theorem for univalent maps, U is approximately circular and $\mathcal{F}(f)$ takes up a fixed fraction of the area of U. The diameter of U must be small if n is large (Corollary 3.5.4). Thus the Lebesgue differentiation theorem applies to prove the theorem. $\square$

For any $\lambda \neq 0$, the tract of λe^z is a half-plane and therefore satisfies Theorem 4.4.2. Thus $\text{area}(I(\lambda e^z)) = 0$, even in the case $\lambda = 1$ when the Julia set is the whole plane (Theorem 4.3.1). If $0 < \lambda < 1/e$, then $f(z) = \lambda e^z$ has an attractive fixed points that attracts the only singular point, 0. Thus $\text{area}(\mathcal{J}(\lambda e^z)) = 0$ if $0 < \lambda < 1/e$ by

Theorem 4.4.3. We shall see later that the Hausdorff dimension is always 2 for these functions (Theorem **??**, due to McMullen [**95**]).

Theorem **??** can be generalized as follows:

THEOREM 4.4.4. *Assume that $f \in \mathcal{S}$ (finitely many singular values) and that the orbit of every singular value is either*

(1) *pre-periodic*
(2) *converges to an attractive fixed point, or*
(3) *converges to a neutral rational cycle.*

Then either $\mathcal{J}(f) = \mathbb{C}$ or $\mathrm{area}(\mathcal{J}(f) \setminus I(f)) = 0$.

We leave the proof as a exercise. Hint: ...?

The version stated in [**49**] is slightly more general; it only requires the set where $|f|$ is small to be large on average, instead of uniformly. This is also left as an exercise. Hint:...?

THEOREM 4.4.5. *Suppose $f \in \mathcal{B}$ satisfies*

$$\liminf_{r\to\infty} \frac{1}{\log r} \int_1^r |\{\theta : |f(re^{i\theta})| < R\}| \frac{dt}{t} > 0 \tag{26}$$

for some $R < \infty$. Then $\mathrm{area}(I(f)) = 0$. *Moreover, there is a $M < \infty$ so that for a.e. $z \in \mathbb{C}$,*

$$\liminf_{n\to\infty} |f^n(z)| < M.$$

4.5. A Julia set with positive area

THEOREM 4.5.1. *For any $\lambda \neq 0$, the escaping set (and hence the Julia set) of $f(z) = \lambda \cosh(z) = \frac{\lambda}{2}(e^z + e^{-z})$ has positive area.*

PROOF. We will prove this for $\lambda = 2$, $f(z) = e^z + e^{-z}$. The general case is almost identical and left to the reader.

We will say Q is a "grid square" if it has sides of length 2π parallel to the coordinate axes and corners in the set $2\pi(\mathbb{Z} + i\mathbb{Z})$. We let $Q(n, m)$ be the grid square with lower left corner at $2\pi(n + im)$. The map $\to e^z$ maps this square to the annulus $A_n = \{e^{2\pi n} \leq |z| \leq e^{2\pi(n+1)}\}$ with a derivative that bounded between these same two bounds. If n is large and positive, then $f(z)$ maps $Q(n,m)$ to a region B_n that approximates A_n to within $O(1)$. Let $C_n \subset B_n$ be the subregion of $B_n \setminus \{x+iy : |y| < 2\pi(n+1)\}$ covered by grid squares. See Figure 4.

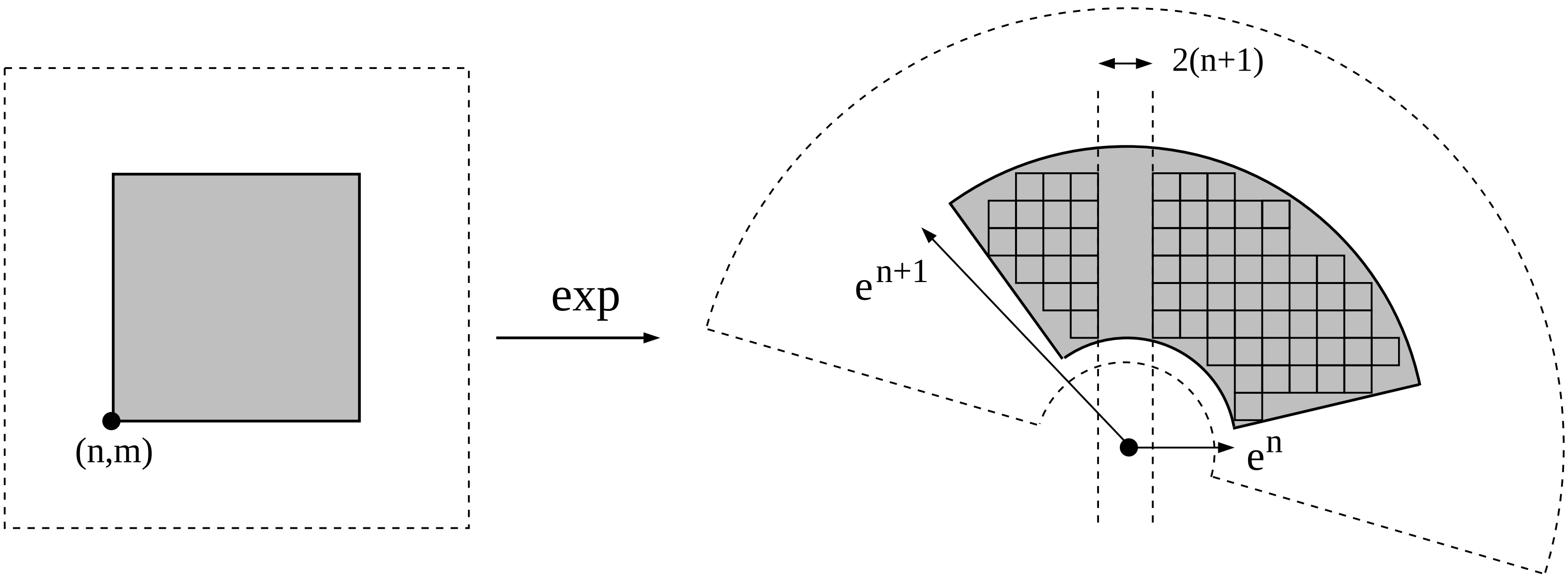


FIGURE 4. A grid square is mapped univalently to a (slit) annulus by the exponential map. The map f maps the same square to a region closely approximates this annulus. Most of the area this region is filled by grid squares, even is we remove a vertical slit of width $O(n)$ along the imaginary axis. Note that the figure is not to scale; in reality, $n \ll e^n$.

The area of C_n is the area of A_n minus the area of the region of the plane that is either within $O(1)$ of ∂A_n or within $O(n+1)$ of the imaginary axis. The area of $A_n = \pi(e^{4\pi(n+1)} - e^{4\pi n}) \simeq \pi e^{4\pi n}$. The area of the region within $O(1)$ of ∂A_n has area comparable to the length of ∂A_n, which is $O(e^{2\pi n}$. The area of B_n that is within distance $O(n+1)$ of the imaginary axis is $O(e^{2\pi n} n$. Thus $B_n \setminus C_n$ has area at most $O(ne^{2\pi n}$. By our derivative estimate on f, $f^{-1}(B_n \setminus C_n) \subset Q$ has area at most $O(ne^{-2\pi n})$. Thus $1 - O(ne^{-2\pi n})$ of the area of $Q(n,m)$ maps under f into a square $Q(p,q)$ with $p \geq n+1$.

Start with some square $Q_0 = Q(n_0, m_0)$ where n_0 is sufficiently large. By the argument above, we can find a collection of subregions $\{W_k\} \subset Q_0$ which are mapped to grid squares $Q(p,q)$ with $p \geq n_0 + 1$ and so that the leftover regions has area less than $O(ne^{2\pi n_0}$. Now apply the argument to each image square to decompose each W_k into subregions that are mapped univalently to grid squares $Q(p,q)$ with $p \geq n_0 + 2$ by two iterations of f. By Koebe's theorem, the derivative of f^2 is comparable to a constant on each of the second generation subregions, so the uncovered portion of the first generation regions is $O((n_0+1)e^{2\pi(n_0+1)}$, with a absolute constant. We proceed in this way, defining generations of subregions which cover all of the previous generation except for area $O((n_0+k)e^{-2\pi(n+k)}$ (the constant does not depend on k since the derivative of the iterated map on each kth generation piece is close to a constant by Koebe's theorem, and this constant does not depend on k). The intersection of all the generations is thus a set of area at least

$$\prod_{k=1}^{\infty} 1 - O((n_0+k)e^{-2\pi(n_0+k)}) > 0,$$

and contains only escaping points. Since f is in the Eremenko-Lyubich class, these points are also in the Julia set. $\square$

The proof above shows that there is a set of positive area where $|f^n(z)| \gtrsim n$, but this is very weak. Instead of throwing out a strip of width $O(n)$ around the imaginary axis, we could have omitted a strip of width $O(a_n \cdot e^{2\pi n})$ for any sequence with $\sum a_n < \infty$ and have obtained a set of positive area. For example, taking $a_n = e^{-\epsilon n}$ gives a set of positive area where $\{|f^n(z)|\}$ grows faster than iterating $e^{(2\pi-\epsilon)x}$.

4.6. A Julia set with dimension 2

The **exponential family** consists of functions of the form

$$f(z) = \lambda e^z, \quad \lambda \neq 0.$$

THEOREM 4.6.1 (McMullen, [**95**]). *The escaping set of any member of the exponential family has Hausdorff dimension* 2.

PROOF. The proof is similar to the discussion in Section 4.5, except that now about half the squares will have to be thrown out at each stage.

Recall that e^z maps the horizontal strip $U = \{x+iy : |y| < \pi/4\}$ conformally on the the sector $\{z : |\arg(z)| < \pi/4\}$ and every vertical translate of the strip is also mapped to the same sector. Let $W = \cup_{k\in\mathbb{Z}}(U + 2\pi i k)$ be the union of such vertical translates.

If $S = S(n,m) = \frac{\pi}{2}[n, n+2] \times 2\pi[m-\frac{1}{8}, m+\frac{1}{8}]$ is a "grid square" inside $W \cap \mathbb{H}_r(h_k)$ then the image of S under f is a truncated sector

$$\{z : \lambda e^{\pi n/2} < |z| < e^{\pi(n+1)/2}, |\arg z| < \pi/4\},$$

Then $f(S) \cap W \cap \mathbb{H}_r(h_{k+1}$ contains $\frac{\lambda^2}{C}\exp(\pi n)$ disjoint squares of the same type for some fixed $C < \infty$ that is independent of n.

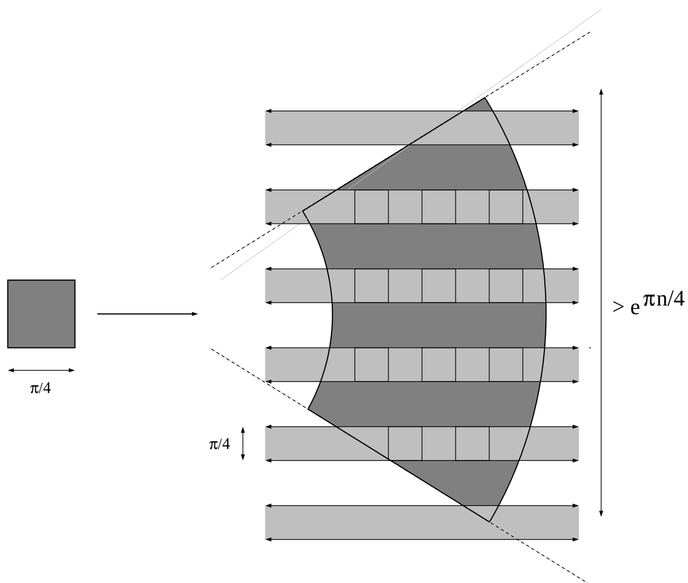


FIGURE 5. A grid square is mapped univalently to a sector as shown. If the square is in $\{x > \pi n/2\}$, then the image region has area comparable to $\exp(\pi n)$ and contains at least $\simeq \exp(\pi n)$ $\frac{\pi}{4}$-squares, where the construction can be iterated.

The preimages of these squares are regions in $S(n,m)$ of size approximately $\exp(-\pi n/2)$. If we give them equal mass, then the mass of each region W is less than

$$C\lambda^{-2}\exp(-\pi n) \leq \exp(-(2-\epsilon)\pi n/2) \leq \operatorname{diam}(W)^{2-\epsilon},$$

where ϵ is close to zero (depending on C and ϵ) if n is large enough.

By iterating this construction, we obtain a measure μ supported on a Cantor subset E that satisfies the Frostman condition

$$\mu(D(x,r)) \leq M \cdot r^{2-\epsilon},$$

(see Lemma A.1.6). By the mass distribution principle (Theorem A.1.4), the set E has dimension $> 2-\epsilon$. Moreover, this set is clearly in the escaping set, since each of our grid squares was exponentially further from the origin than its preimage. Thus the escaping set has Hausdorff dimension $\geq 2-\epsilon$ for every $\epsilon > 0$, and hence has dimension equal to 2.

$\square$

4.7. A Julia set with dimension close to 1

Next we give example in the Eremenko-Lyubich class where the Julia set has Hausdorff dimension that is close to 1. The existence of such functions is due to Gwyneth Stallard []. She showed that for function in $\mathcal{B}$, the Julia set always has Hausdorff dimension strictly greater than one (see Theorem []) and she and Phil Rippon showed that the packing dimension of such a Julia set is always equal to 2 (see Theorem **??**). However, there do exist transcendental Julia sets that have both Hausdorff and packing dimension equal to 1 ([**27**]).

The basic requirement is a function $f \in \mathcal{B}$ which has a single tract

$$\Omega = \{z : |f(z)| > 1\},$$

that approximates a half-strip, as illustrated in Figure 6

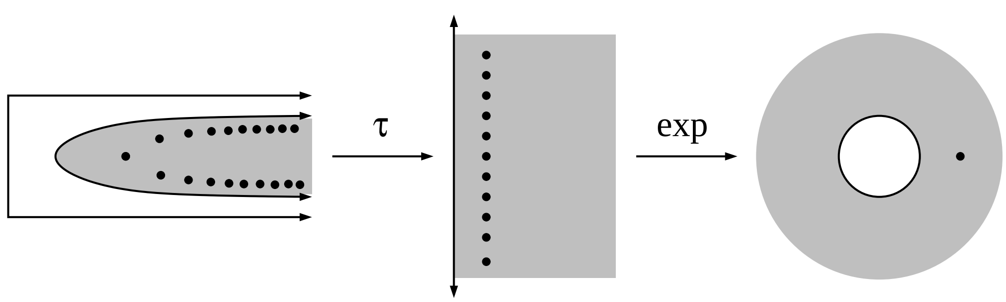


FIGURE 6. A function with small Julia set.

LEMMA 4.7.1. *Suppose* $f \in \mathcal{B}$ *and suppose* $\mathcal{S}(f) \subset \mathbb{D}_R = \{z : |z| < R\}$. *Let* $\Omega_0 = \Omega = \{z : |f(z)| > R\}$ *and* $\Omega_{n+1} = f^{-1}(\Omega_n)$. *Assume* $\overline{\Omega_0} \subset f(\Omega_0)$. *Suppose that for some* $n \geq 0$, $\overline{\Omega_n}$ *can be covered by a collection of sets* $\{D_k\}$ *so that*

$$\sum_{k=1}^{n} \operatorname{diam}(D_k)^{1+\delta} < \infty, \tag{27}$$

and suppose that for any set $D \subset \Omega_n$,

$$\sum \operatorname{diam}(f^{-1}(D)^{1+\delta} < \frac{1}{2}\operatorname{diam}(D)^{1+\delta}, \tag{28}$$

where the sum is over all connected components of $f^{-1}(D)$. Then $\dim(\mathcal{J}(f)) \leq 1+\delta$.

PROOF. The assumption that $\overline{\Omega_0} \subset f(\Omega_0)$ implies the (open) complement of $\overline{\Omega_0}$ is mapped into itself and thus all iterates avoid Ω_0. Thus by Montel's theorem the iterates of f are normal here and so the complement of $\overline{\Omega_0}$ are in the Fatou set. Thus any point that eventually iterates out of Ω_0 is in the Fatou set, so the Julia set is contained in $E = \cap_n \Omega_n$.

If Ω_n covered by a collection of sets, then Ω_{n+1} is covered by the collection of all pre-images of these sets. The two conditions in the lemma implies $1+\delta$ Hausdorff content of Ω_n tends to zero and hence E has $1+\delta$ content zero. Thus $\dim(\mathcal{J}(f)) \leq \dim(E) \leq 1+\delta$. □

We will construct a $f \in \mathcal{B}$ so that for any disk $D = D(z,r) \subset \Omega_0 = \{z : |z| > R\}$ $|z| > 2$, $r < 1$ has f-preimages $\{D_j\}_{-\infty}^{\infty}$ in Ω_0 that satisfy

$$\operatorname{diam}(D_j) = O\left(\frac{\operatorname{diam}(D)}{|z|(\log|z| + |j|)}\right). \tag{29}$$

Suppose this is true. Then the pre-image of D under the map $f_K(z) = f(z) - K$ is the same as the preimage of $D + K$ under the map f, and hence its preimages satisfy

$$\operatorname{diam}(D_j) = O\left(\frac{\operatorname{diam}(D)}{K(1+|j|)}\right). \tag{30}$$

Thus if we fix $\delta > 0$ and then choose K large enough,

$$\begin{aligned}
\sum_j \operatorname{diam}(D_j)^{1+\delta} &\leq C\operatorname{diam}(D)^{1+\delta} \cdot \sum_j \frac{1}{(|z|(\log K + |j||))^{1+\delta}} \\
&\leq \frac{\operatorname{diam}(D)^{1+\delta}}{\delta|z|^{1+\delta}(\log K)^{\delta}} \\
&\leq \frac{1}{8}\operatorname{diam}(D)^{1+\delta},
\end{aligned}$$

if $|z|$ is large enough, depending on δ. Thus we can replace each set D_j by a disk with at most twice the diameter and get a collection of disks that covers f_K^{-1} with a Hausdorff sum that is at most half of what we started with.

We start by covering Ω_0 by disks $\{D(m,2)\}$. The preimages satisfy

$$\begin{aligned}
\sum_{m\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \operatorname{diam}(D_{j,m})^{1+\delta} &\leq C\sum_m m^{-1-\delta}\sum_j (\log m + j)^{-1-\delta} \\
&\leq C\sum_m m^{-1-\delta}\frac{1}{\delta}(\log m)^{-1-\delta} \\
&\leq \frac{C}{\delta}(\log K)^{-\delta}\sum_m m^{-1-\delta} \\
&\leq \frac{C}{\delta^2}(\log K)^{-\delta}.
\end{aligned}$$

Thus Ω_1 is covered by a collection of domains with finite $(1+\delta)$-sum. By Lemma 4.7.1, this completes the proof, except for constructing a function f that satisfies (30).

Let

$$F(z) = \exp(\exp(z)),$$

for $z \in S = \{x+iy : x > 0, |y| < \pi\}$ and set it to zero outside the half-strip S. On the vertical side of the half-strip,

$$F(z) = F(0+iy) = \exp(\exp(iy)) = \exp(\cos y + i\sin y)$$

is bounded. On the top and bottom horizontal sides

$$F(z) = F(x \pm i\pi) = \exp(\exp(x \pm i\pi)) = \exp(-\exp(x))$$

it tends to zero very quickly as $x \nearrow \infty$. In fact, for $\frac{3}{4}\pi < |y| < \frac{5}{4}\pi$, we have

$$|F(z)| = \exp(\Re(\exp(x(\cos y + i\sin y))) = \exp(\exp(x\cos y)) \leq \exp(\exp(-x/\sqrt{2})).$$

Let φ be a positive, smooth, radial bump function of total mass 1 and supported in $D(0,\pi/4)$. Then the convolution of F and φ is a smooth function G that is zero outside a $\pi/4$-neighborhood of S, is equal to F inside the smaller half-strip $S' = \{x+iy : x > \pi/4, |y| < \frac{3}{4}\pi\}$. In particular, G is holomorphic except on a $\pi/4$ neighborhood of ∂S and its $\overline{\partial}$ derivative decays rapidly near infinity. Thus

$$H(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\overline{\partial}G}{w-z}dxdy,$$

is a smooth bounded function such that $\overline{\partial}H = \overline{\partial}G$. Hence $f(z) = G(z) - H(z)$ is holomorphic on the whole plane and we claim it has the desired properties.

Fix a constant M so that $\sup_{S''}|H|, \sup_{S''}|H'| \leq M$. Fix a disk $D = D(w,r) \subset S \cap \overline{\mathbb{D}}_R^c$. If $f(z) \in D$ then $F(z) = G(z) \in D' = D(w, r+M)$. Thus each component W of $f^{-1}(D)$ is contained in a component W' of $F^{-1}(D')$. Moreover, on W' the derivatives $|f'|$ and $|F'|$ differ by at most M, and we can choose R so large that $|f'/F'| = O(1)$ on W'. This implies that $\operatorname{diam}(W) \simeq r/|w|$. Thus

$$\begin{aligned}
\sum_{W \in f^{-1}(D)} \operatorname{diam}(W)^{1+\delta} &\leq C \sum_{j\mathbb{Z}} \frac{r^{1+\delta}}{|z|)^{1+delta}(\log|z| + 2\pi j)^{1+\delta}} \\
&\leq \frac{C}{\delta} |z|^{-1-\delta} (\log|z|)^{-\delta} r^{1+\delta} \\
&\leq \frac{C}{\delta} \frac{r^{1+\delta}}{|R|^{1+\delta}(\log R)^{\delta}}.
\end{aligned}$$

This is $\leq \frac{1}{2} r^{1+\delta}$ if R is large enough, depending on C and δ.

Given R, we can choose K so large that f_K maps By definition, f maps the left half-plane into a bounded set, so for K large enough $f_K = f - K$ maps the left half-plane into itself. Thus the left half-plane is in the Fatou set. Similarly, given $R < \infty$, we can choose K large enough so that the disk $\mathbb{D}_R$ is mapped into the left half-plane (and hence is in the Fatou set too). Thus it suffice to cover $\Omega_0 \setminus \mathbb{D}_R$. If we cover this part of Ω_0 by disks $\{D(n,2)\}_{n=R}^{\infty}$, then the pre-images of $D_n = D(n,2)$ satisfy

$$\begin{aligned}
\sum_{W \in f^{-1}(D_n)} \operatorname{diam}(W)^{1+\delta} &\leq \frac{C}{\delta} |n|^{-1-\delta} \\
&\leq \frac{C}{\delta} |R|^{-\delta}.
\end{aligned}$$

4.8. Dimension of bounded orbits

Recall that $K(f)$ denotes the points with bounded orbits.

THEOREM 4.8.1 (Stallard). *If* $f \in \mathcal{B}$ *then* $\dim(K(f) \cap \mathcal{J}(f)) > 1$.

In particular, $\dim(\mathcal{J}(f)) > 1$ for any $f \in \mathcal{B}$. Before beginning the proof we need to gather some relevant facts. The following is simple and well known fact from real analysis.

LEMMA 4.8.2. *If* h *is increasing on* $[r, \infty)$ *and* $\delta > 0$, *then*

$$h'(x) \leq h(x)^{1+\delta},$$

except on a set E_δ of Lebesgue measure at most $h(r)^{-\delta}/\delta$.

PROOF. If E is the set where this fails, then

$$|E| = \int_E dx \leq \int_E \frac{h'(x)}{h(x)^{1+\delta}} dx \leq \int_r^\infty \frac{h'(x)}{h(x)^{1+\delta}} dx = \frac{1}{\delta h(r)^\delta}.$$

□

We will apply the lemma in the following situation. Let $f \in \mathcal{B}$, let $F : \Omega \to \mathbb{H}_r$ be f in logarithmic coordinates and set

$$h(x) = \max\{\Re F(x+iy) : y \in \mathbb{R}\}.$$

Let $z_x \in \Omega$ be a point where this maximum is attained. Note that h is increasing; we shall apply the previous lemma to h in order to prove:

LEMMA 4.8.3. *Fix $f \in \mathcal{B}$. If $x \in \mathbb{R}$ is large enough and*

$$D = D(F(z_x), \frac{1}{4}h(x)),$$

then there is a collection of subdisks $\{D_j\}$ so that

$$\sum_j \operatorname{diam}(D_j) \geq C\frac{|h(x)|^3}{h'(x)},$$

and F^2 is a conformal map from a domain W_j onto D where $2D_j \subset W_j \subset 10D_j$. In particular, if $\delta > 0$ and x is not in the exceptional set E_δ from Lemma 4.8.2, then

$$\sum_j \operatorname{diam}(D_j) \geq C(\delta)\operatorname{diam}(D)^{2-\delta}.$$

PROOF. By Koebe's $\frac{1}{4}$-theorem, since $2D \subset \mathbb{H}_r$, D has an F-preimage W in Ω that satisfies

$$\operatorname{diam}(W) \simeq \operatorname{diam}(D)/h'(x) \simeq h(x)/h'(x).$$

Let $W_k = W + 2\pi i k$. Then the F-preimages of these are regions V_k in Ω arranged in a "chain" tending to ∞ in both directions so that

$$\operatorname{diam}(V_k) \simeq \operatorname{dist}(V_k, V_{k+1})\operatorname{diam}(W) \simeq \operatorname{dist}(V_k, V_{k+1})h(x)/h'(x).$$

We now replace each V_k by a sub-disk D_k of comparable size (we can do this by the distortion estimates for conformal maps since V_k is an image of a disk under a map that is conformal on the double of that disk).

FIGURE OF PREIMAGES and CHAIN

Since the chain crosses a square $S \subset D$ of comparable size,

$$\sum_{D_k \subset \frac{1}{2}D} \operatorname{diam}(D_k) \geq \operatorname{diam}(D) \cdot \frac{h(x)}{h'(x)} \simeq \frac{h(x)^2}{h'(x)}.$$

Translating the collection $\{D_k \subset \frac{1}{2}D\}$ vertically by $2\pi i j$, $|j| \leq \frac{1}{2}\operatorname{diam}(D)$, gives a collection of disks with the desired properties. $\square$

Proof of Theorem 4.8.1. Take x so large that the disks given by Lemma 4.8.3 satisfy

$$\sum_j \operatorname{diam}(D_j) \geq C(\delta)\operatorname{diam}(D)^{2-\delta} > 10\operatorname{diam}(D).$$

Then, by continuity, there is some $s > 1$ so that

$$\sum_j \operatorname{diam}(D_j)^s \geq \operatorname{diam}(D)^s.$$

This defines an iterated function system whose limit set has dimension $\geq s$, consists of points with bounded orbits in the Julia set (by Corollary 4.2.4). $\square$

4.9. Packing dimension is always 2

Suppose f is entire and e is its exceptional point (if one exists). Recall that this means that the iterates of any neighborhood of $\mathcal{J}(f)$ eventually covers $\mathcal{J}(f) \setminus \{e\}$.

Lemma 4.9.1. *Suppose f a transcendental entire function and there are compact sets $K_n \subset \mathcal{J}(f) \setminus \{e\}$ with $\overline{\operatorname{Mdim}}(K_n) \nearrow \alpha$. Then* $\operatorname{Pdim}(\mathcal{J}(f) \geq \alpha$.

Proof. If V is a bounded open set that hits the Julia set, then $f^n(V)$ eventually covers K_n and hence $\overline{\operatorname{Mdim}}(\mathcal{J}(f) \cap V \geq \overline{\operatorname{Mdim}}(K_n)$. Thus $\operatorname{Pdim}(\mathcal{J}(f)) \geq \overline{\operatorname{Mdim}}(K_n)$ by Lemma A.3.2 in the Appendix. The result now follows by taking $n \nearrow \infty$. $\square$

The following is due to Rippon and Stallard in []. It has been generalized to larger classes of functions by Bergweiler [].

Theorem 4.9.2. *If $f \in \mathcal{B}$ then* $\operatorname{Pdim}(\mathcal{J}(f)) = 2$.

Proof. By Lemma 4.9.1, it suffices to construct a compact subsets of $\mathcal{J}(f)$ that have upper Minkowski dimension as close to 2 as we wish and that avoid the exceptional point.

Fix $\delta > 0$. We will build a subset that has upper Minkowski dimension $\geq \frac{2}{1+\delta}$. As before, let $F : \Omega \to \mathbb{H}_r$ be f in logarithmic coordinates. Let

$$h_n(x) = \max\{\Re F^n(x+iy) : y \in \mathbb{R}\}.$$

If we apply Lemma 4.8.2 to each function, we see that the exceptional sets have total measure

$$\frac{1}{\delta}\sum_n h_n(r)^{-\delta},$$

which is finite, since the maximum modulus of f^n increases at least like an iterated polynomial. Thus we can choose an x so that

$$(h_n)'(x) \leq (h_n(x)))^{1+\delta}, \tag{31}$$

for every n. Let $z_n = x + iy_n$ be the point where $\Re F^n(x+iy)$ attains it maximum on the vertical line through x. Let $S_n \subset \mathbb{H}_r$ be a square centered at $F^n(z_n)$ with side length $\frac{1}{4}h_n(x)$. Then there is a univalent branch of F^{-N} that maps S_n conformally to a subdomain W_n of Ω that contains z_n. Note that

$$\operatorname{diam}(W) \simeq h_n(x)/(F^n)'(z_n) \gtrsim h_n(x)/h_n'(x).$$

If we subdivide S_n into $\simeq h_n(x)^2$ subsquares of side-length 2π then each of them hits the escaping set $I(F)$ (by Theorem 1.7.1, $I(f)$ hits every large circle centered at the origin, so $I(f)$ hits every vertical line far enough to the right). Thus W contains $h_n(x)^2$ subdomains each of diameter $\gtrsim 1/h_n'(x)$ and each hitting $I(F)$. Thus using the definition of upper Minkowski dimension and (31),

$$\begin{aligned}
\overline{\mathrm{Mdim}}(I(f) \cap D(z_1, 2\pi) &\geq \limsup_{n\to\infty} \frac{\log h_n(x)^2}{\log h_n'(x)} \\
&\geq 2\limsup_{n\to\infty} \frac{\log h_n(x)}{\log(h_n(x))^{1+\delta}} \\
&= \frac{2}{1+\delta}
\end{aligned}$$

Thus $\mathcal{J}(f) \supset I(f)$ has packing dimension at least $2/(1+\delta)$. Since this holds for any $\delta > 0$, we see that $\mathrm{Pdim}(\mathcal{J}(f)) = 2$. $\square$

Later we shall prove that the fast escaping set $A(f)$ contains an unbounded component and hence also hits every large enough circle centered at the origin. Thus the proof above can be repeated to show that $A(f)$ also has packing dimension 2. This was the result obtained by Rippon and Stallard in [].

4.10. Dimension of slowly escaping points

THEOREM 4.10.1 (Bergweiler and Peter). *Let $f \in \mathcal{B}$ and suppose h is a guage function such that*

$$\lim_{t\to 0} \frac{\log h(t)}{\log t} = 1.$$

Let $\{p_n\}$ be a sequence of positive numbers tending to ∞ and define

$$I(f, \{p_n\}) = \{z : |f^n(z)| \leq p_n \forall n\}.$$

Then $\mathcal{H}_h(I(f, \{p_n\})) = \infty$. In particular, $H_h(I(f)) = \infty$ and $\dim(I(f, \{p_n\})) \geq 1$.

PROOF. So if x_1 is large enough so that $E \cap [x_1, \infty)$ has measure $\leq 1/2$, we can inductively find $x_{k+1} \in [h(x_k), h(x_k) + 1] \setminus E$ so that $x_k \nearrow \infty$ and define disks

$$D_k = D(z_{x_k}, \frac{1}{4}x_k).$$

The argument in the proof of Theorem 4.8.1 shows that each of these disks contains subdisks that define an iterated function scheme that has an invariant Cantor set of some dimension $s_k > 1$. Moreover, each D_k contains a subregion that is conformally mapped by F onto D_{k+1} (in fact, taking vertical translations, there are about x_k such subdomains).

Run the iterated function scheme associated to D_1 for N_1 steps and then switch to D_2 for n_2 steps, and so on. In the limit we obtain a limiting Cantor set that has dimension ≥ 1, and we can arrange for infinite h-measure by waiting long enough (depending on h) before performing each switch. □

THEOREM 4.10.2 (Bergweiler and Peter). *Suppose h is a guage function such that*

$$\lim_{t\to 0} \frac{h(t)}{t} = 0.$$

*Then, using the notation of Theorem **??**, there exists $f \in \mathcal{B}$ such that $\mathcal{H}_h(I(f) \setminus I(f, \{p_n\})) = 0$.*

Thus we can find guage functions h between t and $t^{1+\epsilon}$ for which every function in $\mathcal{B}$ has an infinite h-measure set of slow escaping points, but some functions have a zero h-measure set of "quickly escaping" points.

4.11. $I(e^z)$ is connected

The following result of Lasse Rempe-Gillen illustrates a "exotic" planar sets arise naturally in the of the dynamics of the exponential function.

THEOREM 4.11.1 (Rempe-Gillen, [**111**]). *$I(e^z)$ is connected.*

PROOF. □

The connectedness of the escaping set for other exponential functions has been proven by Rempe-Gillen [**112**] and Jarque [**73**]. In [**96**] Mihaljević-Brandt shows the escaping set is disconnected for various examples including the map $\pi \sinh z$.

CHAPTER 5

Extremal Length

Chapter 1 dealt with hyperbolic distance, one of the most important conformal invariants. In this chapter we introduce another conformal invariant of equal importance: extremal length. The basic idea is to consider a positive function ρ on a domain Ω. We think of ρ as analogous to $|f'|$ where f is a conformal map on Ω. Just as the image area of a set E can be computed by integrating $\int_E |f'|^2 dxdy$, we can use ρ to define areas by $\int_E \rho^2 dxdy$. Similarly, we can define lengths of curves in Γ by $\int_\gamma \rho ds$. The modulus of a family of paths Γ is defined by minimizing the ρ-area of Ω over all ρ's which give every path in Γ length at least 1. The extremal length of the path family is the reciprocal of its modulus (it is sometimes more convenient to consider one of these quantities and sometimes more convenient to use the other).

The importance of modulus and extremal length is that it has a geometric definition and can often be estimated in situations where other conformal invariants cannot be directly computed. By conformally mapping to a situation (such as the unit disk) where different conformal invariants can be compared by explicit formulas, we can obtain geometrically based estimates for other conformal invariants, such as the Ahlfors distortion theorem for harmonic measure.

After discussing some of the basic properties and examples related to extremal length, we use it to prove Caratheodory's boundary extension theorem for conformal maps, and a more technical looking analog that we will need later when studying general quasiconformal mappings. In this chapter we introduce the definition of quasiconformal map, but only prove that "nice" K-quasiconformal maps (e.g., diffeomorphisms and certain piecewise linear maps) preserve extremal length up to factor of K. This suffices for the final result in this chapter: showing that the iterate of a Fatou component of an entire function is a Fatou component with at most one point omitted.

5.1. Quadrilaterals and annuli

A conformal invariant is a number which is invariant under conformal mappings. We are often in the situation where we wish to know the value of some conformal invariant (e.g., that harmonic measure of the edge of a polygon) and are able to estimate some other conformal invariant (e.g., the modulus of some path family in the polygon). Using a known relation between the invariants, we can turn an estimate for one into an estimate for the other.

Probably the most important example of a conformal invariant is the (conformal) modulus.

Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω. We say ρ is admissible for Γ if

$$\ell(\Gamma) = \ell_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho ds \geq 1,$$

and define the modulus of Γ as

$$\mathrm{Mod}(\Gamma) = \inf_\rho \int_M \rho^2 dxdy,$$

where the infimum is over all admissible ρ for Γ. This is a well known conformal invariant whose basic properties are discussed in many sources such as Ahlfors' book [**3**]. Its reciprocal is called the extremal length of the path family and is denoted

$$\lambda(\Gamma) = 1/M(\Gamma).$$

Modulus and extremal length satisfy several properties that are helpful in estimating these quantities.

Lemma 5.1.1 (Conformal invariance). *If $\mathcal{F}$ is a family of curves in a domain Ω and f is a one-to-one analytic mapping from Ω to Ω' then $M(\mathcal{F}) = M(f(\mathcal{F}))$.*

Proof. This is just the change of variables formulas

$$\int_\gamma \rho \circ f |f'| ds = \int_{f(\gamma)} \rho ds,$$

$$\int_\Omega (\rho \circ f)^2 |f'|^2 dxdy = \int_{f(\Omega)} \rho dxdy.$$

These imply that if $\rho \in \mathcal{A}(f(\mathcal{F}))$ then $|f'| \cdot \rho \circ f^{-1} \in \mathcal{A}(f(\mathcal{F}))$, and thus $M(f(\mathcal{F})) \leq M(\mathcal{F})$. We get the other direction by considering f^{-1}. $\square$

LEMMA 5.1.2 (Monotonicity). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are collections such that every $\gamma \in \mathcal{F}_1$ contains some curve in $\mathcal{F}_2$ then $M(\mathcal{F}_1) \leq M(\mathcal{F}_2)$ and $\lambda(\mathcal{F}_1) \geq \lambda(\mathcal{F}_2)$.*

The proof is immediate since $\mathcal{A}(\mathcal{F}_1) \supset \mathcal{A}(\mathcal{F}_2)$.

LEMMA 5.1.3 (Grötsch Principle). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains then $M(\mathcal{F}_1 \cup \mathcal{F}_2) = M(\mathcal{F}_1) + M(\mathcal{F}_2)$.*

PROOF. Suppose ρ_1 and ρ_2 are admissible for $\mathcal{F}_1$ and $\mathcal{F}_2$. Take $\rho = \rho_1$ and $\rho = \rho_2$ in their respective domains. Then it is easy to check that ρ is admissible for $\mathcal{F}_1 \cup \mathcal{F}_2$ and $\int \rho^2 = \rho_1^2 + \int \rho_2^2$ so domains then $M(\mathcal{F}_1 \cup \mathcal{F}_2) \leq M(\mathcal{F}_1) + M(\mathcal{F}_2)$. By restricting an admissible metric ρ to each domain, a similar argument proves the other direction. $\square$

LEMMA 5.1.4. *[Series Rule] If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains and every curve of $\mathcal{F}$ contains both a curve from $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\lambda(\mathcal{F}) \geq \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2)$.*

PROOF. If $\rho_1 \in \mathcal{A}(\mathcal{F}_i)$ for $i = 1, 2$, then $\rho = t\rho_1 + (1-t)\rho_2$ is admissible for $\mathcal{F}$. Since the domains are disjoint we may assume $\rho_1\rho_2 = 0$ everywhere so taking For $0 \leq t \leq 1$, take

$$= t^2\rho_1 + (1-t^2)\rho_2.$$

it is easy so see that this is admissible and since the domains are disjoint we may assume $\rho_1\rho_2 = 0$. Integrating ρ^2 then shows

$$M(\mathcal{F}) \leq t^2 M(\mathcal{F}_1) + (1-t^2)M(\mathcal{F}_2),$$

for each t. To find the optimal t set $a = M(\mathcal{F}_1)$, $b = M(\mathcal{F}_2)$, differentiate the right hand side above, and set it equal to zero

$$2at - 2b(1-t) = 0.$$

Solving gives $t = b/(a+b)$ and plugging this in above gives

$$\begin{aligned} M(\mathcal{F}) \quad \leq \quad & t^2 a + (1-t^2)b = \frac{b^2 a a^2 b}{(a+b)^2} \\ & = \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a}+\frac{1}{b}} \end{aligned}$$

or

$$\frac{1}{M(\mathcal{F})} \geq \frac{1}{M(\mathcal{F}_1)} + \frac{1}{M(\mathcal{F}_2)},$$

which, by definition, is the same as

$$\lambda(\mathcal{F}) \geq \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2),$$

$\square$

Given a Jordan domain Ω and two disjoint closed sets $E, F \subset \partial\Omega$, the **extremal distance** between E and F (in Ω) is the extremal length of the path family in Ω connecting E to F (paths in Ω that have one endpoint in E and one endpoint in F). The series result is a sort of "reverse triangle inequality" for extremal distance. See Figure 1.

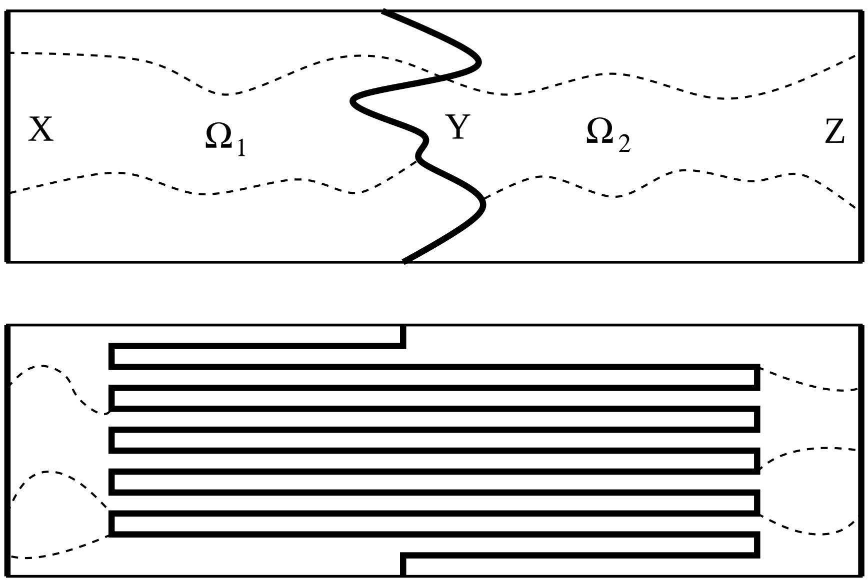


FIGURE 1. The series rule says that the extremal distance from X to Z in the rectangle is greater than the sum the extremal distance from X to Y in Ω_1 plus the extremal distance from Y to Z in Ω_2. The bottom figure show a more extreme case where the extremal distance between opposite sides of the rectangle is much larger than either of the other two terms (that these terms are small will follow from later estimates, e.g. Lemma 5.1.13)

Next we actually compute the modulus of some path family. The fundamental example is to compute the modulus of the path family connecting opposite sides of a

$a \times b$ rectangle; this serves as the model of almost all modulus estimates. So suppose $R = [0,b] \times [0,a]$ is a b wide and a high rectangle and Γ consists of all rectifiable curves in R with one endpoint on each of the sides of length a. Then each such curve has length at least b, so if we let ρ be the constant $1/b$ function on R we have

$$\int_\gamma \rho ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\mathrm{Mod}(\Gamma) \leq \iint_T \rho^2 dxdy = \frac{1}{b^2} ab = \frac{a}{b}.$$

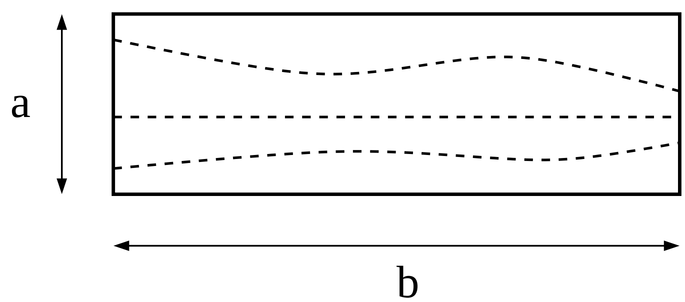


FIGURE 2. The modulus of the path family connecting the two vertical sides of length a is a/b. Thus if $a \ll b$ the family has small modulus and large extremal length.

To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$(\int fgdx)^2 \leq (\int f^2 dx)(\int g^2 dx).$$

To apply this, suppose ρ is an admissible metric on R for γ. Every horizontal segment in R connecting the two sides of length a is in Γ, so since γ is admissible,

$$\int_0^b \rho(x,y)dx \geq 1,$$

and so by Cauchy-Schwarz

$$1 \leq \int_0^b (1 \cdot \rho(x,y))dx \leq \int_0^b 1^2 dx \cdot \int_0^b \rho^2(x,y)dx.$$

Now integrate with respect to y to get

$$\int_0^a 1dy \leq b \int_0^a \int_0^b \rho^2(x,y)dxdy,$$

or

$$\frac{a}{b} \leq \iint_R \rho^2 dxdy,$$

which implies $\mathrm{Mod}(\Gamma) \geq \frac{b}{a}$. Thus we must have equality.

Another useful computation is the modulus of the family of path connecting the inner and out boundaries of the annulus $A = \{z : r < |z| < R\}$. An argument similar to the one above shows that the modulus of this family is $\frac{1}{2\pi} \log \frac{R}{r}$.

Lemma 5.1.5. *If $A = \{z : r < |z| < R\}$ then the modulus of the path family connecting the two boundary components is $\frac{1}{2\pi} \log \frac{R}{r}$. More generally, if $\mathcal{F}$ is the family of paths connecting $r\mathbb{T}$ to a set $E \subset R\mathbb{T}$, then $M(\mathcal{F}) \geq |E| \log \frac{R}{r}$.*

Proof. By conformal invariance, we can rescale and assume $r = 1$. Suppose ρ is admissible for $\mathcal{F}$. Then for each $z \in E \subset \mathbb{T}$,

$$1 \leq (\int_1^R \rho dr)^2 \leq (\int_1^R \frac{dr}{r})(\int_1^R \rho^2 r dr) = \log R \int_1^R \rho^2 r dr$$

so

$$\begin{aligned} \int_0^{2\pi} \int_1^R \rho^2 r dr d\theta &\geq \int_E \int_r^R \rho^2 r dr d\theta \\ &\geq |E| \int_1^R \rho^2 r dr \\ &\geq |E| \log R \end{aligned}$$

□

Given a Jordan domain Ω and two disjoint closed sets $E, F \subset \partial\Omega$, the **extremal distance** between E and F (in Ω) is the extremal length of the path family in Ω connecting E to F (paths in Ω that have one endpoint in E and one endpoint in F). The previous result says that the extremal distance between the two boundary components of a round annulus is $\frac{1}{2\pi} \log \frac{R}{r}$.

When we discuss the modulus of an annulus there are two obvious possible path families we might mean: the family connecting the two boundary components or the family separating the components. See Figures 3 and 4.

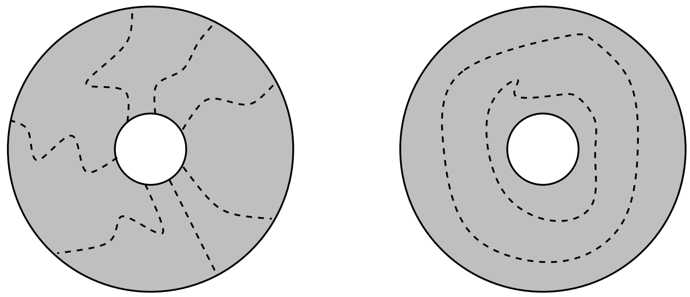


FIGURE 3. Path families that connect boundary components of an annulus and separate the boundary components. We define the modulus of an annulus to be the modulus of separating family (this equals the extremal length of the connecting family).

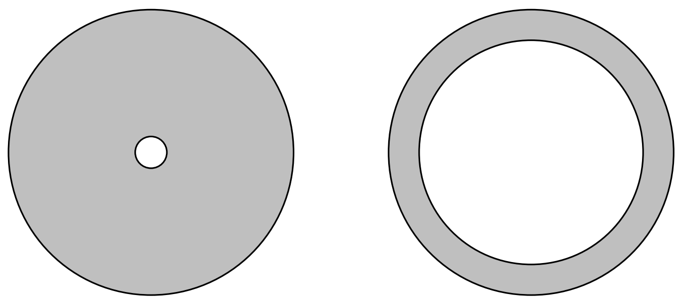


FIGURE 4. The annulus on the left has "big" modulus and one on the right has "small" modulus (these are moduli of the path families separating the boundaries). Equivalently, the extremal distance between the boundary components on the left is large and is small on the right (these are extremal lengths of path families connecting the boundaries).

By $M(A)$ we shall mean the modulus of the family separating the components. Thus a "thick" annulus geometrically will have large modulus. See Figure **??**. Similarly, the modulus $M(Q)$ of a quadrilateral Q with a designated pair of opposite sides will be the modulus of the path family that separates the two sides. This in a quadrilateral of large modulus, the designated sides are "far apart" from each other relative to their diameters.

If $E, F \subset \partial\Omega$ and $\mathcal{F}$ is the family of path connecting E to F (i.e., all paths in Ω with one endpoint in E and the other endpoint in F), there is a reciprocal family $\mathcal{F}^*$ consisting of generalized paths that separate E and F. By a "generalized path" we mean a finite union γ of rectifiable curves $\{\gamma_j\}$ so that each component γ_j has its endpoints on $\partial\Omega \setminus (E \cup F)$ and so that any path connecting E and F must intersect γ. See Figure 5.

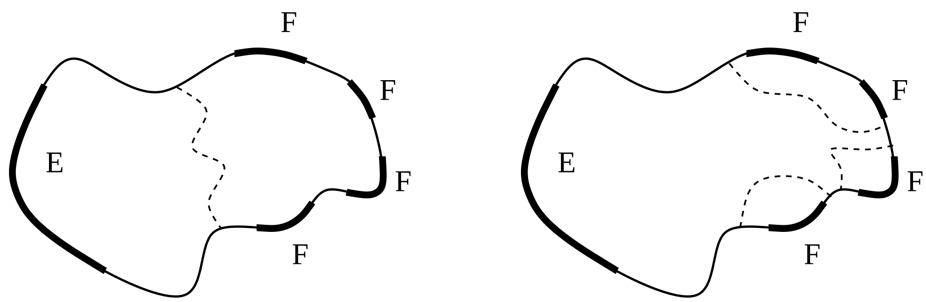


FIGURE 5. Definition of separating paths. Such a path can have a single component (left) or multiple components (right).

Suppose Ω is a Jordan domain, $E, F \subset \partial\Omega$, are each finite unions of closed intervals and $E \subset I$ and $F \subset J$ where I, J are disjoint arcs of $\partial\Omega$ (we will assume below that I and J were chosen to be minimal with this property). Using the Riemann map, we can assume $\Omega = \mathbb{D}$ and define $u(z)$ on $\mathbb{C} \setminus (E \cup F)$ to the solution of the Dirichlet problem with boundary values 0 on E and 1 on F. Then u is constant on E and F and by symmetry has normal derivative zero on $\mathbb{T} \setminus (E \cup F)$ so that its harmonic conjugate v has tangential derivative zero there, hence is constant one each component of $\mathbb{T} \setminus (E \cup F)$. Thus $f = u + iv$ is a holomorphic map of $\mathbb{D}$ to a $1 \times m$ rectangle with horizontal slits removed, each corresponding to components of $I \setminus E$ and $J \setminus F$. See Figure 6. By conformal invariance, the modulus of the path family $\mathcal{F}$ that connects E and F is therefore m (review the proof given for a rectangle above and note that it also works for a rectangle with horizontal slits removed). Similarly, the modulus of the reciprocal family $\mathcal{F}^*$ of generalized paths that separate E and F is $1/m$.

Obtaining an upper bound for the modulus of a path family usually involves choosing a metric; every metric gives an upper bound. Giving a lower bound usually involves a Cauchy-Schwarz type argument, which can be harder to do in general cases. However, in the special case of a path family connecting sets $E, F \subset \partial\Omega$, a lower bound for the modulus can also be computed by giving a upper bound for the reciprocal separating family. Thus estimates of both types can be given by producing metrics (for different families) and this is often the easiest thing to do.

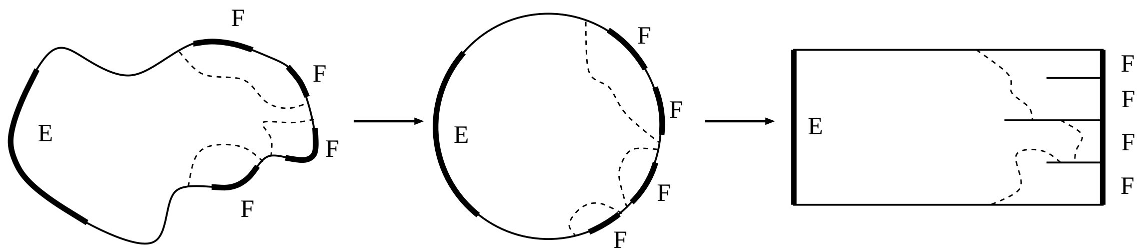


FIGURE 6. A Jordan region with separated sets E, F can be conformal mapped to slit rectangle in which E and F map to opposite sides. Such a path can have a single component (left) or multiple components (right).

It is clear that if $\Omega \subset \mathbb{C}$ is doubly connected and $A = \{1 < |z| < r\} \subset \Omega$, then $M(\Omega) \geq M(A) = \frac{1}{2\pi} \log r$. The following useful result is almost a converse of this:

LEMMA 5.1.6. *For each $m > 0$ there is a $\phi(m) > 0$ so that if Ω is a topological annulus surrounding 0 with modulus $\geq \phi(m)$, then it contains a round annulus with modulus m.*

PROOF. We prove the converse: if one complementary component of Ω contains 0 and 1 and the other contains a point z with $|z| = r = e^{2\pi m} \geq 1$, then the modulus of Ω is less than some number $\phi(r)$.

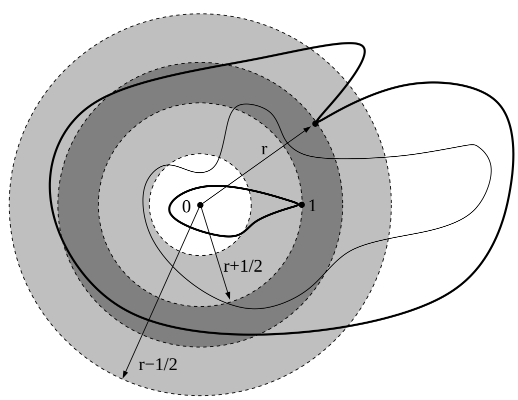


FIGURE 7. Proof of Lemma 5.1.6.

Define a metric by $\rho = 1$ on $A_1 = \{z : \frac{1}{2} < |z| < r + \frac{1}{2}\}$ and let Γ be the path family in Ω separating the two boundary components. We want to show ρ is admissible for Γ. Every such path in Γ contains a point in $A_0 = \{z : 1 < |z| < r\}$. If

γ contains a point in $\{|z| \leq 1/2\}$ then $\gamma \cap A_1$ has length at least 1, so it has ρ-length at least 1. Similarly if γ contains a point in $\{|z| \geq r + \frac{1}{2}\}$. Finally, if γ remains inside A_1 and it must have length $\geq \pi$ and hence has length (and ρ-length) greater than 1 Thus ρ is admissible. Since $\int \rho^2 dxdxy \leq \text{area}(A_1) \leq \pi(r + \frac{1}{2})^2$. Thus the lemma holds with $\phi(m) = \pi(e^{2\pi m} + \frac{1}{2})^2$ (this is not sharp). □

LEMMA 5.1.7. *Suppose $A = U \backslash K \subset \mathbb{C}$ is a topological annulus. Then* $4\text{diam}(K)^2 \leq \text{area}(U)/M(A)$.

PROOF. Take $\rho \equiv 1/2\text{diam}(K)$. This is an admissible metric for the path family separating K from ∂U, so

$$M(A) \leq \frac{1}{4}\text{diam}(K)^{-2}\text{area}(A) \leq \frac{1}{4}\text{diam}(K)^{-2}\text{area}(U),$$

as desired. □

The following result is taken from Milnor's book ([**98**], Corollary B.9) where he credits the result to McMullen.

LEMMA 5.1.8. *Suppose $A = U \setminus K \subset \mathbb{C}$ is a topological annulus. Then* $\text{area}(K) \leq \text{area}(U)/(1 + 4\pi M(A))$.

PROOF. This uses the isoperimetric inequality that says a curve of length L can enclose area at most $L^2/4\pi$. Thus every curve that separates K from ∂U has length at least $C = \sqrt{4\pi \text{area}(K)}$ and so $\rho \equiv 1/C$ is an admissible metric for this path family. Thus

$$M(A) \leq \int_A C^{-2} dxdy = \frac{\text{area}(A)}{4\pi \text{area}(K)},$$

or

$$\text{area}(K) \leq \frac{\text{area}(A)}{4\pi M(A)} M = \frac{\text{area}(U) - \text{area}(K)}{4\pi M(A)},$$

or

$$\text{area}(K) \leq \frac{\text{area}(U)}{1 + 4\pi M(A)}.$$

□

COROLLARY 5.1.9. *With notation as above,* $\text{area}(K) \leq \text{area}(U) \exp(-4\pi M(A))$.

PROOF. Using the conformal map to a round annulus we can cut A into n nested annuli $A_j = U_j \setminus K_j$, each with moduli $M(A)/n$. Applying the previous result we get

$$\text{area}(K_j) \leq \text{area}(K_{j-1})/(1 + 4\pi M(A)/n),$$

and hence

$$\text{area}(K) \leq \text{area}(U)(1 + 4\pi M(A)/n)^{-n},$$

which implies the desired result when we take $n \to \infty$. $\square$

LEMMA 5.1.10. *Suppose $\Omega \subset \mathbb{C}$ is a topological annulus of modulus M whose boundary consists of two Jordan curves γ_1, γ_2 with γ_2 separating γ_1 from ∞. Then* $\text{diam}(\gamma_1) \leq (1-\epsilon)\text{diam}(\gamma_2)$ *where $\epsilon > 0$ depends only on M.*

PROOF. Rescale so $\text{diam}(\gamma_2) = \text{diam}(\Omega) = 1$ and suppose $\text{diam}(\gamma_1) > 1-\epsilon$. Then there are points $a \in \gamma_1$ and $b \in \gamma_2$ with $|a-b| \leq \epsilon$. Let ρ be the metric on Ω defined by $\rho(z) = \frac{1}{|z-a|\log(1/2\epsilon)}$ for $\epsilon < |z-a| < 1/2$. Then any curve $\gamma \subset \Omega$ that separates γ_1 and γ_2 satisfies $\int_\gamma \rho ds \geq 1$ and

$$\int \rho^2 dxdy \leq \frac{\pi}{4} \log^{-2} \frac{1}{2\epsilon}.$$

Thus the modulus of the path family separating the boundary components is bounded above by the right hand side, and the modulus of the reciprocal family connecting the boundary components is bounded below by $\frac{\pi}{4}\log^2\frac{1}{2\epsilon}$. Thus $\epsilon \geq \frac{1}{2}\exp(-\sqrt{\pi M/4})$. $\square$

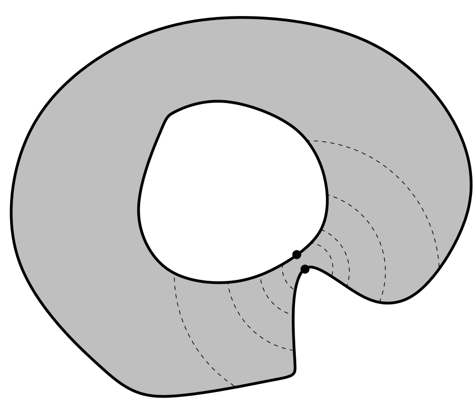


FIGURE 8. Proof of Lemma 5.1.10.

LEMMA 5.1.11. *Suppose $Q \subset \mathbb{H}$ is a quadrilateral with a pair of opposite sides being intervals $I, J \subset \mathbb{R}$. Let A be the topological annulus formed by taking $Q \cup I \cup J \cup Q^*$ (where Q^* is the reflection of Q across $\mathbb{R}$. Then $M(A) = \frac{1}{2}M(Q)$.*

PROOF. If $f : Q \to R = [0,1] \times [0,m]$ is conformal with I, J mapping to the horizontal sides of R, then $g(z) = \exp(\pi f(z)/m))$ maps Q conformally to an half-annulus in $\mathbb{H}$ with I, J mapping to intervals on $\mathbb{R}$. By Schwarz reflection, this extends to a conformal map of A to a round annulus with inner radius 1 and outer radius $\exp(\pi/m)$, so $M(A) = \frac{1}{2\pi}\frac{\pi}{m} = \frac{1}{2m} = \frac{1}{2}M(Q)$. See Figure 9. $\square$

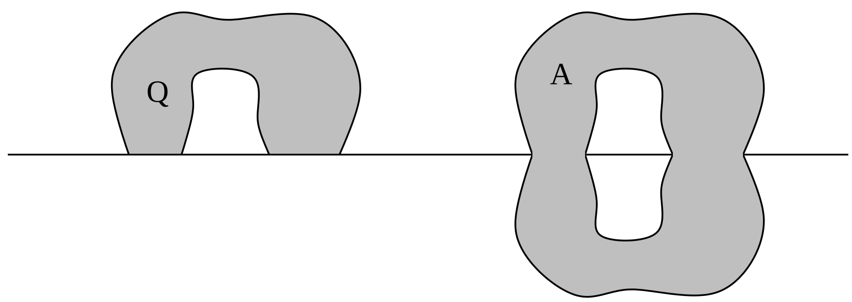


FIGURE 9. Reflecting the quadrilateral Q across the line gives an annulus with half the modulus.

COROLLARY 5.1.12. *If $\{Q_k\}$ are disjoint quadrilaterals in $\mathbb{D} \setminus \{0\}$, such that*

1. $\frac{1}{M} \leq M(Q_k) \leq M$,
2. *Q_k separates Q_{k+1} from 0,*
3. *a pair of opposite sides of Q_k consist of arcs on $\mathbb{T}$.*

Then $\operatorname{diam}(Q_k) = O(\lambda^k)$ *for some $\lambda < 1$ that only depends on M.*

PROOF. Use reflection across the circle to turn each quadrilateral into an annulus of comparable modulus and then apply Lemma 5.1.10. $\square$

LEMMA 5.1.13. *Suppose Q is a quadrilateral with opposite pairs of sides E, F and C, D. Assume*

1. *E and F can be connected in Q by a curve of diameter $\leq \epsilon$,*
2. *any curve connecting C and D in Q has diameter at least 1.*

Then the modulus of the path family connecting E and F in Q is larger than $M(\epsilon)$ where $M(\epsilon) \to \infty$ as $\epsilon \to 0$.

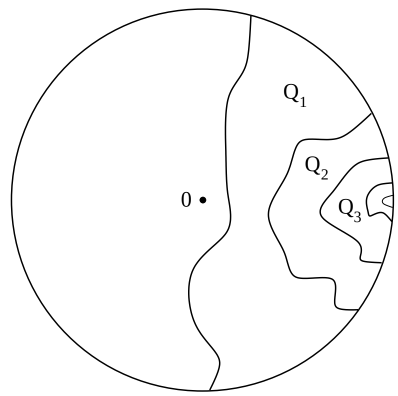


FIGURE 10. Proof of Lemma 5.1.12 .

PROOF. This is very similar to the proof of Lemma 5.1.10, so we only sketch it. There is a segment $(a,b) \subset Q$ with $|a-b| \leq \epsilon$ and $a \in E$ and $b \in F$. Define a metric on Q by $\rho(z) = \frac{1}{2}|z-a|^{-1}/\log(1/2\epsilon)$ for $\epsilon < |z-a| < 1/2$. Any curve γ connecting C and D must cross S and since γ has diameter ≥ 1 it must leave the annulus where ρ is non-zero. As before this shows that the modulus of the path family in Q separating E and F is small, hence the modulus of the family connecting them is large. $\square$

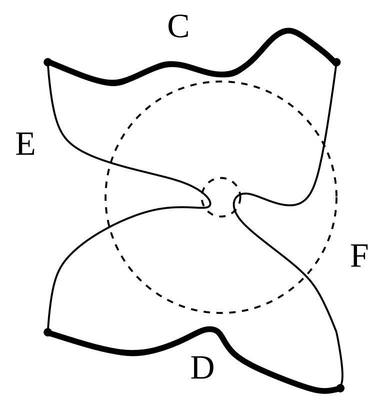


FIGURE 11. Proof of Lemma 5.1.13.

In [**60**] Gehring and Hayman proved the following fundamental inequality that says that the hyperbolic geodesic is (up to a constant factor) the most efficient way to connect two points in a simply connected plane domain. The result has been generalized in many directions (e.g., [**35**], [**12**], [**67**], [**68**], [**71**], [**82**], [**139**]).

THEOREM 5.1.14 (Gehring-Hayman inequality). *There is an absolute constant $C < \infty$ to that the following holds. Suppose $\Omega \subset \mathbb{C}$ is hyperbolic and simply connected. Given two points in Ω, let γ be the hyperbolic geodesic connecting these two points and let γ' be any other curve in Ω connecting them. Then $\ell(\gamma) \leq C\ell(\gamma')$.*

PROOF. Let $f : \mathbb{D} \to \Omega$ be conformal, normalized so that γ is the image of $I = [0, r] \subset \mathbb{D}$ for some $0 < r < 1$. Without loss of generality we may assume $r = z_{N+1}$ for some N (if not we truncate a segment of the form $J = [z_{N+1}, r]$ and use Koebe's theorem to compare the lengths of $f(J)$ and $\gamma' \cap f(Q_{N+1})$). Let

$$Q_n = \{z \in \mathbb{D} : 2^{-n-1} < |z - 1| < 2^{-n}\},$$

and let

$$\gamma_n = \{z \in \mathbb{D} : |z - 1| = 2^{-n}\},$$

$$z_n = \gamma_n \cap [0, 1).$$

Let $Q'_n \subset Q_n$ be the sub-quadrilateral of points with $|\arg(z - 1)| < \pi/6$. Each of these has bounded hyperbolic diameter and hence by Koebe's theorem its image is bounded by four arcs of diameter $\simeq d_n$ and opposite sides are $\simeq d_n$ apart. In particular, this means that any curve in $f(Q_n)$ separating γ_n and γ_{n+1} must cross $f(Q'_n)$ and hence has diameter $\gtrsim d_n$. Since Q_n has bounded modulus, so does $f(Q_n)$ and so Lemma 5.1.13 says that the shortest curve in $f(Q_n)$ connecting γ_n and γ_{n+1} has length $\ell_n \simeq d_n$. Thus any curve γ in Q connecting γ_n and γ_{n+1} has length at least ℓ_n, and so

$$\ell(\gamma) = O(\sum d_n) = O(\sum \ell_n) \leq O(\ell(\gamma')).$$

□

5.2. Logarithmic capacity and Pfluger's theorem

Suppose μ is a positive, finite Borel measure on $\mathbb{R}^2$ and define its potential function as

$$U_\mu(z) = \int \log \frac{2}{|z - w|} d\mu(w).$$

and its energy integral by

$$I(\mu) = \iint \log \frac{2}{|z - w|} d\mu(z) d\mu(w) = \int U_\mu(z) d\mu(z).$$

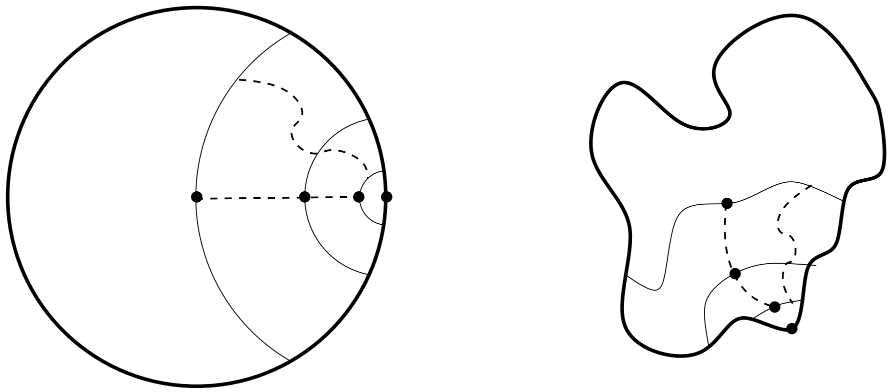


FIGURE 12. Proof of the Gehring-Hayman inequality.

We put the "2" in the numerator so that the integrand is non-negative when $z, w \in \mathbb{T}$, however, this is a non-standard usage.

LEMMA 5.2.1. *U_μ is lower semi-continuous, i.e.,*

$$\liminf_{z \to w} U_\mu(z) \geq U_\mu(w).$$

PROOF. Fatou's lemma. □

LEMMA 5.2.2. *If $\mu_n \to \mu$ weak*, then* $\liminf U_{\mu_n}(z) \geq U_\mu(z)$.

PROOF. If we replace $\varphi = \log \frac{1}{|z-w|}$ by $\varphi_r = \max(r, \varphi)$ in the definition of U to get U^r, then weak convergence implies

$$\lim_n U^r_{\mu_n}(z) = U^r_\mu(z).$$

So for any $\epsilon > 0$ we can choose N so that $n > N$ implies

$$U^r_{\mu_n}(z) \geq U^r_\mu(z) - \epsilon.$$

As $r \to \infty$ $U^r \to U$ (by the monotone convergence theorem), so for r large enough and $n > N$ we have

$$U_{\mu_n}(z) \geq U^r_{\mu_n}(z) \geq U_\mu(z) - 2\epsilon.$$

which proves the result. □

LEMMA 5.2.3. *If $\mu_n \to \mu$ weak*, then* $\liminf I(\mu_n) \geq I(\mu)$.

Proof. The proof is almost the same as for the previous lemma, except that we have to know that if $\{\mu_n\}$ converges weak*, then so does the product measure $\mu_n \times \mu_n$. However, weak convergence of $\{\mu_n\}$ implies convergence of integrals of the form

$$\iint f(x)g(y)d\mu_n(x)d\mu_n(y).$$

and Stone-Weierstrass theorem implies that the finite sums of such product functions are dense in all continuous function on the product space. $\square$

Suppose E is Borel and μ has its closed support inside E. We say μ is admissible for E if $U_\mu \leq 1$ on E and we define the **logarithmic capacity** of E as

$$\text{cap}(E) = \sup\{\|\mu\| : \mu \text{ is admissible for } E\}$$

and we write $\mu \in \mathcal{A}(E)$. We define the **outer capacity** (or exterior capacity) as

$$\text{cap}^*(E) = \inf\{\text{cap}(V) : E \subset V, V \text{open}\}.$$

We say that a set E is **capacitable** if $\text{cap}(E) = \text{cap}^*(E)$.

The logarithmic kernel can be replaced by other functions, e.g., $|z-w|^{-\alpha}$, and there is a different capacity associated to each one. To be precise, we should denote logarithmic capacity as $\text{cap}_{\log}$ or logcap, but to simplify notation we simply use "cap" and will often refer to logarithmic capacity as just "capacity". Since we do not use any other capacities in these notes, this abuse should not cause confusion.

Lemma 5.2.4. *Compact sets are capacitable.*

Proof. Since $\text{cap}(E) \leq \text{cap}^*(E)$ is obvious, we only have to prove the opposite direction. Set $U_n = \{z : \text{dist}(z, E) < 1/n\}$ and choose a measure μ_n supported in U_n with $\|\mu_n\| \geq \text{cap}(U_n) - 1/n$. Let μ be a weak accumulation point of $\{\mu_n\}$ and note

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w) \leq \int \log \frac{2}{|z-w|} d\mu_n(w) \leq 1$$

so μ is admissible in the definition of $\text{cap}(E)$. Thus

$$\text{cap}(E) \geq \limsup \|\mu_n\| = \lim \text{cap}(U_n) = \lim \text{cap}(U_n) = \text{cap}^*(E).$$

$\square$

We prove in Section A.7 that all Borel (indeed all analytic) sets are capacitable. Sets of zero logarithmic capacity must be quite small.

LEMMA 5.2.5. *Suppose E is compact and supports a positive measure μ so that $\mu(D(x,r)) \leq \varphi(r)$, where*

$$\sum_{n=0}^{\infty} n\varphi(2^{-n}) < \infty,$$

Then E has positive capacity.

PROOF. The condition easily implies U_μ is bounded, hence $\mathrm{supp}(\mu)$ has positive capacity. □

COROLLARY 5.2.6. *If $E \subset \mathbb{T}$ is compact and has positive Hausdorff dimension, then it has positive capacity.*

PROOF. This follows from Frostman's theorem (Theorem **??**) since if $\dim(E) > 0$ then E supports a measure that satisfies $\mu(D(x,r)) = O(r^\epsilon)$ for some $\epsilon > 0$ and $\sum_n 2^{-\epsilon n} < \infty$. □

COROLLARY 5.2.7. *If $E \subset \mathbb{T}$ has positive Lebesgue measure, then it has positive capacity. Moreover,*

$$\mathrm{cap}(E) \leq \frac{1}{2\log 2 + 2\log \frac{1}{|E|}}.$$

PROOF. This is the case $d = 1$ of the previous corollary. The estimate follows because if μ is Lebesgue measure restricted to E, then

$$U_\mu(x) \leq 2\int_0^{|E|/2} \log\frac{1}{t}dt = 2(\log\frac{2}{|E|} - \frac{1}{2}|E|) \leq 2\log 2 + 2\log\frac{1}{|E|}.$$

□

It is clear from the definitions that it is monotone

$$E \subset F \quad \Rightarrow \quad \mathrm{cap}(E) \leq \mathrm{cap}(F). \tag{32}$$

and satisfies the regularity condition

$$\mathrm{cap}(E) = \sup\{\mathrm{cap}(K) : K \subset E, K\text{compact}\}. \tag{33}$$

LEMMA 5.2.8 (Sub-additive). *For any sets $\{E_n\}$,*

$$\mathrm{cap}(\cup E_n) \leq \sum \mathrm{cap}(E_n). \tag{34}$$

PROOF. We can write $\mu = \sum \mu_n$ as a sum of singular measures so that μ_n gives full mass to E_n. We can then restrict each μ_n to a compact subset k_n of E_n so that $\mu_n(K_n) \geq (1-\epsilon)\mu(E_n)$. These restrictions are admissible for each E_n and hence

$$\sum \text{cap}(E_n) \geq \sum \mu_n(K_n) \geq (1-\epsilon) \sum \mu_n(E_n) = (1-\epsilon)\|\mu\|.$$

Taking $\epsilon \to 0$ proves the result. □

COROLLARY 5.2.9. *A countable union of zero capacity sets has zero capacity.*

COROLLARY 5.2.10. *Outer capacity is also sub-additive.*

PROOF. Given sets $\{E_n\}$ chose open sets $V_n \supset E_n$ so that $\text{cap}(V_n) \leq \text{cap}^*(E_n) + \epsilon 2^{-n}$. By the sub-additivity of capacity

$$\text{cap}^*(\cup E_n) \leq \text{cap}(\cup V_n) \leq \sum \text{cap}(V_n) \leq \epsilon + \sum \text{cap}^*(E_n).$$

Taking $\epsilon \to$ proves the result. □

Although capacity informally "measures" the size of a set, it is not additive, and hence not a measure.

EXERCISE: Show that if $E, F \subset \mathbb{T}$ are disjoint compact sets, each with positive logarithmic capacity, then $\text{cap}(E) + \text{cap}(F) > \text{cap}(E \cup F)$.

LEMMA 5.2.11. *If E is compact, there exists an admissible μ that attains the maximum mass in the definition of capacity and $U_\mu(z) = 1$ everywhere on E, except possible a set of capacity zero.*

PROOF. Let μ_n be a sequence of measures on E so that $\|\mu_n\| \to \text{cap}(E)$ and $U_n = U_{\mu_n}$ is bounded above by 1 on E (such a sequence exists by the definition of logarithmic capacity). By Lemma 5.2.2 U_μ is also bounded above by 1. Also, by a standard property of weak* convergence $\|\mu\| \leq \liminf \|\mu_n\| = \text{cap}(E)$, and by Lemma 5.2.3,

$$I(\mu) \leq \liminf I(\mu_n) \leq \liminf \|\mu_n\| = \text{cap}(E),$$

so we must have $I(\mu) = \text{cap}(E)$.

First we claim that $U_\mu \geq 1$ except possibly on a set of zero capacity. Otherwise let $T \subset E$ be a set of positive capacity on which $U_\mu < 1 - \epsilon$ and let σ be a non-zero, positive measure on T which potential bounded by 1. Define

$$\mu_t = (1-t)\mu + t\sigma.$$

This is a measure on E so that

$$\begin{aligned}
I(\mu_t) &\leq \int \log \frac{1}{|z-w|}((1-t)d\mu + td\sigma)((1-t)d\mu + td\sigma) \\
&\leq (1-t)^2 I(\mu) + 2t \int U_\mu d\sigma + t^2 I(\sigma) \\
&\leq I(\mu) - 2tI(\mu) + 2t \int U_\mu d\sigma + O(t^2) \\
&\leq I(\mu) - 2tI(\mu) + 2t(1-\epsilon)\|\sigma\| + O(t^2) \\
&< I(\mu),
\end{aligned}$$

if $t > 0$ is small enough. This contradicts minimality of μ.

Next we show that $U_\mu \leq 1$ everywhere on the closed support of μ. By the previous step we know $U_\mu \geq 1$ except on capacity zero, hence except on a set of μ-measure zero. If there is a point z in the support of μ such that $U_\mu(z) > 1$, then by lower semi-continuity of potentials, U_μ is $> 1 + \epsilon$ on some neighborhood of z and this neighborhood has positive μ measure (since z is in the support of μ) and thus $I(\mu) = \int U_\mu d\mu > \|\mu\|$, a contradiction. □

5.3. Pfluger's theorem

THEOREM 5.3.1 (Pfluger's theorem). *If $0 \in K \subset \mathbb{D}$ is a compact connected set with smooth boundary then there are constants C_1, C_2 depending only on K so that following holds. For any $E \subset \mathbb{T}$ that is a finite union of closed intervals, t*

$$\frac{1}{\text{cap}(E)} + C_1 \leq \pi\lambda(\mathcal{F}_E)) \leq \frac{1}{\text{cap}(E)} + C_2,$$

where $\mathcal{F}_E$ is the path family connecting K to E.

PROOF. Let K^* be the reflection of K across $\mathbb{T}$ and let Ω be the connected component of $\mathbb{C} \setminus (E \cup K \cup K^*)$ that has E on its boundary. Let $h(z)$ be the harmonic function in Ω with boundary values 0 on K and K^* and boundary value 1 on E. By the usual theory of the Dirichlet problem, all boundary points are regular (since all boundary components are non-degenerate continua) and hence h extends continuously to the boundary with the correct boundary values. Moreover, h is symmetric with respect to $\mathbb{T}$, and this implies its normal derivative on $\mathbb{T} \setminus E$ is 0.

Clearly $|\nabla h|$ is an admissible metric for $\mathcal{F}$, so

$$M(\mathcal{F}) \le D(h) \equiv \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

We wish to show equality holds.

By Green's theorem and the fact that $h = 1$ on E,

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = -\int_{\mathbb{T}} \frac{\partial h}{\partial n} ds = -\int_{E} \frac{\partial h}{\partial n} ds = -\int_{E} h \frac{\partial h}{\partial n} ds.$$

and this

$$\begin{aligned}
\int_{\partial K} \frac{\partial h}{\partial n} ds &= -\frac{1}{2}\int_{E} \frac{\partial (h^2)}{\partial n} ds \\
&= \frac{1}{2}\int_{\mathbb{T}\setminus E} \frac{\partial (H^2)}{\partial n} ds + \frac{1}{2}\int_{\partial K} \frac{\partial (h^2)}{\partial n} ds + \frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.
\end{aligned}$$

The first term is zero because h has normal derivative zero on$\mathbb{T} \setminus E$, and hence the same is true for h^2. The second term is zero because h is zero on K and so $\frac{\partial (h^2)}{\partial n} h^2 = 2h\frac{\partial h}{\partial n} = 0$. To evaluate the third term, we use the identity

$$\Delta(h^2) = 2h_x{\cdot}h_x + 2h{\cdot}h_{xx} + 2h_y{\cdot}h_y + 2h{\cdot}h_{yy} = 2h\Delta h + 2\nabla h{\cdot}\nabla h = 2h{\cdot}0 + 2|\nabla h|^2 = 2|\nabla h|^2,$$

to deduce

$$\frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy = \int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.$$

Therefore,

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = \int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.$$

Thus the tangential derivative of h's harmonic conjugate has integral $D(h)$ around ∂K and therefore $2\pi h/D(h)$ is the real part of a holomorphic function g on $\mathbb{D} \setminus K$. Then $f = \exp(g)$ maps $\mathbb{D} \setminus K$ into the annulus $A = \{z : 1 < |z| < \exp(2\pi/D(h))$ with the components of E mapping to arcs of the outer circle and the components of $\mathbb{T} \setminus E$ mapping to radial slits. The path family $\mathcal{F}$ maps the the path family connecting the inner and outer circles without hitting the radial slits, and our earlier computations show the modulus of this family is $1/D(h)$.

Now we have to relate $D(h)$ to the logarithmic capacity of E. Let μ be the equilibrium probability measure for E. We know in general that $U_\mu = \gamma$ where $\gamma = 1/\mathrm{cap}(E)$ almost everywhere on E (since sets of zero capacity have zero measure) and is continuous off E, but since U_μ is harmonic in $\mathbb{D}$ and equals the Poisson integral of its boundary values, we can deduce $U_\mu = \gamma$ everywhere on E. Let $v(z) = \frac{1}{2}(U_\mu(z) + U_\mu(1/\overline{z})$. Then since ∂K has positive distance from 0 we there are constants C_1, C_2 so that

$$v + C_1 \leq 0, \qquad v + C_2 \geq 0,$$

on ∂K. Note that $C_1 \geq -\gamma$ by the maximum principle and $c_2 \geq 0$ trivially. Thus by the maximum principle,

$$\frac{v(z) + C_1}{\gamma + C_1} \leq h(z) \leq \frac{v(z) + C_2}{\gamma + C_2}.$$

Since we have equality on E, we get on E

$$\frac{\partial}{\partial n}(\frac{v(z) + C_1}{\gamma + C_1}) \leq \frac{\partial h}{\partial n} \leq \frac{\partial}{\partial n}(\frac{v(z) + C_2}{\gamma + C_2}).$$

When we integrate over E, the middle term is $-D(h)$ (we computed this above) and by Green's theorem

$$\begin{aligned} -\int_E \frac{\partial}{\partial n}\frac{v(z) + C_1}{\gamma + C_1} ds &= \frac{1}{\gamma + C_1}\int_{\mathbb{D}} \Delta(v) dxdy \\ &= \frac{\pi}{\gamma + C_1} \end{aligned}$$

because v is harmonic except for a $\frac{1}{2}\log\frac{1}{|z|}$ pole at the origin. A similar computation holds for the other term and hence

$$\frac{\pi}{\gamma + C_1} \leq D(h) = M(\mathcal{F}) \leq \frac{\pi}{\gamma + C_2},$$

since $D(h) = \int_E \frac{\partial h}{\partial n} ds$. Hence

$$\gamma + C_1 \leq \pi\lambda(\mathcal{F}) \leq \gamma + C_2.$$

This completes the proof of Pfluger's theorem for finite unions of intervals. $\square$

Next we will prove Pfluger's theorem holds for all compact subsets of $\mathbb{T}$. First we need a technical result that implies a continuity property of extremal length.

LEMMA 5.3.2. *Suppose $E \cap \mathbb{T}$ is compact, $K \subset \mathbb{D}$ is compact, connected and contains the origin and $\mathcal{F}$ is the path family connecting K and E in $\mathbb{D} \setminus K$. Fix an admissible metric ρ for $\mathcal{F}$ and for each $z \in \mathbb{T}$, define $f(z) = \inf \int_\gamma \rho ds$ where the infimum is over all paths in $\mathcal{F}$ that connect K to z. Then f is lower semi-continuous.*

PROOF. Suppose $z_0 \in \mathbb{T}$ and use Cauchy-Schwarz to get

$$\begin{aligned}\int_{2^{-n-1}}^{2^{-n}} [\int_{|z|=r} \rho ds]^2 dr &\leq \int_{2^{-n-1}}^{2^{-n}} \int_0^{2\pi} r^2 \rho dr d\theta \\ &\leq \pi 2^{-n} \int_{2^{-n-1}<|z-z_0|<2^{-n}} \rho^2 dxdy \\ &= o(2^{-n}).\end{aligned}$$

Therefore we can choose circular cross-cuts $\{\gamma_n\} \subset \{z : 2^{-n-1} < |z - z_0| < 2^{-n}$ of $\mathbb{D}$ centered at z_0 and with ρ-length ϵ_n tending to 0. By taking s subsequence we may assume $\sum \epsilon_n < \infty$.

Now choose $z_n \to z_0$ with

$$f(z_n) \to \alpha \equiv \liminf_{z \to z_0} f(z).$$

We want to show that there is a path connecting K to z_0 whose ρ-length is as close to α as we wish. Passing to a subsequence we may assume z_n is separated from K by δ_n. Let c_n be the infimum of ρ-lengths of paths connecting γ_n and γ_{n+1}. By considering a path connecting K to z_n, we see that $\sum_1^n c_k \leq f(z_n)$, for all n and hence $\sum_1^\infty c_n \leq \alpha$.

Now choose $\epsilon > 0$ and choose n so that we can connect K to z_n (and hence to γ_n) by a path of ρ-length less than $\alpha + \epsilon$. We can then connect γ_n to z_0 by a infinite concatenation of arcs of γ_k, $k > n$ and paths connecting γ_k to γ_{k+1} that have total length $\sum_n^\infty (\epsilon_n + c_n) = o(1)$. Thus K can be connected to z_0 by a path of ρ-length as close to α as we wish. $\square$

COROLLARY 5.3.3. *Suppose $E \subset \mathbb{T}$ is compact and $\epsilon > 0$. then there is a finite collection of closed intervals F so that $E \subset F$ and*

$$\lambda(\mathcal{F}_E) \leq \lambda(\mathcal{F}_F) + \epsilon.$$

PROOF. By Lemma 5.3.2,

$$V = \{z \in \mathbb{T} : f(z) > r = (\frac{M(\mathcal{F}_E) + \epsilon}{M(\mathcal{F}_E) + 2\epsilon})^{1/2}\},$$

is open, and therefore we can choose a set F of the desired form inside V. Then ρ/r is admissible for $\mathcal{F}_F$, so

$$M(\mathcal{F}_F) \leq \int (\frac{\rho}{r})^2 dxdy = \frac{M(\mathcal{F}_E) + \epsilon}{M(\mathcal{F}_E) + 2\epsilon} \int \rho^2 dxdy \leq M(\mathcal{F}_E) + \epsilon.$$

Thus an inequality in the opposite direction holds for extremal length. $\square$

COROLLARY 5.3.4. *Pfluger's theorem holds for all compact subsets of* $\mathbb{T}$.

PROOF. Suppose $E$is compact and choose sets $E_n \searrow E$ that are finite unions of closed intervals. We have proven both

$$\lambda(\mathcal{F}_{E_n}) \to \lambda(\mathcal{F}_E),$$

and

$$\operatorname{cap}(E_n) \to \operatorname{cap}(E),$$

the inequalities in in Pfluger's theorem extend to E. $\square$

5.4. Estimates of Ahlfors and Beurling

The usefulness of extremal length is its ability to estimate a conformal invariant in terms of geometry (length and area). Using extremal length we can estimate logarithmic capacity, and logarithmic capacity bounds Lebesgue measure, and Lebesgue measure on the disk corresponds to harmonic measure on a simply connected domain. More precisely,

COROLLARY 5.4.1. *Suppose* Ω *is a Jordan domain,* $z_0 \in \Omega$ *with* $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ *and* $E \subset \partial\Omega$. *Let* $\mathcal{F}$ *be the family of curves in* Ω *which separates* $D(z_0, 1/2)$ *from* E. *Then* $\omega(z_0, E, \Omega) \leq C \exp(-\pi M(\mathcal{F}))$. *If* $E \subset \partial\Omega$ *is an arc then the inequality is actually a similarity.*

One of the most famous and most useful applications of this idea is

COROLLARY 5.4.2 (Ahlfors distortion theorem). *Suppose* Ω *is a Jordan domain,* $z_0 \in \Omega$ *with* $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ *and* $x \in \partial\Omega$. *For each* $0 < t < 1$ *let* $\ell(t)$ *be the length of* $\Omega \cap \{|w - x| = t\}$. *Then there is an absolute* $C < \infty$, *so that*

$$\omega(z_0, D(x, r), \Omega) \leq C \exp(-\pi \int_r^1 \frac{dt}{\ell(t)}).$$

PROOF. Let K be the disk of radius $1/2$ around z_0 and let $\mathcal{F}$ be the family of curves in Ω which separate $D(x, r) \cap \partial\Omega$ from K. Let $\mathcal{F}_1 \subset \mathcal{F}$ be the collection of curves of the form

$$L_t = \Omega \cap \{|w - x| = t\}.$$

if ρ is admissible for $\mathcal{F}$ then it is admissible for $\mathcal{F}_1$ and hence

$$1 \leq \int_{L_t} \rho ds \leq (\int_{L_t} \rho^2 ds)\ell(t),$$

so

$$\int_r^1 \int_{L_t} \rho^2 ds dt \geq \int_r^1 \frac{dt}{\ell(t)}.$$

This proves

$$M(\mathcal{F}) \geq \int_r^1 \frac{dt}{\ell(t)},$$

which proves the result by the previous corollary. □

For an alternate version of this using line segments instead of circular arcs, see Exercise **??**.

Corollary 5.4.3 (Beurling's estimate). *There is a $C < \infty$ so that if Ω is simply connected, $z \in \Omega$ and $d = \operatorname{dist}(z, \partial\Omega)$ then for any $0 < r < 1$ and any $x \in \partial\Omega$,*

$$\omega(z, D(x, rd), \Omega) \leq Cr^{1/2}$$

Proof. Apply Corollary 5.4.2 at x and use $\theta(t) \leq 2\pi t$ to get

$$\exp(-\pi \int_{rd}^d \frac{dt}{\theta(t)t}) \leq C \exp(-\frac{1}{2}\log r) \leq C\sqrt{r}.$$

□

An important consequence of the this result is that harmonic measure on a simply connected domain must give zero measure to any set of Hausdorff dimension less than $\frac{1}{2}$. This was the first in a chain of results on planar harmonic measure that eventually led to Makarov's stunning result that harmonic measure for a planar simply connected domain is always one dimensional, i.e., it gives zero measure to any set of dimension < 1 but always gives full measure to some set on dimension $= 1$. See Makarov's paper[**88**] (and [**?**], [**59**] for surveys of developments since Makarov's original result).

The following version of Beurling's theorem will be useful later.

Lemma 5.4.4. *Suppose U is simply connected, K is compact, non-trivial and connected, and $f : U \to V = \mathbb{C} \setminus V$ is holomorphic. Fix $x \in K$ and $r > 0$ and set $E_r = \{w \in \partial U : |f(w) - x| < r\}$. Then*

$$\omega(E_r, z_0, U \setminus E_r) \leq \max(1, Cr^{1/2}),$$

where C depends only $\operatorname{diam}(K)$ *and a lower bound for* $\operatorname{dist}(f(z_0), K)$.

PROOF. Let $v(z) = \omega(D(x, 2r), z, V)$ and $z_1 = f(z_0)$. By Beurling's theorem $v(z_1) \leq Cr^{1/2}$, where C depends only on lower bounds for $\operatorname{diam}(K)$ and $\operatorname{dist}(z_1, K)$. Also, $v > c > 0$ on $D(z, r)$, since $\omega(z, D(x, 2r), V)$, is bounded away from zero on $D(x, r) \cap V$. Thus $u(z) = u(f(z))$ is harmonic in U,

$$u(z_0) = v(z_1) \leq Cr^{1/2},$$

and $u(z) > c$ on E_r. Thus

$$\omega(z_0, E_r, U \setminus E_r) \leq \frac{1}{c} Cr^{1/2}.$$

□

The following is a stronger version of Ahlfors' estimate, and is proved by different techniques. We shall not give a proof here; see Appendix G of [**59**].

THEOREM 5.4.5 (Carleman's estimate). *Let $\Omega \subset \mathbb{C}$ be a domain and let*

$$\Omega_x = \Omega \cap \{y : x + iy \in \Omega\}$$

be the intersection of Ω with the vertical line through x. Let $E_b = \partial\Omega \cap \{x+iy : x \geq b\}$. Suppose each set Ω_x has length less than M and let $\ell(x)$ denote the length of the longest segment in Ω_x. Suppose $z = a + iy \in \Omega$, $d = \operatorname{dist}(z, \partial\Omega)$ and $b > a$. Then

$$\omega(a + iy, E_b, \Omega) \leq \frac{2\pi d}{9M^2} \int_a^b \exp(2\pi \int_a^t \frac{dx}{\ell(x)}) dt)^{-1/2}.$$

This result is very helpful when considering the harmonic measure of a boundary set that can be approached in several different ways through the domain Ω, e.g., see Figure 13. We sketch the proof of Carleman's inequality as a series of exercises.

EXERCISE: prove Wirtinger's inequality: if g, g' are real valued and $g(a) = g(b) = 0$, then

$$\int_a^b (g')^2 dx \leq (\frac{\pi}{b-a})^2 \int_a^b g^2 dx$$

EXERCISE: Set $\omega(z) = \omega(z, E_b, \Omega$ and

$$A(t) = \int_0^t \int_{\Omega_x} |\nabla\omega|^2 dy dx.$$

Use Green's theorem to prove that for $0 < x < b$,

$$A'(x) \geq 2\pi A(x)/\ell(x).$$

EXERCISE: Define $\varphi(x) = \int_{\Omega_x} \omega^2 dy$. Show

$$\frac{\varphi''(x)}{\varphi'(x)} \leq 2\pi/\ell(x).$$

EXERCISE: Set $\mu(x) = 2\pi/\ell(x)$ and

$$\psi(x) = \int_0^x \exp(\int_0^t d\mu) dt.$$

Show that

$$(\log \varphi' - \log \psi')' = \frac{\varphi''}{\varphi'} - \frac{\psi''}{\psi'} \geq 0,$$

and deduce by integration that $\varphi(x)\psi(t) \leq \varphi(t)\psi(x)$ whenever $0 < x < t$.

EXERCISE: Prove

$$\begin{aligned} \psi(b) &= \psi(a) + \int_a^b \exp(\int_0^a \mu(s)ds) \exp(\int_a^t \mu(s)ds) dt \\ &\geq \psi(a)(1 + \frac{2\pi}{M} \int_a^b \exp(\int_a^b \mu(s)ds)dt). \end{aligned}$$

EXERCISE: Prove

$$\omega(z)^2 \leq \frac{\varphi(z)}{d}(1 + \frac{2\pi}{M} \int_a^b \exp(\int_a^b \mu(s)ds)dt).$$

Deduce Carleman's inequality.

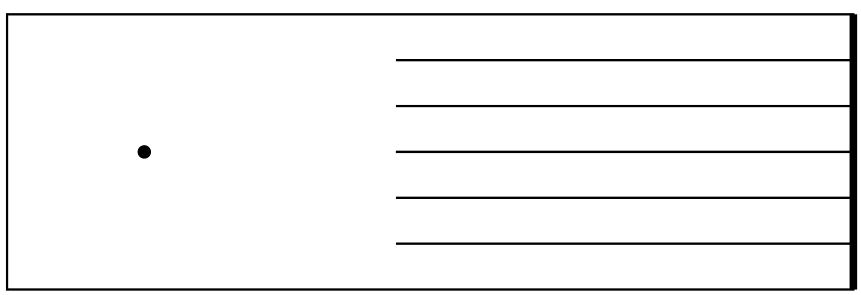


FIGURE 13. When estimating the harmonic measure of the right-hand vertical side, Ahlfors' estimate ignores the slits, but Carleman's estimate does not and gives a smaller upper bound

5.5. Boundary values of conformal maps

Suppose $\partial\Omega$ is bounded in $\mathbb{R}^2$ and $f : \mathbb{D} \to \Omega$ is conformal. For $0 < r < 1$, let

$$a_f(r) = \text{area}(\Omega \setminus f(D(0,r))).$$

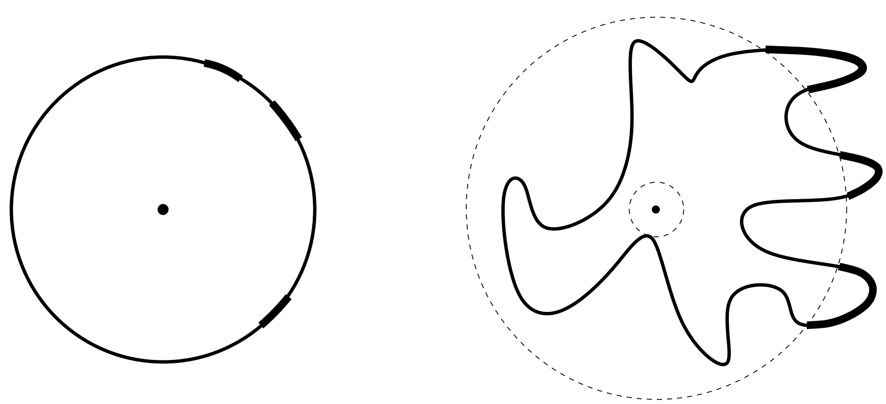
Since $\partial\Omega$ is compact it is easy to see that this tends to zero as $r \to 1$.

LEMMA 5.5.1. *Suppose $f : \mathbb{D} \to \Omega$ is conformal and for $R \geq 1$,*

$$E = \{x \in \mathbb{T} : |f(x)| \geq R\,\mathrm{dist}(f(0), \partial\Omega)\}.$$

Then E has capacity $\leq 2\pi/\log R$. Then $|E| \leq CR^{-1/2}$ (with C independent of Ω).

PROOF. Assume $f(0) = 0$ and $\mathrm{dist}(0, \partial\Omega) = 1$ and let $\rho(z) = |z|^{-1}/\log R$ for $z \in \Omega \cap \{1 < |z| < R\}$. Then ρ is admissible for the path family connecting $D(0, 1/2)$ to $\partial\Omega \setminus D(0, R)$ and $\iint \rho^2 dxdy \leq 2\pi/\log R$. By the Koebe distortion theorem $f^{-1}(D(0, 1/2))$ is contained in a compact subset of $\mathbb{D}$, independent of Ω. The result follows by Theorem **??**. $\square$



COROLLARY 5.5.2. *Suppose $f : \mathbb{D} \to \Omega$ is conformal and $a \in \mathbb{C} \cup \{\infty\}$. Then the set where f has radial limit a has zero capacity.*

PROOF. When $a = \infty$, this is immediate from the previous result. If $a \in \partial\Omega \setminus \{\infty\}$, we can reduce to the case $a = \infty$ by applying the conformal transformation $z \to 1/(a - z)$. The cases $a \notin \partial\Omega$ are trivial. $\square$

Lemma 5.5.1 also follows from a stronger result of Balogh and Bonk in [**11**].

LEMMA 5.5.3. *There is a $C < \infty$ so that the following holds. Suppose $f : \mathbb{D} \to \Omega$ and $\frac{1}{2} \leq r < 1$. Let $E = \{x \in \mathbb{T} : |f(sx) - f(rx)| \geq \delta$ for some $r < s < 1\}$. Then the extremal length of the path family $\mathcal{P}$ connecting $D(0, r)$ to E is bounded below by $\delta^2/Ca(r)$.*

Proof. Suppose $z, w \in \Omega$, suppose γ is the hyperbolic geodesic connecting z and w and suppose $\tilde{\gamma}$ is any path in Ω connecting these points. By the Gehring-Hayman inequality (Theorem 5.1.14) there is a universal $C < \infty$ such that $\ell(\gamma) \leq C\ell(\tilde{\gamma})$ (here $\ell(\gamma)$ denotes the length of γ). In other words, up to a constant, the hyperbolic geodesic has the shortest Euclidean length amongst all curves in Ω connecting the two points.

Now suppose we apply this with $z = f(sx)$ and $w \in f(D(0,r))$. Then the length of any curve from w to z is at least $1/C$ times the length of the hyperbolic geodesic γ between them. But this geodesic has a segment γ_0 that lies within a uniformly bounded distance of the geodesic γ_1 from $f(rx)$ to z. By the Koebe distortion theorem γ_0 and γ_1 have comparable Euclidean lengths, and clearly the length of γ_1 is at least δ. Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least δ/C. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and 0 elsewhere. Then ρ is admissible for $f(\mathcal{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$. Thus $\lambda(\mathcal{P}) \geq \frac{\delta^2}{C^2 a(r)}$. $\square$

Corollary 5.5.4. *If $f : \mathbb{D} \to \Omega$ is conformal, then f has radial limits except on a set of zero capacity (and hence has finite radial limits a.e. on $\mathbb{T}$).*

Proof. Let $E_{r,\delta} \subset \mathbb{T}$ be the set of $x \in \mathbb{T}$ so that $\operatorname{diam}(f(rx, x)) > \delta$, and let $E_\delta = \cap_{0<r<1} E_{r,\delta}$. If f does not have a radial limit at $x \in \mathbb{T}$, then $x \in E_\delta$ for some $\delta > 0$, and this has zero capacity by the previous result. Taking the union over a sequence of δ's tending to zero proves the result. The set where f has a radial limit ∞ has zero capacity by Lemma 5.5.1, so we deduce f has finite radial limits except on zero capacity. $\square$

We will need the following corollary of Lemma 5.5.1 in the chapter on quasiconformal mappings (it is used in the proof that 1-quasiconformal maps are conformal).

Corollary 5.5.5. *There is an absolute $C < \infty$ so that the following holds. Suppose φ is a conformal map from a $\epsilon \times 1$ rectangle R to a Jordan domain that contains no disk larger than δ. Then for every $y \in [0,1]$ there is a $t \in (0,1)$ with $|t - y| < \epsilon$ and such that the horizontal cross-cut of R at height t maps to a arc of length $C\delta$.*

Proof. First assume $y \in (\frac{\epsilon}{2}, 1 - \frac{\epsilon}{2})$ and choose a conformal map $\psi : \mathbb{D} \to R$ that sends 0 to $(\frac{\epsilon}{2}, y)$. Then apply the previous result to $\Phi = \varphi \circ \psi$ to deduce that except for a set of small measure in $I = [y - \frac{\epsilon}{4}, y + \frac{\epsilon}{4}]$, all the horizontal cross-cuts corresponding to this interval have length bounded by $|\Phi'(0)| \leq C\delta$ (by the Koebe theorem). $\square$

THEOREM 5.5.6 (Caratheodory). *If $f : \mathbb{D} \to \Omega$ is a conformal map onto a Jordan domain, then f extends continuously to the boundary.*

PROOF. Since $\partial\Omega$ is a Jordan curve, it is the continuous image of the unit circle $\mathbb{T}$ under a map g. Since $\mathbb{T}$ is compact, g is uniformly continuous. This means that for any $\epsilon > 0$ there is an $\eta > 0$ so that $|x - y| \leq \eta$ implies $|g(x) - g(y)| \leq \epsilon$. Let $\sigma = \min\{|g(x) - g(y)| : |x - y| = \eta\}$. Then if $z, w \in \partial\Omega$ satisfy $|z - w| < \sigma$, then $z = g(x), w = g(y)$ for points x, y within η of each other. Thus the smaller arc between z and w has diameter less than ϵ.

Fix δ small. Choose n so large that $\Omega \setminus f(\mathbb{D}_{1/n})$ contains no disk of radius δ. Choose $\{z_j\}$ to be n equally spaced points on the unit circle and choose interlaced points $\{w_j\}$ so that $|f(w_j) - f(rw_j)| \leq C\delta$ where $r = 1 - 1/n$. Then

$$|f(w_j)-f(w_{j+1})| \leq |f(w_j)-f(rwj)|+|f(rw_j)-f(rw_{j+1})|+|f(rw_{j+1})-f(w_{j+1})| \leq C\delta,$$

where the center term is bounded by Koebe's theorem and the other two by definition.

Thus if δ is so small that $C\delta < \sigma$, the shorter arc of $\partial\Omega$ with endpoints $f(w_j)$ and $f(w_{j+1})$ must have diameter $\leq \epsilon$. Thus the f image of the Carleson square with base I_j (the arc between w_j and w_{j+1} has diameter at most $O(\epsilon)$. This implies f has a continuous extension to the boundary. $\square$

All we really need in the previous proof is that Ω has the property that for every $\epsilon > 0$ there is a $\delta > 0$ so that if γ is a cross-cut of Ω with length $< \delta$, then there is a component of Ω that has diameter $< \epsilon$. The converse, statement is obvious, so this condition on Ω characterizes the planar domains for which the conformal map extends continuously to the boundary. This condition can be shown to be equivalent to $\partial\Omega$ being locally connected, which is also equivalent to $\partial\Omega$ being the continuous image of $\mathbb{T}$.

It is an unfortunate fact that just because a sequence on conformal maps $\{f_n\}$ on $\mathbb{D}$ converges uniformly on compact sets to a conformal map f, the boundary values need not converge on the boundary; see the example illustrated in Figure **??**. However, the conformal boundary values will converge if there is any parameterization of the boundaries that converge:

LEMMA 5.5.7. *Suppose* $\{f_n\}$ *are conformal maps of* $\mathbb{D} \to \Omega_n$ *that converge uniformly on compact subsets of* $\mathbb{D}$ *to a conformal map* $f : \mathbb{D} \to \Omega$*. Suppose that the boundary of each* Ω_n *is the homeomorphic image* $\partial\Omega_n = \sigma_n(\mathbb{T})$ *and that* $\{\sigma_n\}$ *converges uniformly on* $\mathbb{T}$ *to a homeomorphism* $\sigma : \mathbb{T} \to \partial\Omega$*. Then* $f_n \to f$ *uniformly on the* $\overline{\mathbb{D}}$*.*

PROOF. Fix $\epsilon > 0$ and choose n so large that if we divide $\mathbb{T}$ into n equal sized intervals J_j, then σ maps each of them to a set I_j of diameter $< \epsilon$. Let $I_j^k = f_k(J_j)$. Because $\sigma_k \to \sigma$ uniformly, the sets I_j all have diameter $< \epsilon$ too, if k is large enough.

Next choose $\delta > 0$ so small that if $k, m > 1/\delta$ and $\sigma_m(J_j)$ and $\sigma_k(J_i)$ contain points at most distance $C\delta$ apart, then J_i and J_k are the same or adjacent to each other (the absolute constant C will be determined below). We can do this because of the uniform convergence and the fact that σ is 1-to-1. By passing to the limit the same property holds if we replace σ_m by σ.

Next choose m so large that $f(\mathbb{D}) \setminus f((1-\frac{1}{m})\mathbb{D})$ is contained in an δ-neighborhood of $\partial\Omega$. Choose m points $\{z_j\}$ equally spaced on the circle $|z| = 1-\frac{1}{m}$, and let $K_j \subset \mathbb{T}$ be the arc centered at $z_j/|z_j|$ of length $4\pi/m$. By Lemma 5.5.1 choose a point $w_j \in K_j$ so that $|w_j - z_j| \leq 2/m$ and $|f(j_m) - f(j_m)| \leq C\epsilon$. Similarly, choose points $w_j^k \in K_j$ so that $|f(w_j^k) - f_k(z_j)| \leq 2C\delta$. This is possible since $f_k \to f$ uniformly on the compact set $\{|z| \leq 1 - \frac{1}{m}\}$ and thus $\partial f_k(\mathbb{D})$ is contained in an 2δ-neighborhood of $\partial\Omega$ for k large enough and $\partial\Omega_k$ is contained in a δ-neighborhood of $\partial\Omega$ because of the uniform convergence of the parameterizations.

By taking m larger, if necessary, we can also arrange that each I_j contains at least one of the points $f(z_m/|z_m|)$. Thus each $f(K_j)$ is mapped into the union of at most 2 of the I_j and hence its image has diameter at most 2ϵ. Also, the points $f(w_m^p)$ and $f(w_m^{p+1})$ are at most $C\delta$ apart, so belong to the same or adjacent sets I_j. Thus $f_k(K_p)$ is a union of at most 4 such adjacent sets and hence has diameter $O(\epsilon)$.

For each w_p^k there is an arc J_j so that $f_k(w_p^k)\sigma_k(J_j)$. Similarly, there is an arc J_i so that $f(w_p) \in I_i = \sigma(J_i)$. Since $f_k \to f$ uniformly on the finite set $\{z_n\}$, we have, for k sufficiently large

$$|f_k(w_n^k)-f(w_n)| \leq |f_k(w_n^k)-f_k(z_n)|+|f_k(z_n)-f(z_n)|+|f(z_n)-f(w_n)| \leq (2C+1+C)\delta.$$

Since $f_k(w_i^k) \in I_j$ and $f(w_i^k) \in I_i$, I_j and I_i are within $C\delta$ of each other, and hence J_i and J_j are either the same or adjacent. Since I_i and I_j each have diameter $< \epsilon$, there union has diameter $< 2\epsilon$ and the union of the intervals adjacent to these is at most 4ϵ. Similarly for I_i^k and J_j^k. Thus $f_k(K_p)$ and $f(K_p)$ are contained in $O(\epsilon)$-neighborhoods of each other. Thus $f_k \to f$ uniformly on $\mathbb{T}$. By the maximum principle, this implies uniform convergence on the closed disk, as desired. □

Even without the convergence of parameterizations, uniform convergence on compact sets implies convergence of a subsequence on on "most" of the boundary. See [**?**].

(*reminder -Cite Lundberg and David Hamilton*)

Another inconvenient fact is that conformal maps do not have to extend continuously to the boundary. We noted above however, that radial do exist almost everywhere. Another convenient substitute for full continuity says that every conformal map is continuous on a subdomain of $\mathbb{D}$ whose boundary hits "most of" $\partial\mathbb{D}$. The precise statement requires a new definition.

Given a compact set $E \subset \mathbb{T}$ we will now define the associated "sawtooth" region W_E Suppose $\{I_n\}$ are the connected components of $\mathbb{T} \setminus E$ and for each n let $\gamma_n(\theta)$ be the circular arc in $\mathbb{D}$ with the same endpoints as I_n and which makes angle θ with I_n (so $\gamma_n(0) = I_n$ and $\gamma_n(\pi/2)$ is the hyperbolic geodesic with the same endpoints as I_n). Let $C_n(\theta)$ be the region bounded by I_n and $\gamma_n(\theta)$, and let $W_E(\theta) = \mathbb{D} \setminus \cup_n C_n(\theta)$.

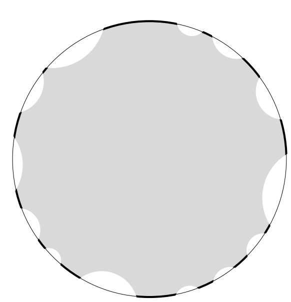


FIGURE 14. The sawtooth domain W_E

Let $W_E = W_E(\pi/8)$ (and let $W_E^* \subset \mathbb{D}^*$ be its reflection across $\mathbb{T}$).

If $f : \mathbb{D} \to \Omega$ and $0 < r < 1$, then define

$$d_f(r) = \sup\{|f(z) - f(w)| : |z| = |w| = r \text{ and } |z - w| \le 1 - r\}.$$

If $\partial\Omega$ is bounded in the plane, then it is easy to see this goes to zero as $r \nearrow 1$, since otherwise any neighborhood of $\partial\Omega$ would contain infinitely many disjoint disks of a fixed, positive size.

LEMMA 5.5.8. *Suppose $f : \mathbb{D} \to \Omega \subset S^2$ is conformal. Then for any $\epsilon > 0$ there is a compact set $E \subset \mathbb{T}$ with $\mathrm{cap}(\mathbb{T} \setminus E) < \epsilon$ such that f is continuous on $\overline{W_E}$.*

PROOF. By applying a square root and a Möbius transformation, we may assume that $\partial\Omega$ is bounded in the plane. Given $r < 1$ let

$$E(\epsilon, r) = \{x \in \mathbb{T} : |f(sx) - f(tx)| > \epsilon \text{ for some } r < s < t < 1\}$$

and note that by Lemmas **??** and 5.5.3

$$|E(\epsilon, r)| \le \exp(-\pi\epsilon^2/Ca(r)).$$

Moreover, this set is open since f is continuous at the points sx and tx. So if we take $\epsilon_n = 2^{-n}$, we can choose r_n so close to 1 that $|E(\epsilon_n, r_n)| \le \epsilon 2^{-n}$. If we define $E = \mathbb{T} \setminus \cup_{n>1} E_n$, then E is closed and $\mathbb{T} \setminus E$ has length $\le \epsilon$ by subadditivity .

To show f is continuous at every $x \in \overline{W_E}$, we want to show that $|x - y|$ small implies $|f(x) - f(y)|$ is small. We only have to consider points $x \in \partial W_E \cap \mathbb{T}$. First suppose $y \in \partial W_E \cap \mathbb{T}$. Choose the maximal n so that $s = |x - y| \le 1 - r_n$. Then $x, y \notin E_n$, so

$$|f(x) - f(y)| \le |f(x) - f(sx)| + |f(sx) - f(sy)| + |f(sy) - f(y)|.$$

The first and last terms on the right are $\le \epsilon_{n-1}$ by the definition of E. The middle term is at most $d_f(1 - s)$ (which tends to 0 as $s \to 0$). Thus $|f(x) - f(y)|$ is small if $|x - y|$ is.

Now suppose $x \in \partial W_E \cap \mathbb{T}$, $y \in \partial W_E \setminus \mathbb{T}$. From the definition of W_E it is easy to see there is a point $w \in \partial W_E \cap \mathbb{T}$ such that $|w - y| \le 2(1 - |y|) \le 2|x - y|$. For the point w we know by the argument above that $|f(x) - f(w)|$ is small. On the other hand, if $t = 1 - |y|$, then

$$|f(y) - f(w)| \le |f(y) - f(tw)| + |f(tw) - f(w)|.$$

The first term is bounded by $Cd_f(1-t)$ and the second is small since $w \notin E_n$. Thus $|f(x) - f(y)|$ is small depending only on $|x-y|$. Hence f is continuous on $\overline{W_E}$. □

reminder - check this:

LEMMA 5.5.9. *If $\{f_n\}$ is a sequence of K-quasiconformal maps that converges uniformly to a homeomorphism f and $\mu_{f_n} \to 0$ almost everywhere, then f is conformal.*

PROOF. □

5.6. Maps between Fatou components

We showed earlier (Lemma 2.6.2) that a multiply connected Fatou component of f is always mapped onto a Fatou component by f. Our next result says that this is almost true for simply connected components; there is at most one omitted point. This result is due to Herring [**70**] and independently, Bergweiler and Rohde [**20**]. See also [**33**], [**34**]. We first need a fact about covering maps and capacity.

LEMMA 5.6.1. *If $p : \mathbb{D} \to \mathbb{C}^{**}$ is the covering map of the twice puncture sphere, then there is a set $E \subset \mathbb{T}$ of positive capacity and a subdomain $\Omega \subset \mathbb{D}$ with $\partial\Omega \cap \partial\mathbb{D} = E$ so that p is bounded and bounded away from 0 and 1 as z approaches E through Ω.*

PROOF. Lift "figure 8" subdomain to disk and show by explicit construction that its cover is quasiconformally equivalent to a slit rectangle. □

COROLLARY 5.6.2. *If $g : \mathbb{D} \to \mathbb{D}$ is holomorphic and $\mathbb{D} \setminus g(\mathbb{D})$ contains at least two points $\{a, b\}$, then there is a set of positive logarithmic capacity in $\partial\mathbb{D}$ where $|g|$ does not have radial limit 1.*

PROOF. Factor g has $h \circ p$, where $p : \mathbb{D} \to \mathbb{D} \setminus \{a, b\}$ is the universal covering map and $h : \mathbb{D} \to \mathbb{D}$ is holomorphic.

By the previous lemma there is a domain $\Omega \subset \mathbb{D}$ and path family $\mathcal{F}$ of finite extremal length in Ω connecting an arc $K \subset \partial\Omega$ to a set $E \subset \mathbb{T} \cap \partial\Omega$, so that p stays bounded away from $\mathbb{T}$, a and b in Ω.

Consider $h^{-1}(\Omega)$. Consider defining h^{-1} on different paths $\gamma \in \mathcal{F}$ starting at the endpoint on K. The inverse is well defined and conformal in a neighborhood of each

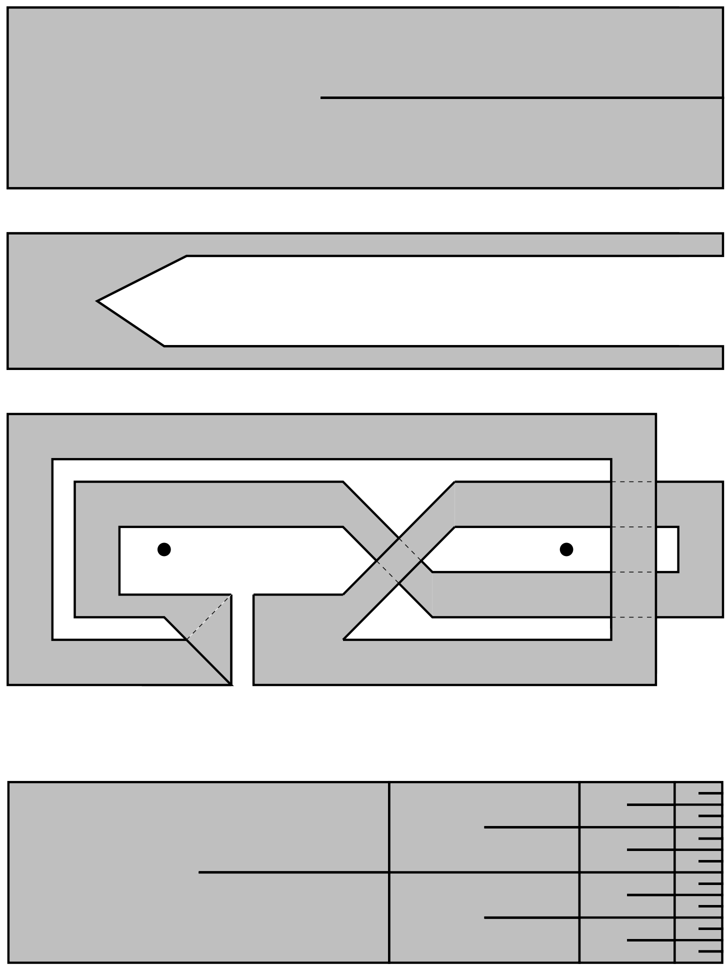


FIGURE 15. The slit rectangle on the top can be QC mapped to the "tuning fork" below it, which in turn can be QC mapped to the Riemann surface where each tine of the fork wraps around the two shown points in different ways but both attach to the starting segment. By adjoining maps in this way we can construct a map of the region Ω on the bottom into the twice punctured disk so that the path family of horizontal lines in Ω is mapped to a family of paths that wrap around the two omitted points in the disk infinitely often. This path family has positive extremal length. Thus lifting it to the disk gives a path family that hits the unit circle in positive capacity.

point w unless w is a singular value of h or the inverse image of γ up to w tends to the unit circle. The first case can only happen countable often and removing the remainder of γ from Ω leaves a new domain Ω' and path family that still has finite extremal length. In the second case, the inverse image of γ up to w is a path that hits the unit circle and along which f approached $p(w) \in \mathbb{D}$. If the set of curves where this happens has finite extremal length we are done. It is does not, then the set of paths where h^{-1} is well defined had finite extremal length, and along these paths $|f|$ is bounded away from 1.

Thus the lemma holds in either case. $\square$

THEOREM 5.6.3. *If U, V are Fatou components of f such that $f(U) \subset V$ then $V \setminus f(U)$ contains at most one point.*

PROOF. If $w \in V \setminus f(U)$, then f has w as a asymptotic value along a curve in U and thus U, V are both simply connected by Corollary 2.6.8. Thus there are conformal maps $\varphi : \mathbb{D} \to U$ and $\psi : \mathbb{D} \to V$ so that $\psi^{-1} \circ \varphi$ is a holomorphic map $g : \mathbb{D} \to \mathbb{D}$.

By Theorem **??** φ has a finite radial limit everywhere except a set of zero capacity and the limits lie on $\partial U = \partial\varphi(\mathbb{D})$. Thus, except for a set of capacity zero, $f \circ \varphi$ has limits in $f(\partial U)$. The latter set is contained in both the Julia set of f and the closure of V, so must be contained in ∂V. Thus, except for capacity zero, $|g(re^{i\theta})| = |\psi^{-1} \circ f \circ \varphi(re^{i\theta})|$ tends to 1 as $r \nearrow 1$ limit on the same set. But Lemma **??** shows that if $\mathbb{D} \setminus g(\mathbb{D})$ contains two points, then there is a set of positive capacity on which g does not tend to 1, a contradiction. $\square$

A holomorphic function f on $\mathbb{D}$ is called a an **inner function** if $|f| \leq 1$ on $\mathbb{D}$ and $|f|$ has non-tangential limit 1 almost everywhere on $\partial\mathbb{D}$. Such functions play a crucial role in function theory on $\mathbb{D}$, e.g., see [**58**]

LEMMA 5.6.4. *If U, V are simply connected Fatou components, $f(U) \subset V$ and $\varphi : \mathbb{D} \to U$ and $\psi : \mathbb{D} \to V$ are conformal maps, then $g = \psi^{-1} \circ f \circ \varphi$ is an inner function.*

PROOF. If not, then there is a set $E \subset \partial\mathbb{D}$ of positive measure where $|g|$ has non-tangential limit $< 1-\epsilon < 1$ and φ has well defined limits. This means $\varphi(E) \subset \partial U$ is a set of positive harmonic measure that maps under f into V (not onto ∂V). However, these would be Julia set points mapping into the Fatou set, which is impossible. $\square$

LEMMA 5.6.5. *If U is a forward invariant Fatou component of f, f is not univalent on U and $E \subset \partial U$ is invariant, i.e., $f^{-1}(E) \cap \partial U = E$ then E either has harmonic measure zero or 1. Under the same hypotheses, $\varphi(E)$ either has zero capacity or full capacity (where φ is a conformal map of U to $\mathbb{D}$.)*

PROOF. Follows from ergodicity of inner functions. Neuwrith 1978 ergodicity of some mappings of the circle and line, [**103**] Pommerenke 1981 Ergodic properties of inner functions [**107**]. □

5.7. $I(f) \cup \{\infty\}$ is connected

In this section we give some results of Rippon and Stallard from "Boundaries of escaping Fatou components" [**117**].

THEOREM 5.7.1. *Suppose f is a transcendental entire function and Ω is a wandering component of the Fatou set that escapes. Then almost every point of $\partial\Omega$ with respect to harmonic measure escapes.*

PROOF. If Ω is multiply connected, then every boundary point is escaping (Lemma**??**), so we may assume that Ω is simply connected. Set $\Omega_n = f^n(\Omega)$. Choose $z_0 \in \Omega = \Omega_0$, set $z_n = f^n(z_0)$ and let $\varphi : \mathbb{D} \to \Omega$ be a conformal map with $\varphi(0) = z_0$ and let $\varphi_n = f^n \circ \varphi$. Let

$$E_n^m = \{w \in \mathbb{T} : |\varphi_n(w)| < m\},$$
$$E^m = \cap_k \cup_{n>k} E_n^m,$$
$$E = \cup_m E^m.$$

The set E maps under φ to the boundary points that do not escape, so it suffices to show this set has zero measure. Therefore it suffices to show that each E^n has zero measure. To prove this, it suffices to show that for each fixed m,

$$\sum_n |E_n^m| < \infty.$$

Since z_n escapes, we can choose N so large that $n \geq N$ implies $|z_n| > 2m$. For each such n, let γ_n be the union of crosscuts in $\Omega_n \cap \{|z|\}$ that separate z_n from $\Omega_n \cap \{|z| < m\}$. Since the Ω_n's are disjoint, so are the crosscuts γ_n. Let $U_m = \{|z| > m\} \cup \{\infty\}$. Clearly

$$|E_n^m| \leq \omega(z_n, \gamma_n, \Omega_n) \leq \omega(z_n, \gamma_n, U_n),$$

and by Harnack's inequality,

$$\omega(z_n, \gamma_n, U_m) \simeq \omega(\infty, \gamma_n, U_m),$$

and the latter is sums to at most 1. Thus almost every point of $\partial\Omega$ (with respect to harmonic measure in Ω) is escaping. $\square$

COROLLARY 5.7.2. *If f is a transcendental entire function, then any bounded component of $I(f)$ meets $\mathcal{J}(f)$.*

PROOF. If E is bounded component of $I(f)$ that does not meet $\mathcal{J}(f)$ then it is contained in a Fatou component, and hence is an escaping Fatou component with no escaping boundary points. Since I is escaping and bounded, it must also be wandering, so this is impossible by Theorem 5.7.1. $\square$

LEMMA 5.7.3. *Suppose f is a transcendental entire function. If U is a bounded simply connected domain that hits the Julia set, then ∂U contains an escaping point.*

PROOF. Let γ_n be the outer boundary of $f^n(U)$. Since U hits the Julia set, iterates of U cover every compact set in the plane (Lemma **??**), so γ_n eventually surrounds 0 and $\text{dist}(\gamma_n, 0) \to \infty$. The compact sets $K_n = \{z \in \partial U : f^n(z) \in \gamma_n\}$ thus form a nested sequence of non-empty compact sets and hence has non-empty intersection. Any point in the intersection is escaping, which proves the result. $\square$

THEOREM 5.7.4. *If f is a transcendental entire function, then $I(f) \cup \{\infty\}$ is connected.*

PROOF. If $K = I(f) \cup \{\infty\}$ is disconnected, then there are disjoint open sets U, V that cover K and both hit K. One of these contains ∞, say V. Then U is bounded, and by adding in the bounded complementary components we may assume U is simply connected and that $\partial U \cap I(f) = \emptyset$. By Lemma 5.7.3, U can't hit the Julia set, so it is contained in a Fatou component, and this component must be escaping (since U hits $I(f)$). If U were not the whole component, its boundary would intersect the component and hence contain an escaping point, a contradiction.. Thus U must be a bounded, escaping component of the Fatou set. Hence it is a wandering component, so by Theorem 5.7.1 $\partial U \cap I(f) \neq \emptyset$, another contradiction. Thus K must be connected. $\square$

LEMMA 5.7.5. *If V is a Fatou component with $\partial V \subset I(f)$ then $V \subset I(f)$.*

LEMMA 5.7.6. *If V is a Fatou component and $\partial V \cap I(f)$ has positive harmonic measure. Then $V \subset I(f)$.*

PROOF. Suppose V is not escaping. Then V is simply connected and by Egorov's theorem (Theorem **??**), there is a compact set $E \subset V$ of positive harmonic measure on which $|f^n| \to \infty$ uniformly. By Lemma **??**, $Julia(f)$ contains a non-trivial continuum K. Choose $a \in K$ and define

$$u_n(z) = \log|f^n(z) - a|,$$

$$F_n^m = \{z \in \partial V : |f^n(z) - a| < 2^{-m}\}.$$

The $\{u_n\}$ are harmonic functions on V. Fix some $z_0 \in V$. By Beurling's projection theorem (see Lemma 5.4.4),

$$\omega(z_0, F_n^m, V) \leq \max(1, C2^{-m/2}),$$

where the constant C depends on $\operatorname{diam}(K)$ and $\operatorname{dist}(z_0, \partial V)$, but nothing else. Therefore on the set where u_n is negative.

$$\int_{F_0} u_n(w) d\omega(w, z_0, V) \geq -C \sum_{m \geq 0} m 2^{-m/2} \geq -C > -\infty.$$

On the other hand

$$\int_E u_n(w)\omega(w, z_0, V) \to +\infty,$$

so $u_n(z_0) \to \infty$. Thus every point of V is escaping. $\square$

LEMMA 5.7.7. *Suppose U is a bounded domain with $\partial U \subset I(f)$. Then $U \setminus I(f) \neq \emptyset$ iff $U \setminus \mathcal{J}(f) \neq \emptyset$.*

PROOF. If U hits the Julia set, it contains non-escaping points since $\mathcal{J}(f) = \partial I(f)$ (Lemma**??**). Conversely, if U does not hit the Julia set, then it is contained in a Fatou component V. If $U \neq V$, then $V \cap \partial U \neq \emptyset$, so V contains an escaping point, hence all of V (and hence all of U) is escaping. If $U = V$ then then $\partial V = \partial U \subset I(f)$, and this implies V is escaping by Lemma 5.7.5. Thus $U \setminus I(f) = \emptyset$. $\square$

We say $E \subset \mathbb{C}$ **has a hole** if there is bounded domain Usuch that $\partial U \subset E$ but $U \setminus E \neq \emptyset$. We say E is an (infinite) **spider's web** if there is a sequence $\{U_n\}$ of bounded, nested, simply connected domains with $\partial U_n \subset E$.

THEOREM 5.7.8. *If $I(f)$ has a hole, then $I(f)$ is a spider's web.*

PROOF. By hypothesis, there is a bounded domain U with $\partial U \subset I(f)$ and $U \setminus I(f) \neq \emptyset$. By Lemma 5.7.7 this implies $U \cap \mathcal{J}(f) \neq$. Let γ_n denote the outer boundary of $f^n(U)$. Since $\gamma_n \subset f^n(\partial U)$, $\gamma_n \subset I(f)$, and by the "blowing-up property" of the Julia set (Corollary 3.5.3), γ_n eventually surrounds every point of the plane. Taking U_n to the be region bounded by γ_n, proves $I(f)$ is a spider's web, once we show that $I(f)$ is connected.

Suppose $I(f)$ is not connected and let V_1, V_2 be two disjoint open sets that each hit $I(f)$ and together cover $I(f)$. In particular, neither ∂V_1 nor ∂V_2 intersects $I(f)$. V_1 and V_2 can never hit the same set γ_n as above, since γ_n is a connected subset of $I(f)$ (if they both hit γ_n, this would prove γ_n is disconnected). Thus at least one of them is bounded. Assume V_1 is bounded. By adding its bounded complementary components, we may assume it is also simply connected. Suppose z_0 is an escaping point in V_1 and choose N_m so that $n \geq N_m$ implies $f^n(z_0)$ is outside γ_m (we can do this since z_0 is escaping). For such a n, $\partial f^n(V_1) \subset f^n(\partial V_1) \subset \mathbb{C} \setminus I(f)$. Moreover, since V_1 is connected, so is $f^n(\partial V_1)$ and it does not intersect γ_m (since the latter set escapes and the former does not). Thus $\overline{f^n(V_1)}$ lies outside γ_m and since this occurs for every m, we see that $\overline{V_1} \subset I(f)$. This implies $\partial V_1 \subset I(f)$, contrary to assumption. $\square$

COROLLARY 5.7.9. *If f has a multiply connected Fatou component, then $I(f)$ is connected.*

An alternate proof will be given in Corollary 6.8.13 Both proofs are due to Rippon and Stallard, though they credit the idea of the proof given above to Noel Baker. In 2010 they found the above corollary in a notebook Baker used for rough work and although he did not provide a proof there, his notes led them the proof above. The alternate proof we give later is based on properties of the fast escaping set and was given by Rippon and Stallard in 2005.

QUESTION: Suppose f is a transcendental entire function and Ω is a wandering component of the Fatou set that escapes. Does $\partial\Omega \setminus I(f)$ have zero capacity with respect to Ω? Since it is invariant it must have zero capacity or full capacity on the boundary.

EXERCISE: Suppose f is a transcendental entire function. Then $I(f)$ either is connected or has infinitely many connected components. (Corollary 5.1, [**117**]).

This is actually a special case of a more general fact.

EXERCISE: Suppose f is a transcendental entire function. Suppose E is completely invariant under f and $\mathcal{J}(f) = \overline{E \cap \mathcal{J}(f)}$. The exactly one of the following holds:

(1) E is connected.
(2) E has two components, one of which is a fixed point of f that is an exceptional point and in the Fatou set.

(Theorem 5.2, [**117**]).

EXERCISE: $f(z) = \frac{1}{2}z^2 \exp(2 - z)$ shows case (2) above can occur.

EXERCISE: In case (3) above, each neighborhood of a point in $\mathcal{J}(f)$ hits infinitely many components of E. More generally, if A and B are completely invariant sets for a transcendental entire function and D is a neighborhood of a point of $\mathcal{J}(f)$ so that $A \cap D$ meets only finitely many components of B, then A meets only finitely many components of B (Theorem 5.3, [**117**]).

OPEN PROBLEM: If $I(f)$ has infinitely many components, must it have uncountably many components? The Julia set has this property.

CHAPTER 6

The fast escaping set

In both polynomial and transcendental dynamics, the Julia set is the boundary of the escaping set, but this fact seems to play a more crucial role in transcendental dynamics. For polynomials, all escaping points escape at essentially the same rate, but for transcendental functions, there can be a wide range of escape rates, and the escaping set can be partitioned in various interesting ways according to these escape rates. This additional structure is interesting in it own right, but can also provide insight into problems not originally formulated in terms of rates of escape. For example, we shall see that that one can prove the escaping set always has an unbounded component, by showing the so-called "fast escaping points" have an unbounded component.

Given an entire function f, recall that $M(r, f) = \max\{|f(z)| : |z| = r\}$. This is a function from $[0, \infty)$ into itself, and so can be iterated. Let $M^n(r, f)$, be the nth such iterate. It is easy to see that if $|z| = r$, then

$$|f^n(z)| \leq M^n(r, f),$$

since the left-hand side corresponds to following the orbit of a single point z, whereas the right hand side can replace each orbit point by another point on the same circle around the origin before applying f again. The extra freedom would seem to allow the right-hand side to grow much more quickly than the left-hand side. Indeed, the only way that the left-hand side could "keep up" was of every iterate of z landed on a circle where $|f|$ was approximately constant, so that moving to a different point of the circle does not help. Remarkably, it is always possible to choose an initial z that does this. This argument is due to Eremenko, and was the original proof that escaping points exist. He used a result of Wiman and Valiron that for an entire function f, $|f|$ is close to constant on "most" large circles centered at the origin and that f closely resembles a polynomial on these circles. This approach is based on power series and is described in Section A.11 of the appendix. We shall take alternative approach due to Bergweiler, Rippon and Stallard that applies to functions holomorphic in simply connected, unbounded Jordan subdomains of $\mathbb{C}$. There method shows that if f is entire and $\{z : |f(z)| > R\}$ has more than one component, we can find escaping orbits that stay inside any of these components, or that switch between components on any prescribed schedule.

As an application of fast escaping points to a non-escaping type problem, we finish the chapter with a result of Bergweiler about the Julia sets of maps from the once punctured plane to itself.

6.1. Subharmonic functions

The proof of existence of fast escaping points depends crucially on estimates of the growth entire functions, or on the growth of holomorphic functions defined on certain unbounded subdomains called tracts. These estimates reduce to growth estimates for the subharmonic function $v = \log |f|$, so we start the chapter with a review of some of the basic facts about subharmonic functions.

Recall that a real-valued function f on Ω is **upper semi-continuous** if $\{z : f(z) < \alpha\}$ is open for every $\alpha \in \mathbb{R}$.

EXERCISE: Show f is upper semi-continuous iff

$$\limsup_{z \to w} f(z) \leq f(w).$$

EXERCISE: Show that f is upper semi-continuous on Ω iff for every closed set $E \subset \Omega$, f is the pointwise limit of decreasing sequence of continuous functions.

EXERCISE: Show the maximum of two upper semi-continuous functions is also upper semi-continuous.

EXERCISE: Show that if f is upper-semi-continuous on a closed set E and $f \geq M$ on a dense subset of E then $f \geq M$ everywhere on E.

EXERCISE: If f is upper semi-continuous on Ω then f is bounded above on any compact subset of Ω.

A $[-\infty, \infty)$-valued function v is **subharmonic** on Ω if it is upper semi-continuous and satisfies the sub-mean-value value inequality

$$v(z) \leq \frac{1}{2\pi} \int_0^{2\pi} v(z + re^{i\theta}) d\theta,$$

whenever r is sufficiently small, depending on z. We will call this the "small circles inequality. Subharmonic functions actually satisfy the same inequality whenever $\overline{D(z,r)} \subset \Omega$, but this will require a proof (some texts take this to be the definition).

LEMMA 6.1.1 (The Maximum Principle). *If v is subharmonic on a connected open set Ω and it attains a maximum in Ω, then v is constant.*

PROOF. The proof is the same as for harmonic functions. Suppose v attains a maximum M at $a \in \Omega$ and consider a disk $D(z,r)$ small enough so that

$$M = v(a) \leq \frac{1}{2\pi} \int_0^{2\pi} v(a + re^{i\theta}) d\theta.$$

By assumption $v(a + re^{i\theta}) \leq M$ so we must have equality almost everywhere, and hence on a dense set on the circle and hence everywhere on the circle. Thus v equals M on an open disk around z and so $\{z : v(z) = M\}$ is open and non-empty. On the other hand $\{z : v(z) = M\} = \{z : v(z) \geq M\} = \{z : v(z) < M\}^c$ is closed. Since Ω is connected, $v = M$ on all of Ω. $\square$

The Poisson kernel on the unit circle with respect to the point $a \in \mathbb{D}$ is given by the formula

$$P_a(\theta) = \frac{1 - |a|^2}{|e^{i\theta} - a^2|} = \frac{1 - |a|^2}{1 - 2|a| \cos(\theta - \phi) + |a|^2},$$

where $a = |a|e^{i\phi}$. Given an L^1 function f on the unit circle

$$u(z) = \frac{1}{2\pi} \int f(e^{i\theta}) P_z(e^{i\theta}) d\theta$$

defines a harmonic function on the unit disk. If f is continous it is standard fact that u has a continuous extenstion to the closed unit disk.

LEMMA 6.1.2. *If f is upper semi-continuous on the unit circle and u is the Poisson extension of f to the unit disk, then together u and f define an upper semi-continuous function on the closed unit disk.*

PROOF. Since f is upper semi-continuous on $\mathbb{T}$, there are continuous functions $\{f_n\}$ that converge pointwise downwards to f. The harmonic extensions $\{u_n\}$ of these functions are continuous and converge downwards to an upper semi-continouous limit v on the closed disk that equals f on the boundary. If we can show $v = u$ on the interior, we are done. However this follows from either the Monotone Convergence Theorem or the Lebesgue Dominated Convegence Theorem:

$$\begin{aligned} v(z) \le \lim_n u_n(z) &= \lim_n \int f_n(e^{i\theta}) P_z(e^{i\theta}) d\theta \\ &= \int \lim_n f_n(e^{i\theta}) P_z(e^{i\theta}) d\theta \\ &= \int f(e^{i\theta}(P_z(e^{i\theta}) d\theta \\ = u(z). \end{aligned}$$

□

LEMMA 6.1.3. *If v is subharmonic on the closure of $D(a,r)$ and u is the Poisson extension of v on $\{z : |z-a| = r\}$ to the interior of the disk, then $v \le u$. (This justifies the name "subharmonic".)*

PROOF. Suppose $\{g_n\}$ are continuous functions converging downwards to v and let u_n be the harmonic extensions to the disk. Then $v - u_n$ is subharmonic on the open disk and limsup is $v(x) - g_n(x) \le 0$ as we approach any boundary point, so $v \le u_n$ on the whole disk. Thus $v(z) \le \lim_n u_n(z)$. The prove that $v = u$ is just as above. □

LEMMA 6.1.4. *Suppose v is subharmonic on a domain Ω and $\overline{D(z,r)} \subset \Omega$. If we replace v in $D(x,r)$ by its Poisson extension, the resulting function u is sub-harmonic on Ω.*

PROOF. The upper semi-continuity is an earlier exercise. The "small circles" mean value inequality is obvious except on the boundary of the circle, but we know that $v \le u$ in the disk, so for a point x on $\partial D(a,r)$,

$$u(x) = v(x) \le \frac{1}{2\pi} \int_0^{2\pi} v(a + re^{i\theta}) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}).$$

□

EXERCISE: The maximum of two subharmonic functions is subharmonic.

EXERCISE: The supremum of a collection of subharmonic functions is subharmonic.

EXERCISE: If $v(x+iy)$ is subharmonic on $\mathbb{C}$, but only depends on x, then v is a convex function of x.

The main example of subharmonic functions we will encounter is $\log |f(z)|$ where f is holomorphic on Ω. This is harmonic except where f vanishes, and at these points it equals $-\infty$, and clearly satisfies the sub-mean-value inequality at such points.

COROLLARY 6.1.5. *If v is subharmonic on Ω, then v satisfies the mean value inequality on all disks with closure in Ω.*

PROOF. Suppose $\overline{D(z,r)} \subset \Omega$ and let g_n, u_n and u be as the proof of Lemma 6.1.2. Then by its harmonic extension u in $D(x,r)$. Then

$$v(a) \leq u(a) = \lim_n \int_0^{2\pi} g_n(a+re^{i\theta})d\theta = \int_0^{2\pi} v(a+re^{i\theta})d\theta. \quad \square$$

LEMMA 6.1.6. *v is subharmonic on $D(a,r)$, then*

$$I_v(t) = \int_0^{2\pi} v(a+te^{i\theta})d\theta$$

is an increasing function of $t \in (0,r)$.

PROOF. Suppose $0 < t < s < r$. If we replace v on $D(a,t)$ by the Poisson extension of it boundary values we get another subharmonic function u that is harmonic on $D(a,t)$ and subharmonic on $D(0,r)$. Thus $u = v$ on both$\{|z| = t\}$ and $\{|z| = s\}$, so

$$I_v(t) = I_u(t) = u(a) \leq I_u(s) = I_v(s). \quad \square$$

COROLLARY 6.1.7. *If v is subharmonic and smooth on Ω, then*

$$\int_0^{2\pi} \frac{\partial v}{\partial n}(a+te^{i\theta})d\theta \geq 0$$

whenever $\overline{D(a,t)} \subset \Omega$; here n denotes the outward pointing normal for the circle..

LEMMA 6.1.8. *If ψ is a smooth, radial function supported in $D(0,\epsilon)$ and v is subharmonic on Ω then the convolution of ψ and v*

$$u(z) = \iint \psi(z-w)v(w)dxdy,$$

is subharmonic on $\Omega_\epsilon = \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) > \epsilon\}$, is smooth and $u \geq v$.

PROOF. If is a standard fact that the convolution is smooth if ψ is smooht. Next, for $a \in \Omega_\epsilon$, that fact that ψ is radial and has mass one implies

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} u(a+te^{i\phi}d\phi) &= \frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{\epsilon}\int \psi(re^{i\theta})v(a+te^{i\phi}+re^{i\theta})rdrd\theta]d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\epsilon}[\frac{1}{2\pi}\psi(re^{i\theta})\frac{1}{2\pi}\int_0^{2\pi} v(a+te^{i\phi}+re^{i\theta})d\phi]rdrd\theta] \\
&\geq \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\epsilon} v(a+re^{i\theta})\psi(re^{i\theta})d\theta]rdr \\
&= u(a) \\
&= \int_0^{\epsilon}\psi(re^{i\theta})[\frac{1}{2\pi}\int_0^{2\pi} v(a+re^{i\theta})d\theta]rdr \\
&\geq v(a)\cdot\int_0^{\epsilon}\psi(re^{i\theta}) \\
&= v(a)\cdot 1.
\end{aligned}$$

This shows both that u is subharmonic and the $u \geq v$. $\square$

Just as harmonic function are a 2-dimensional analog of linear functions, subharmonic functions are an analog of convex functions, and like convex functions, they have positive measures as their (distributional) Laplacians.

LEMMA 6.1.9. *If v is subharmonic on $\mathbb{C}$ then Δv is a positive measure.*

PROOF. If v is a smooth subharmonic function on the closure of smooth domain Ω, then Green's theorem says

$$\iint_\Omega u\Delta v - v\Delta u dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}ds.$$

Taking $\Omega = D(x,r)$ a disk $u \equiv 1$, gives

$$\iint_{|z-a|<r}\Delta v = \int_{|z-a|=r}\frac{\partial v}{\partial n} \geq 0,$$

so $\Delta v \geq 0$. In general, we can smooth v by convolving by a smooth radial bump function of small support and to get approximations v_n that converge downward to v (downwards because v is subharmonic). If u is a non-negative compactly supported test function in Ω then the boundary integrals in Green's theorem vanish and

$$0 \leq \iint_\Omega u\Delta v_n dxdy = -\iint_\Omega v_n\Delta u dxdy \to -\iint_\Omega v\Delta u dxdy,$$

which means $\Delta v \geq 0$ in the sense of distributions. A positive distribution is a positive measure (e.g., see Folland's book [**?**], Theorem ??). □

The positive measure associated to v will be denoted μ_v or Δv. This is called the **Riesz measure** associated to v.

EXERCISE: If μ is a finite positive measure supported on a compact set E, then

$$U_\mu(z) = \int \log|z-w|d\mu(w),$$

is a subharmonic function on the plane with $\Delta U_\mu = \mu$.

The subharmonic functions we will consider are mostly of the form

$$v(z) = \max(\log R, \log|f(z)|),$$

where f is entire. In this case, Δv is just $|f'(z)|ds$ where ds is arc-length measure on $\{z : |f(z)| = R\}$. We will generalize this idea further down below.

LEMMA 6.1.10 (Jensen's formula for subharmonic functions). *Let v be subharmonic $a, \in \mathbb{C}$, $r > 0$. Then*

$$\begin{aligned} v(a) &= \frac{1}{2\pi}\int_0^{2\pi} v(a+re^{i\theta})d\theta - \int_{|z-a|<r} \log\frac{r}{|z-a|}d\Delta v \\ &= \frac{1}{2\pi}\int_0^{2\pi} v(a+re^{i\theta})d\theta - \int_0^r \Delta v(\overline{D(a,t)})\frac{1}{t}dt. \end{aligned}$$

PROOF. The first equality is Green's theorem, the second follows by switching to polar coordinates and integrating by parts. □

It is sometime convenient to rewrite Jensen's formula as follows. For $t > 0$ let

$$n(a,t,u) = \mu_v(\overline{D(r,t)}).$$

Then Jensen's formula becomes

$$\frac{1}{2\pi}\int_0^{2\pi} v(a+re^{i\theta})d\theta = \int_0^r n(z,t,v)\frac{dt}{t} + v(a).$$

In the special case that $v = \log|f|$ and f is holomorphic, then v has a delta-mass point measure at each zero of f, so, in this case, $n(z,t,v)$ counts the number of zeros of f (according to multiplicity) inside the disk $\overline{D(z,t)}$. In particular, this mass is always a non-negative integer. Surprisingly, this behavior only requires $v = \log|f|$ near the boundary of the disk; f need not holomorphic, or even defined, in the whole interior of the disk:

LEMMA 6.1.11. *Suppose u is subharmonic on $\mathbb{C}$ and let γ be a closed, piecewise smooth Jordan curve with interior domain W. Suppose γ has a neighborhood U where $u = \log|f|$ for some non-vanishing holomorphic function f. Then $\mu_u(W)$ is a non-negative integer. $\in \mathbb{N} = \{1, 2, 3, \dots\}$. If, in addition, we know $|f| = R$ on γ and $|f| > R$ on the component of $U \setminus \gamma$ disjoint from W the $\mu(W) > 0$ (so $\mu(W) \in \mathbb{N} = \{1, 2, \dots\}$.*

PROOF. We can write

$$u(z) = \int_E \log \frac{1}{|z-w|} d\mu_u(w) + h(z),$$

where h is harmonic and E is a compact subset of W. By the Cauchy-Riemann equations, if $u = \Re(\log|f| + i \arg f) = \mathrm{Log} f$ locally, then

$$\nabla u = \nabla \log|f| = (\partial_x + i\partial_y) \log|f| = \partial_x(\log|f| + i \arg f) = (\mathrm{Log} f)' = f'/f.$$

We may assume $U \cup W$ is simply connected and for $z \in U$,

$$\frac{f'(z)}{f(z)} = \int_E \frac{1}{z-w} d\mu_u(w) + g(z),$$

where g is holomorphic in $U \cup W$. Integrating along γ, the left side gives the change in argument of $\log f$ along γ and hence is purely imaginary integer multiple of 2π. Integrating the right side gives $2\pi i \mu_u(E)$. The first conclusion follows.

If in addition we know $|f| = R$ and $|f| > R$ in $U \setminus \overline{W}$, then $\log|f|$ has zero tangential deriviative along γ and non-negative outward normal along γ. Moreover, the normal deriviative is non-zero except on a finite set. Thus $\arg f'$ has a non-negative tangential deriviative along γ that is positive except on a finite set. Thus the total change in the argument can't be zero. Since we know it most be an integer, it is a positive integer. □

6.2. Direct tracts

Suppose Ω is a planar domain with piecewise smooth boundaries and that it has an unbounded complement. Suppose that f is a holomorphic function on Ω that extends continuously to the boundary and $|f| = R$ on the boundary and $> R$ on the interior. Then Ω is called a **direct tract** of f. If the map $f : \Omega \to \overline{\mathbb{D}}_R^c$ is a universal covering map, then Ω is called a **logarithmic tract** of f. We have already seen that all Eremenko-Lyubich functions have logarithmic tracts, e.g., Section 5.4.2. This is not true for all entire functions, but following is.

LEMMA 6.2.1. *Every transcendental entire function has a direct tract.*

PROOF. Just let Ω be a connected component of $\{|f(z)| > 1\}$. This has smooth boundary, except possibly at critical points of f. It is unbounded, for otherwise $\overline{\Omega}$ would be compact and $|f|$ would take a maximum > 1, contrary to the maximum principle. Finally, the complement of Ω is unbounded, since by Picard's theorem, f takes most values in $\mathbb{D}$ infinitely often in every neighborhood of infty. $\square$

Because of this, the following results on direct tracts all apply to transcendental entire functions. The following generalize results from Chapter 1.

Recall that a function $\log g(r)$ is convex as a function of $\log r$ means that $\log g(e^t)$ is convex as a function of t. For example if $r_k = \exp(t_k)$, $k = 1, 2$, we get

$$\log g(e^{(t_1+t_2)/2}) \leq \frac{1}{2}(\log g(e^{t_1}) + \log g(e^{t_2}),$$

or

$$g(\sqrt{r_1 r_2}) \leq \sqrt{g(r_1)g(r_2)}. \tag{35}$$

LEMMA 6.2.2. *If f is entire then $\log M(r, f) = \max\{\log |f(z)| : |z| = r\}$ is an increasing, convex function of $\log r$.*

PROOF. Clearly $M(r, f)$ increases with r by the maximum principle. Since $g(z) = f(e^z)$ is entire,

$$\log M(e^x, f) = \sup\{\log |g(x + iy))| : y \in \mathbb{R}\},$$

is a subharmonic function of x alone and hence convex (a supremum of subharmonic functions is subharmonic, and a subharmonic function of x alone is convex). $\square$

LEMMA 6.2.3. *Suppose Ω is a direct tract for f and*

$$M(f, r) = \max_{|z|=r} |f(z)|.$$

Then

$$\lim_{r \to 0} \frac{\log M(f, r)}{\log r} = \infty.$$

PROOF. There are two cases depending on the geometry of the tract Ω. First suppose $\mathbb{C}\setminus\Omega$ has at least one unbounded component K and let $\Omega' = \mathbb{C}\setminus L \supset \Omega$. Then Ω' is simply connected complement of K (since K is connected). Let $r_0 = \mathrm{dist}(K, \{0\})$. Then Beurling's estimate (Corollary 5.4.3) says that for $z \in \Omega \cap D(0, r_0)$, and $r > 2r_0$,

$$\begin{aligned}\omega(z, \partial\Omega \cap \{|w| > r\}, \Omega) &\leq \omega(z, \partial\Omega' \cap \{|w| > r\}, \Omega)\\ &\leq C\exp(-\pi \int_{r_0}^{r} \frac{dt}{2\pi})\\ &\leq C(\frac{r_0}{r})^{1/2}.\end{aligned}$$

Since $v(z) = \max(0, \log|f(z)|)$ is subharmonic,

$$0 < v(z) \leq (\max_{|z|=r} \log|f(z)|)\cdot \omega(z, \partial\Omega \cap \{|w| > r\}, \Omega) \leq (\max_{|z|=r}\log|f(z)|)\cdot C(\frac{r_0}{r})^{1/2},$$

so

$$\max_{|z|=r} \log|f(z)| > Cr^{1/2} \gg \log r,$$

if r is large enough.

In the second case, Ω has no unbounded complementary components, so it has infinitely many bounded components In this case we use Jensen's formula

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} v(a + re^{i\theta})d\theta &= v(a) + \int_{|z-a|<r} \log\frac{r}{|z-a|} d\Delta v\\ &\geq v(a) + \frac{1}{2}\int_{|z-a|<\sqrt{r}} \log r d\Delta v\\ &\geq v(a) + \frac{1}{2}\log r d\Delta v(D(z, \sqrt{r})).\end{aligned}$$

Since $v \geq 0$, this gives

$$\frac{\max_{|z|=r} v(z)}{\log r} \geq \Delta v(D(z, \sqrt{r})) \to \infty,$$

where we have used Lemma 6.1.11 to say each complementary component of Ω in $D(z, \sqrt{r})$ contributes at least 1 to the Riesz measure of $D(z, \sqrt{r})$. $\square$

LEMMA 6.2.4. *Suppose the notation in Lemma 6.2.3 still holds. Then for any $C < \infty$ there is a r_0 (depending on f and C) so that $r \geq r_0$ implies*

$$M(f, 2r) \geq CM(f, r).$$

PROOF. Taking $r_1 = r/2$ and $r_2 = 2r$ and using (**??**) gives

$$M(r, f) \leq \sqrt{M(2r, f)M(r/2, f)}.$$

Squaring this and rearranging,

$$M(r, f)\frac{M(r, f)}{M(r/2, f)} \leq M(2r, f).$$

Since $\log M(r, f)$ is an increasing and convex function of $\log r$ and $\log M(r, f)/\log r \to \infty$, the derivative of $\log M$ with respect to $\log r$ increases to ∞. Thus

$$\log \frac{M(r, f)}{M(r/2, f)} = \log M(\exp(\log r), f) - \log M(\exp(\log r - \log 2), f) \to \infty.$$

Hence this is eventually bigger than $\log C$ and this proves the lemma. □

6.3. Logarithmic measure

If $E \subset [1, \infty)$ is a measurable set, we let $\log(E)$ denote the image of E under the map and define $\log E = \log t : t \in E\}$ and we call

$$|\log E| == \int_E \frac{dt}{t}$$

the logarithmic measure of E. We introduce this because many interesting properties of $M(r, f)$ fail to hold for all r, but do hold except on a set of finite logarithmic measure.

LEMMA 6.3.1. *Suppose $0 < r_n \nearrow \infty$ and $0 < \rho_n \nearrow L < \infty$. Then the complement of $\cup_n(\rho_n r_n, \rho_n r_{n+1})$ has finite logarithmic measure.*

PROOF. The complement is covered by intervals of the form $[r_n\rho_{n-1}, r_n, \rho_n]$ that have logarithmic length $\log \rho_{n+1} - \log \rho_n$. Summing the telescoping series shows the complement has total logarithmic length $\leq L - \rho_0$. □

LEMMA 6.3.2. *Suppose $N(x)$ is an increasing function on $I = [r_0, \infty)$ and $\lim_{x\to\infty} N(x) = \infty$. Fix numbers $\frac{1}{2} < \beta < \alpha < 1$. Then there is a set E of finite logarithmic measure so that $|h| < N(r)^{-\alpha}$ implies*

$$N(r + h) \leq (1 + cN(r)^{-\beta}) \cdot N(r), \quad \textit{for } r \in I \setminus E,$$

where $c = 2^{\frac{1}{\beta}-1}\beta^{-1}$.

PROOF. Let $X \subset I$ be a compact set where the inequality fails and assume $N \geq 1$ on X (the part of X that lies in the bounded interval where $N \leq 1$ obviously have finite measure). Each point of X is contained in an interval $(a, b) = (r - |h|, r + |h|)$ so that $N(b) > (1 + cN(r)^{-\beta})N(a)$. By the easy form of Vitali's covering lemma (Lemma 12.1.1), there is a finite, disjoint subcollection of these intervals that covers a fixed fraction of the length of $\log X$. Ordering the intervals as $\{I_k\} = \{(a_k, b_k)\}_1^K$ from left to right and setting $t_k = N(a_k) \geq 1$, we get

$$t_{k+1} = N(a_{k+1}) \geq N(b_k) \geq N(a_k) + cN(a_k)^{1-\beta} = t_k + ct_k^{1-\beta}.$$

We claim this implies

$$t_k \geq k^{1/\beta}.$$

To prove the claim, compare the sequence $\{t_k\}$ to the sequence $\{s_k\} = \{k^{1/\beta}\}$. Since $1/\beta > 1$, the function $k^{1/\beta}$ is increasing and convex up it stays above any of its tangent lines. Considering the tangent line at $k+1$ gives

$$\begin{aligned} s_{k+1} - s_k &\leq \frac{1}{\beta}(k+1)^{\frac{1}{\beta}-1} \\ &\leq ck^{\frac{1}{\beta}-1} \\ &\leq cs_k^{1-\beta}. \end{aligned}$$

By assumption $t_1 \geq 1 = s_1$, so by induction we deduce

$$s_{k+1} - s_k \leq ct_k^{1-\beta} \leq t_{k+1} - t_k,$$

and hence $t_k \geq s_k$ for all k, as claimed.

Since $\alpha > \beta$,

$$\sum |I_k| \leq \sum_k t_k^{-\alpha} \leq \sum_k k^{-\alpha/\beta} < \infty.$$

Thus $\log X$ has finite linear measure, as desired. □

LEMMA 6.3.3. *Suppose Φ is increasing and convex up on $[x_0, \infty)$ and let $\frac{1}{2} < \alpha < 1$. Then there is a set E of finite measure such that*

$$\Phi(x + h) \leq \Phi(x) + \Phi'(x)h + o(1)$$

for all $|h| \leq \Phi'(x)^{-\alpha}$ and $x \notin E$.

PROOF. We apply the previous lemma to the function $N(x) = \Phi'(x)$ (since Φ is increasing and convex, N is positive and increasing and well defined except at possibly countable many points where we can define it by making it continuous from the left).

Choose β with $\frac{1}{2} < \beta < \alpha$. Suppose $0 < h < \Phi'(x)^{-\alpha}$. Then by Lemma 6.3.2 and using $\beta + \alpha > 1$,

$$\begin{aligned}
\Phi(x+h) &= \Phi(x) + \int_x^{x+h} \Phi'(t)dt \\
&\leq \Phi(x) + \Phi'(x+h)h \\
&\leq \Phi(x) + \Phi'(x + c\Phi'(x)^{-\alpha})h \\
&\leq \Phi(x) + (1 + c\Phi'(x)^{-\beta})\Phi'(x)h \\
&\leq \Phi(x) + c\Phi'(x)^{1-\beta})h + \Phi'(x)h \\
&\leq \Phi(x) + c\Phi'(x)^{1-\alpha-\beta} + \Phi'(x)h \\
&\leq \Phi(x) + o(1) + \Phi'(x)h.
\end{aligned}$$

The case for $h < 0$ is similar. $\square$

Suppose f is defined on a direct tract and we let $v = \log\max(R, |f|)$. Then v is subharmonic on the plane. Define

$$B(r, v) = \max_{|z|=r} v(z).$$

By Lemma 6.2.2, B is increasing and convex up as a function of $\log r$ (i.e., $B(e^x, v)$ is convex as a function of x). Thus B' is positive and exists except on a countable set and and we set

$$a(r, v) = \frac{dB(r, v)}{d\log r} = rB'(r, v)$$

whenever the derivative is defined. By Lemma 6.2.3 we know

$$\frac{\log B(r)}{\log r} \nearrow \infty.$$

Applying Lemma 6.3.3 gives:

LEMMA 6.3.4. *Suppose v is as above and $\alpha > 0$. Then there is a set of finite logarithmic measure so that if $|\log s/r| \leq a(r, v)^{-\alpha}$, then*

$$B(s, v) \leq B(r, v) = a(r, v)\log\frac{s}{r} + o(1),$$

uniformly as $r \to \infty$, $r \notin F$.

PROOF. We apply Lemma 6.3.3 to $\Phi(x) = B(e^x, v)$. Setting $r = e^x$ and $s = e^{xs+h}$ we get

$$B(s,v)\Phi(x+s) \leq \Phi(x) + \Phi'(x)h + o(1) \leq B(r,v) + a(r,v)\log\frac{s}{r} + o(1),$$

if

$$|\log\frac{s}{r}| = |h| \leq \Phi'(x)^{-\alpha} \leq a(r,v)^{-\alpha}.$$

$\square$

6.4. A Wiman-Valiron type estimate in tracts

For transcendental entire functions, Wiman and Valiron proved that for most $r > 0$ there are points z_r with $|z_r| = r$ so that

$$f(z) \sim Cz^a,$$

on a "large" neighborhood of z_r (here C, a are constants that depend on r). Their original approach is based on power series and is sketched in an appendix. In this section we prove a similar result for functions defined in a direct tract, following the proof of Bergweiler, Rippon and Stallard in [**19**].

We start with the following result that contains most of the hard work.

THEOREM 6.4.1. *Let Ω be a direct tract of f, let v be as above and let $\tau > 1/2$. Then there is a set F of finite logarithmic measure such that for $r \in [1,\infty) \setminus F$ we have $D_\rho \subset \Omega$ where $D_\rho = D(z_r, r \cdot a(r,v)^{-\tau})$ and z_r is a point on $\{|z| = r\} \cap \Omega$ where $|f|$ attains its maximal value.*

EXERCISE: Check this lemma holds for e^z.

PROOF. Fix $\frac{1}{2} < \alpha < \tau$ and choose $\epsilon > 0$ so that $(1-\alpha)(1+\epsilon) < 1$. Choose $\delta > 0$ so that $1-\delta = (1-\alpha)(1+\epsilon)$. Let F be the union of the exceptional sets from Lemma 6.3.4 applied with α and Lemma 4.8.2 applied with to $h = B(r,v)$ with ϵ replacing δ in that lemma. Thus for r outside this exceptional set, we have both

$$B(s,v) \leq B(r,v) + a(r,v)\log\frac{s}{r} + o(1),$$

for $|\log s/r| \leq a(r,v)^{-\alpha}$ and

$$a(r,v) \leq B(r,v)^{1+\epsilon}.$$

Set $\rho = 2ra(r,v)^{-\tau}$.

Consider

$$
\begin{aligned}
u(z) &= v(z) - v(z_r) - a(r,v)\log\frac{|z|}{r} \\
&= v(z) - B(r,v) - a(r,v)\log\frac{|z|}{r}. \qquad (36)
\end{aligned}
$$

For $z \in D_{512\rho} = \{w : |w - z_r| \leq 512\rho\}$,

$$
|\log\frac{|z|}{r}| = |\log(1 + \frac{|z - z_r|}{r})| \leq 2\frac{|z - z_r|}{r} = O(a(r,v)^{-\tau}) \leq a(r,v)^{-\alpha}
$$

if r is large enough. Thus Lemma 6.3.4 applies to such z's and we deduce that if $s = |z|$,

$$
\begin{aligned}
u(z) &= v(z) - B(r,v) - a(r,v)\log\frac{|z|}{r} \\
&\leq B(|z|,v) - B(r,v) - a(r,v)\log\frac{|z|}{r} \\
&\leq [B(r,v) + a(r,v)\log\frac{|z|}{r} + o(1)] - B(r,v) - a(r,v)\log\frac{|z|}{r} \\
&= o(1) \qquad (37)
\end{aligned}
$$

on $D_{512\rho}$.

We claim that $D_\rho \subset \Omega$ if r is sufficiently large. If not, there is a point $w \in D_\rho \setminus \Omega$. Let K be the connected component of $\mathbb{C} \setminus \Omega$ that contains w. We consider two cases, depending on whether or not K is contained in $D_{256\rho}$.

Case 1: K is not contained in $D_{256\rho}$. Then K intersects each of circles centered at z_r with radii between ρ and 256ρ. Let V be the component of $\Omega \cap D_{256\rho}$ that contains z_r and let $E = \partial V \cap D_{256\rho}$. The Beurling projection theorem then implies that $\omega(z_r, E, V) \geq 1/2$ (see Corollary 5.4.3). Moreover, $v = 0$ on the set E, and, by the definition of U, we have

$$
\begin{aligned}
u(z) &= -B(r,v) - a(r,v)\log\frac{|z|}{r} \\
&\leq -B(r,v) + a(r,v)^{1-\alpha} \\
&\leq -B(r,v) + B(r,v)^{(1-\alpha)(1+\epsilon)} \\
&\leq -B(r,v) + B(r,v)^{1-\delta} \\
&\leq -\frac{1}{2}B(r,v)
\end{aligned}
$$

Hence at the center z_r of the disk $D_{256\rho}$ we have

$$\begin{aligned} 0 = u(z_r) &\leq -\frac{1}{2}B(r,v)\omega(z_r, E, \Omega \cap D_{256\rho}) + \max_{z\in\partial V} u(z)\cdot\omega(z_r, E^c, \Omega\cap D_{256\rho}) \\ &\leq -\frac{1}{4}B(r,v) + o(1) \end{aligned}$$

since u is bounded by 1 on V if r is large enough. This is impossible since $B(r,v)$ tends to ∞ as r does. Thus Case 1 cannot occur.

Case 2: K is contained in $D_{256\rho}$. For large r, $D_{256\rho}$ does not contain the origin, and this implies that $u - v$ is harmonic in $D_{256\rho}$. Thus their Riesz measures agree in this disk. Lemma 6.1.11 implies the Riesz measure of K is a positive integer and so

$$n(z_r, t, u) = n(z_r, t, v) \geq 1$$

for $t \geq 256\rho$. Hence

$$\int_0^{512\rho} n(z_r, t, u)\frac{dt}{t} \geq \int_{256\rho}^{512\rho} n(z_r, t, u)\frac{dt}{t} \geq \log 2.$$

However, by Jensen's formula for subharmonic functions (Lemma 6.1.10)

$$\int_0^{512\rho} n(z_r, t, u)\frac{dt}{t} \leq \frac{1}{1024\pi\rho}\int_{|z|=512\rho} u(z)ds = o(1),$$

as $r \to \infty$, $r \notin F$. This is a contradiction, so case 2 does not occur either, proving that D_ρ is contained in Ω. □

See [**14**] for related work by Bergweiler that gives a sharper estimate for the size of the Wiman-Valiron disks. He shows that the estimate holds on a disk $D(z_r, r/\sqrt{\psi(a_f(r,D))})$ if

$$\int_T^\infty \frac{dt}{\psi(t)} < \infty$$

$$K \leq \frac{t\psi'(t)}{\psi(t)} \leq L,$$

for some constants $T > 0$, $K > 0$ and $L < 2$.

THEOREM 6.4.2. *With notation as in Theorem 6.4.1,*

$$f(z) \sim f(z_r)(\frac{z}{z_r})^{a(r,v)},$$

and

$$|f(z)| \sim M(|z|, f),$$

for $z \in D_\rho = D(z_r, r\cdot a(r,v)^{-\tau})$ as $r \to \infty$, $r \notin F$.

PROOF. Define a holomorphic function g on D_ρ by

$$g(z) = \log \frac{f(z)}{f(z_r)} - a(r,v)\log\frac{z}{z_r} = \log\left(\frac{f(z)}{f(z_r)}\left(\frac{z_r}{z}\right)^{a(r,v)}\right),$$

with the branches of the logarithm chosen so $g(z_r) = 0$. Since $u = \Re g$, the Borel-Caratheodory estimate (Lemma 1.1.6) gives

$$\max_{|z-z_r|=t} |g(z)| \le 4 \max_{|z-z_r|=2t} u(z),$$

for $0 < t < \rho/2$. Thus $g \to 0$ on $D_{\rho/2}$ uniformly as $r \to \infty$, $r \notin F$ by (37). This gives the first claim.

To prove the second claim, note that

$$M(|z|, f) \ge |f(z)| \ge (1 - o(1))|f(z_r)| \cdot \left|\frac{z}{z_r}\right|^{a(r,v)},$$

and

$$\begin{aligned} M(|z|, f) &= \exp(B(|z|, v)) \\ &\le \exp\left(B(r,v) + a(r,v)\log\frac{|z|}{r} + o(1)\right) \\ &= (1 + o(1))|f(z_r)| \cdot \left|\frac{z}{z_r}\right|^{a(r,v)}, \end{aligned}$$

for $z \in D$. These two inequalities give

$$\frac{1 - o(1)}{1 + o(1)} M(|z|, f) \le |f(z)| \le (1 + o(1))M(|z|, f),$$

which is the second claim. $\square$

COROLLARY 6.4.3. *For all $\beta > 0$, there is an $\alpha > 1$ and a set $F \subset [1, \infty)$ of finite logarithmic measure so that for large enough $r \notin F$,*

$$\{z : \frac{|f(z_r)|}{\beta} \le |z| \le \beta|f(z_r)|\} \subset f(D(z_r, \frac{\alpha}{a(r,v)})).$$

Thus $\log f(D(z_r, \frac{\alpha}{a(r,v)}))$ contains the rectangle

$$|\Re(z) - \log(z_r)| \le \log\beta, \qquad |\Im(z) - \arg(f(z_r))| \le \gamma.$$

6.5. Fast escaping points exist

THEOREM 6.5.1. *Let D be a direct tract of f. Then there exists $w_0 \in D$ such that $w_n = f^n(w_0) \in D$ for all n and $w_n \to \infty$.*

PROOF. Let F be as in Theorem 6.4.2. Since F has finite logarithmic measure, we can choose a sequence $\{r_n\}$ increasing to ∞, so that $r_{n+1} \in I_n \setminus F$ where

$$I_n = [\frac{1}{2}|f(z_n)|, 2|f(z_n)|],$$

and where $z_n = z_{r_n}$ is the point on $\Omega \cap \{|z| = r_n\}$ where $|f(z)|$ is maximized. Take $\alpha > 1$ as in Corollary 6.4.3 and set $D_n = \overline{D(z_n, \alpha r_n/a(r_n, v))}$. Then by Corollary 6.4.3, $f(D_n)$ contains the annulus $\{z : |z| \in 2I_{n+1}\}$ and hence $f(D_n)$ contains D_{n+1}. Let $C_0 = D_0$ and inductively choose C_n to be a component of $f^{-n}(D_n)$ contained in C_{n-1}. These are nested compact sets, so have a common point w_0 whose iterations clearly remain in D forever. This prove the theorem. $\square$

For use later, note that the iterates of w_0 satisfy $|f^{n+1}(w_0)| \in I_{n+1}$ and hence

$$|f^{n+1}(w_0)| \geq \frac{1}{2}M(|f^n(w_0)|, f). \tag{38}$$

where, as before,

$$M(r, f) = \max\{|f(z)| : z \in \Omega, |z| = r\}$$

and M_Ω^n is the nth iteration of this function.

By the proof of Theorem 6.5.1, $|w_{n+1}| > \frac{1}{2}\max\{|f(z)| : z \in \Omega, |z| \leq |w_n|\}$ so this sequence grows at essentially the fastest possible rate at each step. We can make this observation a bit more precise by defining the “fast escaping set” and the “fast escaping through Ω” set as:

$$A^0(f, \rho) = \{z : \text{ and } |f^n(z)| \geq M^n(\rho) \forall n \in \mathbb{N}\}.$$

$$A^0(f, \Omega, \rho) = \{z \in D : f^n(z) \in \Omega \text{ and } |f^n(z)| \geq M^n(\rho) \forall n \in \mathbb{N}\}.$$

We use the first set for entire functions when we don't care which component of $\{|f| > R\}$ our orbit lies in; we use the second set when f is only defined on a direct tract, or when f is entire and we want to restrict attention to orbits that stay in a single component of $\{|f| > R\}$. The “0” in A^0 is used because this set will be a special case of a set $A^L(f, \Omega, \rho)$ that we will introduce in the next section.

THEOREM 6.5.2 (Bergweiler, Rippon, Stallard [**19**]). *$A^0(f, \Omega, \rho)$ is non-empty.*

PROOF. By Lemma 6.2.4 we have $M(2r) > 4M(r)$ for r large enough. By the proof of Theorem 6.5.1 and (38), there is a sequence $\{w_n\} = \{f^n(w_0)\}$ in Ω so that

$$|z_0| \geq 2\rho = 2M^0(\rho) \quad \text{and} \quad |w_{n+1}| \geq \frac{1}{2}M(|w_n|).$$

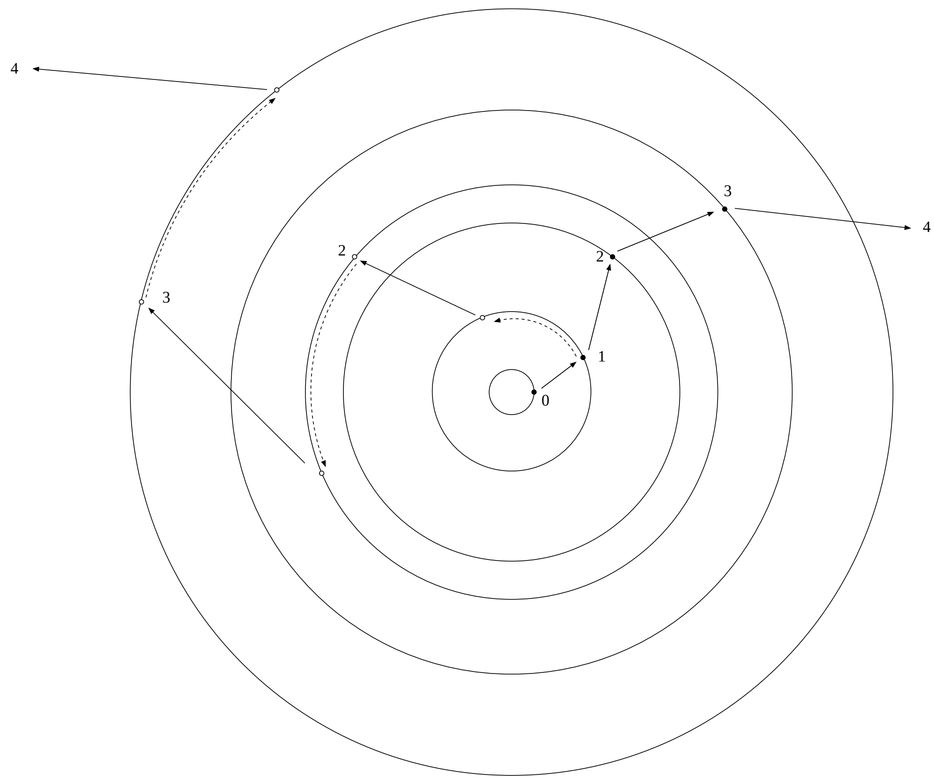


FIGURE 1. The difference between $M(r, f^n)$ and $M^n(r, f)$. The former is the largest possible value of $|f^n(z)|$ where $|z| = r$. This is obtained by starting on the circle $\{|z| = r\}$ and following a single orbit for n steps. $M^n(r, f)$ is the nth iterate of the real value function $M(r, f)$. It is obtained by applying f to a the point on $\{|z| = r\}$ where $|f|$ is maximized, but instead of applying f to $f(z)$ we can apply f to any point on the circle $\{|w| = |f(z)|\}$. Because we can choose the point that we iterate at each stage, this grows at least as fast (and often faster) than following a fixed orbit. Fast point of f are points whose orbits grow as quickly as the iterates $M^n(r, f)$. The existence of such points is not obvious, but see Theorem 6.5.2.

We claim that

$$|w_{n+1}| \geq M^{n+1}(\rho),$$

for all n. We prove this by induction. The case $n = 0$ is stated on the left above. The induction step is

$$|w_{n+1}| \geq \frac{1}{2}M(|w_n|) \geq 2M(\frac{1}{2}|w_n|) \geq 2M(M^n(\rho)) = 2M^{n+1}(\rho).$$

Thus $w_0 \in A^0(f, \Omega, \rho)$, proving the set is non-empty. $\square$

Let f be a transcendental entire function and $\epsilon > 0$. Then the argument above shows that for r sufficiently large there is a point

$$w \in \{z : r \le |z| \le (1+\epsilon)r\} \cap A(f)$$

with $|f^n(w)| > M^n(r, f)$ for all $n \in \mathbb{N}$. Moreover, we can choose w so that there are sequences $\{z_n\} \subset \mathbb{C}$ and $\{k_n\} \subset (0, \infty)$ so that if

$$D_n = \{z : |z - z_n| < \frac{\epsilon}{4}|z_n|\},$$
$$A_n = \{z : k_n^{-1} M(|z_n|, f) \le |z| \le k_n M(|z_n|, f),$$

then,

(1) $f^n(w) \in D_n$, $A_n \subset f(D_n)$ and $|f(z_n)| = M(|z_n|, f) \ge M^{n+1}(|z_0|, f)$.
(2) $D_n \subset A_n \cap \{z : |z| \ge M(|z_{n-1}|, f)\}$,
(3) $w = \cap B_n$ where $B_0 = \overline{D_0}$ and B_n is a component of $f^{-n}(\overline{D_n})$,
(4) $k_n \nearrow \infty$.

A point satisfying all these properties is called an **Eremenko point** by Rippon and Stallard in [**118**].

6.6. Levels of the fast escaping set

Suppose Ω is a direct tract of f and define

$$\begin{aligned} A^L(f, \Omega, \rho) &= \{z : f^L(z) \in A^0(f, D, \rho)\} \\ &= \{z : \forall n \ge \max(0, -L), f^{n+L}(z) \ge M^n(\rho, f)\} \end{aligned}$$

and

$$\begin{aligned} A(f, \Omega, \rho) &= \{z : \exists L \in \mathbb{N} \text{ s.t. } f^L(z) \in A^0(f, D, \rho) \\ &= \cup_{L \in \mathbb{Z}} A^L(f, \Omega, \rho) \\ &= \cup_{L \in \mathbb{N}} A^L(f, \Omega, \rho) \end{aligned}$$

The last equality holds since $s < t$ implies $A^s \subset A^t$. Note that each set A^L is closed, and hence A is an increasing union of closed sets.

We can make (but won't write down) similar definitions for $A^L(f, \rho)$ and $A(f, \rho)$. These are the same as above, except that if f is defined on multiple tracts, then we don't require the orbit of a point to stay in a single tract. Because of this extra freedom, such orbits can escape more quickly to ∞, e.g., if f is an entire function and $\||f| > 1\}$ has connected components $\{\Omega_j\}$ it is possible for

$$A(f, \rho) \cap A(f, \Omega_j, \rho) = \emptyset,$$

for every j, i.e., unrestricted orbits tend to infinity at a faster rate than orbits restricted to any single tract of f.

EXERCISE: Construct an example where $A(f, \rho) \cap A(f, \Omega_j, \rho) = \emptyset$, as described above. This will be easier later the the text when we have developed quasiconformal methods for constructing entire functions. The basic idea is to build a function with two tracts that are "wide" and "narrow" at different scale, and thus the growth of f in each tract is different, growing faster in a tract when that tract is narrow. An orbit that switches between tracts so as to always land in a "narrow" section will iterate to ∞ faster than an orbit that is required to stay in one tract and experience both slow and fast growth of f.

EXERCISE: Show that if f is an entire function so that $\{z : |f(z)| > R\}$ has $N < \infty$ connected components $\{\Omega_n\}_1^N$, then for any sequence $\{s_n\}$ with $s_n \in \{1, \dots, N\}$ there is a point $z_0 \in A^0(f, \rho)$ so that $f^n(z_0) \in \Omega_{s_n}$.

EXERCISE: Show that if f has infinitely many tracts $\{\Omega_n\}$, then not all sequences occur. (Hint let $\omega_n = \min\{|z| : z \in \Omega_n\}$ and choose a sequence so that this grows too quickly to be attained by any orbit).

EXERCISE: Take $f(z) = \exp(-z) + \exp(\exp(z))$ and $R > e + 1$. Show there is one direct tract Ω_0 is the left half-plane and infinitely many in the right hand plane, including one, Ω_1 that includes all large enough real numbers. Show $A(f, \Omega_0) \cap A(f) = \emptyset$, but $A(f, \Omega_1) \subset A(f)$.

EXERCISE: if $g = h^{-1} \circ f \circ h$ where h is linear, then $A(f) = h(A(g))$. (Theorem 2.2d of [**118**]).

EXERCISE: $A(f) = A(f^m)$ (Theorem 2.6 of [**118**]).

EXERCISE: Fix $\epsilon > 0$ and let $\mu(r) = \epsilon M(r, f)$. Then if $r > r_0$ implies $\mu(r) > r$ and $|z| > r_0$ satisfies $|f^n(z)| \geq \mu^n(r_0)$ for all n, then $z \in A(f)$. (Theorem 2.7 of [**118**]).

To simplify notation, it would be nice to get rid of the ρ in $A(f, \rho)$. We can do this with the following lemma:

LEMMA 6.6.1. *$A(f,\Omega,\rho)$ is independent of ρ for ρ large enough. Similarly for $A(f,\rho)$.*

PROOF. Assume that $\rho > \rho_0$ implies that $M(\rho,f) > rho$; this insures that $M^n(\rho,f) \nearrow \infty$. Given two values $\rho_0 < s < t$, there is some k so that $M^k(s,f) > M(t,f)$, and hence $M^{n+k}(s,f) > M^n(t,f)$. This implies

$$A^{L+k}(f,\Omega,t) \subset A^{L+k}(f,\Omega,s) \subset A^L(f,\Omega,t),$$

hence, taking the union over all $L \in \mathbb{Z}$,

$$A(f,\Omega,s) = A(f,\Omega,t).$$

$\square$

Since $A(f,\Omega,\rho)$ is independent of ρ for $\rho > \rho_0$, we denote this set as $A(f,\Omega)$. This is the **fast escaping set through** Ω. If f is entire, and we don't care about which components of $\{|f| > \rho\}$ the orbits visit, we define $A(f)$ similarly, and call it the **fast escaping set** of f.

The definition of $A(f)$ given above is due to Rippon and Stallard [**118**], but the original definition, due to Bergweiler and Hinkkanen [**17**] is slightly different:

$$C_R^L(f) = \{z : \exists L \in \mathbb{N} \text{ s.t. } |f^{n+L}(z)| \geq M(R,f^n), \forall n \in \mathbb{N}, \},$$
$$C_R(f) = \cup_L C_R^L,$$

and in [**116**] Rippon and Stallard considered another variant

$$B_R^L(f) = \{z : \exists L \in \mathbb{N} \text{ s.t. } |f^{n+L}(z)| \notin \text{hull}(f^n(\mathbb{D}_R)), \forall n \in \mathbb{N}, \},$$
$$B_R(f) = \cup_L B_R^L,$$

where D is any disk hitting $\mathcal{J}(f)$.

THEOREM 6.6.2 (Rippon-Stallard []). *$A(f) = B_R(f) = C_R(f)$.*

PROOF. We claim that if R is sufficiently large, then

$$\begin{aligned} \{z : |z| \leq M^n(R/2,f) &\subset \text{hull}(f^n(\mathbb{D}_R)) \\ &\subset \{z : |z| \leq M(R,f^n)\} \\ &\subset \{z : |z| \leq M^n(R,f)\} \\ &\subset \{z : |z| \leq M^{n+1}(R,f)\} \end{aligned}$$

All the inclusions are obvious except the first one. To prove this, note that if R is sufficiently large then Eremenko's application of the Wiman-Valiron implies that there is a point z in the annulus $A = \{z : \frac{R}{2} < |w| < R\}$, with a neighborhood $U \subset A$ so that $f^n(U)$ contains the circle $\{|z| = M^n(R/2, f)\}$. This proves the first containment.

These containments imply

$$A^L_{R/2}(f) \supset B^L_R(f) \supset C^L_R(f) \supset A^{L-1}_R(f).$$

Taking unions over L gives

$$A(f) \supset B_R(f) \supset C_R(f) \supset A(f).$$

□

EXERCISE: Show directly that $C(f)$ does not depend on the choice of R, as long at $R > \min_{z \in \mathcal{J}(f)} |z|$. Show that $B(D)$ does not depend on the choice of D.

EXERCISE: [**?**] The quite fast escaping set is defined by Rippon and Stallard as

$$Q_\epsilon(f) = \{z : \exists k \in \mathbb{N} \text{ such that } |f^{n+k}(z)| \leq \mu^n_\epsilon(R) \text{ for } n \in \mathbb{N}\},$$

where $\mu_\epsilon(r) = M(r)^\epsilon$. They give examples where this agrees with the fast escaping set and examples where they differ.

6.7. Connected components of $A(f)$ are unbounded

We start by reviewing some properties of connected sets.

A set is connected if it cannot be written as a disjoint open sets.

LEMMA 6.7.1. *If K is connected, so is its closure $\overline{K}$.*

LEMMA 6.7.2. *If $E_1 \supset E_2 \supset \dots$ are non-empty, compact, connected sets, then so is $E = \cap_n E_n$.*

PROOF. Compactness is clear and non-emptiness is the Cantor intersection theorem. If U, V are any pair of disjoint open sets then $\{E_n \setminus (U \cup V)$ is a nested sequence of compact, non-empty sets, hence has a non-empty intersection. Thus E is not covered by U and V and so is connected. □

LEMMA 6.7.3. *Suppose K a connected component of a compact set E and $E(\epsilon)$ is the closed ϵ-neighborhood of E. Let K_ϵ be the connected component of $E(\epsilon)$ that contains K. Then $\cap_{n>0} K_{1/n} = K$. Moreover, for any $\delta > 0$ there is an $n > 0$ so that $K_{1/n} \subset K(\delta)$.*

PROOF. K is clearly in the intersection, since it is in each K_ϵ. Moreover, the intersection is closed, connected subset of E that contains K, so must be K, since K is a component of E. To prove the final claim, note that otherwise there is an $\delta > 0$ so that $K_{1/n} \setminus K(\delta)$ forms a nested sequence of non-empty compact sets, hence has non-empty intersection, contradicting the first part of the lemma. □

The arguments in the previous section shows that $A^0(f, \Omega, \rho)$ is unbounded. We want to show every connected component of this set is also unbounded, but first we need a few topological facts about connected sets.

LEMMA 6.7.4. *Suppose X is a compact, compact connected subset of the Riemann sphere that contains ∞. Then each connected component K of $E = X \cap \{z \leq R\}$ hits $\{|z| = R\}$.*

PROOF. Suppose K is a component of E that does not hit $\{|z| = R\}$. Since K is closed, there is an $\epsilon > 0$, so that K_ϵ (as defined above) is contained in $D(0, R)$. Since no point of E hits ∂K_ϵ (such points are distance ϵ from E, $E_1 = E \cap K_\epsilon$ and $E_2 = E \setminus \overline{K_\epsilon}$ are disjoint open sets covering E. Thus E_1 and $E_2 \cup X \setminus \overline{D(0, R)}$ form a disjoint, open cover of X, contradicting the connectedness of X. Thus K must hit $\{|z| = R\}$. □

THEOREM 6.7.5 (Boundary bumping lemma). *Suppose X is a compact, compact connected subset of the Riemann sphere that contains ∞. Then each connected component of $E = X \setminus \{\infty\}$ is unbounded.*

PROOF. Suppose $K \subset D(0, R)$ were a bounded connected component of E. Then $\overline{K}$ is closed, bounded and contains K, so $K = \overline{K}$ must be closed. Let $V = E \cap D(0, 2R)$ and let B the component of $\overline{V}$ that contains K. By Lemma 6.7.4, B hits $E \cap \{|z| > R\}$ so is not equal to K. Therefore B is a connected subset of E that strictly contains K, a contradiction. Hence K cannot be bounded. □

The preceding result may seem so obvious that the reader may wonder if any proof is really needed (or, at least, if the proof was longer than needed). To illustrate the problem involved we remind the reader of an old result of Knaster and Kuratowski:

LEMMA 6.7.6. *There is a connected set X on the 2-sphere so $X \setminus \{\infty\}$ is totally disconnected.*

PROOF. Let C be the standard middle thirds Cantor set, $E \subset C$ the countable set of endpoints of components of $\mathbb{R} \setminus C$ and $F = C \setminus E$. Let

$$Y = \{x + iy : x \in E, y \in \mathbb{Q}, y \geq 0\} \cup \{x + iy : x \in F, y \in \mathbb{R} \setminus \mathbb{Q}, y > 0\}.$$

and $X = Y \cup \{\infty\}$. We claim that X is connected and Y is totally disconnected. The latter claim is easy: any two vertical lines through C are contained in disjoint open half-planes, so any connected component of Y would have to be contained in one such line L. However, $Y \cap L$ has only point components, so the same is true for Y.

Next we show X is connected. Suppose U, V are disjoint open sets in the sphere such that $X \subset U \cup V$. Without loss of generality we assume $\infty \in U$. For each $Rx \in C$ let $f(x) = \sup\{y : x + iy \in V\}$ (set $f(x) = 0$ is there are no such points y). This is clearly bounded (since U covers a neighborhood of ∞) and $x \in F$ implies $f(x) \in \mathbb{Q}$ (since V is open). For each rational $r > 0$ let

$$T_r = \{x \in C : f(x) = r\} = \overline{V} \cap \{y = r\},$$

and let $S = \{x : f(x) = 0\}$. Then $\{T_r\}$ is a countable collection of closed sets. Moreover, each of these sets is disjoint from E since $f(x)$ must be irrational for $x \in E$. Thus each T_r is nowhere dense in C; otherwise, being closed, they would contain points of E (which is dense in C).

Since $C = S \cup E \cup \cup_r T_r$, Baires' theorem implies S is dense. The vertical lines through S therefore intersect every open sets that hits Y, so U intersects every open set that hits Y. Therefore V is the empty and we conclude X is connected. $\square$

It might seem that this is an artificial example, but similar sets arise naturally in dynamics:

THEOREM 6.7.7 (Mayer, [**93**]). *For every exponential map $z \to \lambda e^z$ with an attracting fixed point, the landing points of dynamic rays in $\mathbb{C}$ are totally disconnected, but become a connected set on the sphere when we add $\{\infty\}$.*

PROOF ????

A useful consequence of the boundary bumping lemma is

LEMMA 6.7.8. *If K is a bounded connected component of a closed set $X \subset \mathbb{C}$ then there is a Jordan curve $\gamma \subset \mathbb{C} \setminus X$ that surround K.*

PROOF. Consider the usual grid of $\frac{1}{n} \times \frac{1}{n}$ closed squares in the plane and let X_n be the union of these squares that hit X and let K_n be the connected component of X_n that hits K. If K_n is bounded, then there is a component of the curve $\{z : \mathrm{dist}(z, K_n) = (10n)^{-1}\}$ that surrounds K and is disjoint from X.

If the result fails, then $E = \cap_n \{K_n \cup \{\infty\}\}$ is a closed connected set on the sphere that contains K. By the boundary bumping lemma the component F of $E \setminus \{\infty\}$ that contains K must be unbounded. Thus the component of X that contains K must be unbounded. But K is itself this component, so K is unbounded. $\square$

Now we apply these topological observations to the fast escaping set.

THEOREM 6.7.9. *Every connected component of $A^L(f, \rho)$ is unbounded.*

FIRST PROOF. Fix $L \in \mathbb{Z}$ and $z_0 \in A^L(f, \rho)$. If $n+L \geq 0$, then let $R_n = M^n(f, \rho)$ and $X_n = \{z : |z| \geq R_n\}$ and note that $z_0 \in f^{-n-L}(X_n)$. Thus z_0 is in a connected component L_n of this closed set. Clearly L_n is closed in $\mathbb{C}$. If it were bounded, then $|f^{n+L}|$ would have to be bounded on L_n and since it equals R on ∂L_n the maximum principle implies it is constant; a contradiction. Thus L_n is closed and unbounded.

Since $|f^{n+L}(z)| \leq R_n$ implies $|f^{n+1+L}(z)| \leq R_{n+1}$, we also have $|f^{n+1+L}(z)| \geq R_n$ implies $|f^{n+L}(z)| \geq R_n$. Hence $L_{n+1} \subset L_n$. Thus

$$K = \cap_{n \in \mathbb{N}, n+L \geq 0} (L_n \cup \{\infty\}),$$

is a closed connected set on the sphere that contains both z_0 and ∞. By the boundary bumping lemma (Lemma **??**), each component of $K \setminus \{\infty\}$ is unbounded.

Each finite point z of K is in every L_n with $n + L \geq 0$, so $|f^{n+L}(z)| \geq R_n = M^n(f, \rho)$ for all $n \geq \max(0, -L)$. Thus $z \in A^L(f, \rho)$ and hence $K \setminus \{\infty\} \subset A^L(f, \rho)$.

Since every component of $A^L(f,\rho)$ contains a component of $K\backslash\{\infty\}$, every component of $A^L(f,\rho 0)$ is also unbounded. □

SECOND PROOF OF THEOREM 6.7.9. Suppose $A^L(f,\rho)$ has a bounded component K. By Lemma 6.7.8, there is a Jordan curve γ that misses $A^L(f,\rho)$ and surrounds K. For each $n = 1, 2, 3, \dots$ define

$$\gamma_n = \{z \in \gamma : |f^n(z)| \geq M^{n+L}(\rho)\}.$$

Since K is inside γ the maximum principle says that each γ_n is non-empty, and they are clearly closed and nested, i.e., $\gamma_{n+1} \subset \gamma_n$. Thus the intersection is non-empty and consists of points in $A^L(f,\rho)$, contradicting the fact that γ is disjoint from this set. Therefore there are no bounded components of $A^L(f,\rho)$. □

LEMMA 6.7.10. *If $E_1 \subset E_2 \subset \dots$ are connected sets, then $\cup_n E_n$ is connected.*

PROOF. Suppose U, V are disjoint open sets that disconnect E, i.e., they cover E and $U \cap E$ and $V \cap E$ are both non-empty. By assumption, each E_n is contained in either U or V. By passing to a subsequence, we can assume they are all contained in U, but this implies $V \cap E = \emptyset$. □

COROLLARY 6.7.11. *Every connected component of $A(f)$ is unbounded.*

PROOF. If $z \in A(f)$ then $z \in A^L(f,\rho) \subset A(f)$ for some L and ρ. By Theorem 6.7.9, the component of $A^L(f,\rho)$ containing z is closed and unbounded, thus the component of $A(f)$ containing z is also unbounded. □

THEOREM 6.7.12. *For any transcendental entire function f, $I(f)$ has an unbounded component.*

PROOF. Every component of $A(f)$ is unbounded and there is at least one such component K (since $A(f) \neq \emptyset$). Thus the connected component of $I(f)$ that contains K is also unbounded. □

Eremenko's conjecture asks if every component of $I(f)$ is unbounded. At this writing, this is still open. A stronger versions asks if every every path component of $I(f)$ is unbounded. In Chapter 10 we will show this is true in the special case of hyperbolic Eremenko-Lyubich functions of finite order. In Theorem 9.3.1 we give a hyperbolic example of infinite order where the strong Eremenko conjecture fails within the Eremenko-Lyubich class.

6.8. Fast escaping Fatou components

We define that set of orbits that tend to infinity faster than iterating any polynomial:

$$Z(f) = \{z \in I(f) : \lim_{n\to\infty} \frac{1}{n} \log\log |f^n(z)| = \infty\}.$$

(The "Z" is for "zipping" to infinity.)

LEMMA 6.8.1. $A(f,\Omega) \subset Z(f)$.

PROOF. This follows from Lemma 6.2.3. EXPLAIN FURTHER □

LEMMA 6.8.2. *A Fatou component U that hits $Z(f)$ must be wandering.*

PROOF. We may assume U is simply connected (since otherwise it is wandering by Corollary 2.6.4). Suppose it is not wandering, i.e., $f^n(U) \subset U$. Replacing f by $g = f^n$ we assume $g(U) \subset U$. Take any point in U and connect z to $g(z)$ by a curve γ. Then $g^n(z0$ is connected to $g^{n+1}(z)$ by $\gamma_n = g^n(\gamma)$, and the hyperbolic diameter of γ_n is no greater than that of γ. Since U is simply connected there is a constant $C < \infty$ so that

$$|g^{n+1}(z)| \leq C|g^n(z)|,$$

when $|g^n(z)|$ is large enough. Thus

$$|g^{n+k}(z)| \leq C^k|g^n(z)|,$$

when $|g^n(z)|$ is large enough, which implies $z \notin Z(g)$. This easily implies $z \notin Z(f)$ as well, and hence U must have been wandering. □

COROLLARY 6.8.3. *A Fatou component that hits $A(f)$ must be wandering.*

THEOREM 6.8.4. *If U is a Fatou component that hits $A(f)$ then $\overline{U} \subset A(f)$*

PROOF. Suppose f is a transcendental entire function and $R > 0$ satisfies $M(r,f) > r$ for all $r > R$. If U hits $A(f)$ then it hits $A_R^L(f)$ for some L, and it suffices to show $\overline{U} \subset A_R^{L-1}(f)$. Let $z_0 \in U \cap A_R^L(f)$. Then, by definition,

$$|f^n(z_0)| \geq M^{n+L}(R,f),$$

for all $n \in \mathbb{N}$ such that $n + K \geq 0$. Suppose $z_1 \in U$, by normality $z_1 \in I(f)$, but we need a stronger statement. Since the Julia set contain at least two distinct points and f^n never takes these values on U, Schottky's Lemma (Lemma 1.5.7 says that there is a C so that

$$|f^n(z_1)| \geq |f^n(z_0)|^{1/C} \geq (M^{n+L}(R, f))^{1/C} \geq M^{n+L-1}(R, f),$$

if R is large enough (the last inequality follows from Lemma 1.6.2 that says that the slope of $\log M(r, f)/\log r \nearrow \infty$). Thus $U \subset A_R^{L-1}$. Since the latter is a closed set, it also contains the closure of U. □

It is not true that if a Fatou component hits $I(f)$ then it closure lies in $I(f)$. We shall prove later (see Example 7.5.1) that

$$f(z) = 1 + z + e^{-z},$$

has a single Fatou component U and this component is in $I(f)$. However, in this case $\partial U = \mathcal{J}(f)$ and hence contains periodic points, which are clearly not escaping.

If f has a multiply connected Fatou component, then such components eventually surround every point, so must hit any unbounded connected set. In particular, they must hit $A(f)$, and hence:

Corollary 6.8.5. *Suppose f is a transcendental entire function. Then the closure of every multiply connected Fatou component is in $A(f)$.*

Corollary 6.8.6. $A(f) \cap \mathcal{J}(f) \neq \emptyset$.

Proof. If not, then $A(f)$ is a non-empty subset of $\mathcal{F}(f)$, and by Theorem 6.8.4 there is a Fatou component U so that $\partial U \subset \overline{U} \subset A(f)$. But $\partial U \subset \mathcal{J}(f)$. □

Corollary 6.8.7 (Rippon-Stallard [**116**]). *Suppose f is a transcendental entire function that has a multiply connected Fatou component. Then $A(f)$ is connected.*

Proof. In this case, any two components of $A(f)$ hit some common Fatou component and thus are the same component. □

The following generalizes Corollary 2.6.8.

Lemma 6.8.8. *If f is entire and $|f(z)| = O(|z|^C)$ along some curve γ tending to infinity, then all the Fatou components of f are simply connected.*

PROOF. If there is a multiply connected Fatou component U, then it is the fast escaping set $A(f)$ and hence in the $Z(f)$ and hence for any $c < \infty$, we have $|f(z)| > |z|^c$ in all large enough iterates of U. However, every large enough iterate of U hits γ, which is a contradiction. $\square$

LEMMA 6.8.9. *If f is transcendental entire, then $\mathcal{J}(f) = \partial A(f)$.*

PROOF. If $\partial A(f)$ contains a point $z \in \mathcal{F}(f)$, then $A(f)$ also hits U and hence $U \subset A(f)$, which implies z is not a boundary point of $A(f)$. Thus $\partial A(f) \subset \mathcal{J}(f)$.

On the other hand, by Lemma 2.4.9, the Julia set is contained in the accumulation set of $f^{-n}(z)$ for every z, except possibly one. Since $A(f)$ is non-empty and contains an escaping orbit, it contains a non-exceptional z. Since the inverse images of this point are all in $A(f)$, we deduce $\mathcal{J}(f) \subset \overline{A(f)}$. If D is a disk in $A(f)$ then $f^n(D)$ omits every pre-periodic point of f; since there are always at least two such points, f^n is normal on D and hence $D \subset \mathcal{F}(f)$. Thus the interior of $A(f)$ is in $\mathcal{F}(f)$ and hence $\mathcal{J}(f) \subset \partial A(f)$. $\square$

COROLLARY 6.8.10. *If f is transcendental entire, then $\mathcal{J}(f) = \overline{\mathcal{J}(f) \cap A(f)}$.*

PROOF. Since $\mathcal{J}(f) \cap A(f)$ is completely invariant and contains infinitely many points $\mathcal{J}(f) \subset \overline{\mathcal{J}(f) \cap A(f)}$. The reverse containment is obvious since $J(f)$ is closed. $\square$

LEMMA 6.8.11. *If f has no wandering domains, then $\mathcal{J}(f) = \overline{A(f)}$.*

PROOF. By the previous lemma, the Julia set is contained in the closure of $A(f)$. If we did not have equality, then $A(f)$ hits some Fatou component, which then must be wandering by Corollary 6.8.3. $\square$

LEMMA 6.8.12. *If E is connected and $E \subset F \subset \overline{E}$, then F is connected.*

PROOF. Suppose U, V are disjoint open sets that cover F. Then E hits only one of them, say U. Then $\overline{E} \cap V = \emptyset$ and hence $F \cap V = \emptyset$. $\square$

We can now give a second proof Corollary 5.7.9

COROLLARY 6.8.13 (Rippon-Stallard [**116**]). *Suppose f is a transcendental entire function that has a multiply connected Fatou component. Then $I(f)$ is connected.*

PROOF. Since $\mathcal{J}(f) = \partial A(f)$,

$$A(f) \subset A(f) \cup (\mathcal{J}(f) \cap I(f)) \subset \overline{A(f)}.$$

We proved in Corollary 6.8.7 $A(f)$ is connected in this case, so this implies $A(f) \cup (\mathcal{J}(f) \cap I(f))$ is also connected by Lemma 6.8.12.

On the other hand, if V is a Fatou component in $I(f)$ and $z_0 \in V$ then eventually $f^n(z_0)$ lies outside any bounded set and hence outside any fixed multiply connected Fatou component. Thus all of V (and $\overline{V}$) lie outside this component and hence $\overline{V} \subset I(f)$. Thus V is in the same component of $I(f)$ as the connected set $A(f) \cup (\mathcal{J}(f) \cap I(f))$ and hence $I(f)$ is connected. □

EXERCISE: Show, in contrast to Theorem 6.8.4, that if a Fatou component hits $I(f)$, its closure need not be in $I(f)$ (consider $f(z) = z + 1 + e^{-z}$ which has a Baker domain).

EXERCISE: In the proof of Theorem 6.8.4, show that $\overline{U} \subset A^L_R(f)$ if f is simply connected. (Hint: replace Schottky's lemma by the fact that $|f^n(z_0)|$ and $|f^n(z_1)|$ must be comparable.)

EXERCISE: If all components of $\mathcal{F}(f)$ are simply connected, then $\partial A^L_R(f) \subset \mathcal{J}(f)$.

OPEN PROBLEM: Can there be unbounded Fatou components in $A(f)$?

THEOREM 6.8.14. *Suppose f is a transcendental entire function and $R > 0$ satisfies $M(r, f) > r$ for all $r > R$. All the components of $A(f) \cap \mathcal{J}(f)$ are unbounded iff f has no multiply connected Fatou components.*

PROOF. One direction is easy: if there are multiply connected Fatou components then such components eventually surround every point of the plane, and hence $\mathcal{J}(f)$ has no unbounded components.

For the other direction, suppose all Fatou components are simply connected. It suffices to prove that all components of $A^L_R(f) \cap \mathcal{J}(f)$ are unbounded. Let $z_0 \in A^L_R(f) \cap \mathcal{J}(f)$. Then by the proof of Theorem 6.7.9 z_0 lies in an unbounded component K of $A^L_R(f)$. By the previous EXERCISE, $\partial K \subset \mathcal{J}(f)$, so z_0 lies in some connected component S of ∂K.

We claim S is unbounded. Suppose not. Then there is an bounded open set U such that $S \subset U$ but $\partial U \cap \partial K = \emptyset$. Since K contains S and is unbounded it must hit ∂U. Thus ∂U must be contained in the interior of K, hence in the interior of the escaping set and hence in the Fatou set. But U contains a point of the Julia set and hence ∂U belongs to a multiply connected Fatou component. The contradiction implies no component of $A_R^L(f) \cap \mathcal{J}(f)$ is bounded. $\square$

CHAPTER 7

Baker domains

In polynomial dynamics a Fatou component can have a rationally neutral periodic point on its boundary so that points of the component iterate towards the (finite) orbit of the periodic point. This can also happen for transcendental functions, but in this case, it is also possible to have periodic Fatou components consisting of escaping points. These are called Baker domains. As we saw in Section **??**, these cannot occur in the Eremenko-Lyubich class, but they can occur in general. In this chapter we shall describe a number of examples, and prove some basic properties of Baker domains. One of the most interesting is that although a Baker domain need not contain singular values, there must always be singular values at "all scales"; making this precise provides another proof that functions with bounded singular sets (i.e., the Eremenko-Lyubich class) do not have Baker domains.

Since several interesting examples of Baker domains arise using from maps of the punctured plane, we start by considering such maps. This also provides an application of the some of our results about fast escaping points from the last chapter.

7.1. Iteration in $\mathbb{C}^*$

If we consider holomorphic maps $f : \Omega \to \Omega$ where Ω is a subset of the Riemann sphere, Montel's theorem implies the iterates of f will be normal on all of Ω if this domain omits three points of the sphere. Thus the most interesting cases are the sphere, the plane and the punctured plane. The first corresponds to iteration of rational maps, the second to iteration of entire functions (the subject of these notes), and the third to the iteration of maps $f : \mathbb{C}^* \to \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. This case was first considered by Radström [**109**] in 1953 and in a number of more recent papers (e.g., [**8**], [**9**], [**13**], [**15**], [**26**], [**54**], [**53**], [**52**], [**51**], [**79**], [**78**], [**84**], [**83**], [**90**], [**89**], [**101**].

Let $\Pi(z) = \exp(az)$ for some $a \neq 0$. This is a holomorphic covering map $\Pi : \mathbb{C} \to \mathbb{C}*$, so if $g : \mathbb{C}* \to \mathbb{C}*$ is holomorphic, it can be lifted to an entire function f so that $\exp(af(z)) = g(\exp(az))$.

LEMMA 7.1.1 (Bergweiler [**15**]). $\mathcal{J}(f) = \Pi^{-1}(\mathcal{J}(g))$.

PROOF. Since $g(\exp(z)) = \exp(f(z))$, applying g to both sides gives

$$g(g(\exp(z)) = g(\exp(f(z)) = \exp(f(f(z)))),$$

or $g^2(\exp(z)) = \exp(f^2(z))$. Induction gives $g^n(\exp(z)) = \exp(f^n(z))$. Suppose $U \subset \mathbb{C}$ is an open set such that $V = \Pi(U)$ is a subset of a a Fatou component of g. Then $\{g^n\}$ has a convergent subsequence g^{n_k} on V. If it converges to ∞ then $\Re(f^{n_k}) \to \infty$ on U and hence $f^{n_k} \to \infty$ on U and hence $U \subset \mathcal{F}(f)$. Similarly if $g^{n_k} \to 0$. Otherwise $g^{n_k} \to \varphi$ where φ is holomorphic and non-zero on U(by Hurwitz's theorem). Thus there is a sequence of integers m_k so that

$$|f^{n_k} - \frac{2\pi i}{a} m_k - \log \varphi| \to 0.$$

If only finitely many different values of m_k occur, then one value m occurs infinitely often and this gives a convergent subsequence of iterates converging to $\frac{2\pi i}{a} m - \log \varphi$. Otherwise we can choose a subsequence where $|m_k| \to \infty$, and this gives subsequence of iterates converging to ∞. Thus $\Pi^{-1}(\mathcal{F}(g)) \subset \mathcal{F}(f)$ (or, if you prefer, $\mathcal{J}(F) \subset \Pi^{-1}(\mathcal{J}(g))$).

Suppose equality does not hold. Then there is a point $z \in \mathcal{F}(f)$ with $w \in \Pi(z) \in \mathcal{J}(g)$. For any neighborhood U of z, $\Pi(U)$ is a neighborhood of w and by Theorem 3.4.2 it contains a periodic point w_1. (Theorem 3.4.2 was stated for transcendental entire functions, but its proof applies whenever there are at least five points where the iterates of f are not normal; this holds for transcendental self-maps of $\mathbb{C}^*$.)

By Theorem 6.8.1, it also contains a point $w_2 \in Z(g)$. If $U \subset \mathcal{F}(f)$, we shall show that U cannot contains lifts of both types of points. Theorem 6.8.1 was stated for functions with a direct tract, but it is easy to see that a transcendental self-map of $\mathbb{C}^*$ has a direct tract at either 0 or ∞, and by conjugating with $1/z$, if necessary, we can assume the direct tract is at ∞. Thus Theorem 6.8.1 applies.

First note that since

$$\exp(f(z + 2\pi i))g(\exp(z + 2\pi i)) = g(\exp(z)) = \exp(f(z)),$$

we must have

$$f(z+2\pi i) = f(z) + k2\pi i + f(z),$$

and since the choice of k is both discrete and continuous in z, it must be the same for all z.

$$f(z+2n\pi i) = f(z) + kn2\pi i + f(z).$$

From this we see that

$$f(z+it) = f(z) + O(|t|),$$

and hence all the components of $\mathcal{F}(f)$ must be simply connected by Lemma 6.8.8.

Now consider $z_1 \in U$ so that $\Pi(z_1) = w_1$. There is an m so that $g^m(w_1) = w_1$, so there is a k so that $f^m(z_1) = km2\pi i + f(z_1)$. Hence the f iterates of z_1 either remain bounded (if $k = 0$) or tend to infinity in a vertical strip (wide enough to include $z_1, \dots, f^{m-1}(z_1)$) at a rate $O(m)$. Since z_2 is in the same simply connected Fatou component as z_1, this implies the f-iterates of z_2 are also $O(m)$ by Lemma 1.4.4. On the other hand, since $w_2 \in Z(g)$, the iterates of z_2 have real parts that tend to $+\infty$ faster than $\exp(cn)$, for any $c > 0$. The contradiction proves not part of $\mathcal{F}(f)$ can project onto $\mathcal{J}(g)$, which completes the proof. □

7.2. Rate of escape

THEOREM 7.2.1. *If f is a transcendental entire function and $z \in \Omega$, a Baker domain for f, then there is a $1 \leq C < \infty$ so that*

$$\frac{C}{|z|} \leq |f(z)| \leq C|z|$$

. Thus

$$\log |f^n(z)| = O(n).$$

7.3. Classification of Baker domains

Baker domains are simply connected. (otherwise would be wandering).

Invariant domain; f corresponds to iteration of inner function converging to boundary.

Denjoy-Wolff theorem

Classification based on conjugacy.

THEOREM 7.3.1 (Baker-Domínguez). *Let Ω be a Baker domain on which f is not univalent. Then Ω has uncountably many different ends to ∞ and $\partial\Omega$ has infinitely many components.*

7.4. Singular points

We have already seen (Theorem **??**) that if $S(f)$ is bounded, then f has no Baker domains. In fact, if there are Baker domains, then there large singular points must be fairly common in the following sense.

THEOREM 7.4.1 (Bargmann). *If f is a transcendental entire function that has a Baker domain, then $S(f) \cap \{z : r < |z| < Cr\} \neq \emptyset$ for all sufficiently large $r > 0$.*

LEMMA 7.4.2. *There is a f transcendental entire function that has a Baker domain Ω that contains no singular values.*

THEOREM 7.4.3 (Bergweiler). *If f is a transcendental entire function that has a Baker domain Ω such that $\Omega \cap S(f) = \emptyset$, then there is a sequence of complex numbers $p_n \to \infty$ such that $p_n \in P(f)$ (the post-singular set of f), $|P_{n+1}/p_n| \to 1$ and $\mathrm{dist}(p_n, \Omega) = o(|p_n|)$.*

THEOREM 7.4.4 (Bergweiler). *Let f be transcendental entire of finite order such that*

$$f(z) = z + a + o(1),$$

as $z \to \infty$, $|\arg z| \leq \eta$ for some $a, \eta > 0$. Then f has an invariant Baker domain Ω which contains

$$\{z = x + iy : |\arg z| \leq \eta, > x > R\},$$

for some $R > 0$ and $\Omega \cap S(f)$ is unbounded.

7.5. Examples

EXAMPLE 7.5.1. [**55**], *Example 1. $f(z) = z + 1 + e^{-z}$. If $z = x + iy$ we have*

$$\Re(f(z)) = x + 1 + \cos(y)e^{-x} \geq x + 1 - e^{-x} > x = \Re(z),$$

whenever $\Re(z) = x > 0$, so the right half=plane is contained in a Baker domain. Fatou showed this was the only Baker domain for this example.

EXAMPLE 7.5.2. *Baker* [**6**] *showed that*

$$f(z) = z + \frac{\sin(\sqrt{z})}{\sqrt{z}} + a$$

has a Baker domain for $a \in \mathbb{R}$ large enough and Fleischmann [**56**] *showed this is true for all $a > 0$. For large a, points of the real axis iterate to ∞ and one can show that there is a parabolic shaped region*

$$V = \{x + iy : y^2 < 4(x+1), x > a^2\}$$

that is mapped into itself; thus it must be contained in a Baker domain.

To see this, note that the square root map sends the parabolic region into a half-strip centered on the positive axis and $|\sin(z)/z|$ is bounded by some b/x on this half-strip. Hence

$$\Re(f(z)) \geq x + a - b/x,$$

and

$$|\Im(f(z))| \leq |y| + b/x$$

so

$$|\Im(f(z))|^2 \leq 2|y|^2 + 2b^2/x^2 \leq 4(x+1) + 3 \leq 4(\Re(z) + 1)$$

if $x^2 > 2b^2$ and $a - b/x > 1$.

Baker [**6**], *that there are no Baker domains if f has order $\frac{1}{2}$, minimal type, and this example shows this is very close to sharp. Here, the* **order** *of a function is*

$$\rho(f) = \limsup_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|}.$$

If $0 < \rho(f) = \rho < \infty$, we define the **type** *as*

$$\tau(f) = \limsup_{z\to\infty} \frac{\log|f(z)|}{|z|^\rho}.$$

The function is called **minimal type** *if $\tau(f) = 0$,* **maximal type** *if $\tau(f) = \infty$ and* **mean type** *otherwise. The function f above is order /12, mean type. Baker* [**6**] *noted there are examples with Baker domains of arbitrarily small positive types. Baker's conjecture asks if this sharp: if f is order $1/2$, minimal type, then every Fatou component is bounded.*

EXAMPLE Rippon and Stallard [**115**] showed

$$f(z) = az(1 + \exp(-z^p),$$

with $a > 1$, $p \in \mathbb{N}$ has p invariant Baker domains, one in each sector $\{z : |\arg(z) - 2\pi\frac{k}{p}| < \frac{\pi}{p}\}$ for $k = 0, \ldots, p-1$. Moreover, $g(z) = e^{2\pi i/p} f(z)$ has a p-cycle of Baker domains. See also [**100**] for a similar example by Morosawa.

7.6. Logarithmic examples

Let $\Pi(z) = \exp(az)$ for some $a \neq 0$. This is a holomorphic covering map from $\mathbb{C}* = \mathbb{C} \setminus \{0\}$ to $\mathbb{C}$, so if $g : \mathbb{C}* \to \mathbb{C}*$ is holomorphic, it can be lifted to an entire function f so that $\exp(af(z)) = g(\exp(az))$.

LEMMA 7.6.1 (Bergweiler [**15**]). $\mathcal{J}(f) = \Pi^{-1}(\mathcal{J}(g))$.

PROOF. $\square$

EXAMPLE Apply this to $g(z) = cz\exp(-z)$, $\Pi(z) = \exp(-z)$ to obtain $f(z) = z - \log c + e^{-z}$ (this contains Fatou's example when $c- = 1/e$. If $|c| < 1$ then g has an attracting fixed point at 0, so f has an invariant half-plane in the right half-plane where $f^n \to \infty$.

EXAMPLE

Baker and Dominguez [**9**] take $g(z) = z\exp(-z)$ and $\Pi(z) = \exp(-z)$ which gives

THEOREM 7.6.2 (Baker Dominguez 1999 [**10**]). *If U is an invariant Baker domain and f is not univalent, then ∞ is dense in ∂U (more precisely U is simply connected and there is a dense set of radial segments in the disk that the conformal map to U take to curves limiting on ∞).*

CHAPTER 8

Models for Eremenko-Lyubich functions

In this section we describe a method of constructing many examples in the Eremenko-Lyubich class $\mathcal{B}$ with particular geometrical and dynamical properties . The main idea is to reduce to constructing certain "model functions" that are only defined on certain unbounded subsets of the plane. There is a great deal of freedom in building the models, so it is often easier to make the iterations of a model function behave as desired then to do the same directly with an entire function. To pass from the model to the entire functions two steps are needed, both non-obvious. First, we must prove that every model function can be approximated in an appropriate sense by an Eremenko-Lyubich function. Second, we must prove that this approximations implies that the model and its approximating entire function have similar dynamics. Indeed, in many cases of interest, we will show that the functions are conjugate on their Julia sets; in these cases the two Julia sets are images of each other under a quasiconformal mapping of the plane. One particular example we shall work out in detail is the construction of a function in $\mathcal{B}$ where the strong version of Eremenko's conjecture fails: $I(f) \cup \{\infty\}$ is not path connected.

8.1. Model domains

Let $\Omega \subset \mathbb{C}$ be a union of disjoint, unbounded simply connected domains such that

(1) each Ω is bounded by a Jordan curve,
(2) $\overline{\Omega} \cap \overline{\mathbb{D}} = \emptyset$, and
(3) sequences of components of Ω accumulate only at ∞.

Such an Ω will be called a **model domain**.

If Ω is a model domain, suppose that τ on Ω is a holomorphic map $\tau : \Omega \to \mathbb{H}_r = \{\Re z > 0\}$, such that

(1) the restriction τ_j to a component Ω_j of Ω is a conformal map between Ω_j and $\mathbb{H}_r$,
(2) if $\tau(z) \to \infty$ then $z \to \infty$.

Then $f(z) = \exp(\tau(z))$ is a **model function**.

Given a model function, when is there an entire function in $\mathcal{B}$ or $\mathcal{S}$ with the same dynamics as the model? We have more flexibility building a model function (it doesn't have to be defined everywhere), so it may be easier to build counterexamples using models than using entire functions. If we knew that every model could be approximated by an entire function in an appropriate sense, then we could transfer the counterexample from the model setting to an entire function. This does work, but to state a theorem, we need to be more precise about the sense in which the entire function approximates the model.

One possibility is uniform approximation on the set $\overline{\Omega}$; here both the model and the entire function are defined. Moreover, Arakelian's theorem (Theorem **??**) is available to make such approximations and many interesting examples have been created in this way, e.g., by Eremenko and Lyubich in [**48**].

However, we choose to use a different form of approximation that is more appropriate for constructing functions in $\mathcal{B}$. We say that two entire functions are **quasiconformally equivalent** if there are quasiconformal maps φ, ψ of the plane so that

$$\psi(f(z)) = g(\varphi(z)). \tag{39}$$

If the maps ψ and φ are the same then we have

$$\psi(f(z)) = g(\psi(z)),$$

or

$$f = \psi^{-1} \circ g\psi,$$

which immediately implies

$$f^n = \psi^{-1} \circ g^n\psi.$$

In this case we say the functions are **quasiconformally conjugate** and we have $\mathcal{J}(g) = \psi(\mathcal{J}(f))$; conjugate the dynamical point of view. ne

However, functions that are merely equivalent but not conjugate can behave quite differently under iteration. For example if we take

$$g(z) = e^z, \quad f(z) = \frac{1}{10}e^z, \quad \varphi(z) = z - \log 10, \quad \psi(z) = z,$$

then we have $\psi \circ f = g \circ \varphi$, so f and g are quasiconformally equivalent (in fact, they are conformally equivalent in the sense that φ, ψ are both conformal). However $\mathcal{J}(g) = \mathbb{C}$ by Theorem 4.3.1, and $\mathcal{J}(f) \neq \mathbb{C}$ since f has an attracting fixed at the smaller solution of $10x = e^x$.

An even weaker version of quasiconformal equivalence is to require that (39) only hold on a subset of the plane. We say the functions are **quasiconformally equivalent near** ∞ if there are quasiconformal maps ψ, φ of the plane and a $0 < R < \infty$ so that this holds on the $\{z : |f(z)| > R\} \cup \{z : |g(z)| > R\}$. Note that we need to include both sets, since it possible that one of these sets could be equivalent to a subset of the other, e.g., one of these sets has one component and the other has two or more. We are assuming that the tract-system for each function can be mapped to the other a quasiconformal homeomorphism of the plane, and hence they have the same number of components and have the same "shape" in a rough sense.

The relation "quasiconformally equivalent near ∞" clearly makes sense for comparing an entire function g with a model function f. If $f \in \mathcal{B}$ and R is chosen so large that the singular set of f is inside $D(0, R0$, then the restriction of f to $\{z : |f(z)| \geq R\}$ is clearly a model function (i.e., "forgetting" the small part of f creates a model of f near ∞). Can we reverse this? Given a model of what f should look like near ∞, is there an entire function that, in fact, look like this? The answer is yes.

THEOREM 8.1.1 (All models occur). *Suppose (Ω, F) is a model and $0 < \rho \leq 1$. Then there is $f \in \mathcal{B}$ and a quasiconformal $\varphi : \mathbb{C} \to \mathbb{C}$ so that $F = f \circ \varphi$ on $\Omega(2\rho)$. In addition,*

(1) *$|f \circ \varphi| \leq e^{2\rho}$ off $\Omega(2\rho)$ and $|f \circ \varphi| \leq e^{\rho}$ off $\Omega(\rho)$. Thus the components of $\{z : |f(z)| > e^{\rho}\}$ are in 1-to-1 correspondence to the components of Ω via φ.*
(2) *$S(f) \subset D(0, e^{\rho})$.*
(3) *the quasiconstant of φ is $O(\rho^{-2})$ with a constant independent of F and Ω,*
(4) *φ^{-1} is conformal except on the set $\Omega(\frac{\rho}{2}, 2\rho)$.*

This result is somewhat surprising: the model function F can be defined on an open set Ω with many (even infinitely many) connected components, and the definition of F on each component involves scaling parameters that can be chosen independently on different components. Nevertheless, it is always possible to extend F into the gaps between its components (no matter how narrow the gaps) to create a quasiregular function that can then be made holomorphic by a quasiconformal change of variable. Moreover, we shall later state a stronger version of Theorem **??** that bounds the quasi-constant of the map ϕ in the equivalence, and shows that ϕ may be taken to be conformal on most of Ω.

We sketch the proof of Theorem 8.1.1 quickly here to give the basic idea. Let $W = \mathbb{C} \setminus \overline{\Omega(\rho)}$. It is simply connected, non-empty and not the whole plane, so there is a conformal map $\Psi : W \to \mathbb{D}$. Since Ψ maps ∂W to the unit circle, if we knew that $F = f|_\Omega$ for some entire function f, then $B = e^{-\rho} \cdot F \circ \Psi^{-1}$ would be an inner function on $\mathbb{D}$ (i.e., a holomorphic function on $\mathbb{D}$ so that $|B| = 1$ almost everywhere on the boundary).

The proof of Theorem 8.1.1 reverses this observation. Given the model and the corresponding domain W and conformal map Ψ we construct a Blaschke product B (a special type of inner function) on the disk so that $G = B \circ \Psi$ approximates $F = e^\tau$ on $\partial\Omega(\rho)$ (the precise nature of the approximation will be described later). This step is fairly straightforward using standard estimates of the Poisson kernel on the disk. We then "glue" G to F across ∂W to get a quasi-regular function g that agrees with F on $\Omega(2\rho)$ and agrees with G on W. This takes several (individually easy) steps to accomplish. We then use the measurable Riemann mapping theorem to define a quasiconformal mapping $\phi : \mathbb{C} \to \mathbb{C}$ so that $f = g \circ \phi$ is holomorphic on the whole plane. The only critical points of g correspond to critical points of B, and critical points introduced into $\Omega(\rho, 2\rho)$ by the gluing process. We will show that both types of critical values have absolute value $\leq e^\rho$. A different argument shows that any finite asymptotic value of f must correspond to a limit of B along a curve in $\mathbb{D}$, so all finite asymptotic values of f are also bounded by e^ρ. Thus $f \in \mathcal{B}$. Since g is only non-holomorphic in $\Omega(\rho, 2\rho)$, we will also get that ϕ^{-1} is conformal everywhere except in $\Omega(\rho, 2\rho)$.

8.2. Reduction of Theorem 8.1.1 to the case $\rho = 1$

We start the proof of Theorem 8.1.1 with the observation that it suffices to prove the result for $\rho = 1$.

To do this we define two quasiconformal maps, ψ_ρ and φ_ρ. Define

$$L(x) = \begin{cases} x, & 0 < x < \rho/2, \\ (\frac{2-\rho}{\rho})(x - \rho/2) + \rho/2 & \rho/2 \le x \le \rho, \\ x/\rho & \rho \le x \le 2\rho. \end{cases}$$

This is a piecewise linear map that sends $[\rho/2, \rho]$ to $[\rho/2, 1]$ and sends $[\rho, 2\rho]$ to $[1, 2]$. The slope on both intervals is less than $2/\rho$. For $z = x + iy \in \mathbb{H}_r$, define

$$\sigma_\rho(z) = \begin{cases} L(x) + iy & 0 < x \le 2\rho, \\ z + 2 - 2\rho & x > 2\rho. \end{cases}$$

This is quasiconformal $\mathbb{H}_r \to \mathbb{H}_r$ with quasiconstant $K \le 2/\rho$. Then set

$$\psi_\rho(z) = \begin{cases} z, & z \notin \Omega \\ \tau_j^{-1} \circ \sigma_\rho \circ \tau_j(z), & z \in \Omega_j. \end{cases}$$

Note that ψ_ρ is the identity near $\partial\Omega$, so ψ_ρ is quasiconformal on the whole plane by the Royden gluing lemma, e.g., Lemma 2 of [**21**], Lemma I.2 of [**44**] on page 303, or [**114**]. (Actually, since ψ_ρ is the identity off $\Omega(\rho/2)$ which has a smooth boundary, one can use a weaker version of the gluing lemma.)

Next, define

$$\varphi_\rho(z) = \begin{cases} z, & |z| < e^{\rho/2} \\ \exp(\sigma_\rho(\log(z))), & |z| \ge e^{\rho/2} \end{cases}.$$

Note that even though $\log(z)$ is multi-valued, the function σ_ρ does not change the imaginary part of its argument, so the exponential of $\sigma_\rho(\log(z))$ is well defined. This is clearly a quasiconformal map of the plane with quasiconstant $2/\rho$. Note also that these functions were chosen so that if $F = \exp \circ \tau$ is the model function associated to Ω and τ, then on Ω_j

$$\begin{aligned} F \circ \psi_\rho &= \exp \circ \tau_j \circ \tau_j^{-1} \circ \sigma_\rho \circ \tau_j \\ &= \exp \circ \sigma_\rho \circ \log \circ \exp \circ \tau_j \qquad (40) \\ &= \varphi_\rho \circ F. \end{aligned}$$

Now apply Theorem 8.1.1 to the model (Ω, F) with $\rho = 1$ to get a $f \in \mathcal{B}$ and a quasiconformal map $\Phi : \mathbb{C} \to \mathbb{C}$ so that $f \circ \Phi = F$ on $\Omega(2)$ and $S(f) \subset D(0, e^1)$. Let $g_\rho = \varphi_\rho^{-1} \circ f \circ \Phi \circ \psi_\rho$. This is an entire function pre and post-composed with quasiconformal maps of the plane, so it is quasiregular. By the measurable Riemann mapping theorem, there is a quasiconformal $\Phi_\rho : \mathbb{C} \to \mathbb{C}$ so that $f_\rho = g_\rho \circ \Phi_\rho^{-1}$ is entire and clearly

$$S(f_\rho) = S(g_\rho) \subset \varphi_\rho^{-1}(S(f)) \subset \varphi_\rho^{-1}(D(0, e)) = D(0, e^\rho).$$

For $z \in \Omega(2\rho)$, $\psi_\rho(z) \in \Omega(2)$, so using this and (40)

$$\begin{aligned} f_\rho \circ \Phi_\rho(z) &= g_\rho(z) \\ &= \varphi_\rho^{-1}(f(\Phi((\psi_\rho(z)))) \\ &= \varphi_\rho^{-1}(F(\psi_\rho(z))) \\ &= F(z). \end{aligned}$$

Similarly, $|f_\rho \circ \Phi_\rho| = |g_\rho|$ is bounded by $e^{2\rho}$ off $\Omega(2\rho)$. The quasiconstant of Φ_ρ is, at worst, the product of the constants for Φ, ψ_ρ and φ_ρ, which is $K_1 \cdot 4\rho^{-2}$, where K_1 is the upper bound for the quasiconstant in Theorem 8.1.1 in the case $\rho = 1$.

Finally, our construction in the next section will show that Φ is conformal except on $\Omega(1, 2)$ and that F has a quasiregular extension to the plane that is holomorphic except on $\Omega(1, 2)$ and is bounded by e off $\Omega(1)$ and by e^2 off $\Omega(2)$. This implies that g_ρ is holomorphic except on $\Omega(\rho/2, 2\rho)$ (since ψ_ρ is holomorphic off $\Omega(\rho/2, 2\rho)$ and φ_ρ^{-1} is holomorphic off $\{e^{\rho/2} < |z| < e^2\}$.) This, in turn, implies that Φ_ρ is conformal except on $\Omega(\rho/2, 2\rho)$, as desired. Thus f_ρ satisfies Theorem 8.1.1 for the model (Ω, F) and the given $\rho > 0$.

8.3. The proof of Theorem 8.1.1

In this section we give the proof of Theorem 8.1.1 for $\rho = 1$, stating certain facts as lemmas to be proven in later sections.

Let $W = \mathbb{C} \setminus \overline{\Omega(1)}$. This is an open, connected, simply connected domain that is bounded by analytic arcs $\{\gamma_j\}$ that are each unbounded in both directions. See Figure 1. The same comments hold for the larger domain $W_2 = \mathbb{C} \setminus \overline{\Omega(2)}$.

Let $L_1 = \{x + iy : x = 1\}$ and $L_2 = \{x + iy : x = 2\}$. The vertical strip between these two lines will be denoted S. Note that L_1 is partitioned into intervals

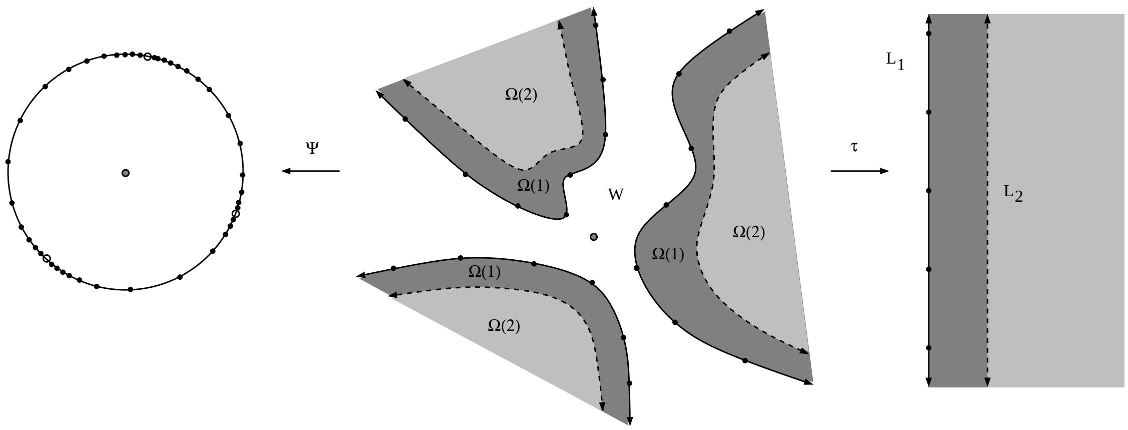


FIGURE 1. W is the complement of $\Omega(1)$; it is simply connected and bounded by smooth curves. We are given the holomorphic function $F = e^\tau$ on $\Omega(2)$ and we will define a holomorphic function on W using the Riemann map Ψ of W to the unit disk, and a specially chosen infinite Blaschke product B on the disk. We will then interpolate these functions in $\Omega(2) \setminus \Omega(1)$ by a quasiregular function. Each component of this set is mapped to a vertical strip by τ, and it is in these strips that we construct the interpolating functions. Note that the integer partition on the boundary of the half-plane pulls back under τ to a partition of each component of $\partial\Omega(1)$, and that Ψ maps these to a partition of the unit circle (minus the singular set of Ψ). The Blaschke product B will be constructed so that $B^{-1}(1)$ approximates this partition of the circle.

of length 2π by the points $1+2\pi i\mathbb{Z}$. This partition of L_1 will be denoted $\mathcal{J}$. Note that $\tau_j(\gamma_j) = L_1$, so each curve γ_j is partitioned by the image of $\mathcal{J}$ under τ_j^{-1}. We denote this partition of γ_j by $\mathcal{J}_j$. Because elements of $\mathcal{J}_j$ are all images of a fixed interval $J \in L_1 \subset \mathbb{H}_r$ under some conformal map of $\mathbb{H}_r$, the distortion theorem (e.g., Theorem I.4.5 of [**59**]) implies they all lie in a compact family of smooth arcs and that adjacent elements of $\mathcal{J}_j$ have comparable lengths with a uniform constant, independent of j, Ω and F.

Let $\Psi : W \to \mathbb{D}$ be a conformal map given by the Riemann mapping theorem. We claim that Ψ can be analytically continued from W to W_2 across γ_j. Let R_1 denote reflection across L_1 and for $z \in \Omega_j \cap W = \tau_j^{-1}(\{x+iy : 0 < x < 1\})$ let $T = \tau_j^{-1} \circ R_1 \circ \tau_j$; this defines an anti-holomorphic 1-to-1 map from $\Omega_j(0,1)$ to

$\Omega_j(1,2)$ that fixes each point of γ_j. We can then extend Ψ by the formula

$$\Psi(T(z)) = 1/\overline{\Psi(z)},$$

(where the right hand side denotes reflection of $\Psi(z)$ across the unit circle). The Schwarz reflection principle says this is an analytic continuation of Ψ to W_2.

Thus Ψ is a smooth map of each γ_j onto an arc I_j of the unit circle $\mathbb{T} = \partial\mathbb{D} = \{|z| = 1\}$. The complement of these arcs is a closed set $E \subset \mathbb{T}$. It is a standard fact of conformal mappings that since E is the set where a conformal map fails to have a finite limit, it has zero Lebesgue, indeed, zero logarithmic capacity. We will not need this fact, although we will use the easier fact that E can't contain an interval (i.e., a conformal map can't have infinite limits on an interval).

The partition $\mathcal{J}_j$ of γ_j transfers, via Ψ to a partition of $I_j \subset \mathbb{T}$ into infinitely many intervals $\{J_k^j\}, k \in \mathbb{Z}$. We will let $\mathcal{K} = \cup_{j,k} J_k^j$ denote the collection of all intervals that occur this way. Thus $\mathbb{T} = E \bigcup \cup_{K \in \mathcal{K}} K$.

Because Ψ conformally extends from W to W_2, $|\Psi'|$ has comparable minimum and maximum on each partition element of γ_j (with uniform constants). Thus the corresponding intervals $\{J_k^j\}$ have the property that adjacent intervals have comparable lengths (again with a uniform bound).

The hyperbolic distance between two points $z_1, z_2 \in \mathbb{D}$ is defined as

$$\rho(z_1, z_2) = \inf_\gamma \int_\gamma \frac{|dz|}{1 - |z|^2}.$$

See Chapter 1 of [**59**] for the basic properties of the hyperbolic metric. Here we will mostly need the facts that it is invariant under Möbius self-maps of the disk, that hyperbolic geodesics are circular arcs in $\mathbb{D}$ that are perpendicular to $\mathbb{T}$, and that points hyperbolic distance r from 0 are Euclidean distance

$$\frac{2}{\exp(2r) + 1} = O(\exp(-r)),$$

from the unit circle.

For any proper sub-interval $I \subset \mathbb{T}$, let γ_I be the hyperbolic geodesic with the same endpoints as I and let a_I be the point on γ_i that is closest to the origin (closest in either the Euclidean or hyperbolic metrics; it is the same point).

Since $\mathcal{K}$ are disjoint intervals on the circle,

$$\sum_{K\in\mathcal{K}}(1-|a_K|)<\infty,$$

and so

$$B(z)=\prod_{\mathcal{K}}\frac{|a_K|}{a_K}\frac{a_K-z}{1-\overline{a_K}z},$$

defines a convergent Blaschke product (see Theorem II.2.2 of [**58**]). Thus B is a bounded, non-constant, holomorphic function on $\mathbb{D}$ that vanishes exactly on the set $\{a_n\}$. Also, $|B|$ has radial limits 1 almost everywhere. Moreover, B extends meromorphically to $\mathbb{C}\setminus E$, where E is the accumulation set of its zeros on $\mathbb{T}$; this is the same set E as defined above using the map Ψ (the zeros accumulate at both endpoints of every component of $\mathbb{T}\setminus E$, and since these points are dense in E, the accumulation set of the zeros is the whole singular set E). The poles of the extension are precisely the points in the exterior of the unit disk that are the reflections across $\mathbb{T}$ of the zeros.

Any subset $\mathcal{M}$ of $\mathcal{K}$ also defines a convergent Blaschke product. Fix such a subset. The corresponding Blaschke product $B_{\mathcal{M}}$ induces a partition of each I_j with endpoints given by the set $\{e^{i\theta}:B_{\mathcal{M}}(e^{i\theta})=1\}$ and this induces a partition $\mathcal{H}_j$ of each γ_j via the map Ψ. This in turn, induces a partition $\mathcal{L}_j$ of L_1 via τ_j.

We would like to say that the partitions $\mathcal{L}_j$ and $\mathcal{J}$ are "almost the same". The first step to making this precise is a lemma that we will prove in Section 8.4:

LEMMA 8.3.1. *There is a subset $\mathcal{M}\subset\mathcal{K}$ so that if B is the Blaschke product corresponding to $\mathcal{M}$ and $\mathcal{L}_j$ is the partition of L_1 corresponding to B via $\tau_j\circ\Psi^{-1}$, then each element of $\mathcal{J}$ hits at least 2 elements of $\mathcal{L}_j$ and at most M elements of $\mathcal{L}_j$, where M is uniform. In particular, no element of $\mathcal{J}$ can hit both endpoints of any element of $\mathcal{L}_j$ (elements of each partition are considered as closed intervals).*

In Section 8.5 we will prove

LEMMA 8.3.2. *Suppose $K=[1+ia,1+ib]\in\mathcal{L}_j$ and define*

$$\alpha(1+iy)=\frac{1}{2\pi}\arg(B\circ\Psi\circ\tau_j^{-1}(1+iy)),$$

where we choose a branch of α so $\alpha(1+ia)=0$ (recall that $B(\Psi(\tau_j^{-1}(1+ia)))=1\in\mathbb{R}$). Set

$$\psi_1(z)=1+i(a(1-\alpha(z))+b\alpha(z))=1+i(a+(b-a)\alpha(z)).$$

Then ψ_1 is a homeomorphism from K to itself so that $\alpha \circ \psi_1^{-1} : K \to [0,1]$ is linear and ψ_1 can be extended to a quasiconformal homeomorphism of $R = K \times [1,2]$ to itself that is the identity on the $\partial R \setminus K$ (i.e., it fixes points on the top, bottom and right side of R).

The main point of the proof is to show that $\arg(B \circ \Psi \circ \tau_j^{-1}) : K \to [0, 2\pi]$ is biLipschitz with uniform bounds.

Roughly, Lemma 8.3.1 says there are more elements of $\mathcal{J}$ than there are of $\mathcal{L}_j$. This is made a little more precise by the following:

LEMMA 8.3.3. *There is a* 1*-to-*1*, order preserving map of $\mathcal{L}_j$ into (but not necessarily onto) $\mathcal{J}$ so that each interval $K \in \mathcal{L}_j$ is sent to an interval J with* $\mathrm{dist}(K, J) \leq 2\pi$*. Moreover, adjacent elements of $\mathcal{L}_j$ map to elements of $\mathcal{J}$ that are either adjacent or are separated by an even number of elements of $\mathcal{J}$.*

This will be proven in Section 8.6. Again, the proof is quite elementary.

Partition $\mathcal{J} = \mathcal{J}_1^j \cup \mathcal{J}_2^j$ according to whether the interval is associated to some element of $\mathcal{L}_j$ by Lemma 8.3.3 (i.e., $\mathcal{J}_1^j$ is the image of $\mathcal{L}_j$ under the map in the lemma). The maximal chains of adjacent elements of $\mathcal{J}_2^j$ will be called blocks. By the lemma, each block has an even number of elements. We will say that the block associated to an element $J \in \mathcal{J}_1^j$ is the block immediately above J.

Thus each interval K in $\mathcal{L}_j$ is associated to an interval J' that consists of the corresponding J given by Lemma 8.3.3 and its associated block. K and J' have comparable lengths and are close to each other, so the orientation preserving linear map from J' to K defines a piecewise linear map $\tilde{\psi}_2 : \mathbb{R} \to \mathbb{R}$ that is biLipschitz with a uniform constant. Using linear interpolation we can extend this to a biLipschitz map ψ_2 of the strip $S = \{x + iy : 1 < x < 2\}$ to itself that equals $\tilde{\psi}_2$ on L_1 (the left boundary) and is the identity on L_2 (the right side).

Each element $J \in \mathcal{J}_2^j$ is paired with a distinct element $J^* \in \mathcal{J}_2^j$ that belongs to the same block. The two outer-most elements of the block are paired, as are the pair adjacent to these, and so on. Similarly, each point z is paired with the other point z^* in the block that has the same distance to the boundary (the center of the block is an endpoint of $\mathcal{J}$ and is paired with itself).

For each $K \in \mathcal{L}_j$, let J_K be the corresponding element of $\mathcal{J}_1^j$ and let I_K be the union of J_K and its corresponding block. Let $R_K = [1, 2] \times I_K$. Let $U_K = R_K \setminus X_K$, where X_K is the closed segment connecting the upper left corner of R_K to the center of R_K. See Figure 2.

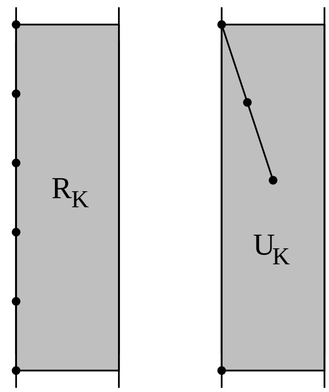


FIGURE 2. Definition of U_K

LEMMA 8.3.4 (Simple folding). *There is a quasiconformal map $\psi_3 : U_K \to R_K$ so that (ψ_3 depends on j and on K, but we drop these parameters from the notation)*

(1) *ψ_3 is the identity on $\partial R_K \setminus L_1$ (i.e., it is the identity on the the top, bottom and right side of R_K),*
(2) *ψ_3^{-1} extends continuously to the boundary and is linear on each element of $\mathcal{J}$ lying in I_K,*
(3) *ψ_3 maps I_K (linearly) to J_K,*
(4) *for each $z \in I_K$, $\psi_3^{-1}(z) = \psi_3^{-1}(z^*) \in X_k$ (i.e., ψ_3 maps opposite sides of X_k to paired points in I_k),*
(5) *the quasiconstant of ψ_3 depends only on $|I_K|/|J_K|$, i.e., on the number of elements in the block associated to K. It is independent of the original model and of the choice of j and K.*

We call this "simple folding" because it is a simple analog of a more complicated folding procedure given in [**31**]. In the lemma above, the image domain is a rectangle with a slit removed and the quasiconstant of ψ_3 is allowed to grow with n, the number of block elements. This growth is not important in this paper because here we only apply the folding construction in cases where this number n is uniformly bounded (this will occur in our application because of Lemma 8.3.1). In [**31**], the corresponding values may be arbitrarily large but the folding construction there must give a map with uniformly bounded quasiconstant regardless. The construction in [**31**] removes a collection of finite trees from R_k and does so in a way that keeps the quasiconstant of ψ_3 bounded independent of n (there are also complications involving how the construction on adjacent rectangles are merged).

We want to treat the boundary intervals in $\mathcal{J}_1$ and $\mathcal{J}_2$ slightly differently. The precise mechanism for doing this is:

LEMMA 8.3.5 (exp-cosh interpolation). *There is a quasiregular map $\sigma_j : S \to D(0, e^2)$ so that*

$$\sigma_j(z) = \begin{cases} \exp(z), & z \in J \in \mathcal{J}_1^j, \\ e \cdot \cosh(z-1), & z \in J \in \mathcal{J}_2^j, \\ \exp(z), & z \in \mathbb{H}_r + 2. \end{cases}$$

The quasiconstant of ϕ_j is uniformly bounded, independent of all our choices.

This lemma will be proven in Section 8.8 and is completely elementary.

We now have all the individual pieces needed to construct the interpolation g_j between e^z on L_2 and $B \circ \Psi \circ \tau_j^{-1}$ on L_1. Let U_j be S minus all the segments X_K where $K \in \mathcal{L}_j$ as in Lemma 8.3.4. Define a quasiconformal map $\psi : U_j \to S$ by

$$\psi = \psi_1 \circ \psi_2 \circ \psi_3,$$

and let $g_j = \sigma_j \circ \psi$ map U_j into $D(0, e^2)$. By definition, each ψ_i, $i = 1, 2, 3$ is the identity on L_2, so $g_j(z) = e^z$ on L_2. For any $K \in \mathcal{L}_j$, the map ψ sends the boundary segments of ∂U_K that lie on some X_K linearly onto elements of $\mathcal{J}_2^j$, so boundary points on opposite sides of X_K get mapped to points that are equidistant from $2\pi i\mathbb{Z}$ and cosh agrees at any two such points. Thus g_j extends continuously across each slit X_K. Finally, the map ψ was designed so that g_j is continuous on S and agrees with $B \circ \Psi \circ \tau_j^{-1}$ on L_1. Thus $g_j \circ \tau_j$ continuously interpolates between $B \circ \Psi$ on W and F on $\Omega(2)$ and so defines a quasiregular g on the whole plane with a uniformly bounded constant. Thus by the measurable Riemann mapping theorem there is a quasiconformal $\varphi : \mathbb{C} \to \mathbb{C}$ so that $f = g \circ \varphi$ is entire.

The singular values of f are the same as for g. On $\Omega(2)$, $g = F = e^\tau$, so g has no critical points in this region. In U_j, $g = g_j$ is locally 1-to-1, so has no critical points there either. Thus the only critical points of g in $\Omega(1)$ are on the slits X_K, then these are mapped by g onto the circle of radius e around the origin. Thus every critical value of g (and hence f) must lie in $D(0,e)$.

If g has a finite asymptotic value outside $\overline{D(0,e)}$, then it must be the limit of g along some curve Γ contained in a single component of Ω. Then e^z has a finite limit along $\tau(\Gamma) \subset \mathbb{H}_r$; this is impossible, so f has no finite asymptotic values outside $\overline{D(0,e)}$. Thus $S(f) \subset \overline{D(0,e)}$, and so $f \in \mathcal{B}$.

This proves Theorem 8.1.1 except for the proof of the lemmas.

8.4. Blaschke partitions

In this section we prove Lemma 8.3.1. We start by recalling some basic properties of the Poisson kernel and harmonic measure in the unit disk $\mathbb{D}$.

The Poisson kernel on the unit circle with respect to the point $a \in \mathbb{D}$ is given by the formula

$$P_a(\theta) = \frac{1-|a|^2}{|e^{i\theta}-a^2|} = \frac{1-|a|^2}{1-2|a|\cos(\theta-\phi)+|a|^2},$$

where $a = |a|e^{i\phi}$. This is the same as $|\sigma'|$ where σ is any Möbius transformation of the disk to itself that sends a to zero. If $E \subset \mathbb{T}$, we write

$$\omega(E,a,\mathbb{D}) = \frac{1}{2\pi}\int_E P_a(e^{i\theta})d\theta,$$

and call this the harmonic measure of E with respect to a. This is the same as the (normalized) Lebesgue measure of $\sigma(E) \subset \mathbb{T}$ where $\sigma : \mathbb{D} \to \mathbb{D}$ is any Möbius transformation sending a to 0. It is also the same as the first hitting distribution on $\mathbb{T}$ of a Brownian motion started at a (although we will not use this characterization).

Suppose $I \subset \mathbb{T}$ is any proper arc, and, as before, let γ_I be the hyperbolic geodesic in $\mathbb{D}$ with the same endpoints as I; then γ_I is a circular arc in $\mathbb{D}$ that is perpendicular to $\mathbb{T}$ at its endpoints. Let a_I denote the point of γ_I that is closest to the origin.

Lemma 8.4.1. $\omega(I,a_I,\mathbb{D}) = \frac{1}{2}$.

Proof. Apply a Möbius transformation of $\mathbb{D}$ that sends a_I to the origin. Then γ_I must map to a diameter of the disk and I maps to a semi-circle. $\square$

Given two disjoint arcs I, J in $\mathbb{T}$, let γ_I, γ_J be the two corresponding hyperbolic geodesics and let a_I^J be the point on γ_I that is closest to J and let a_J^I be the point on γ_J that is closest to I.

LEMMA 8.4.2. $\omega(I, a_J^I, \mathbb{D}) = \omega(J, a_I^J, \mathbb{D})$

PROOF. Everything is invariant under Möbius maps of the unit disk to itself, so use such a map to send I, J to antipodal arcs. Then the conclusion is obvious. $\square$

LEMMA 8.4.3. *If $z, w \in \mathbb{D}$ and $I \subset \mathbb{T}$, then*

$$\frac{\omega(I, z, \mathbb{D})}{\omega(I, w, \mathbb{D})} \leq C$$

where the constant C depends only on the hyperbolic distance between z and w.

PROOF. Suppose $\sigma(z) = (z-w)/(1-\overline{w}z)$ maps w to 0. Then $u(z) = \omega(I, \sigma(z), \mathbb{D})$ is a positive harmonic function on $\mathbb{D}$, so the lemma is just Harnack's inequality applied to u. $\square$

Suppose $I, J, \subset \mathbb{T}$ are disjoint closed arcs and $\operatorname{dist}(I, J) \geq \epsilon \max(|I|, |J|)$. Then we call I and J ϵ-separated. This implies the hyperbolic geodesics γ_I, γ_J are separated in the hyperbolic metric (with a lower bounded depending only on ϵ), but the converse is not true.

LEMMA 8.4.4. *If $I, J \subset \mathbb{T}$ are ϵ-separated, then the hyperbolic distance between a_I and a_I^J is bounded, depending only on ϵ.*

PROOF. Assume I is the longer arc and consider hyperbolic geodesic S that connects a_I^J and a_J^I. Then S is perpendicular to γ_I at a_I^J, so if $1 - |a_I^J| \ll 1 - |a_I|$, S will hit the unit circle without hitting γ_j. See Figure 3. $\square$

LEMMA 8.4.5. *Suppose that I, J are ϵ-separated. Then*

$$\omega(I, a_J, \mathbb{D}) \simeq \omega(J, a_I, \mathbb{D}),$$

where the constant depends only on ϵ.

PROOF. This follows immediately from our earlier results. $\square$

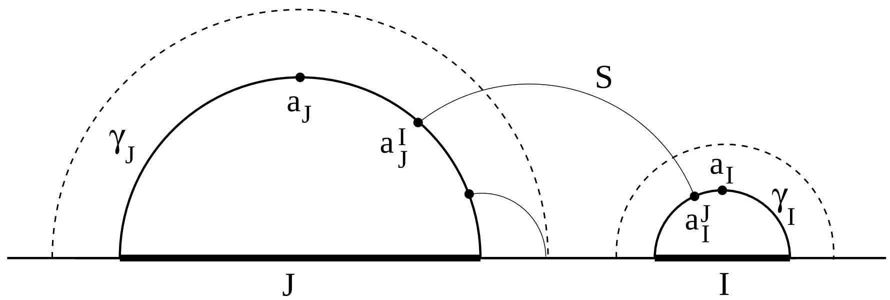


FIGURE 3. If the intervals I and J are ϵ-separated, then a shortest path between γ_I and γ_J must hit each geodesic near the "top" points. A perpendicular geodesic that starts too "low" on γ_J will hit the unit circle without hitting γ_I.

LEMMA 8.4.6. *Suppose that I and J are ϵ-separated and that a_J, a_I are at least distance R apart in the hyperbolic metric. Then*

$$\omega(J, a_I, \mathbb{D}) \leq C(\epsilon)e^{-R}.$$

PROOF. Since the intervals are ϵ-separated, the hyperbolic distance between a_I and a_J is the same as the distance between a_I^J and a_J^I, up to a bounded additive factor. Thus if we apply a Möbius transformation of $\mathbb{D}$ so that $a_J = 0$, a_I is mapped to a point w with $1 - |w| = O(e^{-R})$, which implies $\omega(I, a_J, \mathbb{D}) = O(e^{-R})$. Since the intervals are ϵ-separated, the reverse inequality also holds by Lemma 8.4.5. $\square$

Fix $M < \infty$ and suppose $\mathcal{K}$ is a collection of disjoint (except possibly for endpoints) closed intervals on $\mathbb{T}$ so that any two adjacent intervals have length ratio at most M. We say that two intervals I, J are S steps apart if there is a chain of $S+1$ adjacent intervals $J_0, \dots, J_S$ so that $I = J_0$ and $J = J_S$.

Note that if $I, J \in \mathcal{K}$ are adjacent, then a_I, a_J are at bounded hyperbolic distance T apart (and T depends only on M). Also, if $I, J \in \mathcal{K}$ are not adjacent, then they are ϵ-separated for some $\epsilon > 0$ that depends only on M.

LEMMA 8.4.7. *For any $R > 0$ there is a collection $\mathcal{N} \subset \mathcal{K}$ so that*

(1) *for any $I \in \mathcal{K}$, there is a $J \in \mathcal{N}$ with $\rho(a_J, a_I) \leq R$*
(2) *for any $I, J \in \mathcal{N}$, $\rho(a_J, a_I) \geq R$.*

PROOF. Just let $\mathcal{N}$ correspond to a maximal collection of the points $\{a_K\}$ with the property that any two of them are hyperbolic distance $\geq R$ apart. $\square$

Fix a positive integer S. For each $J \in \mathcal{N}$ choose the shortest element of $\mathcal{K}$ that is at most S steps away from J. Let $\mathcal{M} \subset \mathcal{K}$ be the corresponding collection of intervals.

LEMMA 8.4.8. *Suppose R, S, T are as above and $R \geq 4ST$. If $\mathcal{K}$ and $\mathcal{M}$ are as above, then for all $K \in \mathcal{K}$,*

$$\epsilon \leq \sum_{J \in \mathcal{M}} \omega(K, a_J, \mathbb{D}) \leq \mu,$$

where $\epsilon > 0$ depends only on R and $\mu \to 1/2$ as $S \to \infty$.

PROOF. The left-hand inequality is easier and we do it first. Fix $K \in \mathcal{K}$. There is a $I \in \mathcal{N}$ with $\rho(a_I, a_K) \leq R$, and since adjacent elements of $\mathcal{K}$ have points that are only T apart in the hyperbolic metric, there is an element $J \in \mathcal{M}$ with $\rho(a_K, a_J) \leq R + ST \leq \frac{5}{4}R$. This implies $|J| \simeq |K| \simeq \text{dist}(J, K)$ and these imply $\omega(K, a_J, \mathbb{D}) \geq \epsilon$ with ϵ depending only on ρ. Thus every element of $\mathcal{K}$ has harmonic measure bounded below with respect to some point corresponding to a single element of $\mathcal{M}$ and hence the sum of harmonic measures over all elements of $\mathcal{M}$ is also bounded away from zero uniformly.

Now we prove the right-hand inequality. By our choice of R, points a_J corresponding to distinct intervals in $\mathcal{M}$ are at least distance $R/2$ apart. Fix $K \in \mathcal{K}$. There is at most one point within hyperbolic distance $R/4$ of a_K and the harmonic measure it assigns K is at most $1/2$ since the point lies on or outside the geodesic γ_K.

All other points associated to elements of $\mathcal{M}$ are Euclidean distance $\geq \exp(R/8)|K|$ away from K or are within this distance of K, and are within Euclidean distance $\exp(-R/8)|K|$ of the unit circle (this is because of the Euclidean geometry of hyperbolic balls in the half-space). We call these two disjoint sets $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

Using Lemma 8.4.5 we see that the

$$\sum_{J \in \mathcal{M}_1} \omega(K, a_J, \mathbb{D}) = O(\sum_{J \in \mathcal{M}_1} \omega(J, a_K, \mathbb{D})) = O(\exp(-R/8)).$$

To bound the sum over $\mathcal{M}_2$, we note that each interval in $\mathcal{M}_2$, is the endpoint of a chain of S adjacent intervals that are each at least as long as J. Since

$$|J| \leq \exp(-R/8)|K|,$$

and

$$\mathrm{dist}(J, K) \gtrsim |K|,$$

we can deduce

$$\omega(J, a_K, \mathbb{D}) \leq O(\frac{1}{S})\omega(a_K, J, \mathbb{D}),$$

so since the J's are all disjoint intervals,

$$\sum_{J\in\mathcal{M}_2} \omega(K, a_J, \mathbb{D}) = O(\frac{1}{S} \sum_{J\in\mathcal{M}_2} \omega(J, a_K, \mathbb{D})) = O(\frac{1}{S}).$$

Choosing first S large, and then R large (depending on S and separation constant of $\mathcal{K}$), both sums are as small as we wish, which proves the lemma. □

COROLLARY 8.4.9. *Suppose B is as above and $K \in \mathcal{K}$. Then*

$$\epsilon \leq \frac{1}{|K|}\frac{\partial B}{\partial \theta} \leq C.$$

PROOF. If I, J are ϵ-separated, then it is easy to verify that

$$\sup_{z\in J} P_{a_I}(z), \qquad \inf_{z\in J} P_{a_I}(z),$$

are comparable up to a bounded multiplicative factor that depends only on ϵ. The lemma then follows from our earlier estimates. □

We have now essentially proven Lemma 8.3.1; it just remains to reinterpret the terminology a little. For the reader's convenience we restate the lemma.

LEMMA 8.4.10 (The Blaschke partition). *There is a subset $\mathcal{M} \subset \mathcal{K}$ so that if B is the Blaschke product corresponding to $\mathcal{M}$ and $\mathcal{L}_j$ is the partition of L_1 corresponding to B via $\tau_j \circ \Psi^{-1}$, then each element of $\mathcal{J}$ hits at least 2 elements of $\mathcal{L}_j$ and at most M elements of $\mathcal{L}_j$, where M is uniform. In particular, no element of $\mathcal{J}$ can hit both endpoints of any element of $\mathcal{L}_j$ (elements of each partition are considered as closed intervals).*

PROOF. A computation shows that for the Blaschke product

$$B(z) = \prod_n \frac{|a_n|}{a_n}\frac{z - a_n}{1 - \bar{a}_n z},$$

the derivative satisfies

$$|\frac{\partial B}{\partial \theta}(e^{i\theta})| = \sum_n P_{a_n}(e^{i\theta}),$$

and the convergence is absolute and uniform on any compact set K disjoint from the singular set E of B (since B is a product of Möbius transformations, and the derivative of a Möbius transformation is a Poisson kernel, this formula is simply the limit of the n-term product formula for derivatives).

Lemma 8.4.8 now says we can choose $\mathcal{M}$ so that

$$2\pi\epsilon \leq \int_J |\frac{\partial}{\partial\theta} B| d\theta \leq \frac{3}{4}\cdot 2\pi = \frac{3\pi}{2}.$$

Since the integral over an element of $\mathcal{L}$ has integral exactly 2π, the lower bound means that an element of $\mathcal{L}$ can contain at most $1/\epsilon$ elements of $\mathcal{J}$ and hence can intersect at most $2+\frac{1}{\epsilon}$ elements of $\mathcal{J}$. The upper bound says that each element K of $\mathcal{L}$ must hit at least 2 elements of $\mathcal{J}$. Hence it is not contained in any single element of $\mathcal{J}$, and so no single element of $\mathcal{J}$ can hit both endpoints of K. $\square$

8.5. Straightening a biLipschitz map

LEMMA 8.5.1. *Suppose $K = [1+ia, 1+ib] \in \mathcal{L}_j$ and define*

$$\alpha(1+iy) = \frac{1}{2\pi}\arg(B\circ\Psi\circ\tau_j^{-1}(1+iy)),$$

where we choose a branch of α so $\alpha(1+ia)=0$ (recall that $B(\Psi(\tau_j^{-1}(1+ia))) = 1 \in \mathbb{R}$). Set

$$\psi_1(z) = 1 + i(a(1-\alpha(z)) + b\alpha(z)) = 1 + i(a + (b-a)\alpha(z)).$$

Then ψ_1 is a homeomorphism from K to itself so that $\alpha\circ\psi_1^{-1} : K \to [0,1]$ is linear and ψ_1 can be extended to a quasiconformal homeomorphism of $R = K\times[1,2]$ to itself that is the identity on the $\partial R \setminus K$ (i.e., it fixes points on the top, bottom and right side of R).

PROOF. The linearizing property of ψ_1 is clear from its definition, so we need only verify the quasiconformal extensions property.

Corollary 8.4.9 implies α' is bounded above and below by absolute constants. Let $R = K\times[1,2]$ and define an extension of ψ_1 by

$$\psi_1(x+iy) = u(x,y) + iv(x,y) = x + i[(2-x)\psi_1(1+iy) + (x-1)y)].$$

i.e., take the linear interpolation between ψ_1 on L_1 and the identity on L_2. We can easily compute

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y - \psi(y) & (2-x)(b-a)\alpha'(y) + (x-1) \end{pmatrix}.$$

Note that $|y - h(y)| \leq |K|$ is absolutely bounded. Also, since $|b-a||\alpha'|$ is bounded above and away from 0, so is v_y. Thus the derivative matrix lies in a compact subset of the invertible 2×2 matrices and hence ψ_1 is quasiconformal (with only a little more work we could compute an explicit bound for the quasiconstant, and even prove that the extension is actually biLipschitz). □

8.6. Aligning partitions

Now we prove Lemma 8.3.3, which we restate for convenience.

LEMMA 8.6.1. *There is a* 1*-to-*1*, order preserving map of $\mathcal{L}_j$ into (but not necessarily onto) $\mathcal{J}$ so that each interval $K \in \mathcal{L}_j$ is sent to an interval J with* $\mathrm{dist}(K, J) \leq$ 2π*. Moreover, adjacent elements of $\mathcal{L}_j$ map to elements of $\mathcal{J}$ that are either adjacent or are separated by an even number of elements of $\mathcal{J}$.*

PROOF. For each $K \in \mathcal{K}$ choose $J \in \mathcal{J}$ so that J contains the lower endpoint of K (if two such intervals contain the endpoint, choose the upper one). No interval J is chosen twice, since Lemma 8.3.1 says that no J can hit both endpoints of any element of $\mathcal{L}$.

Fix an order preserving labeling of the chosen $\mathcal{J}$ by $\mathbb{Z}$ and denote it $\{J_n\}$. By the gap between J_n and J_{n+1} we mean the number of unselected elements of $\mathcal{J}$ that separate these two intervals. The position of J_0 is fixed. If the gap between J_0 and J_1 is even (including no gap), we leave J_1 where it is. If the gap is odd, there is a least one separating interval and we replace J_1 by the adjacent interval in $\mathcal{J}$ that is closer to J_0. If the gap between (the new) J_1 and J_2 is even, we leave J_2 alone; otherwise, we move it one interval closer to J_0. Continuing in this way, we can guarantee that for all $n \geq 0$, gaps are even and each J_n is either in its original position or adjacent to its original position. Thus its distance to the associated element of $\mathcal{K}$ is at most 2π. The argument for negative indices is identical. □

8.7. Foldings

Now we prove Lemma 8.3.4. This is the step that makes the gluing procedure a little different from a standard quasiconformal surgery.

LEMMA 8.7.1 (Simple folding). *There is a quasiconformal map $\psi_3 : U_K \to R_K$ so that (ψ_3 depends on j and on K, but we drop these parameters from the notation)*

(1) *ψ_3 is the identity on $\partial R_K \setminus L_1$ (i.e., it is the identity on the the top, bottom and right side of R_K),*
(2) *ψ_3^{-1} extends continuously to the boundary and is linear on each element of $\mathcal{J}$ lying in I_K,*
(3) *ψ_3 maps I_K (linearly) to J_K,*
(4) *for each $z \in I_K$, $\psi_3^{-1}(z) = \psi_3^{-1}(z^*) \in X_k$ (i.e., ψ_3 maps opposite sides of X_k to paired points in I_k),*
(5) *the quasiconstant of ψ_3 depends only on $|I_K|/|J_K|$, i.e., on the number of elements in the block associated to K. It is independent of the original model and of the choice of j and K.*

PROOF. The proof is a picture, namely Figure 4. The map is defined by giving compatible finite triangulations of R_k and U_k (compatible means that there is 1-to-1 map between vertices of the triangulations that preserves adjacencies along edges). Such a map defines linear maps between corresponding triangles that are continuous across edges. Since each such map is non-degenerate, it is quasiconformal and hence the piecewise linear map defined between U_k and R_K is quasiconformal (with quasiconstant given by the worst quasiconstant of the finitely many triangles). The other properties are evident. $\square$

8.8. Interpolating between exp and cosh

LEMMA 8.8.1 (exp-cosh interpolation). *There is a quasiregular map $\sigma_j : S \to D(0, e^2)$ so that*

$$\sigma_j(z) = \begin{cases} \exp(z), & z \in J \in \mathcal{J}_1^j, \\ e \cdot \cosh(z-1), & z \in J \in \mathcal{J}_2^j, \\ \exp(z), & z \in \mathbb{H}_r + 2. \end{cases}$$

The quasiconstant of σ_j is uniformly bounded, independent of all our choices.

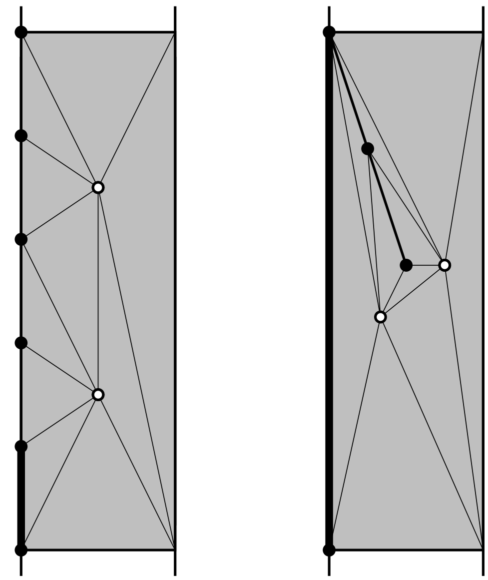


FIGURE 4. The pictorial proof of Lemma 8.7.1 for $n = 5$.

PROOF. As with the previous lemma, the proof is basically a picture; see Figure 5. Suppose $J \in \mathcal{J}$ and let $R = J \times [1, 2]$. The exponential map sends R to the annulus $A = \{e < |z| < e^2\}$, with the left side of R mapping to the inner circle and the top and bottom edges of R mapping to the real segment $[e, e^2]$.

Now define a quasiconformal map $\phi : A \to D(0, e^2)$ that is the identity on $\{|z| = e^2\}$ and on $[e, e^2]$, but that maps $\{|z| = e\}$ onto $[-e, e]$ by $z \to \frac{1}{2}(z + \frac{e^2}{z})$ (this is just a rescaled version of the Joukowsky map $\frac{1}{2}(z + \frac{1}{z})$ that maps the unit circle to $[-1, 1]$, identifying complex conjugate points).

In $\mathbb{H}_r + 2$ and in rectangles of the form $J \times [1, 2]$ for $J \in \mathcal{J}_1$ we set $\sigma_j(z) = \exp(z)$. In the rectangles corresponding to elements of $\mathcal{J}_2$ we let $\sigma_j(z) = \phi(\exp(z))$. This clearly has the desired properties. $\square$

Actually, the cosh function in the lemma can be replaced by any function $h : J \to [-1, 1]$ that has the property that $h(z)$ only depends on the distance from z to the endpoint of J. This will ensure that after applying a folding map, points that started on opposite sides of some slit X_k will end up being identified by h, which is all we need.

This completes the proof of Theorem 8.1.1.

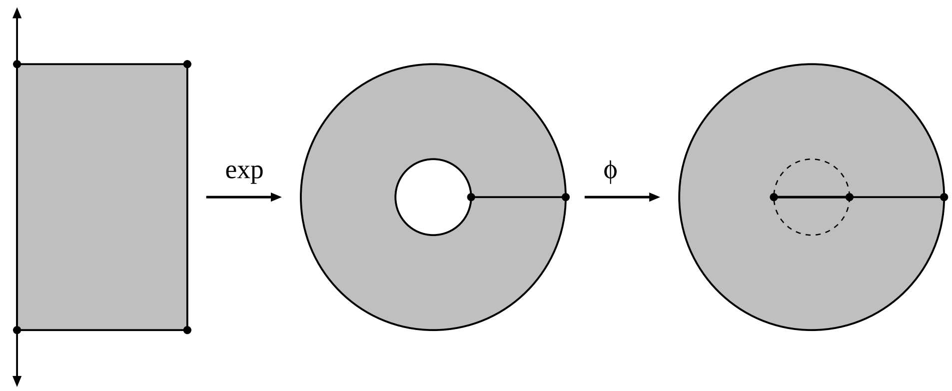


FIGURE 5. The exponential function maps the rectangle $[1,2] \times J$ conformally to the slit annulus $\{e < |z| < e^2\} \setminus [e, e^2]$. The map ϕ is chosen to map the annulus A=$\{e < |z| < e^2\}$ to the slit disk $\{|z| < e^2\} \setminus [-e, e]$ so that it equals the identity on $\{|z| = e^2\}$ and equals $\frac{1}{2}(z + \frac{e^2}{z})$ on $\{|z| = e\}$.

CHAPTER 9

Exotic examples in the Eremenko-Lyubich class

9.1. Rempe Rigidity

We saw above that even quasiconformal equivalence on the whole plane does not imply that two functions have homeomorphic Julia sets, so seems to be little reason to expect that equivalence near ∞ allows us to compare the dynamics of a model and its approximating entire function. The second surprise is a remarkable result of Lasse Rempe that, in fact, they are closely related:

THEOREM 9.1.1 (Rempe rigidity). *Suppose $f \in \mathcal{B}$ and g is a model function that are quasiconformally equivalent near ∞. Then there is an $R > 0$ and a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ so that $f \circ \phi = \phi \circ g$ on*

$$\mathcal{J}_R(f) = \cap_{n=1}^{\infty}\{z : |f^n(z)| \geq R\}.$$

THEOREM 9.1.2 (Rempe rigidity for disjoint type). *Suppose $f \in \mathcal{B}$ and g is a model function with tract Ω and $f = g \circ \varphi$, on Ω, where ψ is quasiconformal on $\mathbb{C}$ and conformal off Ω. Suppose that f and g are disjoint type, i.e., the closures of both tracts are disjoint from $\overline{\mathbb{D}}$. Then there quasiconformal map Φ of the plane so that*

$$\Phi \circ f = g \circ \Phi,$$

on Ω. In particular $\mathcal{J}(g) = \Phi(\mathcal{J}(f))$.

PROOF. Replace Ω by $\{z : |f(z)| > 1 + \epsilon\}$. Then $\partial\Omega$ is smooth and the other assumptions are still valid. Let $W = \varphi(\Omega)$. Since $\overline{W} \cap \overline{\mathbb{D}} = \emptyset$, φ is conformal on a neighborhood U of $\overline{\mathbb{D}}$ and hence smooth there. Thus we can replace φ by another QC map ϕ that equals φ outside U and is the identity on $\mathbb{D}$.

Now define a sequence of quasiconformal maps $\{\Phi_n\}$ on $\mathbb{C}$ by setting Φ_0 to be the identity and, in general,

$$\Phi_{n+1} = g^{-1} \circ \Phi_n \circ f,$$

on each tract of f and $\Phi_{n+1} = \phi$ on $\mathbb{C} \setminus \Omega$.

We can show that each Φ_n is quasiconformal (with the same constant as ϕ) and $\Phi_n = \Phi_{n+1}$ outside $f^{-n}(D(0, 1+\epsilon))$ so the sequence stabilizes off $\mathcal{J}(f)$. Since this set is dense, the sequence of maps converge, uniformly on compact sets to a quasiconformal map Φ such that

$$g \circ \Phi = \Phi \circ f,$$

on Ω. □

Clearly $\mathcal{J}_R(f)$ is an intersection of closed sets and is hence closed. If $\Omega = \{z : |f(z) > R\}$ and $\overline{\Omega} \cap \overline{D(0,R)} = \emptyset$, then an orbit that leaves $\overline{\Omega}$ stays bounded forever. Thus $\mathbb{C} \setminus \mathcal{J}_R(f)$ is in the Fatou set and $\mathcal{J}_R(f) \subset \mathcal{J}(f)$. Moreover, $I(f) \subset \mathcal{J}_R(f)$, hence

$$\mathcal{J}(f) \subset \overline{I(f)} \subset \overline{\mathcal{J}_R(f)} = \mathcal{J}_R(f) \subset \mathcal{J}(f),$$

so $\mathcal{J}_R(f) = \mathcal{J}(f)$. Thus Rempe's theorem says that if an element of $\mathcal{B}$ is quasiconformally equivalent near ∞ to model function, g, $\mathcal{J}(f)$ is the quasiconformal image (under a map of the whole plane) of the Julia set of g. Thus $\mathcal{J}(f)$ and $\mathcal{J}(g)$ share any properties that are preserved by quasiconformal maps. Similarly for $I(f)$ and $I(g)$.

Consider, for example, Eremenko's question: can any point of the escaping set be connected to ∞ by a path inside the escaping set? The quasiconformal image of a path to ∞ is another path to ∞, so if there is a model function that is a counterexample, then there is also an entire function that is a counterexample.

In the next section we will use the approximation and rigidity results stated above to build a counterexample in $\mathcal{B}$ to the strong Eremenko conjecture, to illustrate that these results are simple to use in practice. We will then turn to the more difficult task of proving each of them.

9.2. An escaping set of dimension 1

We claim that the examples f_K constructed in the previous section all have escaping sets of dimension 1. We know that this is the minimal possible dimension for the escaping set of any entire function, because the escaping set hits every circle around the origin of sufficiently large radius, Theorem 1.7.1.

Rempe and Stallard showed that if $f, g \in \mathcal{B}$ are affinely equivalent, i.e.,

$$A \circ f = g \circ B$$

for affine maps A, B, then $\dim(I(f)) = \dim(I(g))$. The family $f_K(z) = f(z) + K$ forms a family of affinely equivalent entire functions and hence

$$1 \geq \dim(I(F_0)) = \inf_K \dim(I(F_0 + K)) \leq \inf_K \dim(\mathcal{J}(F_0 + K)) = 1.$$

Thus the escaping set of F_0 has dimension 1 and is strictly smaller than the dimension of its Julia set (which must be > 1 by Theorem 4.8.1).

All that remains is to prove the result of Rempe-Gillen and Stallard:

THEOREM 9.2.1. *Suppose $f, g \in \mathcal{B}$ are affinely equivalent. Then* $\dim(I(f)) = \dim(I(g))$.

LEMMA 9.2.2. *Suppose $f, g \in \mathcal{B}$ are affinely equivalent and $K > 1$. Then there is $R > 0$ and a K-quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi \circ f = g \circ \varphi$ for all $z \in \mathcal{J}_R(f)$.*

Because we can take K close to 1, dimension are distorted as little as we wish if R is large enough. Proof uses version of Rempe Rigidity theorem.

9.3. The strong Eremenko conjecture fails in $\mathcal{B}$

In this section we construct a function f with a bounded singular set, so that no point of the escaping set $I(f)$ can be connected to ∞ by a path that remains inside $I(f)$. This counter example to the strong Eremenko conjecture was first bound by Rottenfusser, Rükert, Rempe and Schleicher in [**120**], and we follow the presentation given there.

Let $S\{x+iy : x > 4, |y| < 4\}$ be a half-infinite horizontal half-strip and let $\Omega \subset S$ be the subdomain illustrated in Figure **??**. Suppose we have two increasing sequences $\{r_k\}$ and $\{s_k\}$ so that $r_1 < s_1 < r_2 < s_2 < \dots$. For each pair $\{r_k, s_k\}$ we define a "switchback" region as shown in Figure 1. It basically consists of a stretched "S" inside the rectangle $[r_k - 9, s_k + 9] \times [-1, 1]$. The vertical lines $\{x = t\}$ each hit the region in three unit length segments for $r_k \leq t \leq s_k$, and hit it in one unit segment centered on the real axis for $r_k - 9 \leq t \leq r_k - 8$ and $s_k + 8 \leq t \leq s_k + 9$ as shown.

We form an unbounded, simple connected region Ω by taking the union of these switchback regions with the connecting rectangles $[s_k + 9, r_{k+1} - 9] \times [-\frac{1}{2}, \frac{1}{2}]$. Let $\tau : \Omega \to \mathbb{H}_r$ be the conformal map that sends ∞ to ∞ and the points $4 \pm \frac{1}{2}i$ to ± 1.

We can take $r_1 < s_1$ to be any two large numbers; the rest of the sequences are defined inductively as follows. There is a cross-cut ρ_1 the connects the top and bottom sides of Ω and that lies between the lines $\{x = r_1 - 9\}$ and $\{x = r_1 - 8\}$. We can assume that this cross-cut is actually a hyperbolic geodesic and that $\tau(\rho_1)$ is a semi-circle in $\mathbb{H}_r$ with its endpoints on the imaginary axis and centered at the origin. Let x_1 be the intersection of this semicircle with the real line (thus also equals the radius of the semicircle). If we take an arc γ of the semicircle centered at x_1 and of length $\simeq 8\exp(-x_1)$, the image of this arc under exp is a smooth curve that closely approximates the vertical line segment on length 8 centered at $\exp(x_1)$. Choose $r_{2+1} = \exp(x_1) - 1$. Then $\{x = r_{2+1}\} \cap S$ and $\exp(\gamma) \cap S$ are both cross-cuts of S and the first is strictly to the left of the second. See Figure 3.

The value S_2 is chosen in a similar manner, except that we start with a cross-cut σ_1 of Ω that lies between $\{x = s_1 + 8\}$ and $\{x = s_1 + 9\}$ that maps to a semi-circle of radius y_1 in $\mathbb{H}_r$ and then take $s_2 = \exp(y_1) + 1$. Then the cross-cut corresponding to $\{x = s_2\}$ lies to the right of the hyperbolic cross-cut passing through $\exp(y_2)$.

Once we have $\{r_2, s_2\}$ we can define the second switchback region, define new cross-cuts at the entrance and exit of this region and use these to define r_3, s_3.

This argument is slightly circular since we are using the map $\tau : \Omega \to \mathbb{H}_r$ before Ω has been defined. To be more careful, we should define Ω_k to be the part of Ω involving the the first k switchbacks, followed by an infinite horizontal tube and use the conformal map τ_k of this domain onto $\mathbb{H}_r$ to define r_{k+1} and s_{k+1}. Then one needs to prove that that the image of the crosscuts ρ_k, σ_k under τ_k and τ are very close.

Thus any path in Ω that connects $\{x = r_{k+1}\}$ to $\{x = s_{k+1}\}$ has a pre-image that connects $\{x = r_k - 8\}$ to $\{x = s_k + 8\}$. Therefore this same preimage curve has two subarcs that connect $\{x = r_k\}$ to $\{x = s_k\}$ (in that order; there is also a third that connects them in the opposite order).

This is the critical property of Ω. For by induction, any path in Ω that connects $\{x = 12r_{k+n}\}$ to $\{x = s_{k+n}\}$ has a n-th order preimage that connects $\{x = r_k\}$ to $\{x = s_k\}$ with 2^n different subarcs. Since this is true for a fixed k and arbitrarily

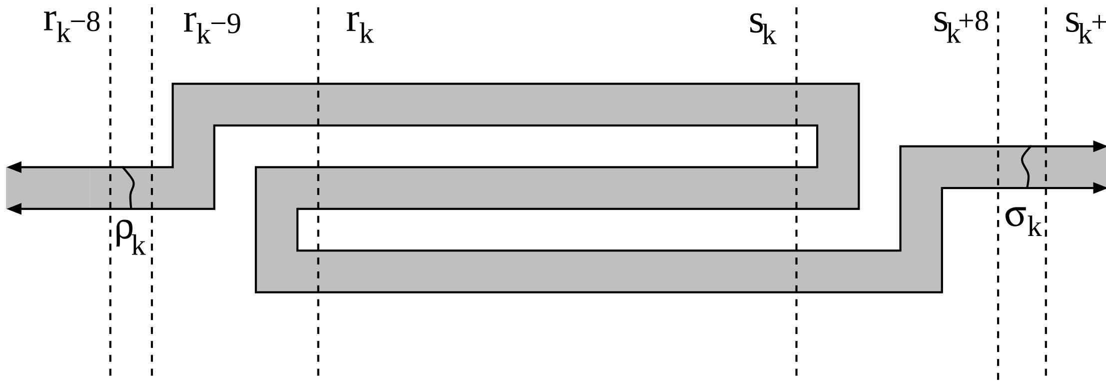


FIGURE 1. The building block of our track. Copies of this are connected by horizontal tubes to define the tract Ω.

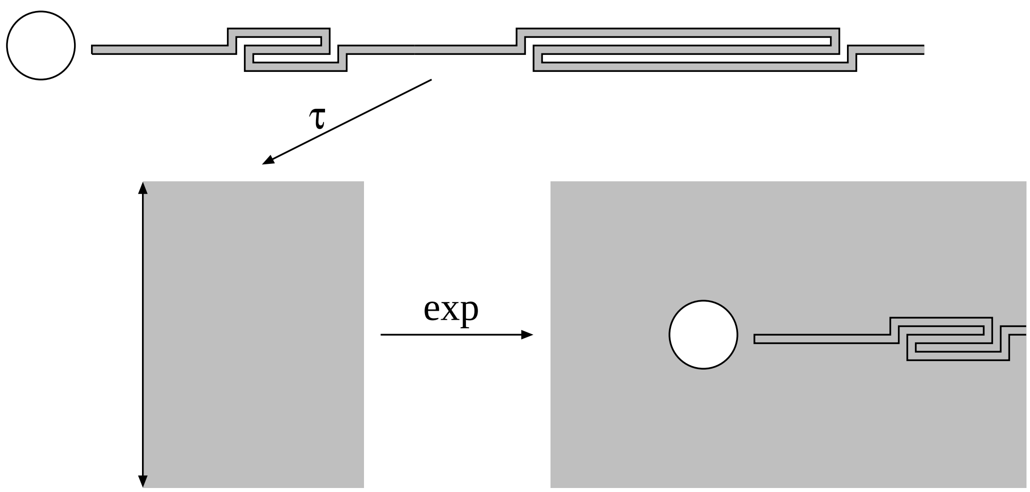


FIGURE 2. The map τ is conformal from Ω to the right half-plane and is followed by the exponential map onto the outside of the unit disk. This covers Ω and so $f^{-1}(\overline{\Omega})$ consists of countable many copies of $\overline{\Omega}$ inside itself. Repeated inverse images gives a sequence of nested close sets whose intersection is the Julia set of the model $F = e^{\tau}$. The building block of our track. Copies of this are connected by horizontal tubes to define the tract Ω.

large n, we see that a path to ∞ in $I(f)$ would have to alternately cross $\{x = r_k\}$ and $\{x = s_k\}$ infinitely often. This is impossible for a path, so $I(f)$ does not contain paths to ∞.

Suppose $w_0 \in \mathcal{J}(F)$. Then $w_0 \in J \subset \mathcal{J}(F)$ where J is a closed connected set obtained by intersecting some sequence of components of $F^{-n}(\overline{\Omega})$. Suppose there is a: Command not found.

FIGURE 3. This is the essential property of the switchbacks: the crosscuts at the entrance and exits map under F to curves that cross the next switchback between $\{x = r_{k+1}\}$ and $\{x = s_{k+1}\}$. This implies that any curve in Ω that connects these two vertical lines, also connects the F images of the two cross-cuts. Hence the F-preimage of such a curve connects ρ_k and σ_k and hence connects $\{x = r_k\}$ and $\{x = s_k\}$ at least three times.

a path γ that connects w_0 to ∞ in J. Choose k so that $|w| < r_k - 9$. The the path γ has a sub-arc that connects $\{x = r_k\}$ to $\{x = s_k\}$, but only a finite number of disjoint such subarcs, say N of them. Let $w_n = F^n(w_0)$. Then $|w_n| \le r_{k+n} - 9$ by the choice of $\{r_k\}$. Since $\gamma_k = F^k(\gamma)$ is a path that connects w_k to ∞, it must connect $\{x = r_{k+n}\}$ to $\{x = s_{k+n}\}$. Thus by the argument above, its nth preimage γ must connect $\{x = r_k\}$ to $\{x = s_k\}$, 2^n times. If we take $2^n > N$, this is a contradiction, proving that no such path γ exists.

Thus the stronger version of Eremenko's conjecture fails for the model function f. But Theorems **??** and **??** say there is an Eremenko-Lyubich class function f and the quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ so that $\mathcal{J}(f) = \phi(\mathcal{J}(F))$. Thus the conjecture also fails for f. More formally, we obtain:

THEOREM 9.3.1. *There is a hyperbolic $f \in \mathcal{B}$ so that no point of $\mathcal{J}(f)$ is connected to ∞ by a path in $\mathcal{J}(f)$.*

9.4. Hyperbolic dimension 2

A set $K \subset \mathcal{J}(f)$ is called hyperbolic if it is compact, $f(K) \subset K$ and $|(f^n)'| \ge \lambda$ on K for some $\lambda > 1$ and all sufficiently large n. The hyperbolic dimension of f is the supremum of the Hausdorff dimensions of all hyperbolic subsets $K \subset \mathcal{J}(f)$. This was introduced by Shishikura in [**128**]. Clearly the hyperbolic dimension is a lower bound for $\dim(\mathcal{J}(f))$, but they need not be the same in general

THEOREM 9.4.1 (Stallard, [**131**]). *There is a hyperbolic $f \in \mathcal{B}$ with hyperbolic dimension $< \dim(\mathcal{J}(f))$.*

THEOREM 9.4.2 (Rempe, [**113**]). *There is a hyperbolic $f \in \mathcal{B}$ with hyperbolic dimension 2.*

CHAPTER 10

Escaping rays for Eremenko-Lyubich functions

One of the main conjectures in transcendental dynamics has been Eremenko's question of whether every connected component of the escaping set is unbounded. The so called "Strong Eremenko" conjecture asked if every point of the escaping set can be connected to infinity by a path inside the escaping set. A counterexample in the Eremenko-Lyubich class to the stronger conjecture was found by Rottenfusser, Rückert, Rempe and Schleicher in [**120**], where they also prove the strong Eremenko conjecture is true for Eremenko-Lyubich function of finite order. In this chapter we give their positive result; the counterexample will be constructed in a later chapter, after we have developed more of the theory of quasiconformal mappings.

10.1. A characterization of arcs

THEOREM 10.1.1. *If E is a compact subset of the Riemann sphere that is totally ordered and the order topology agrees with the subset topology induced from the sphere, then E is an arc (a homeomorphic image of* $[0, 1]$.

PROOF. □

10.2. Tracts and external addresses

Define $\mathcal{B}_{rmlog}$
Define normalize
Define disjoint type
Define external address, $\mathcal{J}_s$,

LEMMA 10.2.1. *f $F \in \mathcal{B}_{rmlog}$ and $z, w \in \mathcal{J}_s$, then*

$$\lim_{k\to\infty} \max(\Re F^k(z), \Re F^k(w)) = \infty$$

PROOF. □

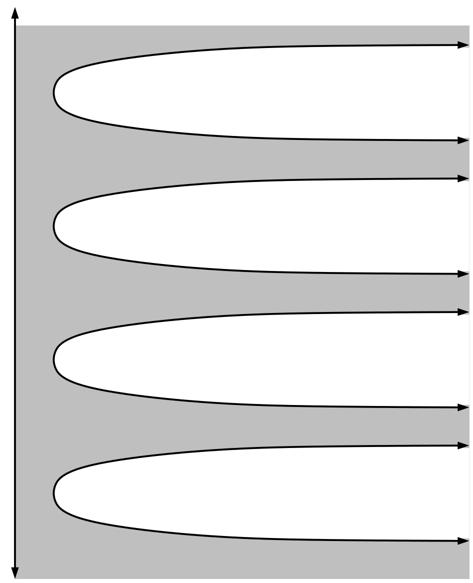


FIGURE 1. An example of logarithmic tracts. Each white component is conformally mapped to the right half-plane by F, which satisfies $F(z) = F(z + 2\pi i)$. These tracts have closure contained in $\mathbb{H}_r$, so define a hyperbolic function of disjoint type.

LEMMA 10.2.2. *For $F \in \mathcal{B}_{rmlog}$ there is a $K \geq 0$ so that if $z \in \mathcal{J}^K(f)$ and z is the external address of $z$then $\mathcal{J}_s$ contains a closed, unbounded connected set with* $\operatorname{dist}(A, z) \leq 2\pi$.

10.3. The head-start condition implies path connected

Suppose U, V are logarithmic tracts of F and $\varphi : \mathbb{R}to\mathbb{R}$ is an increasing continuous function such that $\varphi(x) > x$ for all x. We say that (U, v) satisfy a **head-start condition** for φ if for all $z, w \in \overline{U}$ with $F(z), F(w) \in \overline{V}$

$$\Re z > \varphi(\Re w) \Rightarrow \Re F(z) > \varphi(\Re F(w)) \Rightarrow .$$

An external address s satisfies the head-start condition for φ if

(1) all consecutive pairs of tracts for s satisfy the head-start condition for φ and
(2) for all distinct points $z, w \in \mathcal{J}_s$ there is a positive integer m such that either

$$\Re F^m(z) > \Re F^m(w),$$

or

$$\Re F^m(w) > \Re F^m(z).$$

We say F satisfies the head-start condition if every external address satisfies it for some φ. We say that F satisfies a uniform head-start condition if the same φ works for all external addresses.

If F satisfies a head-start condition, then it is possible to order points with the same external addresses according to their rates of escape to infinity. For $z, w \in \mathcal{J}_s$ we say that $z \succ w$ if there is a positive integer k so that

$$\Re F^k(z) > \varphi(\Re F^k(w)).$$

We also let $\infty \succ z$ for all $z \in \mathcal{J}_s$.

Note that we can never have $z \succ z$. If $a \succ b$ and $b >\succ c$ then, by definition, there are integers k, n so that

$$\Re F^k(a) > \varphi(\Re F^k(b)) > \Re F^k(b),$$

and

$$\Re F^n(b) > \varphi(\Re F^n(c)).$$

If $m \geq max(k, n)$, these equations also both hold for n, so

$$\Re F^n(a) > \varphi(\Re F^n(b)) > \Re F^n(b) > \varphi(\Re F^n(c)),$$

so $a \succ c$. By condition (2) of the head-start condition for external addresses, any two points in J_s can be compared, thus $\succ$ defines a total order on $\mathcal{J}_s$.

Lemma 10.3.1. *For any external address s, the order topology on $\mathcal{J}_s$ agrees with the topology as a subset of the Riemann sphere. Every component of $\mathcal{J}_s$ is homeomorphic to an arc.*

Proof. □

Theorem 10.3.2 (Rottenfusser, Rückert, Rempe, Schleicher [**120**]). *If $F \in \mathcal{B}_{rmlog}$ satisfies a head-start condition, then the orbit of every escaping point z eventually lands on a ray tail.*

Proof. □

10.4. Finite order implies head-start

CHAPTER 11

Quasiconformal mappings: geometric aspects

In this chapter we discuss three definitions of quasiconformal maps: geometric, piecewise C^1 and analytic.

11.1. Angle distortion of linear maps

Conformal maps preserves angles; quasiconformal maps can distort angles, but only in a controlled way. To make this distinction more precise we must have a way to measure angle distortion and we start with a discussion of linear maps.

Consider the linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax+by, cx+dy).$$

Let M^T denote the transpose of the real matrix M, i.e., its reflection over the main diagonal. Then

$$M^T \cdot M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} \equiv \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is positive and symmetric and hence has two positive eigenvalues λ_1, λ_2, assuming M in non-degenerate. The square roots $s_1 = \sqrt{\lambda_1}$, $s_2 = \sqrt{\lambda_2}$ are the singular values of A (without loss of generality we assume $s_1 \geq s_2$). Then

$$M = U \cdot \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \cdot V,$$

where U, V are rotations. Thus M maps the unit circle to an ellipse whose major and minor axes have length s_1 and s_2. Thus M preserves angles iff it maps the unit circle to a circle iff $s_1 = s_2$. Otherwise M distorts angles and we let $D = s_1/s_2$ denote the dilatation of the linear map M. This is the eccentricity of the image ellipse and is ≥ 1, with equality iff M conformal.

The inverse of a linear map with singular values $\{s_1, s_2\}$ has singular values $\{\frac{1}{s_2}, \frac{1}{s_1}\}$ and hence dilatation $D = (1/s_2)/(1/s_1) = s_1/s_2$. Thus the dilatation of a linear map and its inverse are the same.

Given two linear maps M, N with singular values $s_1 \geq s_2$ and $t_1 \geq t_2$ respectively, the singular values of the composition MN are trapped between $s_1 t_1$ and $s_2 t_2$ (this occurs for the maximum singular values since they give the operator norms of the matrices and these are multiplicative; a similar argument works for the minimum singular values and the inverse maps). Thus the dilation is less than $(s_1 t_1)/(s_2 t_2)$ i.e., dilatations satisfy

$$D_{M \circ N} \leq D_M \cdot D_N.$$

The dilatation D can be computed in terms of a, b, c, d as follows. The eigenvalues λ_1, λ_2 are roots of the

$$0 = \det(M^T \cdot M - \lambda I),$$

which is the same as

$$0 = (E - \lambda)(G - \lambda) - F^2 = EG - F^2 - (E + G)\lambda + \lambda^2.$$

Thus

$$\begin{aligned}
\lambda_1 \lambda_2 &= EG - F^2 \\
&= (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 \\
&= a^2 b^2 + a^2 d^2 + c^2 b^2 + d^2 c^2 - (a^2 b^2 + 2abcd + c^2 d^2) \\
&= a^2 d^2 + c^2 b^2 - 2abcd \\
&= (ad - bc)^2
\end{aligned}$$

Similarly,

$$\lambda_1 + \lambda_2 = E + G = a^2 + b^2 + c^2 + d^2.$$

The values of λ_1, λ_2 can be found using the quadratic formula:

$$\begin{aligned}
\{\lambda_1, \lambda_2\} &= \frac{1}{2}[E + G \pm \sqrt{(E + G)^2 - 4(EG - F^2)}] \\
&= \frac{1}{2}[E + G \pm \sqrt{(E - G)^2 + 4F^2}].
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\lambda_1}{\lambda_2} &= \frac{E+G+\sqrt{(E-G)^2+4F^2}}{E+G-\sqrt{(E-G)^2+4F^2}} \\
&= \frac{(E+G+\sqrt{(E-G)^2+4F^2})^2}{(E+G)^2-(E-G)^2-4F^2} \\
&= \frac{(E+G+\sqrt{(E-G)^2+4F^2})^2}{4(EG+F^2)}.
\end{aligned}$$

and hence

$$D = \frac{s_1}{s_2} = \sqrt{\frac{\lambda_1}{\lambda_2}} = \frac{E+G+\sqrt{(E-G)^2+4F^2}}{2\sqrt{EG+F^2}}.$$

This formula can be made simpler by complexifying. Think of the linear map M on $\mathbb{R}^2$ as a map f on $\mathbb{C}$:

$$x+iy \to ax+by+i(cx+dy) = u(x,y)+iv(x,y) = f(x+iy)$$

Then

$$M = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and we define

$$f_z = \frac{1}{2}(f_x - if_y) = \frac{1}{2}(u_x+v_y)+\frac{i}{2}(v_x-u_y),\ f_{\bar z} = \frac{1}{2}(f_x+if_y) = \frac{1}{2}(u_x-v_y)+\frac{i}{2}(v_x+u_y).$$

Some tedious arithmetic now shows that

$$\begin{aligned}
4|f_z|^2 &= (u_x+v_y)^2+(v_x-u_y)^2 \\
&= u_x^2+2u_xv_y+v_y^2+v_x^2-2v_xu_y+u_y^2
\end{aligned}$$

$$\begin{aligned}
4|f_{\bar z}|^2 &= (u_x-v_y)^2+(v_x+u_y)^2 \\
&= u_x^2-2u_xv_y+v_y^2+v_x^2+2v_xu_y+u_y^2
\end{aligned}$$

so

$$(|f_z|+|f_{\bar z}|)(|f_z|-|f_{\bar z}|) = |f_z|^2-|f_{\bar z}|^2 = u_xv_y-v_xu_y = s_1s_2 = \det(M).$$

In particular, if we assume M is orientation preserving and full rank, then $\det(M) > 0$ and we deduce $|f_z| > |f_{\bar z}|$. Similarly,

$$\begin{aligned}(|f_z| + |f_{\bar z}|)^2 + (|f_z| - |f_{\bar z}|)^2 &= 2(|f_z|^2 + |f_{\bar z}|^2)\\ &= u_x^2 + v_x^2 + u_y^2 + v_x^2\\ &= E + G\\ &= \lambda_1 + \lambda_2\\ &= s_1^2 + s_2^2.\end{aligned}$$

From these equations and the facts $s_1 \geq s_2$, $|f_z| > |f_{\bar z}|$ we can deduce

$$s_1 = |f_z| + |f_{\bar z}|, \qquad s_2 = |f_z| - |f_{\bar z}|,$$

and hence

$$D = \frac{s_1}{s_2} = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|}.$$

Note that $D \geq 1$ with equality iff f is a conformal linear map. It is often more convenient to deal with the complex number.

$$\mu = \frac{f_z}{f_{\bar z}},$$

which is called the **complex dilatation** (although sometimes we abuse notation and just call thus the dilatation, if the meaning is clear from context). Since $|f_{\bar z}| < |f_z|$, we have $|\mu| < 1$ and it is easy to verify that

$$D = \frac{1 + |\mu|}{1 - |\mu|}, \qquad |\mu| = \frac{D-1}{D+1},$$

so that either D or $|\mu|$ can be used to measure the degree of non-conformality.

We leave it to the reader to check that the map

$$x + iy \to (ax + by) + i(cx + dy)$$

can also be written as

$$(z, \bar z) \to \alpha z + \beta \bar z,$$

where $z = x + iy$, $\bar z = x - iy$ and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, satisfy

$$\alpha_1 = \frac{a+d}{2}, \quad \alpha_2 = \frac{a-d}{2}, \quad \beta_1 = \frac{c-b}{2}, \quad \beta_2 = \frac{b+c}{2},$$

In this notation $\mu = \beta/\alpha$ and

$$D = \frac{|\beta| + |\alpha|}{|\alpha| - |\beta|}.$$

As noted above, the linear map f sends the unit circle to an ellipse of eccentricity D. What point on the circle is mapped furthest from the origin? Since

$$s_1 = |f_z| + |f_{\bar z}|,$$

the maximum stretching is attained when $f_z z$ and $f_{\bar z}\bar z$ have the same argument, i.e., when

$$0 < \frac{f_z z}{f_{\bar z}\bar z} = \frac{z^2}{\mu|z|^2},$$

or

$$\arg(z) = \frac{1}{2}\arg(\mu),$$

Thus $|\mu|$ encodes the eccentricity of the ellipse and $\arg(\mu)$ encodes the direction of its major axis.

If we follow f by a conformal map g, then the same infinitesimal ellipse is mapped to a circle, so we must have $\mu_{g\circ f} = \mu_f$. If f is preceded by a conformal map g, then the ellipse that is mapped to a circle is the original one rotated by $-\arg(g_z)$, so $\mu_{f\circ g} = (|g_z|/g_z)^2\mu_f$. To obtain the correct formula in general we need to do a little linear algebra. Consider the composition $g\circ f$ and let $w = f(z)$ so that the usual chain rule gives

$$(g\circ f)_z = (g_w\circ f)f_z + (g_{\bar w}\circ f)\bar f_z,$$

$$(g\circ f)_{\bar z} = (g_w\circ f)f_{\bar z} + (g_{\bar w}\circ f)\bar f_{\bar z}.$$

or in vector notation

$$\begin{pmatrix}(g\circ f)_z\\(g\circ f)_z\end{pmatrix} = \begin{pmatrix}f_z & \bar f_z\\ f_{\bar z} & \bar f_{\bar z}\end{pmatrix}\begin{pmatrix}(g_w\circ f)\\(g_{\bar w}\circ)\end{pmatrix}$$

The determinate of the matrix is

$$f_z\bar f_{\bar z} - \bar f_z f_{\bar z} = f_z\overline{f_z} - \overline{f_{\bar z}}f_{\bar z} = |f_z|^2 - |f_{\bar z}|^2 = J,$$

which is the Jacobian of f, so by Cramer's Rule,

$$(g_w\circ f) = \frac{1}{J}[(g\circ f)_z\bar f_{\bar z} - (g\circ f)_{\bar z}\bar f_z],$$

$$(g_{\bar w}\circ f) = \frac{1}{J}[(g\circ f)_{\bar z}f_z - (g\circ f)_z f_{\bar z}],$$

so

$$\begin{aligned}\mu_g \circ f &= \frac{(g\circ f)_{\bar z} f_z - (g\circ f)_z f_{\bar z}}{(g\circ f)_z \bar{f}_{\bar z} - (g\circ f)_{\bar z} \bar{f}_z} \\ &= \frac{\mu_{g\circ f} f_z - f_{\bar z}}{\bar{f}_{\bar z} - \mu_{g\circ f}\bar{f}_z} \\ &= \frac{f_z}{\bar{f}_z}\frac{\mu_{g\circ f} - \mu_f}{1 - \mu_{g\circ f}\overline{\mu_f}}.\end{aligned}$$

Now set $h = g\circ f$ or $g = h \circ f^{-1}$ to get

$$\mu_{h\circ f^{-1}} \circ f = \frac{f_z}{\bar{f}_z}\frac{\mu_h - \mu_f}{1-\mu_h\overline{\mu_f}}.$$

Thus if h and f have the same dilatation μ, then $g = h\circ f^{-1}$ is conformal. We will need this in the case when h is more general than an homeomorphism.

11.2. Piecewise smooth quasiconformal maps

We say that a linear map f is K-**quasiconformal** if $D_f \le K$. The linear map need not be defined on the whole plane. Given two triangles T_1, T_2 with vertices a, b, c and A, B, C, there is a unique affine map $T_1 \to T_2$ taking $a \to A$, $b \to B$ and $c \to C$. The map is orientation preserving if both triangles were labeled in the same orientation.

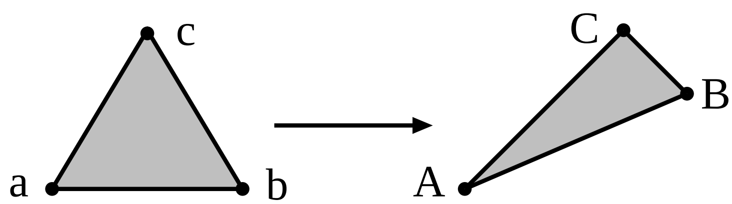


FIGURE 1. A pair of similarly oriented, labeled triangles defines a linear map and has an associated dilatation.

There is an obvious affine map between these triangles and we can easily compute its quasiconformal constant of this map as follows. First use a conformal linear map to send each triangle to one of the form $\{0, 1, a\}$ and $\{0, 1, b\}$. The affine map is then of the form $f(z) \to \alpha z + \beta \bar z$ where $\alpha + \beta = 1$ and $\beta = (b-a)/(a-\bar a)$ and from this we see that

$$K_f = \frac{1+|\mu_f|}{1-|\mu_f|},$$

where
$$\mu_f = \frac{f_{\bar z}}{f_z} = \frac{\beta}{\alpha} = \frac{b-a}{b-\bar a},$$
If the triangle T' is degenerate, or has the opposite orientation as T, we simply give ∞ as our QC bound K.

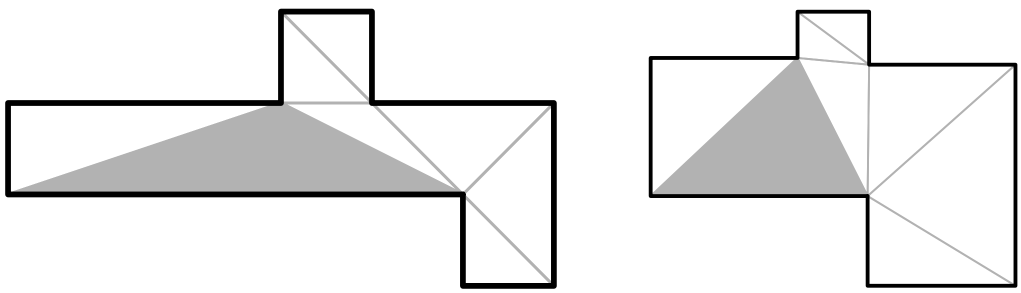


FIGURE 2. Two compatible triangulations of different polygons. The most distorted triangle is shaded; this determines an upper bound for the piecewise linear map between the polygons.

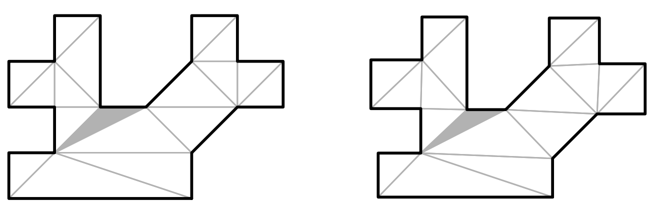


FIGURE 3. Another example of compatible triangulations of different polygons. The most distorted triangle is shaded; in this case the dilatation is bounded by 3.12157. (These examples are from [**30**], where these bounds on dilatation are used to measure the effective of number calculation of Schwarz-Christoffel parameter calculations.)

Two triangulations are compatible if the there is a bijection between the triangles that preserves adjacency. The linear maps between each pair of corresponding triangles defines a continuous, piecewise linear map between the regions covered by the triangulations and we call this map K-quasiconformal if the dilatation of each individual triangle map is bounded by K. For finite triangulations, we always get a K-quasiconformal map for some K, since we only have a finite set of dilatations, but for infinite triangulations, K-quasiconformality is a non-trivial assumption. The class of piecewise linear quasiconformal maps is extremely limited compared to the full class we shall define later, but even these simple maps suffice for many of the applications we will later make to dynamics.

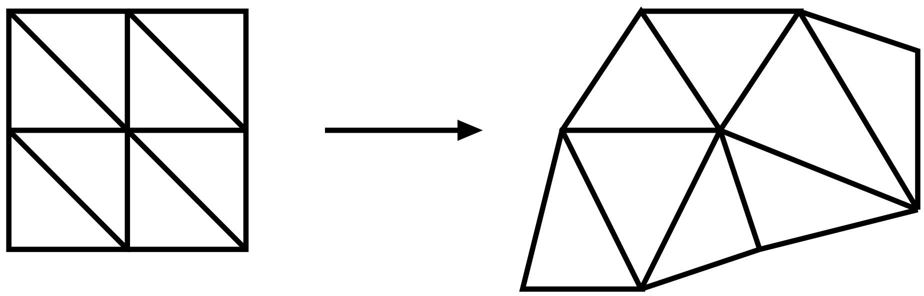


FIGURE 4. A pair of compatible finite triangulations defines a piecewise linear quasiconformal map where the quasiconstant of the triangulation is the maximum of the individual dilatations. A pair of compatible infinite triangulations defines a quasiconformal map if the dilatations are uniformly bounded above.

The piecewise differentiable definition: h is K-quasiconformal on Ω if there are countable many analytic curves whose union is a closed set Γ of Ω such that h is continuously differentiable on each connected component of $\Omega' = \Omega \setminus \Gamma$ and $D_h \leq K$ on Ω'.

The main motivating example is when $\Omega = \mathbb{C}$, Γ is a triangulation of the plane and μ_h is constant on the interior of each triangle.

A quadrilateral Q is a Jordan domain with two disjoint closed arcs on the boundary. By the Riemann mapping theorem and Caratheodory's theorem, there is a conformal map from Q to a $1 \times m$ rectangle that extends continuously to the boundary with the two marked arcs mapping to the two sides of length a. The ratio $M = M(Q) = 1/m$ is called the modulus of the four distinct marked on the boundary and is uniquely determined by Q. The conjugate of Q is the same domain but with the complementary arcs marked. Its modulus is clearly the reciprocal of Q's modulus.

The geometric definition: A homeomorphism h, defined on a planar domain Ω, is K-quasiconformal if the

$$\frac{1}{K}M(Q) \le M(h(Q)) \le KM(Q),$$

for every quadrilateral $Q \subset \Omega$.

Our first goal is to check that the this includes all the maps we previously defined as K-quasiconformal in terms of their dilatations. Suppose $Q \subset \Omega$ and $h(Q)$ are respectively equivalent to $1 \times a$ and $1 \times b$ rectangles and h is has dilatation bounded by K. Since the dilatation is unchanged by composing with conformal maps, it suffices to show

LEMMA 11.2.1. *If we have a piecewise differentiable K-quasiconformal map between a $1 \times a$ and $1 \times b$ rectangle with dilatation $\le K$, then $\frac{a}{K} \le b \le Ka$. Thus the piecewise differentiable definition implies the geometric definition.*

PROOF. By integrating over horizontal lines in the first rectangle, we see

$$b \le \int_0^a (|f_z| + |f_{\bar z}|)dx,$$

and integrating in the other variable,

$$b \le \int_0^1 \int_0^a (|f_z| + |f_{\bar z}|)dxdy.$$

Thus by Cauchy-Schwarz

$$\begin{aligned} b^2 &\le (\int_0^1 \int_0^a (|f_z| + |f_{\bar z}|)(|f_z| - |f_{\bar z}|)dxdy)(\int_0^1 \int_0^a \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|}dxdy) \\ &\le (\int_0^1 \int_0^a (|f_z|^2 - |f_{\bar z}|^2)dxdy)(\int_0^1 \int_0^a \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|}dxdy) \\ &\le (\int_0^1 \int_0^a J_f dxdy)(\int_0^1 \int_0^a D_f dxdy) \\ &\le baK, \end{aligned}$$

so $b \le Ka$. The other direction follows by considering the inverse map. $\square$

In order for the proof to work we need two things: (1) the area of the range to be bounded above by integrating the Jacobian over the domain and (2) each horizontal line segment S to have an image whose length is bounded above by the integral of $|f_z| + |f_{\bar z}|$ over S. This certainly holds if f_z and $f_{\bar z}$ are piecewise continuous on a partition of the plane given by countable many analytic curves, as we have assumed, but it holds much more generally. The geometric definition of quasiconformality actually implies that the map h has partials almost everywhere and is absolutely continuous on almost every line. This, in turn, implies the necessary estimates holds. This will be discussed later in Chapter 12

Next we record the annulus analog of the previous result for rectangles.

Corollary 11.2.2. *If we have a piecewise differentiable K-quasiconformal map f between annuli $A_r = \{1 < |z| < r\}$ and $A_R = \{1 < |z| < R\}$ rectangle with dilatation $\leq K$, then $\frac{1}{K} \log r \log R \leq K \log r$.*

Proof. Slit A_r with $[1, r]$ for form a quadrilateral $Q \subset A_r$ and let $Q' = f(Q) \subset A_R$. See Figure 5 Then $M(A_R) \leq M(Q') \leq KM(Q) = M(A_r)$. The first inequality occurs because of monotonicity of modulus (Lemma 5.1.2); every separating curve for the annulus connects opposite sides of Q' (but there are connecting curves that don't correspond to closed loops). The other direction follows by considering the inverse map. See Figure 11.2.2. $\square$

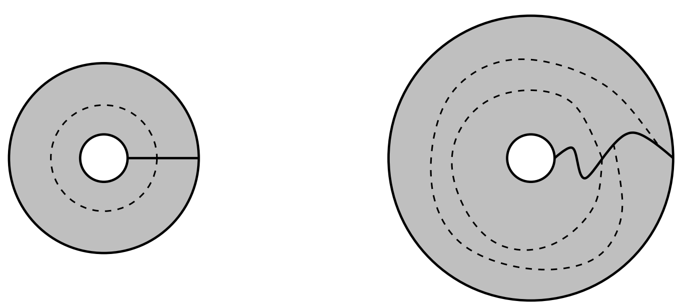


Figure 5. Notation in the proof of 11.2.2.

Lemma 11.2.3. *Suppose f is holomorphic on Ω and ϕ is quasiconformal and C^1 on $\mathbb{C}$. Suppose ψ is quasiconformal on Ω and $\mu_\psi = \mu_{\phi \circ f}$ everywhere that $f' \neq 0$. Then there is a holomorphic function g on $\Omega' = \psi(\Omega)$ so that*

$$g \circ \psi = \phi \circ f.$$

Proof. Let $g = \phi \circ f \circ \psi^{-1}$. Every point where f' is non-zero, there is a disk where the composition is conformal. Thus g is continuous all of Ω' and holomorphic except on a countable set, hence is holomorphic on all of Ω' (since isolated points are removable for bounded holomorphic functions). $\square$

11.3. Compactness and continuity

The Arzela-Ascoli theorem states that a collection of continuous functions is relatively compact if and only if it is equicontinuous and pointwise bounded. In this section we prove that K-quasiconformal maps of the plane, normalized to fix both 0 and 1 have both these properties, and are also closed under uniform convergence on compact sets. Thus normalized K-quasiconformal maps are compact.

Some normalization is necessary; the maps $f_n(z) = nz$ are 1-quasiconformal, but are not pointwise bounded or equicontinuous.

LEMMA 11.3.1. *If $\{f_n\}$ is a sequence of K-quasiconformal maps on Ω that converge uniformly on compact subsets to a homeomorphism f, then f is K-quasiconformal.*

PROOF. Any quadrilateral $Q \subset \Omega$ has compact closure in Ω so $Q' \lim_n f_n(Q)$ is a quadrilateral in $f(\Omega)$ and we need only check that if Q is a quadrilateral then $M(\lim_n f_n(Q)) = \lim_n M(f_n(Q))$. However, this follows from Lemma 5.5.7 of Chapter 5. $\square$

LEMMA 11.3.2. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map that fixes both* 0 *and* 1. *Then $|f(x)|$ is bounded with an estimate depending on $|x|$ and K, but not on f.*

PROOF. First suppose $\Re(x) \leq 1/2$ and consider the topological annulus with boundary component $[0, x]$ and $[1, \infty)$. See Figure **??**. boundary components is bounded below depending only on $|x|$. But if $R = |f(x)|$ then by using the metric $\rho(z) = 1/(|z| \log R)$, we see that the modulus of $f(\mathcal{F})$ is at most $1/\log R$. This is a contradiction if R is too large. $\square$

THEOREM 11.3.3. *A K-quasiconformal map of the plane that fixes both* 0 *and* 1 *is locally Hölder continuous.*

PROOF. Suppose f is as in the lemma and $x, y \in D(0, r)$. By Lemma 11.3.2, $D(0, 2r)$ is mapped into $D(0, R)$ for some $R = R(r, K)$. Surround $\{x, y\}$ by $N =$

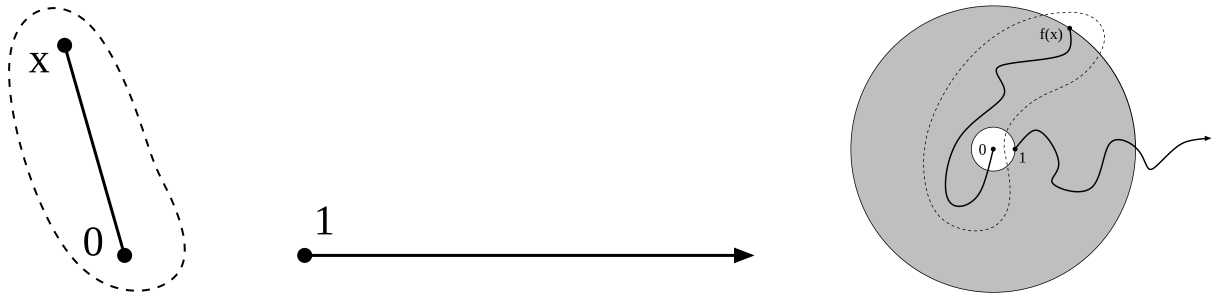


FIGURE 6. If $|f(x)| \gg |x|$ then the modulus of the path family separating $[0, x]$ and $[0, \infty)$ must change by more than a factor of K.

$\lfloor \log_2 \frac{r}{|x-y|} \rfloor$ annuli $\{A_j\}$ of modulus $\log 2$. See Figure 7. The image annuli $\{f(A_j)\}$ have moduli bounded away from zero, and hence $\mathrm{diam}(f(A_{j+1})) \leq (1-\epsilon)\mathrm{diam}(f(A_j))$ by Lemma 5.1.10. Therefore

$$|f(x) - f(y)| \leq R(1-\epsilon)^N \leq R2^{\log_2(1-\epsilon)(1+\log_2 R - \log_2 |x-y|)} \leq C(R)|x-y|^{\log_2(1-\epsilon)}.$$

$\square$

Later we will compute the actual Hölder exponent as $1/K$.

THEOREM 11.3.4. *If f is piecewise differentiable and the dilatation μ satisfies certain estimates of the form*

$$\max_{|x| \leq R, r > 1/R} \frac{1}{r^2} \int_{D(x,r)} D_f(z)dxdy \leq \phi(R),$$

then f has modulus of continuity that depends only on ϕ if $\phi \nearrow \infty$ slowly enough as $R \to \infty$.

PROOF. Repeat the proof of Theorem 11.3.3, only now the moduli of the image annuli can tend to zero. However, as long as ϕ grow slowly enough, then

$$\mathrm{diam}(f(A_j)) \leq \prod_{j=1}^{N} (1 - \epsilon(\phi(R)))$$

where $\epsilon(K)$ is as in Lemma 5.1.10. $\square$

There have been a number of excellent papers written on explicit bounds for this kind of result, but we will only need the "soft" version above. See [**41**], [**?**], [**36**], [**87**], [**138**].

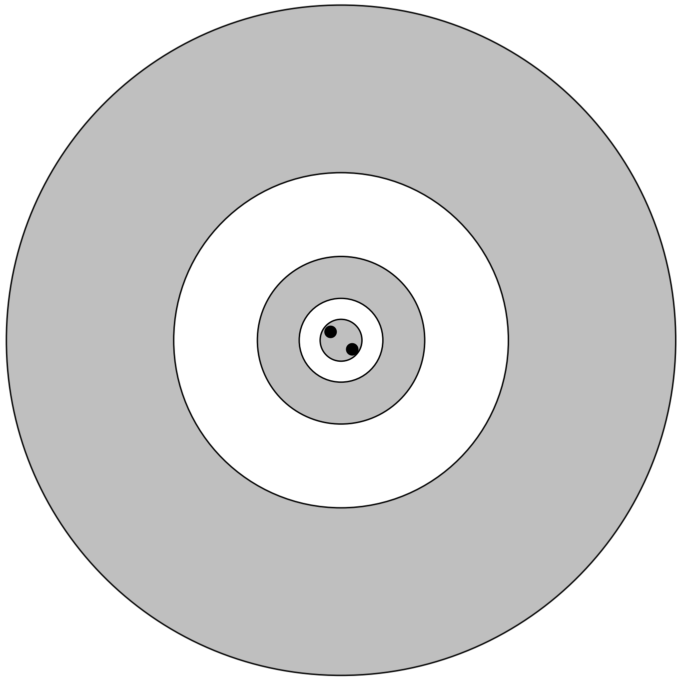


FIGURE 7. Annuli of fixed modulus map to annuli with modulus bounded below, and whose diameters shrink geometrically. Thus f is Hölder continuous.

LEMMA 11.3.5. *If $\varphi : \mathbb{D} \to \mathbb{D}$ is quasiconformal and onto, then φ extends continuously to a homeomorphism of $\mathbb{T} = \partial\mathbb{D}$ to itself.*

PROOF. We may assume $f(0) = 0$; the general case follows after composing with a Möbius transformation.

Suppose $w, z \in \mathbb{D}$. We will show that

$$|f(z) - f(w)| \leq C|z - w|^{\alpha},$$

for constants $C < \infty$, $\alpha > 0$ that depend only on the quasiconstant K of f. This implies f is uniformly continuous and hence has a continuous extension to the boundary of $\mathbb{D}$.

Let $d = |z - w|$ and $r = \min(1 - |z|, 1 - |w|)$. There are several cases depending on the positions of the points z, w and the relative sizes of d and r. See Figure **??**.

To start, note that if $|z - w| \geq \frac{1}{10}$ we can just take $C = 20$ and $\alpha = 1$. So from here on, we assume $|z - w| < 1/10$.

Suppose $r > 1/4$, so $z, w \in \frac{3}{4}\mathbb{D}$. Surround the segment $[z, w]$ by $N \simeq \log d$ annuli with moduli $\simeq 1$. Then just as in the proof of Theorem 11.3.3, the image annuli have moduli $\simeq 1$ (with a constant depending on K) and hence

$$|f(z) - f(w)| \leq (1 - \epsilon(K))^N = O(|z - w|^\alpha),$$

for some $\alpha > 0$ depending only on K.

Next suppose $|z| \geq 3/4$ and $d > r$. Then separate $[z, w]$ from 0 by $N \simeq \log d$ disjoint quadrilaterals with a pair of opposite sides being arcs of $\mathbb{T}$, and all with moduli $\simeq 1$. Since $f(0) = 0$ and the image quadrilaterals have moduli $\simeq 1$, there diameters shrink geometrically, so

$$|z - w| = (1 - \epsilon(K))^N = O(d^\alpha),$$

as desired.

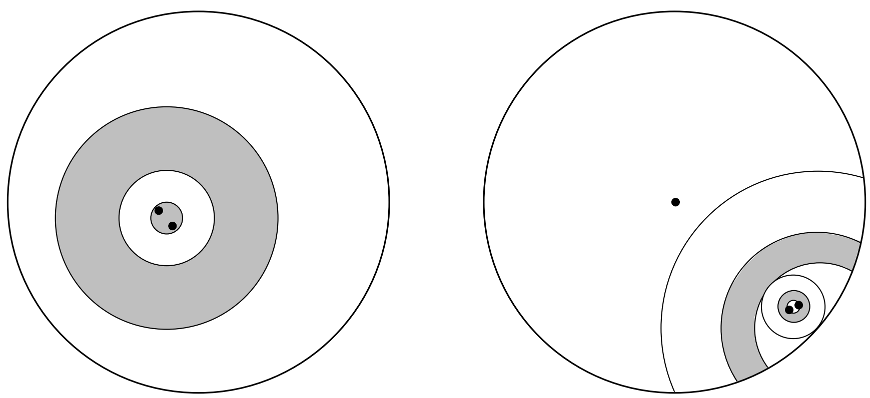


FIGURE 8. The proof of Hölder estimates in the disk is similar to the proof in the plane,except that we need to use quadrilaterals, as well as annuli, if the pair of points in near the boundary.

Finally, if $r \leq d$ we combine the two previous ideas: we start by separating $[z, w]$ from 0 by $\simeq \log d$ quadrilaterals with as above. The smallest quadrilateral then bounds a region of diameter approximately r containing $[z, w]$ and we then construct $\simeq \log r/d$ disjoint annuli with moduli $\simeq 1$ that each separate $[z, w]$ from this smallest quadrilateral. See Figure **??**. The same arguments as before now show

$$|z - w| = (1 - \epsilon(K))^{-\log r}(1 - \epsilon(K))^{\log r/d} = O(d^\alpha) = O(|z - w|^\alpha).$$

$\square$

LEMMA 11.3.6. *For any $\delta > 0$ and and any $r > 0$ there is an $\epsilon > 0$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is $(1+\epsilon)$-quasiconformal and f fixes 0 and 1, then $|z - f(z)| \leq \delta$ for all $|z| < r$.*

PROOF. If not, there is a sequence of $(1+\frac{1}{n})$-quasiconformal maps that all fix 0 and 1 and points $z_n \in D(0,r)$ so that $|z_n - f_n(z_n)| > \delta$. However, there is a subsequence that converges uniformly on compact subsets of the plane to a 1-quasiconformal map that fixes 0 and 1 and that moves some point by at least δ. However a 1-quasiconformal map is conformal on $\mathbb{C}$, hence of form $az + b$ and since it fixes both 0 and 1, it is the identity and hence doesn't move any points, a contradiction. □

LEMMA 11.3.7. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map that fixes both 0 and 1. Then there is a constant $0 < C < \infty$, depending only on K so that if $|z| < 1/C$, then*

$$C^{-1}|z|^K \leq |f(z)| \leq C|z|^{1/K} \leq C|z|^{1/K}.$$

PROOF. Since normalized K-quasiconformal maps form a compact family, there here is a constant $A = A(K)$ so that

$$f(\{|z| = 1\}) \subset \{\frac{1}{A} < |z| < A\}.$$

By rescaling we also get that for any $0 < r < \infty$

$$f(\{|z| = r\}) \subset \{\frac{|f(r)|}{A} < |z| < A|f(r)|\}.$$

Thus if $r < A^{-2}$,

$$\{A|f(r)| < |z| < \frac{1}{A}\}\} \subset f(\{r < |z| < 1\}) \subset \{\frac{|f(r)|}{A} < |z| < A\}\}.$$

Comparing moduli in the first inclusion we get

$$\frac{1}{2\pi} \log \frac{1}{A^2|f(r)|} \leq M(f(\{r < |z| < 1\})) \leq \frac{K}{2\pi} \log \frac{1}{r},$$

which gives

$$|f(r)| \geq r^K / A^2.$$

The second inclusion similarly gives

$$\frac{1}{2\pi} \log \frac{A^2}{|f(r)|} \geq M(f(\{r < |z| < 1\})) \geq \frac{1}{2\pi K} \log \frac{1}{r},$$

which implies $|f(r)| \leq A^2 r^{1/K}$. Taking $C = A^2$ proves the lemma. □

Corollary 11.3.8. *For each $K \geq 1$ there is a $C = C(K) < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and γ is a circle, then there is $w \in \mathbb{C}$ and $r > 0$ so that $f(\gamma) \subset \{z : r \leq |z - w| \leq Cr\}$.*

Proof. Without loss of generality, we can pre and post-compose so that γ is the unit circle and f fixes $0, 1$. By Lemma 11.3.7, $f(\gamma)$ is then contained in an annulus $\{\frac{1}{C} \leq |z| \leq C\}$, and this gives the result. $\square$

11.4. Locally QC implies globally QC

The definition of quasiconformality requires us to check the moduli of all quadrilaterals. In this section we prove that it is enough to verify the definition just on all sufficiently small quadrilaterals.

Lemma 11.4.1. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ that is K-quasiconformal in a neighborhood of each point of Ω, then f is K-quasiconformal on all of Ω.*

Proof. Suppose $Q \subset \Omega$ is a quadrilateral that is conformally equivalent via a map φ to a $1 \times m$ rectangle R and $Q' = f(Q)$ is conformally equivalent a $1 \times m'$ rectangle R'. Divide R into M equal vertical strips $\{S_j\}$ of dimension $1 \times m/M$. We have to choose M sufficiently large that two things happen.

First choose $\delta > 0$ so that f^{-1} is K-quasiconformal on any disk of radius δ centered at any point of Q' (we can do this since Q' has compact closure in Ω). Next, note that the closure of Q' is a union of Jordan arcs γ corresponding via $f \circ \varphi^{-1}$ to vertical line segments in R. By the continuity of $f \circ \phi^{-1}$ there is an $\eta > 0$ so that if $z \in R$ then $f(\phi^{-1}(D(z, \eta)))$ has diameter $\leq \delta$. By the continuity of the inverse map, there is an $\epsilon > 0$ so that $x, y \in Q'$ and $|x - y| < \epsilon$ implies $|\varphi(f^{-1}(x)) - \varphi(f^{-1}(y))| \leq \eta$. Thus for any $\delta > 0$ there is an $\epsilon > 0$ so that if $x, y \in \gamma \subset Q'$ are at most distance ϵ apart, then the arc of γ between then has diameter at most δ (and ϵ is independent of which γ we use).

Choose M so large that each region $Q'_j = f(\varphi^{-1}(S_j))$ contains a disk of radius at most ρ, where ρ will be chosen small depending on ϵ. Map Ω_j conformally to a $1 \times m'_j$ rectangle R'_j. By Lemma 5.5.5 there is an absolute constant C so that every for every $y \in [0, 1]$, there is a $t \in (0, 1)$ with $|t - y| \leq Cm_j$ and so that the horizontal cross-cut of R'_j at height t maps via ϕ_j^{-1} to a Jordan arc of length $\leq C\rho$. Thus we can divide R'_j by horizontal cross-cuts into rectangles $\{R'_{ij}\}$ of modulus $m'_{ij} \simeq 1$ so that the preimages of these rectangles under ϕ_j are quadrilaterals with two opposite sides of length $\leq C\rho$ and which can be connected inside the quadrilateral by a curve of length $\leq C\rho$.

Taking δ as above, choose ϵ as above corresponding to $\delta/4$ and choose ρ so that $3C\rho < \min(\epsilon, \delta/4)$. Then all four sides of the quadrilateral Q'_{ij} have diameter $\leq \delta/4$ and hence Q'_{ij} has diameter less than δ and hence lies in a disk where f^{-1} is K-quasiconformal. Let m_{ij} be the modulus of corresponding preimage quadrilateral $Q_{ij} = f^{-1}(Q'_{ij})$. See Figure 9.

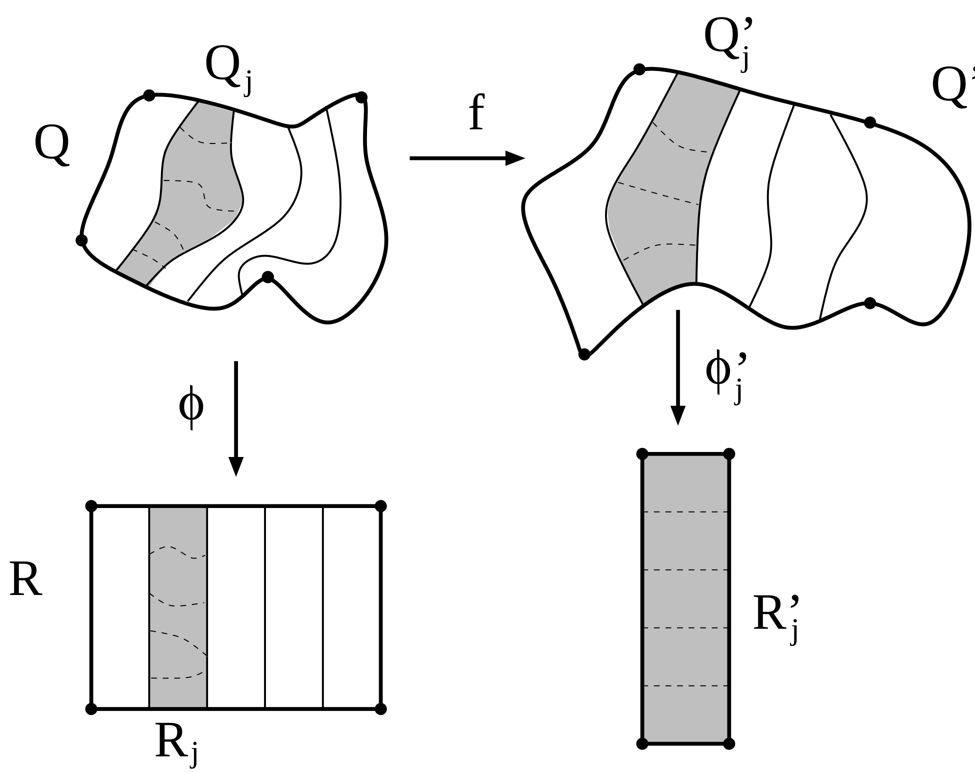


FIGURE 9. Notation in the proof of Theorem 11.4.1.

Then using the rules of extremal length

$$\frac{M}{m} \geq \sum_i \frac{1}{m_{ij}}, \qquad \frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}, \qquad m' \geq \sum_j m'_j,$$

and by the definition of K-quasiconformal,

$$\frac{1}{K} \le \frac{m_{ij}}{m'_{ij}} \le K.$$

Hence

$$\frac{M}{m} \ge \sum_i \frac{1}{m_{ij}} \ge \frac{1}{K} \sum_i \frac{1}{m'_{ij}} = \frac{1}{Km'_j}$$

or

$$\frac{m}{M} \le Km'_j$$

for every j. Thus

$$m \le \sum_{j=1}^{M} \frac{m}{M} \le \sum_j Km'_j \le Km'.$$

Applying the same result to the inverse map shows f is K-quasiconformal. $\square$

If $K = 1$, then $m = m'$ the last line of the above proof becomes

$$m' = m \le \sum_j \frac{m}{M} \le \sum_j m'_j \le m'.$$

so we deduce

$$\sum_j m'_j = m',$$

whereas in general, we only have $\sum_j m'_j \le m'$. We want to use this to deduce that 1-quasiconformal map must be conformal. We start with

LEMMA 11.4.2. *Consider a $1 \times m$ rectangle R that is divided into two quadrilaterals Q_1, Q_2 of modulus m_1 and m_2 by a Jordan arc γ the connects the top and bottom edges of R. Then if $m = m_1 + m_2$, the curve γ is a vertical line segment.*

PROOF. See Figure 10. Let φ_1, φ_2 be the conformal maps of Q_1, Q_2 onto $1 \times m_1$ and $1 \times m_2$ rectangles R_1, R_2 respectively. Set $\rho = |f'_1|$ on Q_1 and $\rho = |f'_2|$ in Q_2 and zero elsewhere. Then each horizontal line is cut by γ into pieces one of which connects the left vertical edge of R to γ, and another that connect γ to the right edge of R. The images of these connect the vertical edges of R_1 and R_2 respectively. Thus the images have lengths at least m_1 and m_2 respectively, there length of the image of the entire horizontal segment in Q is $\geq m_1 + m_2$. If we integrate over all horizontal segments in Q, we see

$$\int_Q (\rho - 1) dxdy \geq m_1 + m_1 - m = 0.$$

Similarly,

$$\int_Q (\rho^2 - 1) dxdy = \text{area}(f_1(Q_1) + \text{area}(f_2(Q_2)) - \text{area}(q) = (m_1 + m_2) - m = 0.$$

Thus

$$\int_Q (\rho - 1)^2 dxdy = \int_Q (\rho^2 - 1) - 2(\rho - 1) dxdy = 0.$$

Since $(\rho - 1)^2 \geq 0$, this implies $\rho = 1$ almost everywhere, i.e., f_1 and f_2 are most linear and the curve γ is a vertical line segment. $\square$

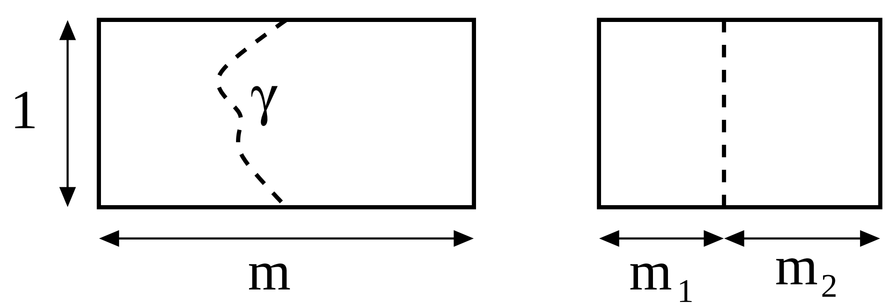


FIGURE 10. A partition of a rectangle as in the proof of Lemma 11.4.3.

LEMMA 11.4.3. *If f is 1-quasiconformal on Ω, then it is conformal on Ω.*

PROOF. If f is 1-quasiconformal in the proof of Theorem 11.4.1, then as noted before Lemma 11.4.2, we must have

$$\frac{M}{m} = \sum_i \frac{1}{m_{ij}}, \qquad \frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}, \qquad m' = \sum_j m'_j,$$

Thus the map $\psi = \varphi' \circ f \circ \varphi^{-1}$ between identical rectangles must be the identity map. Thus $f = (\varphi')^{-1} \circ \varphi$ is a composition of conformal maps, hence conformal. $\square$

11.5. The Perron process and uniformization of planes

The uniformization theorem states that any simply connected Riemann surface is conformally equivalent to either the 2-sphere $\mathbb{S}$, the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. If the surface is non-compact, the sphere is eliminated and surface must be equivalent to either $\mathbb{C}$ or $\mathbb{D}$. These two choices can be distinguished using extremal length: choose a compact connect set K on the surface and consider the set of rectifiable paths that separate K from ∞. If this family has finite modulus, then the surface is equivalent to the disk and otherwise the modulus is infinite and the surface is equivalent to the disk.

For our applications, we only need to use the uniformization theorem in the case when R is built by attaching Euclidean triangles along their edges in a way that is combinatorially identical to the usual triangulation of the plane by identical equilateral triangles. See Figure **??**.

Subharmonic function play an important role in the Perron process for solving the Dirichlet problem on a planer domain or Riemann surface. Suppose Ω is a Riemann surface and we are given a collection of subregions $\{\Omega_j\}$ on which we can solve the Dirichlet problem (e.g., a collection of disks, where we can use the Poisson formula). If $f \in C(\partial\Omega)$ and $\partial\Omega$ is compact, then f has a lower bound M. Let $\mathcal{F}$ be the collection of subharmonic functions v on Ω that have continuous boundary values less than f on $\partial\Omega$. The collection is non-empty since the constant M is in it.

Let $u = \sup\{v : v \in \mathcal{F}\}$. We claim u is harmonic. It is clearly subharmonic since it is a supremum of subharmonic functions. IN each Ω_n we can solve the Dirichlet problem in Ω_n with boundary data u; if u were not harmonic in Ω_n, replacing u with this solution in Ω_n would give a strictly larger element of $\mathcal{F}$.

The final step is to prove that u has the correct boundary values. This requires some assumption on $\partial\Omega$, since it is not true the Dirichlet problem can be solved for every domain.

EXERCISE: Show that if $\Omega = \mathbb{D} \setminus \{0\}$ and we set $f = 1$ on $\mathbb{T}$ and $f = 0$ at 0, then the function u created by the Perron process is the constant 1, and hence does not solve the Dirichlet problem. Indeed, there is no harmonic function on Ω with the given boundary values.

We say a barrier exists at $x \in \partial\Omega$ if there is a $r > 0$ and a non-negative, harmonic function V on $\Omega' = \Omega \cap D(x, r)$ so that

$$\limsup_{z \to x} V(x) \leq 0,$$

but

$$\liminf_{z \to y} V(x) > 0, y \in \partial\Omega' \setminus \{x\}$$

and $V \geq 1$ on $\{|z - x| = r\} \cap \Omega$.

LEMMA 11.5.1. *If there is a barrier at x then the Perron solution u extends continuously to x and equals f there.*

PROOF. First consider the special case when f takes values in $[0, 1]$ and x is the unique point where f takes the minimal value 0. Suppose $v \in \mathcal{F}$. Since v is subharmonic on $\Omega' = \Omega \cap D(x, r)$ and bounded above by 1, it is bounded above by V □

11.6. The measurable Riemann mapping theorem, Part I

The main motivating example is when $\Omega = \mathbb{C}$, Γ is a triangulation of the plane and μ_h is constant on the interior of each triangle. Such maps arise as piecewise linear maps between compatible triangulations, but there are many other examples, as the following shows.

THEOREM 11.6.1. *Suppose Γ is a triangulation of the plane, $0 \leq k < 1$ and $\mu(z)$ is constant on the interior of each triangle with $|\mu| < k$. Then there is a homeomorphism f of the plane with $\mu_f = \mu$.*

PROOF. For each triangle T let A be the affine map with dilatation $\mu(T)$ and T_μ be the image of T under A. Form an Riemann surface by identifying the triangles T_μ along the same edges as in Γ. This defines a Riemann surface that is quasiconformally equivalent to the plane via the map $\Phi : R \to \mathbb{C}$ that is affine on each triangle. By the uniformization theorem, there is also a conformal map $\Psi : R \to \mathbb{C}$ (since R is simply connected and not-compact, it is conformally equivalent to either the disk or the plane and since it quasiconformally equivalent to the plane we know the extremal length of the path family connected an disk to ∞ on R is infinite, and hence it must be conformally equivalent to the plane). Then $\Psi \circ \Phi^{-1} : \mathbb{C} \to \mathbb{C}$ is quasiconformal with dilatation μ. $\square$

THEOREM 11.6.2. *For any measurable μ on the plane with $|\mu| \le k < 1$, there is a quasiconformal map f with $f = \lim_n f_n$ and $\mu_n = \mu_{f_n}$ where $\{\mu_n\}$ satisfy the conditions of Theorem 11.6.1 and $\{f_n\}$ are the corresponding maps.*

PROOF. Take the standard triangulation of the plane (see Figure **??**) and a series of refinements by subdividing each triangle into four sub-triangles. Define a piecewise constant dilatation on the nth triangulation by taking the average of μ on each triangle and let $\{f_n\}$ be the corresponding sequence of quasiconformal maps, normalized to fix $0, 1, \infty$. Since these are all quasiconformal with the same bound, they form an equicontinuous family and we can extract a subsequence that converges uniformly on compact subsets of the plane. The limit function f is also K-quasiconformal by Lemma **??**.

If μ is continuous on a disk D, then the dilatations μ_n converge uniformly to μ on $\frac{1}{2}D$ and so

$\square$

LEMMA 11.6.3. *If the dilatation is symmetric with respect to a circle (or line), the corresponding quasiconformal function can be chosen to be symmetric with respect to the same circle (or line).*

COROLLARY 11.6.4. *If f is piecewise continuous K-quasiconformal on an open set $\Omega \subset \mathbb{C}$ then there is a K-quasiconformal map $g : \mathbb{C} \to \mathbb{C}$ so that $f \circ g$ is conformal on Ω.*

PROOF. The dilatation μ of f is defined on Ω and set it to be zero on the rest of the plane. Apply the construction above to generate a sequence $\{g_n\}$ and limit g. Then $gn \circ f$ $\square$

COROLLARY 11.6.5. *If $f : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal an onto, and we extend f to a map $\mathbb{C} \to \mathbb{C}$ by reflection*

$$f(1/\bar{z}) = 1/\overline{f(z)},$$

then the extension is K-quasiconformal on the whole plane.

Corollary 11.6.6. *If φ is K-quasiconformal on $\mathbb{C}$ and g is holomorphic on Ω then there is a K-quasiconformal map ψ on Ω such that $f = \varphi \circ g \circ \psi^{-1}$ is holomorphic on $\psi(\Omega)$.*

In this case the functional relation can be rewritten as

$$f \circ \psi = \varphi \circ g.$$

If $\Omega = \mathbb{C}$, such a pair of functions f and g are called **quasiconformally equivalent**. We will examine such pairs in more detail in later sections (see Sections **??**, **??**).

Corollary 11.6.7.

11.7. Removable sets for quasiconformal maps

When f is continuously differentiable, it is relatively easy to check whether it is quasiconformal; we just compute the complex dilatation $\mu = f_{\bar{z}}/f_z$ and check that $|\mu| < k < 1$ everywhere. For some applications in dynamics, functions arise that that are homeomorphisms f on $\mathbb{C}$, but which are only C^1 on an open set $\Omega = \mathbb{C} \setminus K$ on an open set $\Omega = \mathbb{C} \setminus K$. If we know the dilatation is bounded on just Ω, can we still deduce that f is quasiconformal? If we can, then we say K is removable for quasiconformal mappings.

This depends on the "size" and "shape" of K. If K has interior, then it is easy to construct counterexamples; choose a disk $D \subset K$ and any non-quasiconformal homeomorphism of the disk to itself that is the identity on the boundary and extend it to be the identity off D. If K has positive area, there are also counterexamples corresponding to applications of the measurable Riemann mapping theorem to a dilatation that is a non-zero constant on K and zero off K. Even if K is quite small, there can be counter examples. For example, given any guage function h such that $h(t) = o(t)$ as $t \searrow 0$, there is a closed Jordan curve γ and a homeomorphism of the sphere that is conformal on both components of $\mathbb{C} \setminus \gamma$ but which is not Möbius (see e.g., [**28**], [**29**], [**76**]). On the other hand, if K has finite or sigma-finite 1-measure then it is removable. These examples show that it is the "shape" rather than the "size" of K that is crucial in most cases of interest.

In this section we give an elegant sufficient condition for K to be removable that is due to Peter Jones and Stas Smirnov [**75**], generalizing an earlier result of Jones [**74**].

A**Whitney decomposition** of an open set Ω consists of a collection of dyadic squares $\{Q_j\}$ contained in Ω so that

(1) the interiors are disjoint,
(2) the union of the closures is all of Ω,
(3) for each Q_j, $\mathrm{diam}(Q_j) \simeq \mathrm{dist}(Q_j, \partial\Omega)$.

The existence of such a collection is easy to verify be taking the set of dyadic squares Q so that

$$\mathrm{diam}(Q) \leq \frac{1}{4}\mathrm{dist}(Q, \partial\Omega),$$

and that are maximal with respect to this property (i.e., the parent square fails this condition).

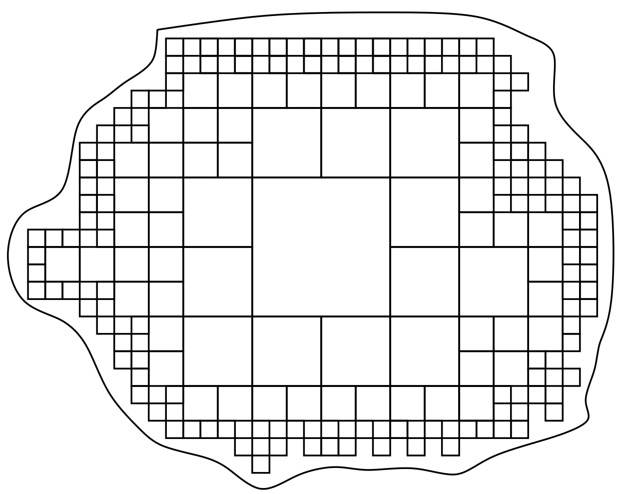


FIGURE 11. A Whitney decomposition.

Suppose K is compact, $\delta > 0$ and for each $x \in K$ let γ_x be a Jordan arc in $\Omega = \mathbb{C} \setminus K$ that connects x to $\Omega_\delta = \{z \in \Omega : \mathrm{dist}(z, K) \geq \delta\}$. For a single x, γ_x may consist of several arcs that connect x to Ω_δ. See Figure 12.

For each Whitney square $Q \subset \Omega$, let

$$S(Q) = \{x \in K : \gamma_x \cap Q \neq \emptyset\}.$$

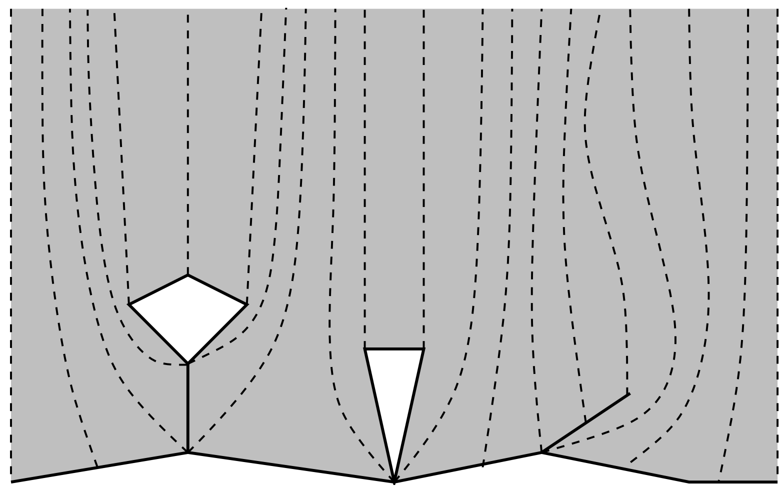


FIGURE 12. Each boundary point is connected to a point distance δ from $\partial\Omega$. Some points may be connected by more than one curve.

This is called the "shadow" of Q on K; the name comes from the special case when K is connected and does not separate the plane and γ_x is a hyperbolic geodesic connecting x to ∞. If we think of ∞ as the "sun" and the geodesics as light rays, then $S(Q)$ is the part of K that blocked from ∞ by Q, i.e., it is Q's shadow. See Figure **??**.

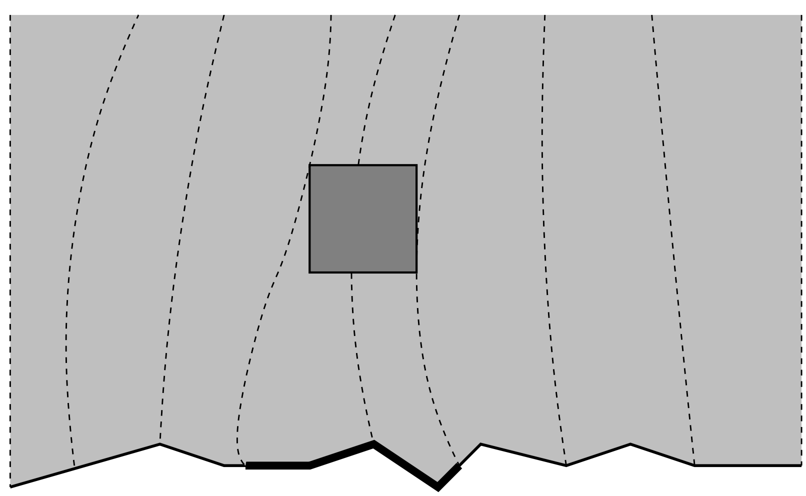


FIGURE 13. The shadow of a Whitney square on the boundary.

The immediate shadow $I(Q) \subset S(Q)$ is the closure of all $x \in S(Q)$ so that Q is the first Whitney square of that size hit by γ_x as we traverse it from x to Ω_δ.

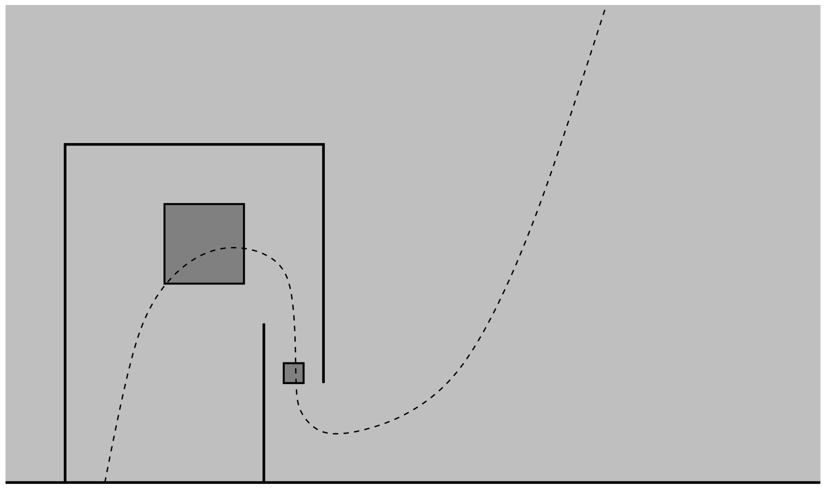


FIGURE 14. The paths connecting a Whitney square to its shadow can sometimes hit larger Whitney squares. However this path will hit a largest square, and there after only hit smaller squares.

Given $x \in I(Q)$, we let $\mathcal{I}(x, Q)$ to be all the dyadic squares for Ω that are hit by γ_x between x and Q, i.e., this is an infinite chain of Whitney squares that starts at Q and accumulates on x and has Q as its unique largest square.

We will assume three things about shadows:

(1) $I(Q)$ is closed.
(2) $\lim_{n\to\infty} \sum_{Q\in\mathcal{D}_n(\Omega)} \mathrm{diam}(I(Q))^2 = 0$ where the sum is over all Whitney squares for Ω of side length 2^{-n}.
(3) $\mathrm{dist}(I(q), Q) \to 0$ as $\mathrm{diam}(Q) \to 0$,

These will hold in most situations we are interested in. For example, if Ω is simply connected and we take γ_x to be arcs of hyperbolic geodesics connecting some base point $z_0 \in \Omega$ to x, then (2) always holds, (3) holds if $\partial\Omega$ is locally connected, and (1) holds if Ω is a John domain.

An open, connected set Ω in $\mathbb{R}^2$ is called a **John domain** if any two points $a, b \in \Omega$ can be connected by a path γ in Ω with the property that $\mathrm{dist}(z, \partial\Omega) \gtrsim \min(|z-a|, |z-b|)$. See Figure 15

LEMMA 11.7.1. *Suppose Q is a square, $\lambda > 1$ and f is K-quasiconformal on λQ. Then*

$$\mathrm{area}(f(Q)) \geq \epsilon \mathrm{diam}(f(Q))^2,$$

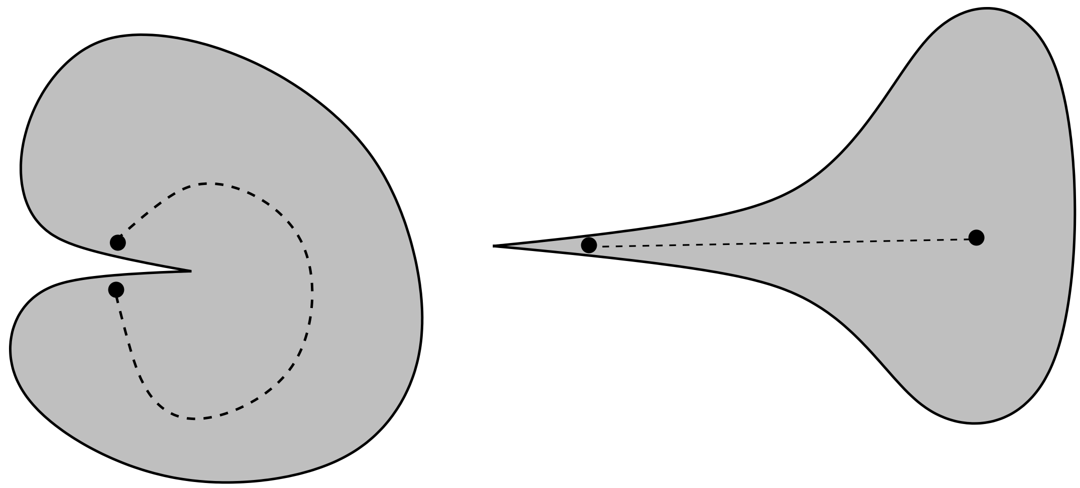


FIGURE 15. The domain on the left is a John domain, but the one on the left is not; inward pointing cusps are OK, but outward pointing cusps are not.

where $\epsilon > 0$ *depends only on* λ *and* K.

PROOF. By rescaling by conformal linear maps we may assume the square Q is $[-2,2] \times [-2,2]$ and the map f fixes 0 and 1. Choose $x \in \partial Q$ and connect x to 0 and connect 1 to $\lambda \partial Q$ by disjoint curves γ_0, γ_1 so that the annular region $\lambda Q \setminus (\gamma_0 \cup \gamma_1)$ has modulus $\simeq 1$ with a constant that depends on λ (and decreases as λ increases. See FIGURE ????.

The image of this annular region has modulus bounded away from 0 and ∞ and this implies $f(\gamma_0)$ is bounded in terms of K (otherwise, as in the proof of Lemma 11.3.2 we could define a metric $\rho(z) = 1/|z|$ on $1 < |z| < R$ and show that the path family separating $f(\gamma_0)$ from $f(\gamma_1)$ has very small modulus). Thus $\mathrm{diam}(f(Q))$ is bounded in terms of K alone.

Now consider the modulus of $A = Q \setminus [0,1]$. Again this is a fixed number $\simeq 1$, so the modulus of $f(A)$ is bounded away from zero. But every curve surrounding $f([0,1])$ has length at least 2, so the metric $\rho = 1/2$ is admissible, so

$$\mathrm{mod}\ (f(A)) \leq \frac{1}{4} \mathrm{area}(f(Q)).$$

Since the left hand side is bounded away from zero depending only on K, so is right hand side. See Figure ???. $\square$

THEOREM 11.7.2. *Suppose* Ω *has a Whitney decomposition so that the corresponding shadow sets satisfy conditions (1)-(3) above. Suppose that* f *is a homeomorphism of the plane that is* K*-quasiconformal on each component of* $\mathbb{R}^2 \setminus \partial\Omega$ *and that there is an* $M < \infty$ *so that*

$$\operatorname{dist}(f(Q_j), f(Q_{j+1})) \leq M \max(\operatorname{diam}(Q_j), \operatorname{diam}(Q_{j+1})), \tag{41}$$

whenever Q_j, Q_{j+1} *are consecutive squares in the chain associated to some* $x \in \partial\Omega$*. Then* f *is a* C*-quasiconformal map on the whole plane where* C *depends only on* K *and* M*.*

If the chain associated to each $x \in \partial\Omega$ consists of adjacent squares (i.e., Q_j touches Q_{j+1}, then the same is true for their images under f, so condition (41) is automatically satisfied. Thus we obtain:

COROLLARY 11.7.3. *Suppose* Ω *has a Whitney decomposition so that the corresponding shadow sets satisfy conditions (1)-(3) above and all the Whitney chains are connected. The* $\partial\Omega$ *is removable to quasiconformal homeomorphisms, i.e., any homeomorphism of the plane that is* K*-QC off* $\partial\Omega$ *is quasiconformal on the whole plane.*

This is the version given by Jones and Smirnov (restricted to the plane). We have stated the more general version with (41) in order to include certain maps arising from groups of circle reflections where we require disconnected Whitney chains, but for which (41) is automatically fulfilled.

In both the theorem and the corollary if if the map f is conformal off $\partial Omega$ (i.e., $K = 1$), then we will show that the extension is conformal everywhere. If the map f is K-quasiconformal off $\partial\Omega$ then we only prove that it is C-quasiconformal for some $C < \infty$. However, it follows from this that f is actually K-quasiconformal on the whole plane. Our hypotheses imply that $\partial\Omega$ has zero area and hence $|\mu_f\| \leq (K-1)/(K+1)$ almost everywhere and this implies f is K-quasiconformal if we use the analytic definition of quasiconformality (which we are delaying until a later chapter). The weaker version will be sufficient for our applications.

PROOF OF THEOREM 11.7.2. Suppose that W is any bounded quadrilateral in the plane, say of modulus m and that $W' = F(W)$ has modulus m'. We want to show that $m' \leq Cm$ where $C < \infty$ depends only on K and M as in the statement of the theorem. We will do this by mimicking the proof of Theorem 11.2.1, that showed that any piecewise differentiable map with bounded dilatation was quasiconformal (in the geometric sense).

Let $\varphi : W \to R = [0, m] \times [0, 1]$ and $\psi : W' \to [0, m'] \times [0, 1]$ be conformal maps of the quadrilaterals Q, Q' to rectangles R, R' of the same modulus. Let $X = \varphi(\partial\Omega \cap W) \subset R$. The main difficulty with the proof is that we are going to consider three different Whitney decompositions: one for W, one for Ω and one for $U = R \setminus X$. To try to differentiate the different Whitney cubes we we let $\{W_j\}$ denote a Whitney decomposition for W, $\{Q_j\}$ a Whitney decomposition for Ω and $\{U_j\}$ a Whitney decomposition for U.

Fix some $\epsilon > 0$. Fix a Whitney cube W_j for W. We assume the decomposition is chosen so that $2W_j \subset W$. Suppose $\delta > 0$ is so small (depending on our choice of W_j) that the following conditions all hold:

(1) If Q_k is a Whitney square for Ω with diameter less than δ and the shadow $I(Q_k)$ hits W_j, then $I(Q_k) \subset 2W_j$ and the entire Whitney chain connecting any point $x \in I(Q_k)$ to Q_k is contained in $2W_j$. This is possible by condition (3) on shadow sets.

(2) Let $\mathcal{S}(W_j)$ denote the collections of all Whitney squares Q_k for Ω so that $\mathrm{diam}(Q_k) \leq \delta$ and $I(Q_k)) \cap W_j \neq \emptyset$. Then

$$\sum_{Q_k \in \mathcal{S}(W_j)} \mathrm{diam}(I(Q_k))^2 \leq \epsilon \mathrm{area}(W_j).$$

This holds for small enough δ, because by condition (2) on shadows, this sum over all Whitney squares for Ω is finite, so removing all the squares bigger than δ gives a sum that tends to 0 as δ tends to zero. Thus we can make is less than $\epsilon\mathrm{area}(W_j)$ by taking δ small enough (depending on W_j).

Let $\mathcal{S} = \cup_{W_j} \mathcal{S}(W_j)$ be the collection of all shadow sets of all Whitney squares for Ω that are in some $\mathcal{S}(w_j)$ for some Whitney square of W.

Claim: $\partial\Omega \cap W_j$ is covered by a finite number of the shadow sets $I(Q_k)$ with $Q_k \in \mathcal{S}(W_j)$.

PROOF OF CLAIM. Each point $x \in \partial\Omega \cap W_j$ is associated to a Whitney chain that contains a square with diameter comparable to δ. There are only finitely many such squares, so their shadows form a finite collection that covers $\partial\Omega \cap W_j$. $\square$

Suppose $L = [a+iy, b+iy]$ is a horizontal segment, compactly contained in the interior of R at height y. We wish to show that

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \leq Cm,$$

where C depends only on K and M. If we can do this, then by letting $a \to 0$ and $b \to m$ we get

$$m' \leq \lim a \to 0, b \to m |g(b+iy) - g(a+iy)|,$$

and hence

$$m' \leq \lim a \to 0, b \to m \int_0^1 |g(b+iy) - g(a+iy)| dy \leq Cm,$$

which is the desired inequality.

Since L is compactly contained in the interior of R and X is relatively closed in the interior of R, $L \cap X$ is compact. Thus $\varphi^{-1}(L \cap X)$ is a compact set of W, hence covered by finitely many whitney squares for W and hence is covered by finitely many shadows sets in $\mathcal{S}$.

Let $\mathcal{X}$ be the image of the elements of $\mathcal{S}$ under φ. Then $L \cap X$ is covered by finitely many elements of $\mathcal{X}$, say $X_1, \ldots X_n$. For $k = 1, \ldots, n$, let $Y_k = [a_k, b_k]$ be the smallest closed interval in L that contains X_n (this is the convex hull of X_k, the interval with the same leftmost and rightmost point as X_k). Then $Y_1, \ldots, Y_n$ also cover $L \cap X$ and we can extract a subcover with the property that $Y_j \cap Y_k \neq \emptyset$ implies $|j - k| \leq 1$.

Since the points a_k, b_k are both in the same set X_k, the preimage points $\varphi^{-1}(a_k), \varphi^{-1}(b_k)$ are both in the same element of $\mathcal{S}$. Thus they are both in the shadow set of some Whitney square for Ω and are associated to a two sided chain of distinct Whitney squares $\{Q_m\}_{-\infty}^{\infty}$ of Whitney squares for Ω. If two chains arising in this way, say from Y_k and Y_m with $m > k$, have a Whitney square in common, then we can combine the chains to form a chain connecting a_k to b_m consisting of distinct squares.

After doing this for all intersections, we end up with a finite collection of closed intervals Z_k in L which covers the same set as the union of the Y_k's and such that the two endpoints of each Z_k correspond to a two-sided Whitney chain in Ω and that different intervals use different Whitney squares (no overlapping chains). Moreover, if Z_k has endpoints c_k, d_k and the corresponding chain is $\{Q_n\}$, then

$$|g(c_k) - g)d_k)| \leq (M+1) \sum_n \mathrm{diam}(\psi(f(Q_n))).$$

The set $V = L \setminus \cup_k Z_k$ consists of finitely many open intervals in $U = R \setminus X$ with their endpoints in X. We break V into countable many sub-intervals by intersecting it with the Whitney squares for U (without loss of generality, we can assume the endpoints of L occur on the boundary of a Whitney square for U). On each Whitney square U_k for U we define the constant function

$$Dg = \frac{\mathrm{diam}(g(U_k))}{\mathrm{diam}(U_k)}.$$

Then if $L_j = L \cap U_j$,

$$\int_{L_j} Dg dx = \mathrm{diam}(g(U_j))/\sqrt{2}.$$

Thus

$$\int_{L \setminus Z_L} Dg dx \simeq \sum_j \mathrm{diam}(g(U_j)),$$

where the sum is over Whitney squares for U that hit L. Thus

$$|g(b+iy) - g(a+iy)| \lesssim \int_{L \cap U} Dg dx + \sum_n \mathrm{diam}(\psi(f(Q_n))).$$

Now integrate in y to get

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \lesssim \iint_U Dg dx + \sum_n \mathrm{diam}(\psi(f(Q_n)))\mu_n,$$

where μ_n is the Lebesgue measure in $[0,1]$ of the set of lines L_y that use the Whitney square Q_n is at least one of the two-sided chains associated to a interval $Z \subset L_y$. The measure of this set is no more than its diameter, which is no more than the diameter of $X_n = \varphi(I(Q_n))$. Thus

$$\int_0^1 |g(b+iy) - g(a+iy)| dy \lesssim \iint_U Dg dx dy + \sum_n \mathrm{diam}(\psi(f(Q_n)))\mathrm{diam}(X_n),$$

We now estimate each term using the Cauchy-Schwarz inequality. First,

$$\begin{aligned}
\sum_n \mathrm{diam}(\psi(f(Q_n)))\mathrm{diam}(X_n) &\le (\sum_n \mathrm{diam}(\psi(f(Q_n)))^2)^{1/2}(\sum_n \mathrm{diam}(X_n)^2)^{1/2} \\
&\le A(\sum_n \mathrm{area}(\psi(f(Q_n))))^{1/2}(\sum_{W_k}\sum_{Q_n\in\mathcal{S}(W_k)}[\frac{\mathrm{diam}(\varphi(W_k)}{\mathrm{diam}(W_k)}\mathrm{diam}(I(Q_n))]^2) \\
&\le A(\sum_n \mathrm{area}(\psi(f(Q_n))))^{1/2}(\sum_{W_k}\sum_{Q_n\in\mathcal{S}(W_k)}[\frac{\mathrm{diam}(\varphi(W_k)}{\mathrm{diam}(W_k)}\epsilon\mathrm{area}(W_k))]^2)^{1/} \\
&\le A(\sum_n \mathrm{area}(R')^{1/2}\epsilon(\mathrm{area}(R)^{1/2}.
\end{aligned}$$

where A just depends on the distortion estimate for conformal maps (Theorem **??**) and ϵ is as small as we wish. Thus this term is small.

The other term is also bounded by Cauchy-Schwarz

$$\begin{aligned}
\iint_U Dgdx &= \sum_k \iint_{U_k} Dgdxdy \\
&\le (\sum_k \iint_{U_k} Dg^2dxdy)^{1/2}(\sum_k \iint_{U_k} dxdy)^{1/2} \\
&\le (\sum_k(\mathrm{diam}(g(U_k))^2)^{1/2}(\mathrm{area}(R))^{1/2} \\
&\le A(\sum_k(\mathrm{area}(g(U_k)))^{1/2}(\mathrm{area}(R))^{1/2} \\
&\le A(area(R')^{1/2}(\mathrm{area}(R))^{1/2} \\
&\le A\sqrt{m'm}.
\end{aligned}$$

Thus

$$\int_0^1 |g(b+iy)-g(a+iy)|dy \lesssim \sqrt{m'm} + O(\epsilon),$$

Taking $\epsilon \to$ gives the desired inequality. □

COROLLARY 11.7.4. *If $\partial\Omega$ has a Whitney decomposition and a collection of shadow sets that satisfy (1)-(3), then any homeomorphism f of the plane that is conformal off $\partial\Omega$ is conformal on the whole plane.*

PROOF. Theorem 11.7.2 implies that f is quasiconformal on the plane, so the point is to show that we can take the quasiconformal constant to be 1. If we redo the proof assuming f is conformal off $\partial\Omega$, then the piecewise constant function Dg can be replaced by the usual derivative $|g'|$. This leads to the inequality

$$m' \leq \sqrt{m'm},$$

or $m' \leq m$, which implies f preserves the modulus of every quadrilateral, hence is 1-quasiconformal, hence is conformal. □

COROLLARY 11.7.5. *If f is a quasiconformal map of the upper half-plane to itself, mapping the real line to itself, then the extension of f to the whole plane by $f(\bar{z}) = \overline{f(z)}$ is quasiconformal in the whole plane.*

PROOF. Immediate from Theorem 11.7.2 since in the upper upper half-plane we can define shadows by vertical projection and these clearly satisfy (1)-(3). □

COROLLARY 11.7.6. *Quasicircles are removable.*

PROOF. If $\Gamma = g(\mathbb{R})$ is a quasiconformal image of the reals and f is a homeomorphism that is quasiconformal on each side of Γ, then $h = f \circ g$ is a homeomorphism that is quasiconformal on each side of $\mathbb{R}$, then quasiconformal on the whole plane. Thus $f = h \circ g^{-1}$ is a composition of quasiconformal maps and hence is quasiconformal. □

LEMMA 11.7.7. *The Riemann map φ from the unit disk to a bounded John domain satisfies*

$$\mathrm{diam}(\varphi(I(Q))) \leq C\mathrm{diam}(\varphi(Q)),$$
$$\mathrm{dist}(\varphi(Q), \varphi(I(Q))) \leq C\mathrm{diam}(\varphi(Q)),$$

for some constant $C < \infty$ and any Whitney square Q and is shadow $I(Q)$.

PROOF. The second inequality follows directly from Lemma 5.5.1 by considering the path family of radial lines connecting Q to I. To prove the first, consider the Whitney-Carleson boxes Q_1 and Q_2 that are adjacent to Q and of the same size. By Lemma 5.5.1 each is connected to its shadow by a radial segment whose image under f has length comparable to $\mathrm{diam}(f(Q))$. Thus there is a geodesic crosscut γ of the disk that passes through Q and whose image has length comparable to $\mathrm{diam}(f(Q))$.

Now suppose x is in the shadow of Q. Any curve connecting 0 to x crosses γ, so any curve Γ connecting $f(0)$ and $f(x)$ crosses $f(\gamma)$ and hence contains a point

$z \in f(\gamma) \cap \Gamma$ that is at most distance $O(\mathrm{diam}(f(Q))$ from $\partial\Omega$. Thus by the definition of John domain, either

$$\mathrm{dist}(f(0), z) = O(\mathrm{diam}(f(Q))),$$

or

$$\mathrm{dist}(f(x), z) = O(\mathrm{diam}(f(Q))).$$

In a bounded domain, the first can only happen for finitely many Qs; for the remainder, the second must hold and hence $f(I(Q))$ is contained in a $O(\mathrm{diam}(f(Q))$ neighborhood of $f(Q)$. □

COROLLARY 11.7.8. *Boundaries of John domains are removable.*

PROOF. The conclusions of the Lemma 11.7.7 easily imply (1)-(3) in Theorem 11.7.2. □

11.8. Definition of quasisymmetric maps

An increasing homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is M-**quasisymmetric** if for all $x \in \mathbb{R}$ and $t > 0$

$$\frac{1}{M} \le \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \le M.$$

This is the same as saying that if I, J are any two intervals with disjoint interiors but a common endpoint then $f(I), f(J)$ have comparable lengths (within a factor of M). We summarize this by saying adjacent intervals of equal length map to comparable intervals. This definition also makes sense for homeomorphisms of the circle to itself, but here we will only deal with the case of homeomorphisms of the real line.

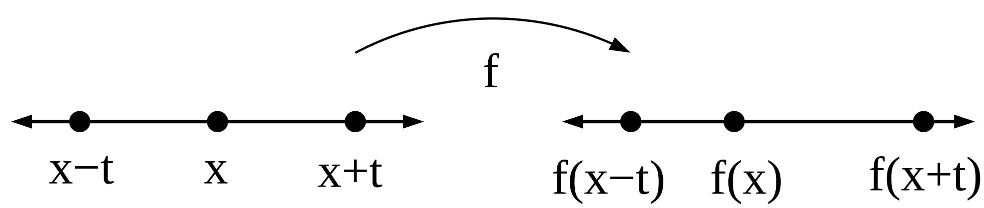


FIGURE 16. The definition of a quasisymmetric homeomorphism of the reals.

LEMMA 11.8.1. *If $f : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping so that $f(\mathbb{R}) = \mathbb{R}$, then the restriction of f to the real line is a quasisymmetric homeomorphism.*

PROOF. After pre- and post-composing by conformal linear maps it is enough to assume that $f(0) = 0$, $f(1) = 1$ and show that $0 < f(\frac{1}{2}) < 1$ is bounded away from both 0 and 1. However, if we consider the topological annuli

$$\Omega_1 = \mathbb{C} \setminus ([0, \frac{1}{2}] \cup [1, \infty)),$$

$$f(\Omega - 1) = \mathbb{C} \setminus ([0, f(\frac{1}{2})] \cup [1, \infty)),$$

we see they must have comparable moduli. The first is a fixed number (the reader can check the modulus is equal to 1 by a symmetry argument) and the second tends to 0 or ∞ if x tends to 1 or 0 respectively. Thus $f(x)$ is bounded away from 0 and 1 in terms of the quasiconstant of f, as desired. See Figure 17. □

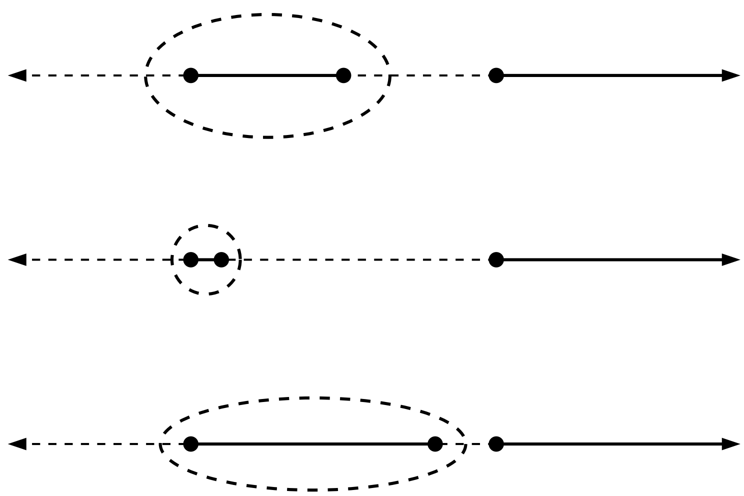


FIGURE 17. A quasiconformal map preserving the line cannot change the modulus of the path family separating $[x - t, x]$ from $[x + t, \infty)$ by more than a bounded factor, from which standard modulus estimates imply that $[x-t, x]$ cannot be either much shorter or much longer than $[x, x + t]$.

It will be helpful to consider a version of quasisymmetry where we only consider comparisons between members of a countable collection of intervals. The most common case is when this collection consists of dyadic intervals, but we will also want to consider a more general case. We say a collection of intervals is A-dyadic if it each interval is divided into two disjoint children with lengths comparable $\geq 1/2A$. When $A = 1$ this gives the usual dyadic intervals. More general examples are given by quasisymmetric images of the dyadic intervals (these are the only examples that we will use).

We will say that an increasing homeomorphism f of the real line is B-quasisymmetric with respect to an A-dyadic family if whenever I is a child of J, then

$$\frac{1}{B} \leq \frac{|f(I)|}{|f(J)|} \cdot \frac{|J|}{|I|} \leq B.$$

We want to check that this implies the usual definition of quasisymmetric, particularly when the constant B is close to 1.

LEMMA 11.8.2. *For any $\epsilon > 0$ and $A < \infty$ there is a $\delta > 0$ so that if $f : \mathbb{R} \to \mathbb{R}$ is $(1+\delta)$-quasisymmetric with respect to an A-dyadic family of intervals, then it is $1+\epsilon$ quasisymetric.*

PROOF. Suppose I, j are adjacent closed intervals of equal length. Without loss of general we may assume this length is 1. Then I contains maximal interval from the A-dyadic collection and this must have length comparable to that of I. Thus I and J are both covered by a bounded number of A-dyadic intervals of comparable length. Now fix a positive integer k and consider kth generation descendents of these intervals. Each has length $O(2^{-k/A})$ so the interval I contains a collection of these intervals that covers all but length $O(2^{-k/A})$. Similarly, J is covered by a collection whose total length is less than $1 + O(2^{-k/A})$. The f images of any two of these intervals are expanded by factors that agree to within $(1+\delta)^k$. Thus

$$\frac{|f(J)|}{|f(I)|} \leq \frac{1+O(2^{-k/A})}{1+O(2^{-k/A})} \cdot (1+\delta)^{O(k)}.$$

Choosing k so large that the first factor on the right is less than $\sqrt{1+\epsilon}$ and then choosing δ so small that the second factor is equally small proves the lemma (since the same argument applies with the roles of I and J reversed). $\square$

LEMMA 11.8.3. *There is a $\epsilon > 0$ so that every $(1+\epsilon)$-dyadic-quasisymmetric map $f : \mathbb{R} \to \mathbb{R}$ has a continuous extension to the closed upper half-plane that is quasiconformal on the open upper half-plane. The quasiconstant K of the extension only depends on ϵ and tends to 1 as $\epsilon \to 0$.*

PROOF. Consider the decomposition of the upper half-plane into dyadic Whitney boxes
$$Q = \{x+iy : x \in I, \frac{1}{2}|I| < y < |I|\},$$
where I ranges over a dyadic decomposition of $\mathbb{R}$. In each such Q we define five vertices: the four obvious corners and the midpoint of the lower edge (which is a corner of two boxes in the next "layer" closer to the boundary). Each such Q can be divided into three triangles by connecting each of its upper corners to the midpoint of its lower side, as shown in Figure **??**.

If $(x+iy)$ is a vertex, then x is the endpoint of two disjoint dyadic intervals I, J of length y. For each vertex define
$$f(x+iy) = f(x) + \min(|f(I)|, |f(J)|),$$
and extend f linearly to each triangle. Each Q contains three triangles. In the "left" and "right" triangle the map we have defined is clearly non-degenerate and preserves orientation. To show this for the center triangle, we need to check that the bottom vertex maps to a point that is lower than either of the two triangle vertices. However, the intervals used to define the images of these points map to intervals at least $2(1+\epsilon)^3$ times as long as the intervals used for the center point. So if ϵ is small enough f is a homeomorphism. Indeed, if $\epsilon < 2^{1/3} - 1$ then the angles of all the image triangles are bounded away from 0 and π so that f is uniformly Q on every triangle.

As ϵ tends to zero, the image triangles tend to Euclidean similarities of the domain triangles and hence the quasiconformal constant tends to 1. $\square$

11.9. Factoring quasisymmetric maps

The proof of extension given above breaks down if the quasisymmetric constant is large. To handle this case we will prove that any quasisymmetric map f can be written as a composition of quasisymmetric maps
$$f = f_n \circ \cdots \circ f_1,$$
all with small constant. Extending these maps and composing the extensions then gives a quasiconformal extension of f. A variety of other proofs are know, e.g., a formula valid for all quasisymmetric maps was given by Alhfors and Beurling (see [**24**] or Chapter IV.B of [**3**]) and a particularly elegant extension is given by Douady and

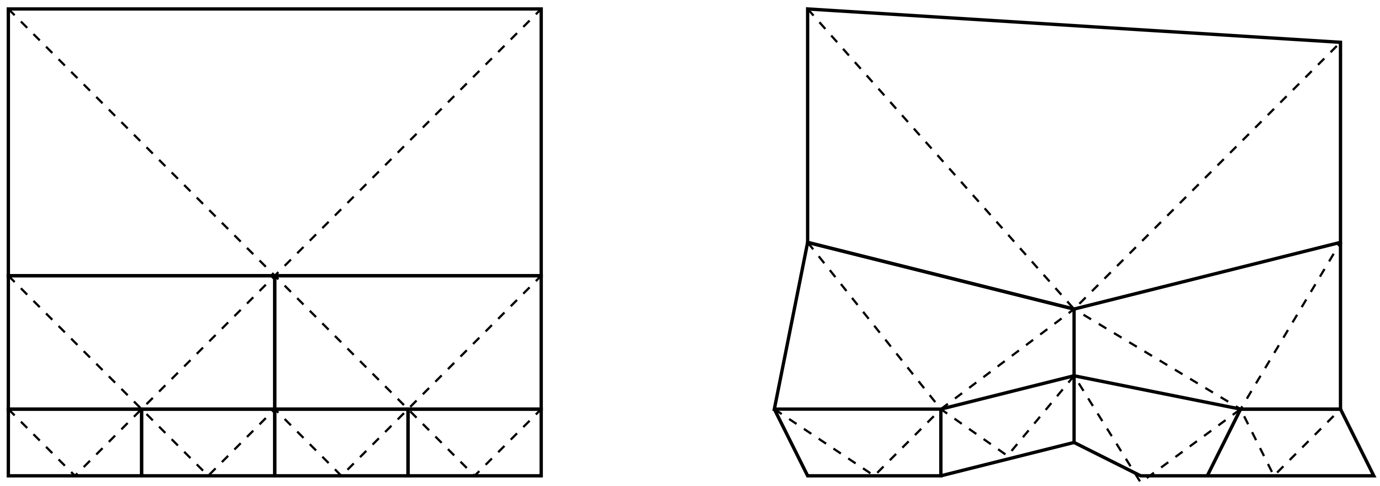


FIGURE 18. A quasisymmetric map with constant close to 1 can easily be extended to a piecewise linear quasiconformal map of the upper half-plane.

Earle in [**43**]. The latter extension commutes with linear fractions transformations of the upper half-plane and this makes if particularly useful for the study of Fuchsian and Kleinian groups.

LEMMA 11.9.1. *Any M-quasisymmetric map $f : \mathbb{R} \to \mathbb{R}$ can be written as a composition of $(1+\epsilon)$-dyadic-quasisymmetric maps.*

PROOF. The proof is essentially just a sequence of pictures that make the statement obvious. The non-obvious idea behind the pictures is to describe homeomorphisms of the real line in terms of hyperbolic earthquakes. This is due to William Thurston [**136**], and has been studied by many others e.g., see [**124**], [**122**] and their references.

An earthquake map in the Euclidean plane would consist of a choice of a line and a map f that is the identity on one side of the line and a translation parallel to the the line on the other side. In the hyperbolic upper half-space, the corresponding map consists of a choice of a hyperbolic geodesic and a map that is the identity on one side of the geodesic and a hyperbolic isometry on the the other side that restricts to a translation on the geodesic. The easiest case occurs when the geodesic is the vertical line $i\mathbb{R}_+$ and the map equals the identity in the second quadrant and equals

$z \to e^{\lambda} z$ in the first quadrant. Here $\lambda \in \mathbb{R}$ is the hyperbolic translation length along the chosen geodesic.

Instead of considering a single geodesic, we want to consider a tesselation of the entire upper half-plane into ideal hyperbolic triangles. We start with a decomposition of the real line into standard dyadic intervals of the form $[j \cdot 2^k, (j+1)2^k)$ where j is any integer and $k \leq 0$. Note that all these intervals have length ≤ 1. Each such dyadic interval is subdivided into two dyadic intervals of half the length that we call its children. Similarly, each dyadic interval is contained in a dyadic interval of twice the length called its parent.

We shall take a slightly non-standard way to define dyadic intervals with length larger than 1. For each $k > 0$, each interval of generation k is a union of two previously defined intervals of generation $k-1$ chosen so that the origin is in the middle half of some kth generation interval. This is clearly possible as illustrated in Figure 19. The point of this is to insure that any two intervals in our collection have a common ancestor. In the standard dyadic system this is not true since intervals in the positive and negative reals never have a common ancestor, but in our system the intervals containing the origin cover the whole line, hence and two bounded intervals are both contained in one of these.

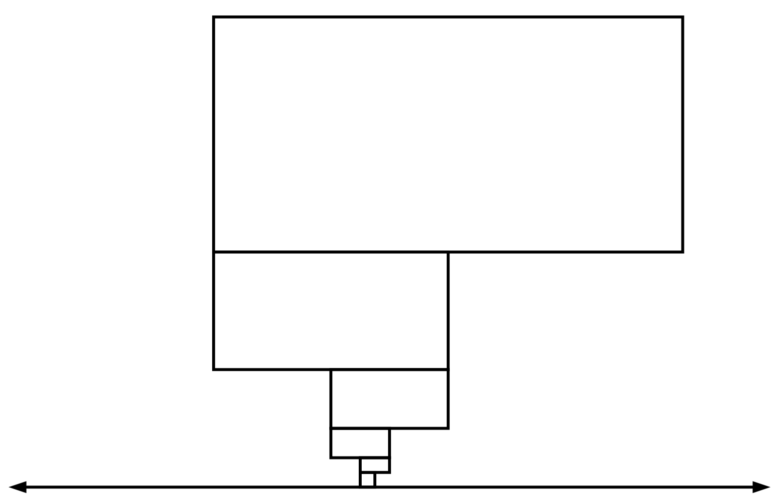


FIGURE 19. We use a non-standard version of the dyadic intervals to insure that any two adjacent intervals have a common ancestor.

To each dyadic interval I in our collection, we have an ideal triangle $T(a, m, b)$ with vertices at the endpoints a, b and midpoint m of I. The collection of all such triangles is partially illustrated in Figure 20.

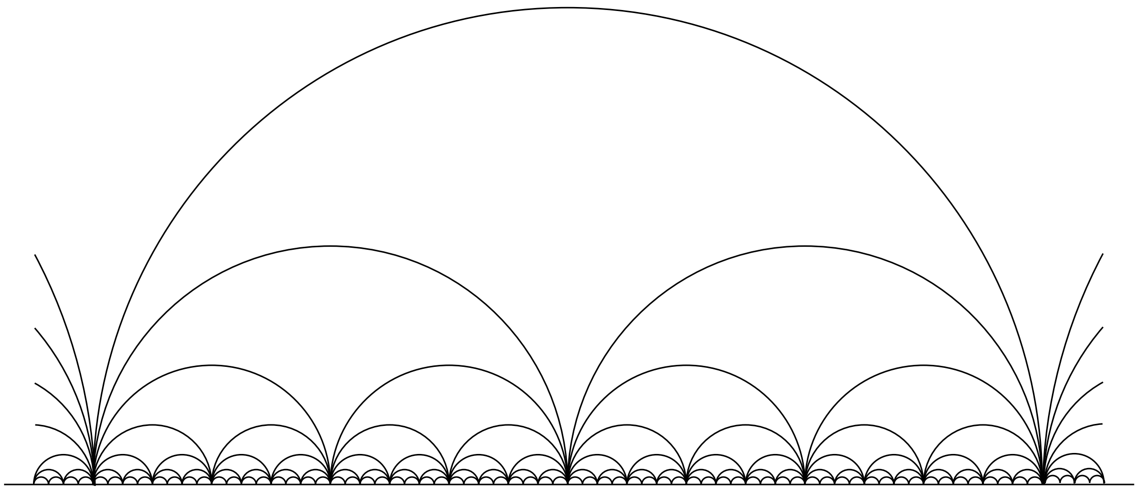


FIGURE 20. A tesselation of the upper half-plane by ideal hyperbolic triangles. The “base” of each triangle on the real line is from the alternating dyadic collection described in the text. Thus any two triangles can be connected by a curve in the upper half-plane that crosses only finitely many other triangles.

If f is an increasing homeomorphism of the real line and I is a interval then there is a hyperbolic isometry τ that sends $T(a, m, b)$ to the ideal triangle $T(f(a), f(m), f(b))$. If σ is the linear map that takes $f(a)$ to a and $f(b)$ to b then $\sigma \circ \tau$ is a hyperbolic isometry that fixes both a and b and maps m to some point between a and b. If f is quasisymmetric then this point is bounded away from both a and b in terms of the quasisymmetric constant. This means that the isometry acts as a bounded hyperbolic translation along the geodesic connecting a and b. Thus a quasisymmetric map gives rise to an earthquake map of the upper half-plane that is isometric on every ideal triangle in our tesselation and so that on two triangles sharing a geodesic edge, the maps differ by a bounded hyperbolic translation. More concisely, quasisymmetric maps give bounded earthquakes.

Conversely, it is easy to see that any bounded earthquake gives a dyadic-quasisymmetric map. Consider two adjacent dyadic intervals, as in Figure 21. By definition, these two intervals have a common ancestor, so they can be connected by a finite chain of adjacent ideal triangles that all have the a common endpoint. If we move this point to ∞ by inversion, then the ideal triangles all have a vertex at infinity (as in 22) and the earthquake maps are are all Euclidean similarities in each ideal triangle. Since I, J have the same length, so do there inverted images, and since $f(I)$ and $f(J)$ have comparable lengths so do their inverted versions (when the their common endpoint is mapped to $infty$.

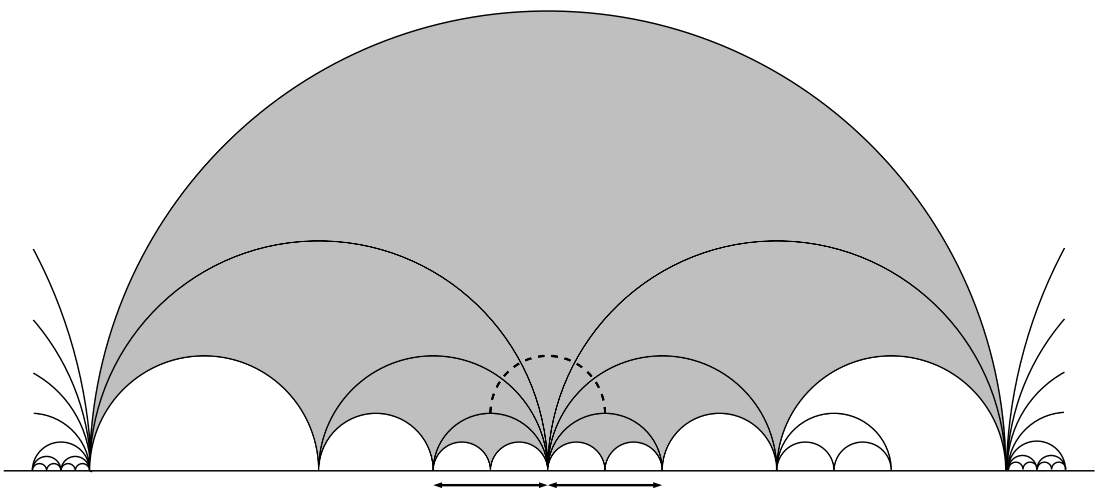


FIGURE 21. Any two adjacent dyadic intervals have a common ancestor and hence the corresponding ideal triangles are connected by a curve that crosses only finitely many triangles in the tesselation.

As we move across the inverted triangles, the earthquake map has a jump corresponding to some Euclidean dilation factor e^{λ_k} between each triangle and the total change is the product of these numbers. Since f is quasisymmetric this product is bounded above and away from zero with a bounds depending only on the dyadic-quasisymmetric constant of f. If we define a new earthquake map by replacing λ_k by $t\lambda_k$ we obtain a new dyadic-quasiconformal map with smaller constant.

Without loss of generality assume f is dyadic-quasiconformal with earthquake data λ and that f fixes both 0 and 1. Fix a positive integer n and for $0 \leq k \leq n$ let let g_k be the dyadic earthquake map that fixes both 0 and 1 and has earthquake data $\frac{k}{n}\lambda$. Let $f_k = g_{k+1} \circ g_k^{-1}$. Since $f_n = f$ and f_0 is the identity, the composition of the g_k's equals f and each of these maps has small earthquake data and hence is dyadic-quasisymmetric with small constant. $\square$

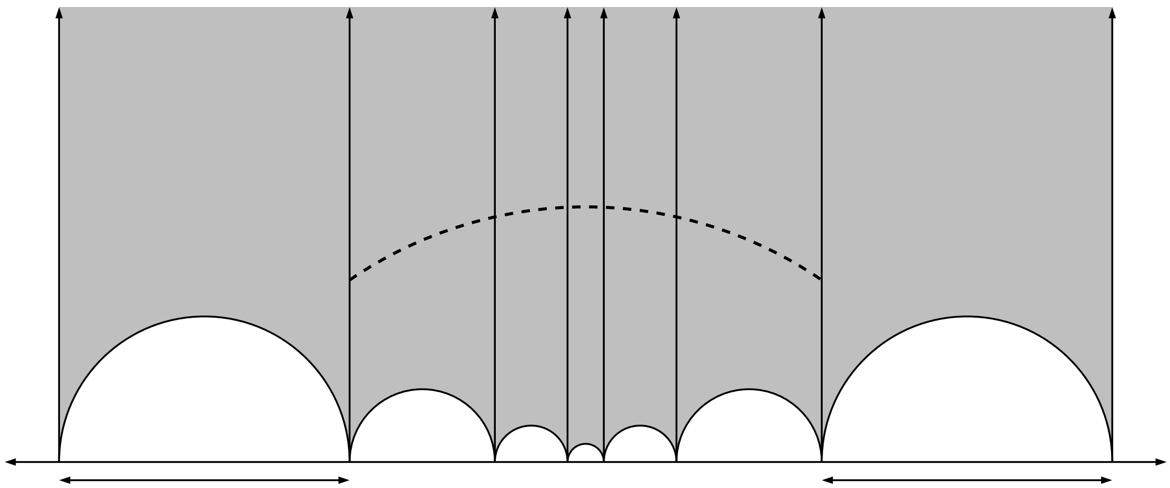


FIGURE 22. The common endpoint of the chain of triangles in Figure 21 is mapped to infinity. The earthquake maps now correspond to Euclidean similarities. Multiplying the hyperbolic translation distance by t corresponds to raising the dilation factor in the similarity to the tth power. This proves a quasisymmetric map can be factored into a composition of quasisymmetric maps with dyadic-constant close to 1.

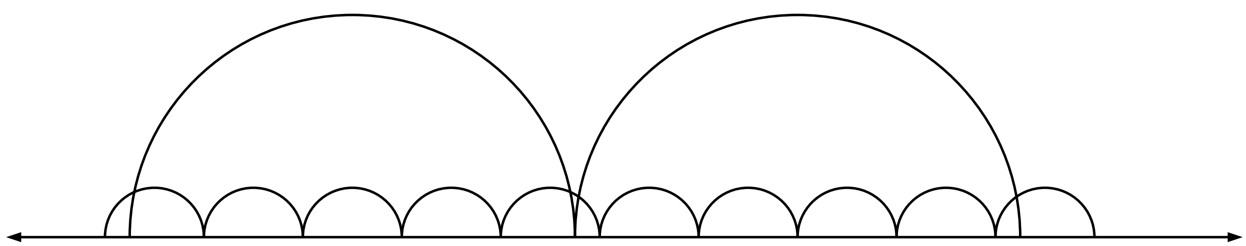


FIGURE 23. Given two adjacent intervals of equal length we approximate them using unions of smaller dyadic intervals. If a map is quasisymmetric with small constant for dyadic intervals, this proves it is also quasisymmetric with small constant for all intervals.

THEOREM 11.9.2. *Any quasisymmetric homeomorphism of the real line is a conformal welding.*

PROOF. By the previous result, any quasisymmetric map can be factored as a composition of quasisymmetric maps all with constant close to 1, and by Lemma **??** each of these can be extended to a quasiconformal map of the upper half-plane. By reflection (Lemma 11.7.5) each of these maps extends to be quasiconformal on the whole plane and the composition of these maps is a quasiconformal map that equals the given quasisymmetric map on the real line. □

11.10. Conformal Welding

Let $\mathbb{D} \subset \mathbb{R}^2$ be the open unit disk, $\mathbb{D}^* = S^2 \setminus \overline{\mathbb{D}}$ and let $\mathbb{T} = \partial\mathbb{D} = \partial\mathbb{D}^*$ be the unit circle. Given a closed Jordan curve Γ, let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be conformal maps onto the bounded and unbounded complementary components of Γ respectively. Then $h = g^{-1} \circ f : \mathbb{T} \to \mathbb{T}$ is a homeomorphism and a homeomorphism is called a conformal welding if it arises in this way. It is known that not every circle homeomorphism is a conformal welding (e.g., see Lemma **??**), but a precise geometric characterization of such homeomorphisms seems like a very difficult problem. See [**?**] for some partial results that show that every circle homeomorphism is "close to" a welding map in a strong sense. For practical purposes the following is often sufficient and is referred to as the "fundamental theorem of conformal welding" (e.g., see [**?**].

THEOREM 11.10.1. *If $f : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism that has a quasiconformal extension to the closed unit disk, then f is a conformal welding map.*

The more usual way to state this is "if f is a quasisymmetric circle homeomorphism then f is a conformal welding". Here quasisymmetric means that any two adjacent intervals of the same length on $\mathbb{T}$ are mapped by f to intervals of comparable length (with a constant M that may depend on f, but not on the intervals). Quasisymmetric circle homeomorphisms are precisely the boundary values of quasiconformal self-maps of the disk, although we have not proved this yet. The fact that the boundary value of quasiconformal map must be quasisymmetric follows from a straightforward modulus estimate. For the other direction, there is a formula for extending a circle homeomorphism to the interior of the disk, and it can be shown that starting with a quasisymmetric map of the circle gives a quasiconformal map of the disk. Both halves of the argument will be discussed in greater detail in Section **??**.

PROOF OF THEOREM 11.10.1. We define a circle chain $\mathcal{C}$ to be a finite union of closed disks $\{D_k\}_1^n$ in $\mathbb{R}^2$ which have pairwise disjoint interiors and such that D_k is tangent to D_{k+1} for $k = 1, \ldots, n-1$, D_n is tangent to D_1 and there are no other tangencies. We also assume the disks are numbered in counterclockwise order. The complement, $X = S^2 \backslash \cup_k D_k$, of a circle chain consists of two disjoint Jordan domains. We shall denote the bounded component by Ω and the unbounded component by Ω^*. Let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be Riemann maps. We shall call (f, g) a normalized circle chain pair if $f(0) = 0$, $g(\infty) = \infty$ and $\text{dist}(0, \partial\Omega) = 1$. Clearly, given a circle chain, we can always obtain a normalized pair by composing with a Möbius transformation.

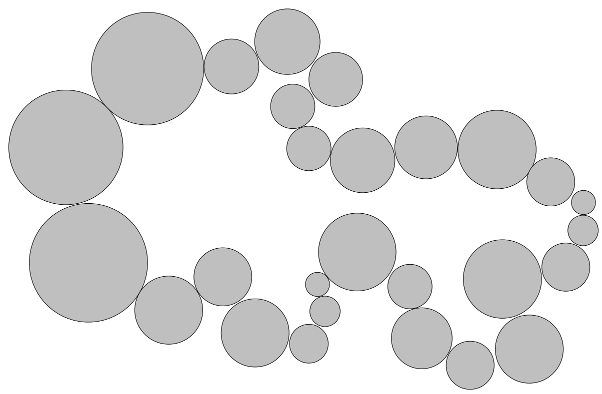


FIGURE 24. A circle chain

LEMMA 11.10.2. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism and suppose $\{x_k\}_1^n \subset \mathbb{T}$ is a finite collection of distinct points listed in counterclockwise order. Let $I_k = (x_k, x_{k+1}), k = 1, \ldots, n$ (modulo n). Then there is a normalized circle chain pair so that for each k,*

$$f(I_k) = \partial D_k \cap \partial\Omega,$$

$$g(h(I_k)) = \partial D_k \cap \partial\Omega^*.$$

We will say that any circle chain that satisfies this conclusion corresponds to h. Another way of stating the lemma is that given any finite positive sequences $\{a_k\}$ and $\{b_k\}$ such that $\sum_{k=1}^n a_k = \sum_{k=1}^n b_n = 1$ we can find a circle chain so that the harmonic measure of each disk in the chain satisfies

$$\omega(D_k, 0, \Omega) = a_k, \quad k = 1, \ldots n,$$

$$\omega(D_k, \infty, \Omega^*) = b_k, \quad k = 1, \ldots n.$$

It is a fact that this circle chain is unique up to Möbius transformations, but we will not need this here. One can prove uniqueness by considering two chains corresponding to the same data. By taking conformal maps between the complements of two such chains and repeatedly extending them by reflection, we can show these maps extend to a homeomorphism of the sphere which is conformal except on a Jordan curve which is the limit set of the Kleinian group generated by reflections in the elements of our circle chain. It is known such a curve is a quasicircle (see Theorem **??**) and hence is removable for conformal maps. Thus the maps extend to be conformal on the whole sphere, i.e., Möbius.

Proof of Lemma 11.10.2. The Koebe circle domain theorem ([**?**], [**?**]; also see [**?**] and its references) states that given any finitely connected domain Ω there is a conformal map $f : \Omega \to \tilde{\Omega}$ onto a domain bounded by circles and points. We shall apply this to a domain $\Omega = \Omega_\epsilon$ constructed as follows. Given n points $\{x_k\}$ on the unit circle $\mathbb{T}$, let $y_k = 2h(x_k) \in 2\mathbb{T} = \{z : |z| = 2\}$. Let γ_n be disjoint smooth Jordan arcs which connect x_k to y_k in the annulus $A = \{z : 1 \leq |z| \leq 2\}$, e.g., the hyperbolic geodesics in A connecting these points. Let $\{I_k\} \subset \mathbb{T}$ be the arcs bounded by the points $\{x_k\}$ and let $\{J_k\}$ be the corresponding arcs on $2\mathbb{T}$. Thus J_k has harmonic measure $|h(I_k)|$ with respect to ∞. Let $\delta = \inf_k |h(I_k)|$ be the smallest of these harmonic measures.

Our domain Ω is the union of $\mathbb{D}$, $2\mathbb{D}^* = \{z : |z| > 2\}$ and an ϵ-neighborhood of each γ_n, where ϵ is assumed to be so small that these neighborhoods are pairwise disjoint and $\partial\Omega$ has n components.

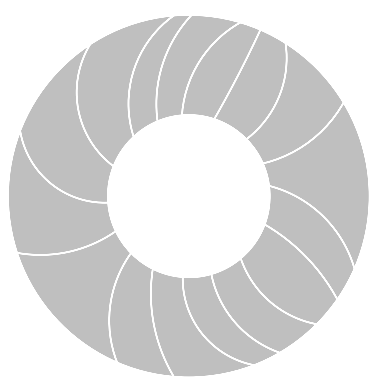


Figure 25. Two disks with connecting tubes

Let $f_\epsilon : \Omega_\epsilon \to \Omega_\epsilon^*$ be the map given by Koebe's theorem. Normalizing by Möbius transformation we may assume $f(0) = 0$, $f(\infty) = \infty$ and $\mathrm{dist}(0, \partial\Omega_\epsilon) = 1$.

We claim that the n circles in the complement of Ω_ϵ^*, are all contained in some disk $D(0, R)$ with R independent of ϵ (but R may depend on h and n). To see this, suppose the union of closed disks satisfies $\cup_k D_k \subset \{1 \leq |z| \leq R\}$ and that it hits both boundary components. Let Ω_1 be the connected component of $f_\epsilon(\Omega_\epsilon \cap D(0, 3/2))$ containing 0. Then for ϵ small enough, each interval I_k has harmonic measure $\geq 1/2n$ in Ω_1 and hence has capacity in Ω_1 which is bounded away from zero depending only on n. Thus by Lemma **??**, every disk must hit $\{|z| \leq M_1\}$, for some M_1 depending only on n. Similarly for Ω_2 (the connected component of $f_\epsilon(\Omega \cap \{|z| > 3/2\})$ containing ∞), i.e., there is a M_2 depending only on δ such that every disk must hit $\{|z| = R/M_2\}$. If R is so large that $R/M_2 > 2M_1$, then every disk in our chain hits both $\{|z| = M_1\}$ and $\{|z| = 2M_1\}$. For large n this contradicts the following simple fact:

LEMMA 11.10.3. *At most 6 disjoint disks can hit both* $\{|z| = 1\}$ *and* $\{|z| = 2\}$.

PROOF. Each such disk has a sub-disk of diameter 1 contained in the annulus $\{1 \leq |z| \leq 2\}$. Each of these intersects the circle $\{|z| = \sqrt{3}/2\}$ in an arc of angle measure $\pi/3$, and hence there can be at most 6 of them. $\square$

Now we can pass to the limit as $\epsilon \to 0$, passing to a subsequence where each disk converges and we are done. $\square$

Now that we have the finite approximations, we want to show they stay bounded as $n \to \infty$. The argument is similar to what we have just done. We will say a circle chain has ϵ-links if every disk has harmonic measure $\leq \epsilon$ with respect to both 0 and ∞.

LEMMA 11.10.4. *Suppose* $h : \mathbb{T} \to \mathbb{T}$ *is an orientation preserving homeomorphism such that for every set* E *of zero logarithmic capacity,* $h(\mathbb{T} \setminus E)$ *has positive logarithmic capacity. Then there is a* $R < \infty$ *and an* $\epsilon > 0$ *(each depending only on* h*) so that for any normalized circle chain corresponding to* h *with* ϵ*-links,*

$$X = S^2 \setminus (\Omega \cup \Omega^*) \subset \{z : 1 \leq |z| \leq R\}.$$

PROOF. Fix $R > 1$ and consider a normalized circle chain such that $X = S^2 \setminus (\Omega \cup \Omega^*) \subset A(1, R)$ and X intersects both boundary components of this annulus. Divide the (closed) disks in the circle chain into three collections: $\mathcal{C}_1$ are the disks which lie inside $D(0, \sqrt{R})$, $\mathcal{C}_2$ are the disks that lie outside $D(0, \frac{1}{2}\sqrt{R})$ and $\mathcal{C}_3$ are all the rest. By Lemma 11.10.3 there are at most 6 elements in $\mathcal{C}_3$. For $i = 1, 2, 3$, let $E_i = f^{-1}(\cup_{D \in \mathcal{C}_i} \partial\Omega_1 \cap D)$. Then E_2 has small logarithmic capacity depending only on R by Lemma **??**, and E_3 has small capacity since it is a union of at most 6 intervals each of length $\leq \epsilon$. Similarly, $h(E_1)$ has small capacity depending only on R.

By choosing ϵ small enough and R large enough we could find such sets where $E_1 \cup E_2 \cup E_3 = \mathbb{T}$ and $\text{cap}(E_2 \cup E_3) + \text{cap}(h(E_1))$ is as small as we wish. But by Lemma **??**, this contradicts our assumption on h, so R must remain bounded as $\epsilon \to 0$. Thus Lemma 11.10.4 is true. $\square$

Given a homeomorphism h and n equidistributed points $\{x_k\}_1^n \subset \mathbb{T}$, let $y_k = h(x_k)$ for $k = 1, \ldots n$ and consider the corresponding circle chain $\mathcal{C}_n$ as given by Lemma 11.10.2. As before, let Ω_n, Ω_n^* denote the bounded and unbounded complementary domains. By reflecting through each circle we obtain a new chain with $n(n-1)$ circles. Continuing in this way we obtain, in the limit, a Jordan curve Γ_n, with complementary components D_n (bounded) and D_n^* (unbounded). See Figure 26 which shows the original chain and the domain Ω_n on the left, three iterations of the reflections in the center and the corresponding domain D_n on the right.

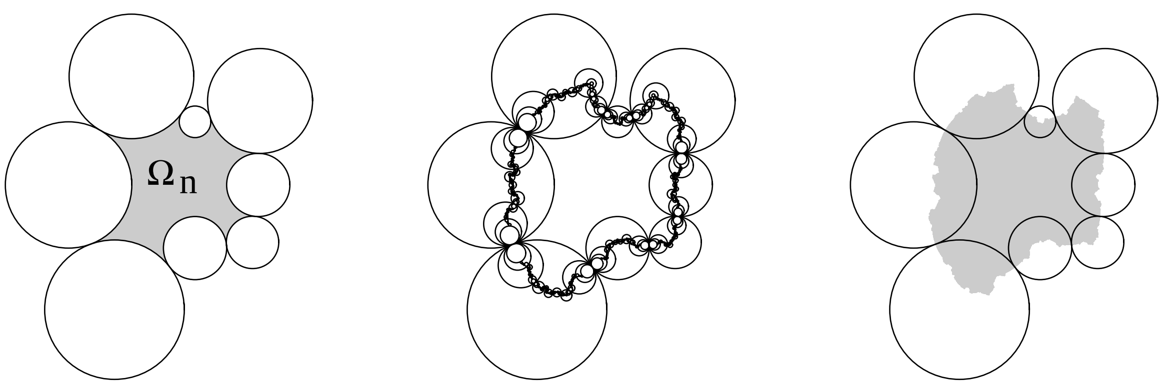


FIGURE 26. Reflections in a circle chain give a curve

Similarly, given a circle chain $\mathcal{D}_n$ of n circles of equal size, with tangent points along the unit circle, we can reflect through the circles, getting a nested sequence of circle chains which limit on the unit circle, as in Figure 27. We claim that if h is the boundary extension of a K-quasiconformal selfmap of the disk, then there is a K-quasiconformal map of the plane sending the circles in Figure 27 to those in Figure 26. We will prove this by constructing the map separately inside and outside the unit circle.

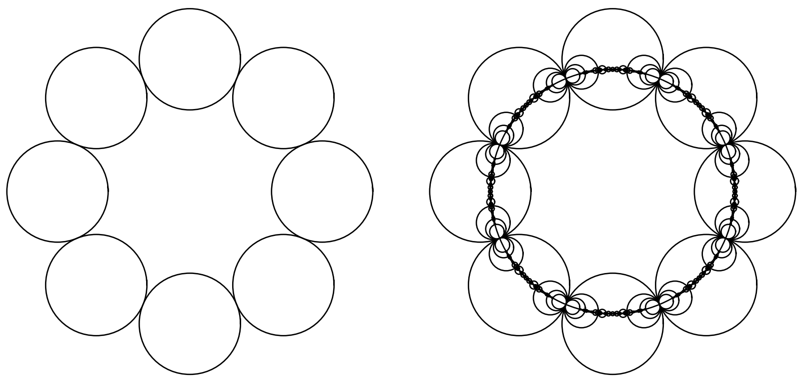


FIGURE 27. A symmetric circle chain with limit $\mathbb{T}$

Let $W_n = S^2 \setminus \{x_1, \dots, x_n\}$. We may assume $n \geq 3$, so there is a universal covering map $\Pi : \mathbb{D} \to W_n$. Let U_n be the component of $\Pi^{-1}(\mathbb{D})$ containing the origin, and note that by symmetry U_n may be chosen to be bounded by hyperbolic geodesics with endpoints at the x_k's (the arcs $\mathbb{T} \setminus \cup\{x_k\}$ are hyperbolic geodesics in W_n; this is even clearer if we map $\mathbb{T}$ to $\mathbb{R}$ by a Möbius transformation). Reflecting these arcs across $\mathbb{T}$ gives the circle chain $\mathcal{D}_n$ in Figure 27 with $\{x_k\}_1^n$ as the points of tangency. The conformal map $f_n \circ \Pi : U_n \to \Omega_n$ can be extended by repeated Schwarz reflection to a conformal map $F_n : \mathbb{D} \to D_n$. See Figure 28.

Similarly, Koebe's theorem gives a conformal map $g_n : \mathbb{D}^* \to \Omega_n^*$. Let $W_n^* = S^2 \setminus \{y_1, \dots, y_n\}$ and consider $\Pi : \mathbb{D}^* \to W_n^*$ as the universal cover of W_n^*. As above, we can lift g_n to map of $\Pi^{-1}(\mathbb{D}^*) \to \Omega_n^*$ and use Schwarz reflection to extend it to a map G_n from $\mathbb{D}^* \to D_n^*$. See Figure 29.

By assumption h is the boundary extension of a K-quasiconformal map of the disk to itself. By reflection we can extend this is a K-quasiconformal map H of S^2 to itself. Then H maps W_n to W_n^* and lifts to a K-quasiconformal map of the universal covers. We can represent these by $\mathbb{D}^*$ so we get a K-quasiconformal map $H_n : \mathbb{D}^* \to \mathbb{D}^*$ which conjugates the covering groups. See Figure 29.

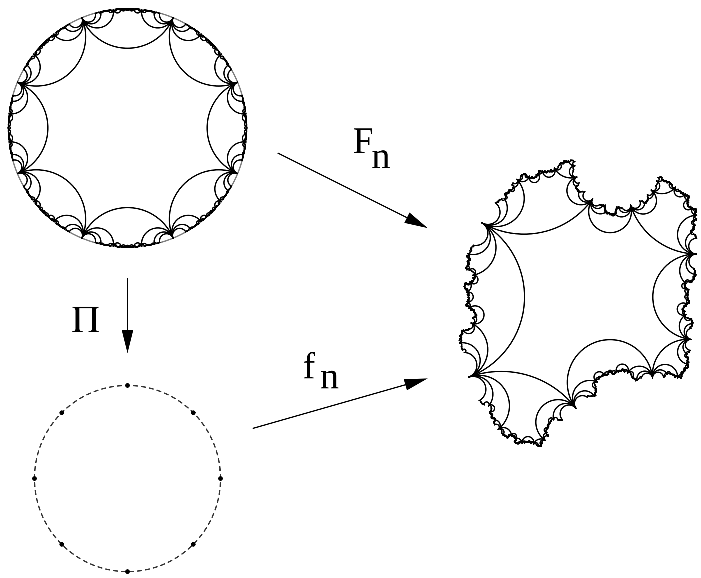


FIGURE 28. Lifting and extending the Koebe map

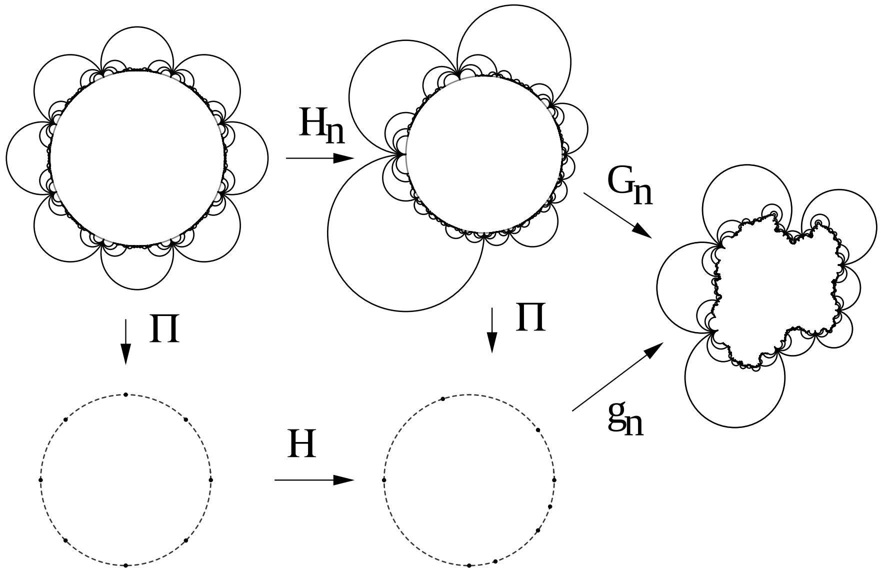


FIGURE 29. Lifting the maps H and g_n.

Thus $G_n \circ H_n$ is a K-quasiconformal map of $\mathbb{D}^*$ to D^* whose boundary values agree with F_n on $\mathbb{T}$, and hence these maps together define a K-quasiconformal map of S^2 (easy to check using the analytic definition of quasiconformal in [**3**]). This map takes $\mathbb{T}$ to Γ_n and the circle chain $\mathcal{D}_n$ to the chain $\mathcal{C}_n$. Taking $n \to \infty$, using the uniform continuity of K-quasiconformal mappings and passing to a subsequence if necessary, we see that our circle chains converge uniformly to a K-quasicircle and that h is the corresponding conformal welding, as desired. □

11.11. Quasicircles

A **quasicircle** is the image of a circle or line under a quasiconformal map of the plane. If the quasicircle is unbounded (it passes through ∞) then we sometimes call it a **quasiline**.

We say a curve γ satisfies Ahlfors three point condition if there is a $M < \infty$ so that

$$|x - z| \leq M|x - y|,$$

for every $x, y \in \gamma$ and every $z \in \gamma$ between x and y. This is equivalent to saying that the diameter of the arc between x and y is $O(|x - y|)$.

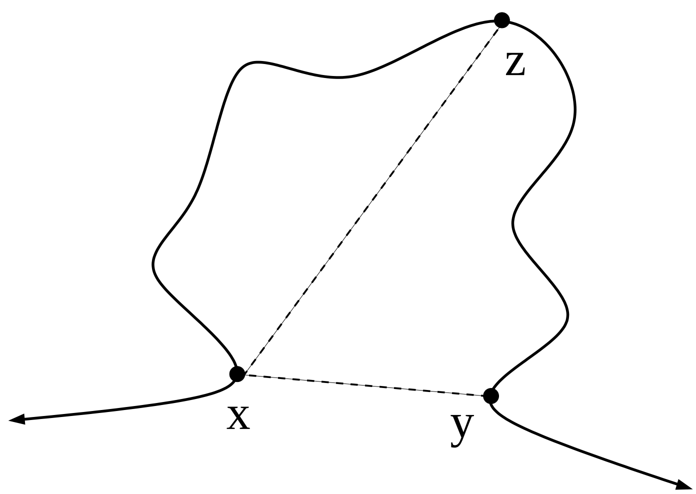


FIGURE 30. Ahlfors' three point condition.

THEOREM 11.11.1. *A Jordan curve γ on the Riemann sphere is a quasiline if and only if it satisfies Ahlfors three point condition.*

PROOF. The necessity of the three point condition is quite easy. Suppose $\gamma = f(\mathbb{R})$ and f is quasiconformal. By pre- and post-composing by conformal linear maps we may assume f fixes both 0 and 1 and that we need only check that the arc of γ connecting 0 to 1 has diameter bounded only in terms of the quasiconformal constant of f. But this is immediate from Lemma 11.3.2 (this was the proof that normalized K-quasiconformal maps are totally bounded, which was part of the proof that they form a compact family).

The sufficiency of the three point condition is a little more involved. Consider conformal maps φ, ψ from the lower half-plane and the upper half-planes to the two sides of γ. We claim that the three point condition implies that the conformal welding map $h = \psi^{-1} \circ \varphi$ is a quasisymmetric homeomorphism of the real line to itself. If the claim is true, then by Theorem **??** h can be extended to a quasiconformal mapping H of the upper half-plane to itself. Then setting $f = \varphi$ in the lower half-plane and setting $f = H \circ \psi$ in the upper half-plane defines quasiconformal maps in both half-planes that both extend continuously to the real line and agree there. Since the line is removable (Theorem 11.7.3) f is actually quasiconformal on the whole plane and hence $\gamma = \varphi(\mathbb{R}) = f(\mathbb{R})$ is a quasiline. Thus it suffices to prove the claim.

To prove the claim fix two points $x, y \in \gamma$ and, without loss of generality, assume there are the images of 0 and 1 under the conformal map φ from the lower half-plane to one side Ω_1 of γ. Let $z = \varphi(\frac{1}{2})$.

Let Γ_1 be the path family in Ω that connects the sub-arc γ_{xz} of γ with endpoints x, z to the disjoint sub-arc γ_y that connects y to infinity. See Figure 31. This family is just the image under φ of the path family in the lower half-plane that connects $[0, \frac{1}{2}]$ to $[1, \infty]$. This path family is the reciprocal to the family that connects $[\frac{1}{2}, 1]$ to $(-\infty, 0]$, hence the product of their moduli is 1. On the other hand, these families correspond under reflection, so they have equal moduli, and hence each has modulus equal to 1. By conformal invariance Γ_1 must also have modulus 1,

Let Γ_2 be the path family in Ω_2 (the other complementary component of γ) that connects the sub-arc of γ with endpoints x, z to the disjoint sub-arc that connects y to infinity. To show that $\psi^{-1} \circ \varphi$ is quasisymmetric, it suffices to show that the modulus of this family is bounded away from both 0 and ∞.

First, we claim that since Γ_1 has modulus 1, $|x - z|/|x - y|$ is bounded away from zero. If not, say $|x - z| < \epsilon |x - y|$, then the three point property implies that the arc of γ between x and z is contained in a ball of radius $O(\epsilon)$ around x. Thus by the monotonicity property of modulus (Lemma 5.1.2) the modulus of Γ_1 is greater than

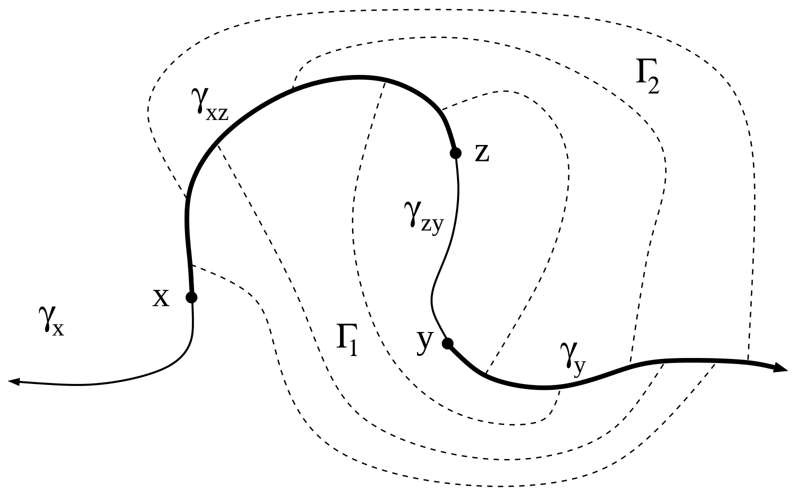


FIGURE 31. The definition of Γ_1 and Γ_2.

the modulus of the path family separating the boundary components of the annulus

$$\{w : O(\epsilon)|x-y| < |x-w| < |x-y|\},$$

which we know to be $(O(1) + \log \frac{1}{\epsilon})/2\pi$. This gives a contradiction if ϵ is too small, so the claim is proved.

Next we claim that the distance between the arc γ_{zy} from z to y and the disjoint arc γ_x connecting x to ∞ is larger than some constant C_1 times $|x-z|$. Otherwise there would be points u, v (one from each of these sets) that were within $\epsilon|x-z|$ of each other. By the three point condition the arc γ_{uv} between them would have diameter $O(\epsilon|x-z|)$, but this arc contains the arc γ_{xz} whose diameter is comparable to $|x-z|$. This is a contradiction if ϵ is small, so the second is also proven.

Now define a metric ρ by setting it to 1 on

$$\{w : \mathrm{dist}(w, \gamma_{zy}) \geq C_1|x-z|\}.$$

Then the *rho*-length of any path connecting γ_{zy} to γ_x is at least $C_1|x-z|$, while the area of ρ is at most the area of a ball of radius $\mathrm{diam}(\gamma zy) + C_1|x-z|$ and hence is $O(|x-z|)$. Thus the modulus of Γ_2 is bounded uniformly above. The modulus of the reciprocal family in Ω_2 is bounded above by the same argument, so we deduce the modulus of Γ_1 is both bounded and bounded away from zero uniformly. That means that $\psi^{-1}(z) = \psi^{-1}(\varphi(\frac{1}{2}))$ is bounded away from both 0 and 1 in terms of the three point constant only. As noted earlier, this implies $h = \psi^{-1} \circ \varphi$ is quasisymmetric and that γ is a quasiline. $\square$

11.12. Finitely connected Fatou components

In this section we give an example of finitely connected wandering domains and show that any connectivity ≥ 2 can occur. Unlike the previous section, where the example was given by a formula, here we give geometric construction. We will build a continuous map that has a wandering domain of finite connectivity and then use a theorem about solutions of the Beltrami equation to deduce that the dynamics this map are conjugate to an entire function; thus the entire function has a wandering domain of finite connectivity.

THEOREM 11.12.1 (Kisaka and Shishikura). *For each $p \geq 2$ there is a transcendental entire function with p-connected wandering Fatou component.*

PROOF. We start with a polynomial $P(z)$ of degree p so that and curve γ_1 that has a single preimage γ_0, and such that the interior domain D of γ_0 has p distinct preimages $\{D_j\}_1^p$ inside D with disjoint closures. This implies the Julia set of P is a Cantor repeller; the inverse images of $W = D \setminus \cup_j D_j$ define nested domains of connectivity $p+1$ that are mapped conformally by P to the next larger domain until they reach Ω which is mapped p-to-1 to the topological annulus between γ_0 and γ_1.

By replacing P by a quasiconformal map in an annular neighborhood of γ_0 we can assume γ_1 is a circle of radius R_1 centered at 0. See Figure **??**.

Let

If $k \notin S$, then Let W_n be the circular annulus $A = \{Z : R_N < |Z| < R_{N+1}\}$. If $k \in S$ then W_n is this annulus with a subdomain U_n removed. U_n is a Jordan domain of diameter $\frac{1}{n} R_n$, that is symmetric with respect to the real line and has a smooth boundary except at the point $\{R_n\} = \partial U_n \cap W_n$ where it has a $90°$ angle . See Figure **??**.

W_n can be conformally mapped to a circular annulus $A(1, s_n)$ (if $k \notin S$ the identity will do and $s_n = R_{n+1}/R_n$ and then to another circular annulus $A(1, s_n^{n+1})$ by composing with the power map $z \to z^{n+1}$.

$\square$

CHAPTER 12

Quasiconformal mappings: analytic aspects

The earlier chapter on geometric aspects of quasiconformal maps suffices for most applications to dynamics, but omits many natural and powerful results. We saw that given a complex dilation μ in the open unit ball of L^∞, we could define an associated quasiconformal map f, but we could only deduce that $\mu = f_{\bar z}/f_z$ under some extra assumptions, such as the the continuity of μ. For a general quasiconformal map, we do not even know if f_z and $f_{\bar z}$ exist, so it is far from clear whether every quasiconformal map arises from some μ.

In this chapter we will show that these partials do exist almost everywhere and that f can be recovered from them by integration, i.e., f is absolutely continuous on (almost all) lines. Thus every quasiconformal map f has a complex dilatation μ, and we will show that f can be recovered from μ via the measurable Riemann mapping theorem. We are also interested in showing that f depends differentiably on μ. A very helpful formula in this regard is Pompeiu's formula (proven for C^1 functions in Chapter **??**):

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_\Omega \frac{f_{\bar z}}{z-w}dxdy. \tag{42}$$

However, it is not even clear whether this formula makes sense for a quasiconformal map; since f is continuous, the first integral is well defined, but it is not clear whether the second integral is well defined in general.

We expect (but have not yet proved) that

$$\text{area}(f(\Omega)) = \int_\Omega J_f dxdy = \int_\Omega |f_z|^2 - |f_{\bar z}|^2 dxdy = \int_\Omega |f_z|^2(1-|\mu_f|^2)dxdy,$$

which would imply f_z and $f_{\bar z}$ are in L^2 locally. However, $|z-w|^{-1}$ is not in L^2, so we can't be sure that the area integral in the Pompeiu formula is convergent. However, $|z-w|^{-1}$ it in L^q locally for every $q<2$, so the integral will be bounded if we can show $f_{\bar z} \in L^p$ locally for some $p>2$. This is a fundamental result of Bojarski in $\mathbb{C}$ []

and of Gehring [] in dimensions ≥ 2 and we will prove it later in this chapter, using the 2-dimensional version of Gehring's proof.

Another problem with proving the Pompeiu formula for quasiconformal maps is a little more subtle. As noted above, we know the formula is valid for smooth functions and to verify it for general quasiconformal maps, we would like to smooth these functions (say by convolution with a smooth, radial bump function) and pass to a limit. In this case, the smoothed functions converge uniformly to the limit, so the boundary integral term converges as desired, but the integrand of the area integral only converges pointwise and we need some extra condition to insure this integral also converges. In this case, we can use Gehring's result and the L^p boundedness of the Hardy-Littlewood maximal theorem to deduce that the sequence integrands coming from the smooth approximations of f is dominated by fixed L^1 function, so the Lebesgue dominated convergence theorem can be applied to verify Pompeiu's formula. Pompeiu's formula can then we applied to prove the differentiable dependence of f on its dilatation μ.

We start with a review of some basic real analysis and them move towards the theorem of Bojarski and Gehring and its consequences.

12.1. Covering lemmas and maximal theorems

THEOREM 12.1.1 (Vitali covering lemma: easy form). *Let $\mathcal{B} = \{B_j\}$ be a finite collection of balls in $\mathbb{R}^d$. Then there is a finite, disjoint subcollection $\mathcal{C} \subset \mathcal{B}$ so that*

$$\cup_{B\in\mathcal{B}} B \subset \cup_{B\in\mathcal{C}} 3B.$$

In particular, the Lebesgue measure of the set covered by the subcollection is at least 3^{-d} times the measure covered by the full collection.

THEOREM 12.1.2 (Vitali covering lemma: harder form). *Suppose $E \subset \mathbb{R}^d$ is a measurable set and $\mathcal{B} = \{B_j\} \subset \mathbb{R}^d$ is a collection of balls so that each point of E is contained in elements of $\mathcal{B}$ of arbitrarily small diameter. Then there is a subcollection $\mathcal{C} \subset \mathcal{B}$ so that $E \setminus \cup_{B\in\mathcal{C}} B$ has zero d-measure.*

The Lebesgue dominated convergence theorem

Egorov's theorem.

(I will always remember Egorov's theorem because when I was a first year graduate student at the University of Chicago and I wanted skip taking the first year analysis course there, I went to Luis Caffarelli's office for an oral exam and the first thing he asked me was to state and prove Egorov's theorem. As is often the case, Egorov's theorem is actually due to Carlo Severini [**126**] who published a proof a year before Egorov [**46**].)

LEMMA 12.1.3 (The Calderon-Zygmund lemma). *) Suppose Q is a square, $u \in L^1(Q, dxdy)$ and suppose*

$$\alpha > \frac{1}{\text{area}(Q)} \int_Q |u| dxdy.$$

Then there is a countable collection of pairwise disjoint open dyadic subsquares of Q so that

$$\alpha \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} |u| dxdy < 4\alpha, \tag{43}$$

$$|u| \leq \alpha \text{ almost everywhere on } Q \setminus \cup_j Q_j, \tag{44}$$

$$\sum \text{area}(Q_j) \leq \frac{1}{\alpha} \int_Q |u| dxdy \tag{45}$$

PROOF. We say a subsquare of Q has property P is the first conclusion above holds and we define a collection of subsquares by iteratively dividing squares that do not have property P into four, equal sized disjoint subsquares, and stopping when property P is achieved. If the average of u over a square is less than α then average over each of the four subsquares is $< 4\alpha$, so every stopped square has property P. Any point not in a stopped square is a limit of squares where the average of u is $< \alpha$, so by the Lebesgue differentiation theorem $u \leq \alpha$ at almost every such point. Finally,

$$\int_Q |u| dxdy \geq \sum_j \alpha \text{area}(Q_j),$$

which proves the third property. □

Hardy-Littlewood maximal function.

Marcinkiewicz interpolation

L^p boundedness of maximal function

Maximal function bounds maximal function of convolution with radial L^1 bump function.

12.2. Absolute continuity on lines

The main type of K-quasiconformal maps used in this text are piecewise C^1 functions that satisfy

$$|f_{\bar{z}}| \leq k|f_z|, \tag{46}$$

where $k - (K-1)/(K+1)$. By itself, this equation is not enough to guarantee a map is quasiconformal. For example, suppose $g : [0,1] \to [0,1]$ is the usual Cantor singular function.e., a continuous function that increases from 0 to 1 on $[0,1]$ and is constant on each complementary component $\{I_j\}$ of the Cantor middle-$\frac{1}{3}$ set E. Then the map $f(x,y) = (x + g(x), y)$, is a homeomorphism of $[0,1] \times [0,1]$ to $[0,2] \times [0,1]$ that is a translation (hence conformal) on each rectangle $I_j \times [0,1]$, where I_j is a complementary interval of the Cantor set. Thus $f_{\bar{z}} = 0$ almost everywhere, but there are several way to check that f is not quasiconformal.

EXERCISE : Find rectangles whose modulus is increased by arbitrarily large factors by f.

EXERCISE: Find a path family Γ of zero modulus, so that $f(\Gamma)$ has positive modulus.

EXERCISE: Show that f map some set of zero area to positive area (later we will prove quasiconformal maps can't do this).

The problem with this example is that it is not absolutely continuous on horizontal lines, and so f cannot be recovered by integrating its partials.

A function f is called **absolutely continuous** on a line L if for every $\epsilon > 0$ there is a $\delta > 0$ so that $m_1(E) < \delta$ implies $m_1(f(E)) < \epsilon$ where m_1 denotes 1-dimensional Hausdorff measure.

THEOREM 12.2.1. *If f is quasiconformal, then f is absolutely continuous on almost every line in any given direction.*

PROOF. After a Euclidean similarity, we may consider horizontal lines in $Q = [0,1]^2$. Define

$$A(y) = \text{area}(f([0,1] \times [0,y])).$$

Then $A(0) = 0$, $A(1) = \text{area}(f(Q)) < \infty$ and A is increasing. Thus A is continuous except on a countable set and has a finite derivative almost everywhere. Fix a value of y where both this things happen, and we will show that f is absolutely continuous on the horizontal line $L_y = [0,1] \times \{y\}$. The main idea is that if this failed, then modulus estimates relating length to area will force $A'(y) = \infty$.

Consider the long, narrow rectangle $R = [0,1] \times [y, y + \frac{1}{n}]$ and divide it into $m << n$ disjoint $\frac{1}{m} \times \frac{1}{n}$ sub-rectangles $\{R_j\}$. Let $R'_j = f(r_j)$ and the the "left", "right", and "bottom" edges of R'_j be the images under f of corresponding edges of R_j. Let b_j be length of $f(L_y \cap \partial R_j)$, i.e., the length of the bottom edge of R'_j. This number might be finite or infinite. Fix $\epsilon > 0$. In the first case, by taking n large enough, we can insure that any curve in $f(R_j)$ than joins the images of the vertical sides of R_j has length $\geq b_j - \epsilon$. In the second case, we can insure these curves all have length $\geq 1/\epsilon$. In both case this follows because as $n \to \infty$, any curve in $f(R_j)$ joining the opposite "vertical" sides limits on the bottom edge and hence the liminf of the lengths of such curves is at least the length of the bottom edge of R'_j.

By quasiconformality we know

$$M(R'_j) \geq M(R_j)/K = \frac{m}{Kn},$$

and using the metric $\rho = 1$ on R'_j, shows

$$M(R'_j) \leq \frac{\text{area}(R'_j)}{b_j^2}.$$

Thus by Cauchy-Schwarz,

$$\begin{aligned}
(\sum_{j=1}^{m} b_j)^2 &\leq (\sum_{j=1}^{m} b_j^2 m)(\sum_{j=1}^{m} \frac{1}{m}) \\
&\leq m \sum_{j=1}^{m} \frac{\mathrm{area}(R'_j)}{M(R'_j)} \\
&\leq m \sum_{j=1}^{m} \frac{\mathrm{area}(R'_j)}{m/Kn} \\
&\leq \sum_{j=1}^{m} \mathrm{area}(R'_j) Kn \\
&\leq K \frac{A(y+\frac{1}{n}) - A(y)}{1/n} \\
&\to KA'(y).
\end{aligned}$$

If any of the b_j's is infinite, so is $A'(y)$, so $f(L_y)$ has finite length for our choice of y. Given a compact set $E \subset L_y$, suppose E is hit by N of the rectangles R_j and that m has been chosen so large that $N/m \leq 2m_1(E)$. Then repeating the argument above, but only summing over the j's so that the bottom edges of R_j hit E,

$$\begin{aligned}
(\sum_{j} b_j)^2 &\leq (\sum_{j} b_j^2 m)(\sum_{j} \frac{1}{m}) \\
&\leq N \sum_{j} \frac{\mathrm{area}(R'_j)}{M(R'_j)} \\
&\leq N \sum_{j} \frac{\mathrm{area}(R'_j)}{m/Kn} \\
&\leq \frac{N}{m} \sum_{j=1}^{m} \mathrm{area}(R'_j) Kn \\
&\leq Km_1(E) \frac{A(y+\frac{1}{n}) - A(y)}{1/n} \\
&\to Km_1(E) A'(y).
\end{aligned}$$

Thus $m_1(E)$ small, implies $\sum b_j$ is small, and hence $f(E)$ has small 1-dimensional measure. Hence f is absolutely continuous on L_y, as desired. $\square$

Basic theorems of real analysis say that if f is absolutely continuous on a line L , then its partial derivative along that lines exists almost everywhere and

$$f((b) - f(a) = \int_a^b f_n ds,$$

where $a, b \in L$ and f_n is the partial in the direction from a to b. Since we have shown that quasiconformal maps are absolutely continuous on almost every horizontal and almost every vertical line, we see that the partial f_x, f_y exist almost everywhere and hence $f_z, f_{\bar{z}}, \mu_f = f_{\bar{z}}/f_z$ are all well defined almost everywhere. Next we want to say that at a point w where these all exist, we have

$$f(z) = f(w) + f_z(w)(z - w) + f_{\bar{z}}(w)(\bar{z} - \bar{w}) + o(|z - w|),$$

i.e., f is differentiable at w. However, as explained in most calculus texts, the existence of partial derivatives at at a point does not imply a function is differentiable there (consider $f(x, y) = x^2y/(x^2 + y^2)$ at the origin).

However, a remarkable theorem of Gehring and Lehto [**61**], says that is implication is true almost everywhere for homeomorphisms. Our proof follows that in [**3**].

THEOREM 12.2.2. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ and has partials almost everywhere, then it is differentiable almost everywhere.*

PROOF. By Egorov's theorem the limits

$$f_x(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h},$$

$$f_y(z) = \lim_{h \to 0} \frac{f(z + ih) - f(z)}{h},$$

are uniform and converge to a continuous functions on a compact set $E \subset \Omega$ so that area$(\Omega \setminus E)$ is as small as we wish.

Almost every point of E is a point of density for the intersection of E with both the vertical and horizontal lines through z_0, so if suffices to proof differentiability at such points. For simplicity we assume 0 is such a point. The proof follows the usual case in calculus where we assume the partials are continuous, except that here we have to replace continuous on a neighborhood of 0 with continuous on a set E that is measure dense around 0.

Because of the continuity and uniform convergence on E, for any $\epsilon > 0$ there is a $\delta > 0$ so that

$$|f_x(0) - f_x(z)|, |f_y(0) - f_y(z)| < \epsilon,$$

if $z \in E \cap D(0,\delta)$-neighborhood of 0 and

$$|f_x(z) - \frac{f(z+h) - f(z)}{h}|, |f_y(z) - \frac{f(z+ih) - f(z)}{h}| < \epsilon,$$

if $z \in E \cap D(0,\delta)$ and $h \in [-\delta, \delta]$.

Note that

$$\begin{aligned} f(z) - f(0) - xf_x(0) - yf_y(0) &= [f(z) - f(x) - yf_y(0)] + [f(x) - f(0) - xf_x(0)] \\ &\quad +[yf_y(x) - f_y(0)] \\ &= I + II + III. \end{aligned}$$

If $|z| < \delta$ and $x \in E$, then by the inequalities above, $I < \epsilon|y|$, $II < \epsilon|x|$ and $III < \epsilon y$, so the term on the far left is bounded by $3\epsilon|z|$, which proves differentiability if $x \in E$. A similar proof works if $iy \in E$.

Fix $\epsilon > 0$ and choose δ so small that if $0 < x < \delta$, then $E \cap (\frac{x}{1+\epsilon}, x) \neq \emptyset$ (this must be possible since $E \cap \mathbb{R}$ has density 1 at 0) and $E \cap (\frac{iy}{1+\epsilon}, y) \neq \emptyset$. Thus if $0 < |x|, |y| \leq \delta/(1+\epsilon)$ can find points $x_1, x_2 \in E \cap (\frac{x}{1+\epsilon}, (1+\epsilon)x)$ and $iy_1, iy_2 \in E \cap i(\frac{y}{1+\epsilon}, (1+\epsilon)y)$ and so that $x + iy$ is inside the rectangle $R = (x_1, x_2) \times (y_1, y_2)$. Since f is a homeomorphism (all we need is that it is continuous and open), $|f|$ takes its maximum on the boundary, so

$$\begin{aligned} &\sup_{z=x+iy\in R} |f(z) - f(0) - xf_x(0) - yf_y(0)| \\ &\qquad \leq \sup_{w=u+iv\in\partial R} |f(zw - f(0) - xf_x(0) - yf_y(0)| \\ &\qquad \leq 3\epsilon|w| + \sup_{w=u+iv\in\partial R} |x-u||f_x(0)| + |y-v||f_y(0)| \\ &\qquad \leq 3\epsilon(1+\epsilon)|z| + \epsilon|f_x(0)||z| + \epsilon|f_y(0)||z|. \end{aligned}$$

$\square$

LEMMA 12.2.3. . *If f is K-quasiconformal then*

$$\int_Q J_f dxdy \leq \int \leq \text{area}(f(Q)) \leq \pi \text{diam}(f(Q))^2,$$

for every square Q.

PROOF. We only use the quasiconformal hypothesis to deduce f is differentiable almost everywhere; the result holds for all such maps. At any point x where f is differentiable we can choose a small square Q_x containing x such that

$$\text{area}(f(Q')) \geq (1-\epsilon) J_f(x) \text{area}(Q'),$$

and by the Lebesgue differentiation theorem, for almost every x we have

$$\int_{Q'} J_f dxdy \leq (1+\epsilon) J_f(x) \text{area}(Q'),$$

for all small enough squares centered at x. Combining these two estimates and using the Vitali covering theorem to extract a collection of disjoint squares $\{Q_j\}$ with centers x_j and with these properties that cover almost every point of Q, we get

$$\begin{aligned}\int_Q J_f dxdy &\leq \sum_j \int_{Q_j} J_f dxdy \\ &\leq (1+\epsilon) J_f(x_j) \text{area}(Q_j) \\ &\leq \frac{1+\epsilon}{1-\epsilon} \text{area}(f(Q_j)) \\ &\leq \frac{1+\epsilon}{1-\epsilon} \text{area}(f(Q)).\end{aligned}$$

Taking $\epsilon \searrow 0$, gives $\text{area}(f(E)) \geq \int_E J_f dxdy$. The inequality area $\leq \pi \text{diam}^2$ is obvious. $\square$

Since $|f_z|^2 \leq J_f/(1-k^2)$, we also get

COROLLARY 12.2.4. *If f is K-quasiconformal then*

$$\int_Q |f_z|^2 dxdy \leq \frac{\pi}{1-k^2} \text{diam}(f(Q))^2,$$

for every square Q.

Next we turn to

LEMMA 12.2.5. *If f is K-quasiconformal, then*

$$\frac{(\int_Q |f_z| dxdy)^2}{\text{area}(Q)} \gtrsim \text{diam}(f(Q))^2.$$

with a uniform constant for every square Q.

PROOF. The path family connecting opposite sides of a square Q has modulus 1, so the image of this family in $f(Q)$ has modulus between K and $1/K$. This implies the shortest path in $f(Q)$ connecting the same sides has length $\simeq \mathrm{diam}(f(Q))$, so the integral of $|f_z|+|f_{\bar z}|$ along any horizontal segment crossing Q is at least $C\mathrm{diam}(f(Q))$ for some fixed $C > 0$ (depending only on K). Since $|f_z| \le |f_z|+|f_{\bar z}| \le 1(1+k)|f_z|$, the same is true for the integral of $|f_z|$. Integrating over all horizontal segments crossing Q gives

$$\int_Q |f_z| dxdy \gtrsim \mathrm{diam}(Q)\mathrm{diam}(f(Q)).$$

Hence

$$\frac{(\int_Q |f_z| dxdy)^2}{\mathrm{area}(Q)} \gtrsim \frac{[\mathrm{diam}(Q)\mathrm{diam}(f(Q))]^2}{\mathrm{area}(Q)} \gtrsim \mathrm{diam}(f(Q))^2.$$

□

Note that for K-quasiconformal maps, $|\mu_f| \le k = (K-1)/(K+1)$ and

$$|f_z|(1-k^2) \le |f_z|^2(1-|\mu|^2) \le |f_z|^2 - |f_{\bar z}|^2 = J_f \le |f_z|^2,$$

so that J_f and $|f_z|^2$ are the same up to a bounded factor. Thus

$$\int_Q |f_z|^2 dxdy \le \int \lesssim \mathrm{diam}(f(Q))^2 \lesssim \frac{(\int_Q |f_z| dxdy)^2}{\mathrm{area}(Q)}$$

or

$$\int_Q |f_z|^2 dxdy \le C\frac{(\int_Q |f_z| dxdy)^2}{\mathrm{area}(Q)}$$

for some constant C that depends only on the quasiconformal constant of f (and not on the choice of the square Q). This is called a reverse Hölder inequality and we shall see in the next section that it has profound implications for the behavior of f_z.

12.3. Gehring's inequality and Bojarski's theorem

Hölder's inequality says that

$$\int fg d\mu \le (\int f^p d\mu)^{1/p}(\int g^q d\mu)^{1/q},$$

where $1 \le p, q \le \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Applying this to a non-negative function on a square Q we get

$$(\frac{1}{\mathrm{area}(Q)} \int_Q v^p dxdy) \ge (\frac{1}{\mathrm{area}(Q)} \int_Q v dxdy)^p,$$

with equality if and only if v is a.e. constant. Thus the "reverse Hölder inequality "

$$(\frac{1}{\mathrm{area}(Q)}\int_Q v^p dxdy) \leq (K\frac{1}{\mathrm{area}(Q)}\int_Q v dxdy)^p,$$

can only hold if $K \geq 1$. If it holds for single Q, this does not say much, except that $v \in L^p \cap L^1$. However, if it holds (with the same K) for all Q's we can deduce that $v \in L^{p+\epsilon}$ for some $\epsilon > 0$. This remarkable "self-improvement" estimate is due to Gehring [], although the proof we give follows the presentation in Garnett's book [**58**].

We start with a technical lemma.

LEMMA 12.3.1. *Suppose that* $p > 1$, $v \geq 0$, $E_\lambda = \{z : v(z) > \lambda\}$, *and*

$$\int_{E_\lambda} v^p dxdy \leq A\lambda^{p-1}\int_{E_\lambda} v dxdy,$$

for all $\lambda \geq 1$. *Then there is* $r > p$ *and* $C < \infty$ *so that*

$$(\int_Q v^r dxdy)^{1/r} \leq C(\int_Q v^p dxdy)^{1/p}.$$

PROOF. This is basically just arithmetic with distribution functions. Note that it suffices to assume $\mathrm{area}(Q) = 1$ and $\int_Q v^p dxdy = 1$. Then

$$\begin{aligned}
\int_{E_1} v^r dxdy &= \int_{E_1} v^p v^{r-p} dxdy \\
&= (r-p)\int_{E_1} v^p(1+\int_1^v \lambda^{r-p-1}d\lambda)dxdy \\
&= (r-p)\int_{E_1} v^p + (r-p)\int_1^\infty \lambda^{r-p-1}\int_{E_\lambda} v^p dxdyd\lambda \\
&\leq (r-p)\int_{E_1} v^p + A(r-p)\int_1^\infty \lambda^{r-2}\int_{E_\lambda} v dxdyd\lambda \\
&\leq (r-p)\int_{E_1} v^p + A(r-p)\int_{E_1} v(\int_0^v \lambda^{r-2}d\lambda)dxdy \\
&\leq (r-p)\int_{E_1} v^p + A\frac{r-p}{r-1}\int_{E_1} v^r dxdy \\
&\leq (r-p)\int_{E_1} v^p + \frac{1}{2}\int_{E_1} v^r dxdy
\end{aligned}$$

where the last inequality holds if r is close enough to p (depending on A and p). Subtracting the last term of the last step from the first step gives

$$\int_{E_1} v^r dxdy \leq 2(r-p) \int_{E_1} v^p dxdy.$$

Off E_1 we have $v \leq 1$ so $v^r \leq v^p$ and hence

$$\int_Q v^r dxdy \leq (1 + 2(r-p)) \int_Q v^p dxdy.$$

Because of our normalizations, this proves the lemma. □

Next we show the reverse Hölder inequality implies the distribution function hypothesis of the previous lemma, and hence Gehring's inequality.

THEOREM 12.3.2. *Let $p > 1$. If $v(x) \geq 0$ and $v \in L^p(Q, dxdy)$, and if the "reverse Hölder inequality"*

$$(\frac{1}{\text{area}(Q)} \int_Q v^p dxdy) \leq (K \frac{1}{\text{area}(Q)} \int_Q v dxdy)^p,$$

holds for all subsquares of a square Q_0, then there is an $r > p$ so that

$$(\frac{1}{\text{area}(Q_0)} \int_{Q_0} v^r dxdy)^{1/r} \leq (C(K,p,r) \frac{1}{\text{area}(Q_0)} \int_{Q_0} v dxdy),$$

PROOF. We need only verify the hypothesis of Lemma 12.3.1. Fix λ and set $\beta = 2K\lambda$. We will split the integral

$$\int_{E_\lambda} v^p dxdy = \int_{E_\lambda \setminus E_\beta} v^p dxdy + \int_{E_\beta} v^p dxdy$$

into two pieces. The second piece is trivial to bound by the correct estimate because

$$\int_{E_\lambda \setminus E_\beta} v^p dxdy \leq \beta^{p-1} \int_{E_\lambda \setminus E_\beta} v dxdy \leq (2K\lambda)^{p-1} \int_{E_\lambda} v dxdy.$$

To bound the other piece of the integral, we use the Calderon-Zygmund lemma (Lemma 12.1.3) to find a sequence of disjoint squares $\{Q_j\}$ so that

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy < 2\beta^p,$$

and $v \leq \beta$ almost everywhere off $\cup Q_j$. Thus $E_\beta \setminus \cup Q_j$ has measure zero and

$$\int_{E_\beta} v^p dxdy \leq \sum_j \int Q_j v^p dxdy \leq 2\beta^p \sum \text{area}(Q_j).$$

We now make use of the reverse Hölder hypothesis to write

$$\beta^p \leq \frac{1}{\text{area}(Q_j)} \int_{Q_j} v^p dxdy \leq (\frac{K}{\text{area}(Q_j)} \int_{Q_j} vdx)^p,$$

hence

$$\begin{aligned}
\text{area}(Q_j) &\leq \frac{K}{\beta} \int_{Q_j} vdxdy \\
&\leq \frac{K}{\beta} (\int_{Q_j \cap E_\lambda} vdxdy + \lambda \text{area}(Q_j) \\
&\leq \frac{K}{\beta} \int_{Q_j \cap E_\lambda} vdxdy + \frac{1}{2} \text{area}(Q_j).
\end{aligned}$$

Solving for $\text{area}(Q_j)$ gives

$$\begin{aligned}
\text{area}(Q_j) &\leq \frac{2K}{\beta} \int_{Q_j} vdxdy \\
&\leq \frac{1}{\lambda} \int_{Q_j} vdxdy.
\end{aligned}$$

Thus by the defining property of the Q_j's,

$$\begin{aligned}
\int_{E_\beta} v^p dxdy &\leq \sum_j \int_{Q_j} v^p dxdy \\
&\leq 2\beta^p \sum_j \text{area}(Q_j) \\
&\leq 2\beta^p \lambda^{-1} \sum_j \int_{Q_j \cap E_\lambda} vdx \\
&\leq 2^{p+1} K^p \lambda^{p-1} \int_{E_\lambda} vdx.
\end{aligned}$$

Thus the hypothesis of Lemma 12.3.1 holds with $A = (2K)^{p-1} + 2^{p+1}K^p$, and we deduce that $v \in L^r(Q, dxdy)$ for some $r > p$. □

To apply Gehring's inequality to the partial derivatives of quasiconformal maps, we have to show that these partial satisfy a reverse Hölder inequality. What we want is

$$\int_Q |f_z|^2 dxdy \leq \frac{C}{\text{area}(Q)} (\int_Q |f_z| dxdy)^2,$$

with a uniform C for all squares in the plane. This was proven in the previous section.

Thus we have proven the theorem of Bojarski and Gehring mentioned earlier:

THEOREM 12.3.3. *If $1 \le K < \infty$, there is a $p > 2$ and $A, B < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $Q \subset \mathbb{C}$ is a square, then*

$$(\frac{1}{\text{area}(Q)} \iint_Q |f_z|^p dxdy)^{1/p} \le A(\frac{1}{\text{area}(Q)} \int_Q |f_z|^2 dxdy)^{1/2} \le B\frac{\text{diam}(f(Q))}{\text{diam}(Q)}$$

LEMMA 12.3.4. *If f fixes $0, 1, \infty$, then*

$$\int_Q |L_f(x) - 1| dxdy \le \epsilon \text{diam}(Q),$$

where $L_f = |f_z| + |f_{\bar z}|$ and $\epsilon \to 0$ as $\|\mu_f\|_\infty \to 0$.

PROOF. Fix a square Q with sides parallel to the axes, let $\ell(Q)$ denote its side length and let S_1, S_2 denote the two vertical sides of S Use fact that as $\|\mu\|_\infty \to 0$, f tends to the identity and

$$\begin{aligned}
0 \le (\frac{1}{\text{area}(Q)} \int_Q |v-1| dxdy)^2 &\le \frac{1}{\text{area}(Q)} \int_Q |v-1|^2 dxdy \\
&\le \frac{1}{\text{area}(Q)} \int_Q (v^2 - 1) - \frac{2}{\text{area}(Q)} \int_Q (v-1) dxdy \\
&\le \frac{1}{\text{area}(Q)} \int_Q (KJ_f - 1) - \frac{2}{\text{area}(Q)} \int_Q (v-1) dxdy \\
&= \frac{1}{\text{area}(Q)} \int_Q (K-1) J_f dxdy + \frac{1}{\text{area}(Q)} \int_Q (J_f - 1) dxdy \\
&\quad - \frac{2}{\text{area}(Q)} \int_Q (v-1) dxdy \\
&\le O(\|\mu\|_\infty) \frac{\text{area}(f(Q))}{\text{area}(Q)} + \frac{\text{area}(f(Q)) - \text{area}(Q)}{\text{area}(Q)} - 2(\frac{\text{dist}(S_1, S_2)}{\ell(Q)} - 1).
\end{aligned}$$

Since f converges uniformly to the identity on Q as $\|\mu\|_\infty \to 0$, each term in the last line tends to zero. □

COROLLARY 12.3.5. *If Ω has a piecewise C^1 boundary and f is quasiconformal on Ω, then*

$$f(w) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{z - w} dz - \frac{1}{\pi} \iint_\Omega \frac{f_{\bar z}}{z - w} dxdy. \tag{47}$$

PROOF. Smooth and take a limit using the L^p boundedness of the the Hardy-Littlewood maximal theorem and the Lebesgue dominated convergence theorem. □

COROLLARY 12.3.6. *If f is quasiconformal, then f maps sets of zero area to zero area and*

$$\text{area}(f(E)) = \int_E J_f dxdy.$$

PROOF. Since $\nu(E) = \text{area}(f(E))$ and $\nu(E) = \int_E J_f dxdy$ are both non-negative Borel measures, it suffices to show that they are equal for some convenient basis of sets, say squares with sides parallel to the coordinate axes. Let Q be such a square.

We have already proved the "$\geq$" direction in Lemma 12.2.3. To prove the other direction, we use the fact that $J_f \in L^p(Q, dxdy)$ for some $p > 1$. Define a smoothed version f_n of f by convolving f with a smooth, non-negative bump function φ_n of total mass 1 and support in $D(0, \frac{1}{n})$. Since f is continuous on $\mathbb{C}$, $f_n \to f$ uniformly on Q. Since convolution is linear, the partials of f_n are the partials of f convolved with φ_n and therefore the supremum over n of these partials is bounded by the Hardy-Littlewood maximal function of f_z, i.e.,

$$\sup_n |(f_n)_z(x)| \leq \mathcal{H}L(f_z)(x),$$

and similarly for $f_{\bar{z}}$. Since the Hardy-Littlewood maximal operator is bounded on L^p for $1 < p < \infty$, and $f_z, f_{\bar{z}} \in L^p$ for some $p > 1$, we see that $\{((f_n)_z)\}$, $\{((f_n)_{\bar{z}})\}$ are dominated by an L^p function and hence by an L^2 function on Q (since $L^p \subset L^2$ on bounded sets). Thus the sequence of Jacobians $\{J_{f_n}\}$ is dominated by an L^1 function on Q, so by the Lebesgue dominated convergence theorem,

$$\int_Q J_{f_n} dxdy \to \int_Q J_f dxdy.$$

Moreover, since f_n is smooth

$$\int_Q J_{f_n} dxdy \geq \text{area}(f_n(Q)),$$

(equality may not hold since we don't known f_n is 1-to-1, and the integral computes area with multiplicity) and since $f_n \to f$ uniformly, $f_n(Q)$ eventually contains any compact subset of $f(Q)$ and hence

$$\limsup_n \text{area}(f_n(Q)) \geq \text{area}(f(Q)).$$

Thus $\text{area}(f(Q)) \leq \int_Q J_f dxdy$, as desired. $\square$

LEMMA 12.3.7. *Suppose $\{g_n\} \in L^p(R, dxdy)$ for some $p > 2$ and*

$$\lim_n \iint_R \frac{g_n(z)}{z-w} dxdy = 0$$

for all $w \in R$. Then $\lim_n \iint_R g_n dxdy = 0$.

PROOF. Fix rectangles $R'' \subset R' \subset R$, each compactly contained in the interior of the next. Using the Cauchy integral formula for the constant function 1 on the curve $\partial R'$ we see that we can uniformly approximate the constant function 1 on R'' by a finite sum $s(z) = \sum \frac{a_k}{z-w_k}$ with $w_k \in \partial R'$ and $\sum |a_k|$ is uniformly bounded. Then

$$\begin{aligned} \iint_R g_n(z)dxdy &= \iint_R g_n(z)s(z)dxdy + \iint_R g_n(z)(1-s(z))dxdy \\ &= o(1) + \iint_{R''} g_n(z)(1-s(z))dxdy + \iint_{R\setminus R''} g_n(z)(1-s(z))dxdy. \end{aligned}$$

For a fixed n, the first integral can be made as close to zero as we wish by taking s close to 1 on R''. The second integral can be made small by taking area$(R \setminus R'') \to 0$; this implies the L^p norm of g_n on $R \setminus R''$ tends to zero (hence so does its L^1 norm) whereas the L^q norm of s remains uniformly bounded (it is a convex combination of L^q functions with bounded norm). Thus we can make $\iint_R g_n dxdy$ as small a we wish if n is large, proving the lemma. □

LEMMA 12.3.8. *If $\{g_n\}$ are K-quasiconformal maps that converge uniformly on compact sets to a quasiconformal map g, then for any rectangle R.*

$$\iint_R [(g_n)_z - g_z]dxdy \to 0,$$

$$\iint_R [(g_n)_{\bar{z}} - g_{\bar{z}}]dxdy \to 0.$$

and $(g_n)_z \to g_z$ and $(g_n)_{\bar{z}} \to g_{\bar{z}}$ weakly.

PROOF. First consider the $\bar{z}$-derivative. Let $h_n = (g_n)_{\bar{z}} - g_{\bar{z}}$. By the Pompeiu formula and the fact that $g_n \to g$ uniformly on R, we deduce that

$$\lim_{n\to\infty} \iint_R \frac{h_n(z)}{z-w} dxdy = 0$$

for any $w \in R$. That

$$\iint_R h_n dxdy \to 0,$$

follows from Lemma **??**. To prove weak conference, take any continuous f of compact support and uniformly approximate it to within ϵ by a function $\tilde{f}$ that is constant on finite union of rectangles. Then

$$\iint f h_n dxdy = \iint (f-\tilde{f}) h_n dxdy + \iint \tilde{f} h_n dxdy.$$

The first integral is bounded by $\epsilon \iint |h_n| dxdy$, which is small since $\|h_n\|_1 \leq C\|h_n\|_p$ is uniformly bounded on a large ball containing the support of both f and $\tilde{f}$. The second integral tends to zero since is a finite linear combination of integrals of h_n over rectangles.

The result for z-derivatives follows from the same proof applied to the complex conjugates of g and $\{g_n\}$, using the fact that $(\bar{f})_{\bar{z}} = \overline{f_z}$. □

12.4. The measurable Riemann mapping theorem, Part II

We proved early using only geometric methods that given any continuous dilatation μ on $\mathbb{C}$ with $\|\mu\|_\infty \leq k < 1$, there was a $K = (k+1)/(k-1)$ quasiconformal map f with dilatation μ. We can extend this from continuous to measurable functions μ.

THEOREM 12.4.1. *Given any measurable function μ on the plane with $\|\mu\|_\infty = k < 1$, there is a $K = (k+1)/(k-1)$ quasiconformal map f with dilatation μ almost everywhere.*

PROOF. Given a measurable μ find a sequence of continuous functions $\{\mu_n\}$ with $\mu_n \to \mu$ pointwise and $\sup_{\mathbb{C}} |\mu_n(z)| \leq k = \|\mu\|_\infty < 1$. Let f_n be the quasiconformal map with dilatation μ_n, normalized to fix both 0 and 1. Then since normalized, K-quasiconformal maps form a compact family (Theorem **??**) there is a subsequence of these maps that converges uniformly on compact sets to a K-quasiconformal map f. This map has a dilatation μ_f. We claim that $\mu_f = \mu$. This will follow from the following lemma. □

LEMMA 12.4.2. *Suppose $\{f_n\}$, f are all K-quasiconformal maps on the plane with dilatations $\{\mu_n\}$, μ_f respectively, that $f_n \to f$ uniformly on compact sets and that $\mu_n \to \mu$ pointwise almost everywhere. Then $\mu_f = \mu$ almost everywhere.*

PROOF. We restrict attention to some domain Ω with compact closure. We know that $f_{\bar{z}} = \mu_f f_z$ almost everywhere and we know that f_z is non-zero almost everywhere, so it suffices to show that

$$f_{\bar z}(w) - \mu(w) f_z(w) = 0,$$

almost everywhere. To prove this it suffices to show that the integral of $f_{\bar z}(w) - \mu(w) f_z(w)$ over any rectangle R is zero (this is an application of the Lebesgue differentiation theorem: at almost every point an integrable function is the limit of its averages over rectangles shrinking down to that point). We re-write this function as

$$\begin{aligned} f_{\bar z}(w) - \mu(w) f_z(w) &= [f_{\bar z}(w) - (f_n)_{\bar z}(w)] \\ +[(f_n)_{\bar z}(w) - \mu_n (f_n)_z(w)] & \\ &\quad +[\mu_n(w)(f_n)_z(w) - \mu(w)(f_n)_z(w)] \\ &\quad +[\mu(w)(f_n)_z(w) - \mu(w) f_z] \\ &= I + II + III + IV. \end{aligned}$$

Term II equals zero almost everywhere, so we need only show that the other three terms tend to zero as n tends to ∞.

Case I: This is Lemma 12.3.8.

Case III: We use Cauchy-Schwarz to show the integral of the third term is bounded by

$$(\iint_R (\mu - \mu_n)^2 dxdy)^{1/2} (\iint_R |(f_n)_x|^2 dxdy)^{1/2},$$

The first integrand tends to zero pointwise and is bounded above by 2 almost everywhere, so the integrals tend to zero by the Lebesgue dominated convergence theorem. On the other hand

$$(\iint_R |(f_n)_x|^2 dxdy)^{1/2} \simeq \mathrm{diam}(f_n(R)),$$

by Lemma 12.2.4, and since $\{f_n\}$ converges uniformly on compact sets, this remains bounded. Thus the integral of III is bounded above by a term tending to zero times a term that is uniformly bounded, and hence it tends to zero.

Case IV: The same lemma as in case I, but applied to $f_z = (\bar f)_{\bar z}$, and using the fact that $(\bar f)_{\bar z} = \overline{(f_z)}$, show that

$$\iint_R (f_z - (f_n)_z) dxdy \to 0$$

for every rectangle R. Now approximate μ in the $L^q(R, dxdy)$ norm by a function ν that is constant on a finite collection of disjoint squares (such functions are dense in

L^q) and we deduce

$$\int_n \iint_R \mu((f_z-(f_n)_z)dxdy = \lim_n \iint_R (\mu-\nu)((f_z-(f_n)_z)dxdy \le \lim_n \|\mu-_\nu\|_q \|(f_z-(f_n)_z)\|_p.$$

The first term is as small as we wish and the second is uniformly bounded, so the product is as small as we wish. Thus the limit must be zero, as desired. □

This completes the proof of the measurable Riemann mapping theorem in the general case.

12.5. The Ahlfors formula

The dependence of f on its dilatation μ is non-linear (there is an explicit power series relationship between the two in terms of certain singular integral operators, see, e.g., [**3**]), but it is possible to give a linear approximation that is valid when $\|\mu\|_\infty$ is small, namely

$$f(w) = w - \frac{1}{\pi}\int_{\mathbb{R}^2} \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty^2),$$

for all $|w| \le 1$, where

$$R(z,w) = \frac{1}{z-w} - \frac{w}{z-1} + \frac{w-1}{z} = \frac{w(w-1)}{z(z-1)(z-w)}.$$

The goal of this section is to prove this formula. The proof is basically a manipulation of the Pompeiu formula

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_\Omega \frac{f_{\bar z}}{z-w}dxdy$$

where we use our L^p estimates on $f_z, f_{\bar z}$ to put certain terms into the error term. We start by showing f is close to the identity in a precise L^p sense when $\|\mu\|$ is small.

Lemma 12.5.1. *If $k < 1$ is small enough then there is a constant $C_3 = C_3(k)$ so that the following holds. Suppose $\|\mu\|_\infty \le k < 1$. Then*

$$\|f_z^\mu - 1\|_{p,1} \equiv (\int_{B_1} |f_z^\mu - 1|^p dxdy)^{1/p} \le C_3\|\mu\|_\infty.$$

for all $2 \le p \le p(k)$.

PROOF. First assume μ is supported in $D(0,R)$ and let $\epsilon = \|\mu\|_\infty$. It is proven on page 100 of [**3**] that $\|F^\mu_z - 1\|_p \le C\|\mu\|_p \le C\epsilon R^{2/p}$ if $p < p(k)$. Since $f^\mu = F^\mu/F^\mu(1)$, Lemma **??** implies

$$\begin{aligned} \|f^\mu_z - 1\|_{p,1} &= \|\frac{F^\mu_z}{F^\mu(1)} - 1\|_{p,1} \\ &= |1 - \frac{1}{F^\mu(1)}| + \frac{1}{F^\mu(1)}\|F^\mu_z - 1\|_{p,1} \\ &\le 2C_2\epsilon + \frac{C(R)}{1 - C\epsilon}\epsilon \\ &\le C\epsilon. \end{aligned}$$

Now write $\check{f}(z) = 1/f(1/z)$. We want to show

$$\|\check{f}^\mu_z - 1\|_{p,R} \le C(R)\epsilon, \tag{48}$$

when μ has support in B_R. Just as above, it suffices to show $\|\check{F}^\mu_z - 1\|_{p,R} \le C\epsilon$. Note that $\check{F}^\mu$ is analytic on $\{z : |z| < 3r\}$ where $r = 1/(3R)$. For an analytic function f on a ball $B(x,r)$ it is easy to see by the mean value property and Hölder's inequality that

$$|f(x)| \le \frac{1}{\pi r^2}\int_{B(x,r)} |f| \le \frac{1}{(\pi r^2)^{1/p}}\|f\|_{L^p(B(x,r))}.$$

Thus by the maximum principle,

$$\begin{aligned} \int_{|z|<r} |\check{F}^\mu_z(z) - 1|^p dxdy &\le C(r) \sup_{|z|=2r} |\check{F}^\mu_z(z) - 1|^p \\ &\le C(r) \int_{r<|z|<3r} |\check{F}^\mu_z(z) - 1|^p dxdy. \end{aligned}$$

On the other hand, changing variables from z to $1/z$ gives

$$\begin{aligned}\int_{r<|z|<R} |\check{F}^{\mu}_z(z) - 1|^p dxdy &= \int_{1/R<|z|<1/r} \left|\frac{z^2 F^{\mu}_z(z)}{F^{\mu}(z)^2} - 1\right|^p \frac{dxdy}{|z|^4} \\ &= \int_{1/R<|z|<1/r} \left|\frac{z^2(F^{\mu}_z(z)-1)}{F^{\mu}(z)^2} + \frac{z^2 - F^{\mu}(z)^2}{F^{\mu}(z)^2}\right|^p \frac{dxdy}{|z|^4} \\ &\le C \int_{1/R<|z|<1/r} \left|\frac{z^2(F^{\mu}_z(z)-1)}{F^{\mu}(z)^2}\right|^p + \left|\frac{z^2 - F^{\mu}(z)^2}{F^{\mu}(z)^2}\right|^p \frac{dxdy}{|z|^4} \\ &\le C(R) \int_{1/R<|z|<1/r} |F^{\mu}_z(z) - 1|^p dxdy \\ &\qquad + C(R) \int_{1/R<|z|<1/r} |z - F^{\mu}(z)|^p dxdy \\ &\le C(R)\epsilon^p.\end{aligned}$$

Since the integral over $\{|z| < 3r\}$ was dominated by a constant (depending only on R) times this estimate, we have proven (48).

The general case now follows just as in [**3**]. Write $f = \check{g} \circ h$ where $\mu_h = \mu_f$ inside the unit disk and $\mu_h = 0$ outside the unit disk. Then

$$\|f_z - 1\|_{p,1} \le \|[(\check{g}_z - 1) \circ h]h_z\|_{p,1} + \|h_z - 1\|_{p,1}.$$

The second term is bounded by $C\epsilon$ by the first paragraph and the first term is bounded using

$$\begin{aligned}\|[(\check{g}_z - 1) \circ h]h_z\|^p_{p,1} &= \int_{B_1} |(\check{g}_z - 1) \circ h|^p |h_z|^p dxdy \\ &\le \frac{1}{1-k^2} \int_{h(B_1)} |\check{g}_z - 1|^p |h_z \circ h^{-1}|^{p-2} dxdy \\ &\le \frac{1}{1-k^2} \left(\int_{h(B_1)} |\check{g}_z - 1|^{2p} dxdy \int_{B_1} |h_z|^{2p-4} dxdy\right)^{1/2}\end{aligned}$$

Clearly $h(B_1) \subset \{|z| < R\}$ for some R depending only on k. Thus using (48), the first integral is bounded by

$$\int_{B_R} |\check{g}_z - 1|^{2p} dxdy \le C\epsilon^{2p},$$

(assuming $2p < p(k)$; but since $p(k) \to \infty$ as $k \to 0$ this holds for some $p > 2$ if k is small enough). On the other hand

$$\int_{B_1} |h_z|^{2p-4} dxdy \le C\left(\int_{B_1} |h_z|^{2p} dxdy\right)^{1-2/p} \le \|\mu_h\|_{p,1} + \|1\|_{p,1} \le C,$$

since $\|h_z - 1\|_p \le C\|\mu_h\|_p$. $\square$

LEMMA 12.5.2. *With notation as above,*

$$f(w) \quad = \quad w - \frac{1}{\pi}\int_{B_1} f_{\bar z}(z)R(z,w)dxdy - \frac{1}{\pi}\int_{B_1}\frac{\check f_{\bar z}(z)}{\check f(z)^2}zS(z,w)dxdy,$$

where $R(z,w) = (\frac{1}{z-w} - \frac{w}{z-1} + \frac{w-1}{z})$ *and* $S(z,w) = \frac{w^2}{1-wz} - \frac{w}{1-z}$.

PROOF. This is where we use the Pompeiu formula. Assume $|w| < 1$ and apply the formula to the unit disk to get

$$f(w) = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_{\mathbb{D}}\frac{f_{\bar z}}{z-w}dxdy. \tag{49}$$

Replace z by $1/z$ in the boundary integral. We claim that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{T}} f(\frac{1}{z})\frac{dz}{z(1-zw)} &= A + Bw + \frac{w^2}{2\pi i}\int_{\mathbb{T}}\frac{dz}{\check f(z)(1-zw)}\\ &= A + Bw - \frac{w^2}{2\pi}\int_{\mathbb{D}}\frac{\check f_{\bar z}(z)dxdy}{\check f(z)^2(1-zw)}.\end{aligned}$$

To see this, first suppose that on $\{|z| = 1\}$ $f(z) = z^n$. Then $f(\frac{1}{z}) = z^{-n}$, and

$$\frac{1}{z(1-zw)} = \frac{1}{z}(1 + zw + (zw)^2 + \dots),$$

so

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathbb{T}} f(\frac{1}{z})\frac{dz}{z(1-zw)}\\ &\quad= \frac{1}{2\pi i}\int_{\mathbb{T}} z^{-n-1})(1 + zw + z^2w^2 + \dots)dz\\ &\quad= \frac{1}{2\pi i}\int_{\mathbb{T}} z^{-n-1})(z^{-n-1} + z^{-n}w + z^{n+1}w^2 + \dot{+}z^{-n-1+j}w^j + \dots)dz.\end{aligned}$$

This integral is only non-zero for the term containing z^{-1}; this corresponds to $j = n$, so for $n \ge 0$,

$$\frac{1}{2\pi i}\int_{\mathbb{T}} f(\frac{1}{z})\frac{dz}{z(1-zw)} = w^n$$

and the integral is zero for $n < 0$. By a similar argument, if $n \ge 2$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\check f(\frac{1}{z})^{-1}zdz}{(1-zw)} &= \frac{1}{2\pi i}\int_{\mathbb{T}} z^{-n+1})(1 + zw + z^2w^2 + \dots)dz\\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} z^{-n-1})(z^{-n+1} + z^{-n}w + z^{n+2}w^2 + \dot{+}z^{-n+1+j}w^j + \dots)dz.\\ &= w^{n-2}\end{aligned}$$

and for $n < 2$ the integral is zero. So the two integrals differ by a factor of w^2 in general and for $n = 0, 1$ the integral on the right can gives $1, w$ but the integral on the left gives 0. Thus

$$\frac{1}{2\pi i}\int_{\mathbb{T}} f(\frac{1}{z})\frac{dz}{z(1-zw)} = A + Bw + \frac{w^2}{2\pi i}\int_{\mathbb{T}}\frac{dz}{\check{f}(z)(1-zw)}. \tag{50}$$

We now rewrite the line integral as an area integral using Pompeiu's formula again with the change of variable $\alpha = wz$

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|z|=1}\frac{\check{f}(z)^{-1}zdz}{1-zw} &= -\frac{1}{2\pi i w^2}\int_{|z|=1}\frac{\check{f}(z)^{-1}wzwdz}{zw-1}\\
&= -\frac{1}{2\pi i w^2}\int_{\alpha=|w|}\frac{\check{f}(\frac{\alpha}{w})^{-1}\alpha d\alpha}{\alpha-1}\\
&= -\frac{1}{w^2}[\check{f}(1) + \frac{1}{\pi}\int_{|\alpha|<|w|}\frac{[\check{f}(\frac{\alpha}{w})^{-1}\alpha]_{\bar{z}}dadb}{\alpha-1}\\
&= -\frac{1}{w^2}[\check{f}(1) + \frac{1}{\pi}\int_{|z|<1}\frac{[\check{f}(z)^{-1}wz]_{\bar{z}}w^2dxdy}{zw-1}\\
&= -\frac{1}{w} + \frac{1}{\pi}\int_{|z|<1}\frac{\check{f}_{\bar{z}}(z)zdxdy}{\check{f}^2(z)(zw-1)}
\end{aligned}$$

Therefore,

$$\frac{1}{2\pi i}\int_{\mathbb{T}} f(\frac{1}{z})\frac{dz}{z(1-zw)} = A + Bw + w - \frac{w^2}{2\pi}\int_{\mathbb{D}}\frac{\check{f}_{\bar{z}}(z)dxdy}{\check{f}(z)^2(1-zw)} \tag{51}$$

as claimed.

Since we know $f(0) = 0$ and $f(1) = 1$ we can solve for the values of A and B. When we do this we get

$$\begin{aligned}
f(w) = w &- \frac{1}{\pi}\int_{|z|<1} f_{\bar{z}}(z)(\frac{1}{z-w} - \frac{w}{z-1} + \frac{w-1}{z})dxdy\\
&- \frac{1}{\pi}\int_{|z|<1}\frac{\check{f}_{\bar{z}}(z)}{\check{f}(z)^2}(\frac{w^2z}{1-wz} - \frac{wz}{1-z})dxdy.
\end{aligned}$$

As a check, the reader can set $w = 0, 1$ and verify that both integrands vanish in these cases. $\square$

LEMMA 12.5.3. *There is a $0 < k < 1$ and a $C_4 < \infty$ so that the following holds. Suppose that f is a quasiconformal mapping of the plane to itself which preserves $\mathbb{H}$,* *fixing* $0, 1$ *and* ∞ *and the Beltrami coefficient of* f *is* μ *with* $\|\mu\|_\infty \le k$. *Then*

$$|f(w) - [w - \frac{1}{\pi}\int_{\mathbb{R}^2} \mu(z) R(z,w)dxdy]| \le C_4 \|\mu\|_\infty^2,$$

for all $|w| \le 1$, *where*

$$R(z,w) = \frac{1}{z-w} - \frac{w}{z-1} + \frac{w-1}{z} = \frac{w(w-1)}{z(z-1)(z-w)}.$$

PROOF. Consider (51). If the first integral, we use

$$f_{\bar z} = \mu f_z = \mu(f_z - 1) + \mu.$$

The first term has L^p norm $O(\|\mu\|_\infty)$ by Lemma **??**, so using Hölder's inequality shows that the first integral equals

$$\begin{aligned} &\frac{1}{\pi}\int_{|z|<1} \mu(z)(\frac{1}{z-w} - \frac{w}{z-1} + \frac{w-1}{z})dxdy + \|\mu\|_\infty^2 \\ &= \frac{1}{\pi}\int_{|z|<1} \mu(z)\frac{w(w-1)}{z(z-1)(z-w)}dxdy + \|\mu\|_\infty^2 \\ &= \frac{1}{\pi}\int_{|z|<1} \mu(z)R(z,w)dxdy + \|\mu\|_\infty^2 \end{aligned}$$

where

$$R(z,w) = \frac{w(w-1)}{z(z-1)(z-w)}.$$

Using the same estimates, second integral is equal to

$$\frac{1}{\pi}\int_{|z|<1} \mu(\frac{1}{z})\bar z^{-2}(\frac{w^2 z}{1-wz} - \frac{wz}{1-z})dxdy + \|\mu\|_\infty^2$$

where $\check\mu(z) = \mu(\frac{1}{z})(z/\bar z)^2$. If we replace z by $1/z$ in the second integral, the integral over the disk transforms into the integral over its complement

$$\begin{aligned} &\frac{1}{\pi}\int_{|1/z|<1} \check\mu(\frac{1}{z})(\frac{w^2/z}{1-w/z} - \frac{w/z}{1-1/z}\frac{-dxdy}{|z|^4} \\ &= \frac{1}{\pi}\int_{|z|>1} \mu(z)(\bar z^2/z^2)(\frac{w^2}{z-w} - \frac{w}{z-1}\frac{1}{z^2\bar z^2}dxdy \\ &= \frac{1}{\pi}\int_{|z|>1} \mu(z)(\frac{w^2(z-1)-(z-w)w}{(z-w)(z-1)z^2}dxdy \\ &= \frac{1}{\pi}\int_{|z|>1} \mu(z)(\frac{w(w-1)}{(z-w)(z-1)z}dxdy. \end{aligned}$$

This integrand has the same form as before which proves the lemma.

NEEDS CHECKING AND FIXING

By Lemma 12.5.2 we can write

$$f(w) \;=\; w - \frac{1}{\pi}\int_{B_1} f_{\bar z}(z)R(z,w)dxdy - \frac{1}{\pi}\int_{B_1} \frac{\check f_{\bar z}(z)}{\check f(z)^2} zS(z,w)dxdy,$$

where $S(z,w) = \frac{w^2}{1-wz} - \frac{w}{1-z}$ and as before $\check f(z) = 1/f(1/z)$. Using $f_{\bar z} = \mu f_z = \mu + \mu(f_z - 1)$, the first integral equals

$$\begin{aligned}\int_{B_1} \mu(z)R(z,w)dxdy \;&+\; \int_{B_1} \mu(z)(f_z(z)-1)R(z,w)dxdy \\ &= \int \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty \|f_z - 1\|_{p,1}\|R\|_{q,1}) \\ &= \int \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty^2),\end{aligned}$$

by Lemma 12.5.1 and the fact that $R \in L^q$, for every $q < 2$ (with a bound depending on q, but not on w for $|w| \le 1$).

The second integral is estimated by writing $\check f_{\bar z} = \check\mu + \check\mu(\check f_z - 1)$ where $\check\mu(z) = (z/\bar z)^2\mu(1/z)$. Repeating the argument above shows the second integral is equal to

$$\begin{aligned}\int_{B_1} \frac{\check\mu(z)}{\check f(z)^2} + &\frac{\check\mu(z)(\check f_z(z)-1)}{\check f(z)^2} zS(z,w)dxdy \\ &= \int_{B_1} \mu(\frac{1}{z})[\frac{1}{\bar z^2} + \frac{z^2 - \check f(z)^2}{\bar z^2 \check f(z)^2}] zS(z,w)dxdy \\ &\qquad + \int_{B_1} \frac{\check\mu(\check f_z - 1)}{\check f(z)^2} zS(z,w)dxdy \\ &= \int_{B_1} \mu(\frac{1}{z})\frac{1}{\bar z^2} zS(z,w)dxdy + I + II\end{aligned}$$

Using Lemma **??**, we see

$$\frac{1}{C}|z|^{1/\alpha} \le |\check f(z)| \le C|z|^\alpha,$$

$$|z - \check f(z)| \le C\|\mu\|_\infty |z|^\alpha,$$

so we can estimate I by

$$\begin{aligned} I &\leq |\int_{B_1} \mu(1/z)\frac{z^2-\check{f}(z)^2}{\bar{z}^2\check{f}(z)^2} zS(z,w)dxdy| \\ &\leq C\|\mu\|_\infty^2 \int_{B_1} |z|^{2\alpha-1-\frac{2}{\alpha}} S(z,w)dxdy \\ &\leq C\|\mu\|_\infty^2 C(\alpha), \end{aligned}$$

if $2\alpha - 1 - \frac{2}{\alpha} > -2$ (recall that we may take α as close to 1 as we wish, if k is small enough). To estimate II, note that for $\frac{1}{p}+\frac{1}{q}=1$, Lemma 12.5.1 implies

$$\begin{aligned} II &= \int_{B_1} \frac{\check{\mu}(z)(\check{f}_z(z)-1)}{\check{f}(z)^2} zS(z,w)dxdy \\ &\leq C\|\mu\|_\infty \|\check{f}_z - 1\|_p \|\frac{zS(z,w)}{\check{f}(z)^2}\|_q \\ &\leq \|\mu\|_\infty^2 \|z^{1-\frac{2}{\alpha}} S(z,w)\|_q. \end{aligned}$$

Fix some $q < 2$, and take k so small that $\alpha > 2q/(2+q)$, which implies the L^q norm is finite (with bound depending only on α, hence only of k). Thus,

$$f(w) = w - \frac{1}{\pi}\int_{B_1} \mu(z)R(z,w)dxdy - \frac{1}{\pi}\int_{B_1} \mu(\frac{1}{z})\frac{1}{\bar{z}^2} zS(z,w)dxdy + O(\|\mu\|_\infty^2).$$

Changing variables from z to $1/z$ in the second integral converts the integrand to the same form as the first (but now over $\{|z|>1\}$). Hence,

$$f(w) = w - \frac{1}{\pi}\int_{\mathbb{R}^2} \mu(z)R(z,w)dxdy + O(\|\mu\|_\infty^2),$$

as desired. $\square$

COROLLARY 12.5.4. *If $\mu(t)$ is continuous in the L^∞ norm, then $f_{\mu(t)}(z)$ is a C^1 curve in $\mathbb{C}$.*

PROOF. Think of the path $\gamma(t) = f_{t\mu}(z)$. The key point is the formula

$$|\gamma(t) - (\gamma(0) + \gamma'(0)t)| \leq Ct^2,$$

holds on an interval $[-\delta, \delta]$ and with a constant C that do not depend on t or z or μ (except for $\|\mu\|_\infty$). Thus using the same estimate at $\gamma(t)$ and time $-t$ gives

$$|\gamma(0) - (\gamma(t) - \gamma'(t)t)| \leq Ct^2.$$

Thus adding the estimates and dividing by t gives

$$|\gamma'(0) - \gamma'(t)| \leq Ct$$

Thus $f_{t\mu}(z)$ has a continuous derivative in t when t is real. When t is multi-variable, the same argument shows we have continuous partial derivatives, and this implies differentiability by the usual calculus argument (e.g., see Rudin's book [**121**]). $\square$

12.6. Non-wandering in the Speiser class

In 1885 Dennis Sullivan proved that a rational map has no wandering domains, a famous open problem dating back to the origins of the subject. Two sets of researchers, Alex Eremenko and Misha Lyubich [**49**], and Lisa Goldberg and Linda Keen [**62**], soon generalized Sullivan's proof from rational functions to the Speiser class:

THEOREM 12.6.1. *If f is an entire function that has a finite singular set, then f has no wandering domains.*

In this section, we will give a proof of this result, first for polynomials and then for the Speiser class (and with minor modifications, the proof would also cover the original case of rational functions).

The main idea is fairly easy to state. If there were a wandering domain U for f, then any dilatation μ on U could be extended to a dilatation on the grand orbit of U so that the corresponding quasiconformal mapping has the property that $g = h \circ f \circ h^{-1}$ is also entire. This gives a continuous map from dilatations on U to entire functions that are quasiconformally conjugate to f, a finite dimensional space by Theorem 12.6.5. By an explicit construction, we can choose a subspace of dilatations with larger dimension on which the map must be 1-to-1 and this violates Brouwer's invariance of domain theorem (you can't map an open subset of $\mathbb{R}^n$ continuously and 1-1 into $\mathbb{R}^{n-1}$). You can avoid the use of Brouwer's theorem by proving that the map from dilatations to entire functions is continuously differentiable and then using the rank theorem instead.

We say that two entire functions f, g are **topologically equivalent** if there are homeomorphisms $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ such that

$$\psi \circ g = f \circ \varphi.$$

The maps are **quasiconformally conjugate** if ψ, φ can be taken to be quasiconformal homeomorphisms.

We say f and g are **topologically conjugate** if there is a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such $\varphi \circ g = f \circ \varphi$. and call the maps **quasiconformally conjugate** if φ can be taken to be quasiconformal. Note that this a stronger condition than equivalence and if f and g are conjugate then

$$f^n = (\varphi \circ g \circ \varphi^{-1})^n = \varphi \circ g^n \circ \varphi^{-1},$$

so that the dynamical behavior of f and g is essentially identical. Obviously the identity map conjugates a map f to itself. The next result says that in some situations, this is the only possible such conjugation.

LEMMA 12.6.2. *Suppose $\{\varphi_t\}$ is a family of quasiconformal maps on $\mathbb{C}$ so that $\varphi_t(z) : X \to \mathbb{C}$ is continuous each fixed z as a function of $t \in X$, X a connected space. Suppose that φ_{t_0} is the identity for some $t_o \in X$. Suppose that $f \in \mathcal{S}$ has the property that $\varphi_t \circ f = f \circ \varphi_t$ for all $t \in X$. Then $\varphi_t(z) = z$ for all $t \in X$ and all $z \in \mathcal{J}(f)$, i.e., every φ_t is the identity when restricted to the Julia set of f.*

PROOF. Because φ_t conjugates the action of f to itself, periodic points are mapped to periodic points with the same period. Since there only countable many such points, they form a discrete set and so $\{\varphi_t(z) : t \in X\}$ must be a single point, since X is connected. Since one of these maps is the identity, every map must fix every periodic point. Finally, since periodic points are dense in the Julia set (Theorem 3.4.2), and quasiconformal maps are continuous, each map φ_t must fix every point in $\mathcal{J}(f)$. $\square$

LEMMA 12.6.3. *If $f : \mathbb{D} \to \Omega \subset \mathbb{C}$ is conformal and $\varphi : \Omega \to \Omega$ is a quasiconformal map that extends continuously to the identity on $\partial\Omega$, then $\Phi = f^{-1} \circ \varphi \circ f$ is a quasiconformal map of the disk to itself that extends to the identity on $\partial\mathbb{D}$.*

PROOF. Clearly $\Phi : \mathbb{D} \to \mathbb{D}$ is quasiconformal and hence extends continuously to a homeomorphism of the unit circle (see Theorem 11.3.5). If the extension of Φ to $\partial\mathbb{D}$ is not the identity, then there is an arc $I \subset \mathbb{T}$ such that $I \cap \Phi(I) = \emptyset$. Choose a point $w \in I$ so that f has a finite radial limit at both z and $\Phi(z)$; we can do this because (1) conformal maps have finite radial limits except on a set of zero capacity (Corollary 5.5.4), and (2) sets of zero capacity map to zero capacity under quasiconformal maps (immediate from Pfluger's theorem).

Take the union of the two radial line segments $[0, w]$ and $[0, \Phi(w)]$. Because φ extends as the identity to $\partial\Omega$, the images of these radial segments under f have the same endpoint on $\partial\Omega$ and hence their union is a a closed Jordan curve γ_w. Now, choose a distinct point $z \in I$ with the same properties and form the closed Jordan curve γ_z. Choose z so that the intersection of γ_z with $\partial\Omega$ is different that the intersection of γ_w with $\partial\Omega$; we can do this because only a set of logarithmic capacity zero on the circle can have the same radial limit. Then $\gamma_z \cap \gamma_w = f(0)$ and γ_z hits both sides of γ_w (since z and $\Phi(z)$ are in different components of $\mathbb{T} \setminus ([0, w] \cup [0, \Phi(w)])$. See Figure **??**. This contradicts the Jordan curve theorem, and thus Φ must extend to the identity on the boundary. □

FIGURE JordanContradiction

LEMMA 12.6.4. *A wandering domain for a polynomial must be simply connected.*

PROOF. The basin of ∞ is periodic, not wandering, so any wandering domain must be bounded and have a bounded orbit. By the maximum principle, the iterates of f are bounded in the interior of any closed curve in the component and hence form a normal family inside the curve. Thus the curve does not surround any Julia points and the component must be simply connected. □

POLYNOMIALS HAVE NO WANDERING DOMAINS. Choose a smooth, non-negative function h on $\mathbb{C}$ supported in $\mathbb{D}$ with gradient bounded by 1 and such that $h(0) > 0$. Define a family of mappings of the upper half-plane to itself by

$$\Phi_t(z)z + th(z).$$

It is easy to check that these are quasiconformal self-maps of $\mathbb{H}$ if we restrict t to a small enough interval $[0, \epsilon]$ and that Φ_0 is the identity. If $t > 0$, then the mapping is definitely not the identity since the cross ratio of the points $-1, 0, 1, \infty$ changes.

Now choose N disjoint intervals $I_k = \{[2k-3, 2k-1]\}_1^N$ and define an N-dimensional family of maps by $\mathbf{t} = (t_1, \ldots, t_N)$, and

$$\Phi_{\mathbf{t}}(z) = z + \sum_{k=1}^{N} t_k h(z - (2k-2)) t_k h(z - (2k-2)).$$

Suppose Ω were a wandering domain for f. Since f has only finitely many critical values, we can replace Ω, if necessary, by an iterate of itself so that neither it nor any iterate contains a critical point. Therefore we may assume f is univalent on Ω and on all forward orbits.

By Lemma 12.6.4 Ω is simply connected, so we can map it conformally by f to $\mathbb{H}$ and define a quasiconformal map $\varphi_{\mathbf{t}} = f^{-1} \circ \Phi_{\mathbf{t}} \circ f$. This defines a dilatation μ on Ω that we extend to the grand orbit of Ω using the composition rule for dilatation so that the corresponding quasiconformal map $\Psi_{\mathbf{t}}$ given by the measurable Riemann mapping theorem has the property that

$$g_{\mathbf{t}} = \Psi_{\mathbf{t}}^{-1} \circ f \circ \Psi_{\mathbf{t}},$$

is entire. Doing the extension backwards is always possible; extending to the forward iterates uses the assumption that f and all its iterates are univalent on Ω.

A consequence of Brouwer's invariance of domains theorem is that any continuous map of an open set in $\mathbb{R}^n$ into $\mathbb{R}^k$ for $k < n$ there is a point $z \in \mathbb{R}^k$ whose preimage has topological dimension ≥ 1 and hence contains a connected set X. Choose some $\mathbf{s} \in X$ and consider the maps $\Psi_{\mathbf{t}} \circ \Psi_{\mathbf{s}}^{-1}$. These conjugate f to itself and one of them is the identity, so by Lemma 12.6.2, they are all the identity on $\mathcal{J}(f)$, hence on $\partial\Omega$ and hence the corresponding maps $\Phi_{\mathbf{t}} \circ \Phi_{\mathbf{s}}^{-1}$ are extend to the identity on $\mathbb{R}$. However, this is manifestly false by construction; the boundary maps are not the identity unless $\mathbf{s} = \mathbf{t}$. Therefore there a polynomial has no wandering domains. □

Next we show how the proof given above for polynomials adapts to entire functions with finite singular sets. By Lemma 4.1.4 Ω is simply connected. for Eremenko-Lyubich functions and hence Speiser class functions.

The only non-trivial new step is to prove that the collection of entire functions with with a given finite singular set is finite dimensional.

By Lemma 4.1.4 Ω is simply connected.

Let M_g denote the collection of all entire functions f that are topologically equivalent to g. An important result of Eremenko and Lyubich [] say that for $g \in \mathcal{S}$, the collection M_g of all f that are topologically equivalent to g form a finite dimensional, complex analytic manifold. We shall just prove a part of this, showing that M_g is finite dimensional in the following sense.

LEMMA 12.6.5. *If $f, g \in \mathcal{S}$ have the same singular values then there is an $\epsilon > 0$ so that the following holds. If*

$$\psi \circ g = f \circ \varphi,$$

where ψ, φ are $(1+\epsilon)$-quasiconformal, then $g(z) = f(az+b)$ for some $a, b \in \mathbb{C}, a \neq 0$.

PROOF. The proof is essential an exercise about covering spaces, and we will need the following lifting lemma that is Theorem 14.3 of Munkres' book [**102**]:

THEOREM 12.6.6 (The general lifting lemma). *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map with $f(y_0) = b_0$. Suppose Y is path connected and locally path connected. The map f can be lifted to a map $F : Y \to E$ such that $F(y_0) = e_0$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)).$$

Here π_1 denotes the fundamental group and f_* is the map between fundamental groups induced by the continuous map f.

In our application, we let $X = \mathbb{C}\backslash S(f) = \mathbb{C}\backslash S(g)$ and let $Y_f = \mathbb{C}\backslash f^{-1}(S(f)), Y_g = \mathbb{C}\backslash g^{-1}(S(g))$. Choose some point $z_0 \in Y_g$. By Lemma **??** $f : Y_f \to X$ and $g : Y_g \to X$ are covering maps.

Since $S(g)$ is a finite set, there is a positive lower bound $\delta > 0$ between any two points in $S(g)$. Since $S(g)$ is bounded, there is an $\epsilon > 0$ so that any $(1 + \epsilon)$-quasiconformal map fixing $0, 1, \infty$ moves each point of $S(f)$ by less than $\delta/10$. Thus if φ is $(1+\epsilon)$-quasiconformal, it is isotopic to the identity via a path of quasiconformal maps that fix each point of $S(g)$. Thus for any closed loop γ in Y_g, the image loop $g(\gamma) = \psi^{-1} \circ f \circ \varphi(\gamma)$ is homotopic to $f \circ \varphi(\gamma)$. Thus $g_*(\pi_1(Y_g, z_0)) \subset f_*(\pi_1(Y_f, \varphi(z_0)))$. In fact, we have equality, since $\pi_1(Y_f)$ is isometric to $\pi_1(Y_g)$ via the homeomorphism φ. By the general lifting lemma we get a homeomorphism $h : F : Y_g \to Y_f$ and this map is locally a composition of g and a branch of f^{-1} and hence is holomorphic. Thus it must be conformal linear, i.e., $h(z) = az + b, a \neq 0$, as claimed. □

12.7. Dilatations with small support

If the dilatation μ of a quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is small, then we expect f to be close to conformal, hence close to linear. If the map is normalized correctly, then we expect it to be close to the identity. There are at least two reasonable senses in which we can ask μ to be small: that $\|\mu\|_\infty$ is small or that $\{z : \mu(z) \neq 0\}$ is small. In this section we consider the latter possibility. To be more precise, we say a measurable set $E \subset \mathbb{R}^2$ is (ϵ, φ)-thin if $\epsilon > 0$ and

$$\text{area}(E \cap D(z,1)) \leq \epsilon\varphi(|z|)$$

where $\varphi : [0,\infty) \to [0,\pi^2]$ is a bounded, decreasing function, such that

$$\int_0^\infty \varphi(r) r^n dr < \infty,$$

for every $n > 1$. If $a > 0$, the function $\varphi(r) = \exp(-ar)$ satisfies this condition, and this example suffices for the applications we will make later.

Recall that a quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is often normalized in one of two ways. For general maps, we can always post-compose with a linear conformal map so that $f(0) = 0$ and $f(1) = 1$; we then say f is normalized to fix 0 and 1. If the dilatation of f is supported on a bounded set, then f is conformal in a neighborhood of ∞ and then we can choose R large and post-compose with a linear conformal map so that

$$|f(z) - z| = O(\frac{1}{|z|}),$$

for $|z| > R/2$. We say that such an f is normalized at ∞. This is also called the **hydrodynamical normalization** of f.

LEMMA 12.7.1. *Suppose F is K-quasiconformal with dilatation μ has bounded support and F is normalized at ∞. Let $E = \{z : \mu(z) \neq 0\}$ and suppose E is (ϵ, φ)-thin. Then for every $z \in \mathbb{R}^2$ and $r \geq 1$,* $\text{diam}(F(D(z,r)) = O(r)$, *i.e., all unit are mapped to sets of comparable or smaller size.*

PROOF. By linking points by chain of unit disks, it suffices to prove this when $r = 1$. Choose $R > 100$ so that $|f(z) - z| \leq 1$, for $|z| > R/8$. The result is clear if $|w| > R/2$, so we may assume $|w| \leq R/2$. Fix such a w.

Let $m = \log R$, so $R = e^m$, and consider the annulus $A = \{z : 1 < |z - w| < e^m\}$. $F(A)$ is a topological annulus and can be conformally mapped to $A_M = \{1 < |z| < e^M\}$ for some $M > 1$. Post-composing F by this conformal map does not change the dilatation, so without loss of generality, we map assume F maps A_m to A_M. If we cut A_m with a radial slit and let $G = \log(F)$, then G maps A_m to a quadrilateral with its vertical sides on $\{x = 0\}$ and $\{x = M\}$. This quadrilateral has area $2\pi M$.

If we integrate over the radial segments in A_m, we get

$$M \leq \int_1^{\exp(m)} (|G_z| + |G_{\bar{z}}|)\frac{dr}{r}$$

so integrating over all angles gives

$$2\pi M \leq \int_0^{2\pi} \int_1^{\exp(m)} (|G_z| + |G_{\bar{z}}|)\frac{dr}{r} d\theta \leq \int_{A_m} (|G_z| + |G_{\bar{z}}|)\frac{dxdy}{r^2}.$$

Thus by Cauchy-Schwarz,

$$\begin{aligned}
(2\pi M)^2 &\leq \left(\int_{A_m} (|G_z| + |G_{\bar{z}}|)(|G_z| - |G_{\bar{z}}|)\frac{dxdy}{r^2}\right)\left(\int_{A_m} \frac{|G_z| + |G_{\bar{z}}|}{|G_z| - |G_{\bar{z}}|}\frac{dxdy}{r^2}\right) \\
&\leq \left(\int_{A_m} J_G dxdy\right)\left(\int_{A_m} D_G \frac{dxdy}{r^2}\right) \\
&\leq 2\pi M\left(\int_{A_m} D_F \frac{dxdy}{r^2}\right)
\end{aligned}$$

since $G(A_m)$ has area $2\pi M$ and $D_G = D_F$ (since log is conformal). Thus

$$\begin{aligned}
M &\leq \frac{1}{2\pi}\int_{A_m} 1 + (D_F - 1)\frac{dxdy}{|z-w|^2} \\
&= m + \frac{1}{2\pi}\int_{A_m} (D_F - 1)\frac{dxdy}{|z-w|^2} \\
&\leq m + \frac{K-1}{2\pi}\int_{A_m} \mathbf{1}_E(z)\frac{dxdy}{|z-w|^2} \\
&= m + O(1)
\end{aligned}$$

where $\mathbf{1}_E$ denotes the indicator function of E (the function that is one on E and zero off E) and we have used the fact that E has finite planar area and $|z-w|^{-1} \leq 1$ on A_m (recall w is the center of the annulus and the inner radius is at least 1.).

By Corollary 11.3.8, the boundary components of $f(A)$ are each closed curves that are contained in round annuli (with concentric circles) of bounded modulus (depending on K). Thus $f(A)$ is contained in a topological annulus with circular boundaries γ_1, γ_2 (not necessarily concentric) whose diameters are comparable to the diameters of the boundary components of $f(A)$. By monotonicity of modulus, this annulus has larger modulus than $f(A)$,and thus

$$\log \frac{\operatorname{diam}(\gamma_2)}{\operatorname{diam}(\gamma_1)} \geq \log R - O(1).$$

Since $|f(z) - z| \le 1$ on $\{|z| = R\}$ we know $\operatorname{diam}(\gamma_2) \simeq R$ and hence

$$f(\{|z - w| = 1\}) \simeq \operatorname{diam}(\gamma_1) = O(1).$$

This proves the lemma. □

Thus if F is as above, Bojarski's theorem (Theorem 12.3.3) says there is a $p > 2$ so that the L^p norm of F_z is uniformly bounded on every unit radius disk. If a region can be covered by n such disks then the L^p norm is $O(n^{1/p})$ with a uniform constant, i.e.,

Corollary 12.7.2. *If F satisfies the conditions of Lemma 12.7.1, and $p = p(K) > 2$ is the value given by Theorem 12.3.3, then $\|F_z \cdot \mathbf{1}_{D(z,r)}\|^p = O(r^{2/p})$ uniformly for all $z \in \mathbb{C}$.*

We now get to the main estimate of this section.

Theorem 12.7.3. *Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is K-quasiconformal, and $E = \{z : \mu(z) \neq 0\}$ is bounded (so F is conformal near ∞) and F is normalized so*

$$|F(z) - z| \le M/|z|,$$

near ∞. Assume E is (ϵ, φ)-thin. Then

$$|F(z) - z| \le \frac{\epsilon^\beta}{|z| + 1},$$

where β depends only on K and φ. In particular, as $\epsilon \to 0$, F converges uniformly to the identity on the whole plane.

Proof. The main idea is to use the Pompeiu formula

$$F(w) = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z)}{z - w} dz - \frac{1}{\pi} \iint_{|z|<r} \frac{F_{\bar z}}{z - w} dxdy. \tag{52}$$

when $\Omega = D(0, r)$, $r > R$; we saw before that the Pompeiu formula is valid even for non-smooth quasiconformal maps because the L^p bounds on F_z given by Gehring's lemma allow us to pass from the smooth case to the general case, i.e., Corollary 12.3.5.

Because of our assumptions on F, the first integral is

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{z + O(1/|z|)}{z - w} dz = w + O(1/r).$$

The left-hand side of (52) and the second integral are both constant for $r > R$ (since the support of $F_{\bar{z}}$ is contained in $D(0, R)$), so the first integral must equal w for all $r > r$. Thus

$$F(w) = w - \frac{1}{\pi} \iint_{|z|<r} \frac{F_{\bar{z}}}{z-w} dxdy = w - \frac{1}{\pi} \iint_{|z|<r} \frac{\mu F_z}{z-w} dxdy.$$

Since $|F_{\bar{z}}| = |\mu F_Z| \leq k|F_z|$, we get

$$|F(w) - w| \leq \frac{k}{\pi} \int_E |\frac{F_z}{z-w}| dxdy.$$

where $k = (K-1)/(K+1)$ is our upper bound for $|\mu|$.

The estimate in the theorem already holds if $|w| \geq R$, so assume $|w| < R$. Let $r = \max(1, |w|/2)$. We will estimate the integral

$$\int_E |\frac{F_z}{z-w}| dxdy,$$

by cutting $D(0, R)$ into four pieces:

$$\begin{aligned} D_1 &= \{z : |z - w| \leq 1\} \\ A &= \{z : 1 \leq |z - w| \leq r\} \\ X &= D(0, R) \setminus (D_1 \cup A \cup D_2), \end{aligned}$$

and showing the integral over each piece is $O(\epsilon^\beta/|w|)$ for some $\beta > 0$ depending only on K.

First consider D_1. With p as in Corollary 12.3.3, the L^p norm of F_z over D_1 is uniformly bounded, so using Hölder's inequality with the conjugate exponents, we get

$$\int_{D_1} |\frac{F_z}{z-w}| dxdy = O(\| \frac{\mathbf{1}_{E \cap D(w,1)}}{|z-w|} \|_q).$$

Since $E \cap D(w, 1)$ has area at most $\varphi(|w|) \leq \varphi(r)$, the L^q norm is bounded above by the case when $E \cap D(w, 1)$ is a disk of radius $s \simeq (\epsilon\varphi(r))^{1/2}$ centered at w . In this case the L^q norm (using polar coordinates and recalling $1 < q < 2$) is

$$O([\int_0^s r^{-q} r dr]^{1/q}) = O(s^{(2-q)/q}) = O((\epsilon\varphi(r))^{\frac{1}{q}-\frac{1}{2}}).$$

Since φ tends to zero faster than any polynomial, this is $= O(\epsilon^{\frac{1}{q}-\frac{1}{2}} \frac{1}{|w|})$. This is the desired estimate with $\beta = \frac{1}{q} - \frac{1}{2} > 0$.

Next consider the integral over A:

$$\begin{aligned}
\int_A |\frac{F_z}{z-w}| dxdy &= \int_A \mathbf{1}_E(z)|F_z| dxdy \\
&= (\int_A \mathbf{1}_E(z)^q dxdy)^{1/q} (\int_A |F_z|^p dxdy)^{1/p} \\
&= O(\text{area}(E \cap A))^{1/q} \|F_z \mathbf{1}_A\|_p \\
&= O((\epsilon r^2 \varphi(r))^{1/q}) r^{2/p} \\
&= O(\epsilon^{1/q} \frac{1}{|w|}),
\end{aligned}$$

since φ decays faster than any power.

Finally, consider the integral over X:

$$\int_X |\frac{F_z}{z-w}| dxdy = O(\frac{1}{|w|} \int_{D_2} \mathbf{1}_E(z)|F_z| dxdy.$$

To estimate the integral, write

$$X = \cup_{k=1}^R X_k = \cup_{k=1}^R X \cap A_k = \cup_{k=1}^R X \cap \{z : k-1 \le |z| < k\},$$

Then

$$\begin{aligned}
\int_{X_k} \mathbf{1}_E(z)|F_z| dxdy &= (\int_{A_k} \mathbf{1}_E(z)^q dxdy)^{1/q} (\int_{A_k} |F_z|^p dxdy)^{1/p} \\
&= (\text{area}(E \cap A_k))^{1/q} (\int_{A_k} |F_z|^p dxdy)^{1/p} \\
&= (\epsilon k \varphi(k))^{1/q} (O(k))^{1/p} \\
&= O(\epsilon^{1/q} \varphi(k))^{1/q} k^{1+1/p}) \\
&= O(\epsilon^{1/q} k^{-2}),
\end{aligned}$$

again since φ decays faster than any power. Summing over k gives the desired estimate. This proves the theorem with $\beta = \frac{1}{q} - \frac{1}{2} > 0$. □

The proof given above shows that the conclusion of Theorem 12.7.3 still holds if $\int_0^\infty \varphi(r) r^n dr < \infty$ for some (large) finite n that depends on K (in particular, it depends on the value $p > 2$ so that $F_z \in L^p$ in Bojarski's theorem). Similarly, we can assume less if we simply want a uniform bound on $|F(w) - w|$, rather than the $O(1/|z|)$ estimate above. We leave these generalizations to the reader.

LEMMA 12.7.4. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is K-quasiconformal (possibly with a dilatation with unbounded support) and is normalized to fix* 0 *and* 1*, and $E = \{z : \mu(z) \neq 0\}$ is (ϵ, φ)-thin. Then*

$$(53) \quad (1 - C\epsilon^\beta)|z - w| - C\epsilon^\beta \leq |f(z) - f(w)| \leq (1 + C\epsilon^\beta)|z - w| + C\epsilon^\beta,$$

where C and β only depend on $k = \|\mu\|_\infty$ and φ.

PROOF. First we note that it suffices to prove this with the additional assumption that μ has bounded support, for a general quasiconformal f is the pointwise limit of such maps (truncate μ_f, apply the measurable Riemann mapping theorem and show the truncated maps converge uniformly on compact subsets to f).

So assume $\mu = \mu$ has bounded support, say inside the disk $D(0, R)$. Then f is conformal outside $D(0, R)$, so we can post-compose by a conformal linear map L to get a quasiconformal map

$$F(z) = z + O(\frac{1}{z}),$$

or

$$|F(z) - z| \leq C/|z|,$$

outside $D(0, 2R)$ with a constant that does not depend on F (this follows from the distortion theorem for conformal maps). We apply Theorem 12.7.3 to get

$$|F(z) - z| \leq C\epsilon^\beta,$$

for all z with constants C, β that depend only on k. Note that

$$f(z) = \frac{F(z) - F(0)}{F(1) - F(0)},$$

and that

$$|F(1) - F(0) - 1| \leq C\epsilon^\beta,$$

so,

$$|f(z) - f(w)| \quad = \quad |\frac{F(z) - F(w)}{F(1) - F(0)}| = \frac{|z - w| + O(\epsilon^\beta)}{1 + O(\epsilon^\beta)},$$

and this implies (53). □

These remarks imply the following technical looking result that will be helpful when we want to construct an entire function in the Eremenko-Lyubich class that has a wandering domain.

LEMMA 12.7.5. *Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is K-quasiconformal, it fixes 0 and 1, maps $\mathbb{R}$ to $\mathbb{R}$, and is conformal in the strip $\{x + iy : |y| < 1\}$. Let $E = \{z : \mu(z) \neq 0\}$ and suppose E is (ϵ, φ)-thin. If ϵ is sufficiently small (depending on k and φ), then $0 < \frac{1}{C} \leq |f'(x)| \leq C < \infty$ for all $x \in \mathbb{R}$, where C depends on K, φ and ϵ is otherwise independent of f. If we fix K and φ and let $\epsilon \to 0$ then $C \to 1$.*

PROOF. For each $x \in \mathbb{R}$, f is conformal on the disk $D(x, 1) \subset S$, so Koebe's $\frac{1}{4}$-theorem says that

$$|f'(x)| \simeq \operatorname{dist}(f(x), \partial f(D(x, 1))).$$

However taking $z = x$ and $w \in \partial D(x, 1)$ in (53) shows that

$$\operatorname{dist}(f(x), \partial f(D(x, 1))) \simeq 1.$$

This gives the first claim. When ϵ is small, then (53) implies that

$$(1 - \delta)S \subset f(S) \subset (1 + \delta),$$

where $\delta > 0$ tends to zero with ϵ (for fixed k and a). Thus as $\epsilon \to 0$, f converges uniformly to the identity on S. In particular, f' converges uniformly to 1 on $\mathbb{R}$. $\square$

CHAPTER 13

Quasiconformal folding

CHAPTER 14

Topological dimension

In the proof of Sullivan's no wandering domains theorem, the main fact that is needed is that if $n > k$, and $f : \mathbb{R}^n \to \mathbb{R}^k$, then there is a point y in the image so that $f^{-1}(y)$ contains a non-trivial continuum. The particular map in question is of the form

$$\mu(t) \to \{f_{\mu(t)}(z_j)\}_{j=1}^k,$$

where $\mu(t)$ is a finite dimensional family of dilatations that varies continuously in the L^∞ norm, f_μ is the normalized quasiconformal map given by the measurable Riemann mapping theorem, and $\{z_j\}$ are a finite number of distinct points in the plane.

In Chapter 12 we developed enough of the analytic theory of quasiconformal maps to prove that $f_{\mu(t)}(z_j)$ is a differentiable function of t and hence the desired result about f^{-1} follows from the rank theorem.

However, even without the analytic theory of quasiconformal maps, it is easy see that $f_{\mu(t)}(z_j)$ is a continuous function of t, and in this case the desired conclusion follows from a purely topological result, Let $I_n = [0, 1]^n$.

Theorem 14.0.1. *If $n > k$ and $f : I_n \to \mathbb{R}^k$ is continuous, then there is a point $y \in f(I_n)$ so that $f^{-1}(y)$ contains a compact connected set with more than one point.*

This result is far from obvious and is closely related to Brouwer's "invariance of domain" theorem that states that $\mathbb{R}^n$ and $\mathbb{R}^k$ are not homeomorphic. I will not attempt to give a complete proof of the result here, but I will give a summary of the proof given in the classic book "Dimension Theory" by Hurewicz and Wallman.

The book is written about separable metric spaces, although for our purposes, it suffices to consider subsets of Euclidean space and I will make this extra assumption to simplify the discussion.

The dimension of a set is defined inductively as follows:

(1) The empty set has dimension -1.

(2) A set X has dimension $\leq n$ if every point has arbitrarily small open neighborhoods whose boundaries have dimension $\leq n-1$.
(3) The set X has dimension $= n$ if it has dimension $\leq n$ but does not have dimension $\leq n-1$.

This says that X has dimension $\leq n$ if there is a basis for the topology of X made up of open sets whose boundaries have dimension $\leq n-1$. We shall let $\mathrm{Dim}(X)$ denote the topological dimension of X, to differentiate it from $\dim(X)$, which we have used throughout these notes for the Hausdorff dimension of X. In the course of this chapter we shall see that the topological dimension has several equivalent formulations, namely, $\mathrm{Dim}(X) \leq n$ if and only if

(1) X can be written as union of $n+1$ sets of dimension ≤ 0,
(2) any $n+1$ pairs of closed subsets of X can be separated by $(n+1)$ closed subsets that have empty intersection,
(3) Every continuous map of X into the n-cube has a stable value (a value that is attained by every continuous function sufficiently close to f in the supremum norm),
(4) every continuous function from any closed subset of X to the n-sphere can be continuously extended to all of X,
(5) X is homeomorphic to a zero $(n+1)$-measure subset of $\mathbb{R}^{2n+1}$.

In this language, the theorem we want follows immediately from two results in of [**72**] (throughout the chapter we shall label results with their names in [**72**] for the convenience of the reader who wishes to consult the original source):

THEOREM 14.0.2 (Proposition II.4.D). *If X is compact, then X has dimension zero iff it is totally disconnected (i.e., contains no non-trivial connected components).*

THEOREM 14.0.3 (Theorem VI.7). *If $f : \mathbb{R}^n \to \mathbb{R}^k$ is a closed mapping (it is continuous and sends closed sets to closed sets), then there is an image point y so that* $\mathrm{Dim}(f^{-1}(y)) \geq n-k$.

The first result is fairly easy to proof from the definitions, but the second is quite involved and uses the Brouwer fixed point, Tietze's extension theorem and Borsuk's theorem on the extension of homotopies. Most of this chapter is devoted to the proof of the second result, given these results from topology.

It interesting to note that the first result can fail if X is not compact. Knaster and Kuratowski constructed a set $X \subset \mathbb{R}^2$ that is totally disconnected, but so that adding a single point $\{a\}$ makes it a connected set $Y = X \cup \{a\}$. Corollary II.3.2 of [**72**] states that adding a point to a set cannot change its dimension, so $\mathrm{Dim}(X) = \mathrm{Dim}(Y)$. Proposition II.2.D says that any zero dimensional set is totally disconnected, so $\mathrm{Dim}(X) = \mathrm{Dim}(Y) \geq 1$. A much harder result (Theorem IV.3) says that a subset of $\mathbb{R}^n$ has dimension n if and only if it contains an open subset, The Knaster-Kuratowski example does not, so X is a totally disconnected set of topological dimension 1.

Suppose $\boldsymbol{C} \subset \mathbb{R}$ is the usual middle thirds Cantor set, let $E \subset \boldsymbol{C}$ be the countable set of endpoints of intervals in $\mathbb{R} \setminus \boldsymbol{C}$ and let $P = \boldsymbol{C} \setminus E$ be the remaining points. Let $a = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2$ and for each $x \in \boldsymbol{C}$, let L_x be the line segment connecting x to a. For $x \in E$, let L_x^* be the points on L_x with rational y-coordinates, and for $x \in P$ let it denote the points on L_x with irrational y-coordinate. Then $X = \cup_{x \in \boldsymbol{C}} L_x^*$ is the desired set. See [**81**] or [**133**]. The point a is called an **explosion point** for the set X.

This phenomenon is particularly interesting in transcendental dynamics, since similar sets arise naturally there: Mayer has shown that the set of landing points of dynamic rays for $\lambda(z)$ s totally disconnected, but becomes connected when we add $\{\infty\}$ [**94**], i.e., $\{\infty\}$ is an explosion point.

14.1. Zero dimensional sets

We say that two subsets $A_1, A_2 \subset X$ can be separated if there are disjoint open subsets U_1, U_2 that contain A_1 and A_2 respectively. We can then describe four properties of X:

(1) X is totally disconnected.
(2) Any two distinct points can be separated.
(3) Any point can be separated from any closed set not containing it.
(4) Any two disjoint closed sets can be separated.

For general X, (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (3) is the definition of X having zero dimension. For compact X all four conditions are equivalent; for general separable sets, (2) and (3) are equivalent, but (1), (2) and (3) are distinct conditions.

We say that A_1, A_s are separated by a set $B \subset X$ if the open sets U_1, U_2 can be chosen to be in different connected components of $X \setminus B$.

LEMMA 14.1.1 (Proposition II.2.E). *If a space X is zero dimensional, then any two closed sets can be separated in X.*

PROOF. Suppose K, L are disjoint closed sets in X. Every $p \in X$ has an open-closed neighborhood that is disjoint from either K or L (or maybe both) and a countable union $\{U_j\}$ of these cover X. Let $V_j = U_j \setminus \sum_{k=1}^{j-1} U_k$; this gives a disjoint open cover of X and each V_j is disjoint from either K or L. Taking the unions of V_j's that hit each of these sets gives disjoint open sets separating them. $\square$

THEOREM 14.1.2 (Theorem II.2). *If $X = \cup X_j$ is a countable union of closed (in X), zero dimensional subsets, then X is also zero dimensional.*

PROOF. It suffices to show that any two closed subsets K, L can be separated (contained in disjoint open sets). Since X_1 is zero dimensional, the sets can be separated in X_1 by Lemma 14.1.1, so X_1 can be divide into two disjoint, closed subsets A_1, B_1 containing $K \cap X_1$ and $L \cap X_1$ respectively. Thus $K \cup A_1$, $L \cup B_1$ are disjoint closed sets in X and hence are contained in disjoint open subsets G_1, H_1 of X that have disjoint closures.

Now repeat the argument replacing K and L by $\overline{G_1}$ and $\overline{H_1}$. By induction we obtain nested sequences of open sets so that

$$G_j \subset \overline{G_j} \subset G_{j+1}, \qquad H_j \subset \overline{H_j} \subset H_{j+1}.$$

Then $\cup G_j$, $\cup H_j$ are open, disjoint subsets of X that contain $K \cap X_j$ and $L \cap X_j$ respectively for every j and hence contain K and L respectively. $\square$

COROLLARY 14.1.3. *A union of two zero dimensional spaces, one of which is closed, is zero dimensional.*

This follows since if A, B are zero dimensional and B is closed, then $X = A \setminus B$ is open in $A \cup B$. But any open set in the separable metric space $A \cup B$ is a countable union of closed sets, and these sets have dimension zero, since they are subsets of A. Thus the corollary follows from the theorem. Since points are closed, we also get:

COROLLARY 14.1.4. *Add a point to a zero dimensional set does not increase its dimension.*

LEMMA 14.1.5. *Let $\mathcal{R}_n^m$ be the set of points in $\mathbb{R}^n$ that have exactly m rational coordinates. Then $\mathcal{R}_m^n$ has dimension zero.*

PROOF. If $n = m$ then $\mathcal{R}_n^m$ is a countable union of points and hence has dimension zero (most small spheres around any point miss a countable set). If $m = 0$, then every point has small neighborhoods that are cubes whose faces have a rational coordinate, and again we get dimension 0.

For $0 < m < n$, fix a choice of m coordinates and fix m rational values and let H be the $k = n - m$ dimensional (in terms of linear algebra) subspace determined by these choices. Then $\mathcal{R}_n^m \cap H$ is a linear image of $\mathcal{R}_k^0$ and hence has dimension 0, and it is a closed subspace of $\mathcal{R}_n^m$ (although not closed in $\mathbb{R}^n$. Thus $\mathcal{R}_n^m$ is a countable union of closed, dimension zero, subspaces of itself, and hence has dimension zero. □

LEMMA 14.1.6 (Proposition II.4.B). *Suppose X is compact and dimension zero, $p \in X$ and $K \subset X$ closed. If p can be separated from each point of K, it can be separated from K by open-closed sets.*

PROOF. Fore each $q \in K$ there are disjoint neighborhoods U and $V$of p and q. Since K is compact, a finite union of V's cover K and the corresponding intersection of the U's is open and disjoint from the union. □

LEMMA 14.1.7 (Proposition II.4.C). *If X is compact and dimension zero, and $p \in X$, then the set $M(p)$ of points that can't be separated from p is connected.*

PROOF. Each point not in $M(p)$ has an open neighborhood disjoint from an neighborhood of p, so $X \setminus M(p)$ is open, so $M(p)$ is closed and contains p. If $M(p)$ were disconnected then $M(p) = K \cup L$ where K, L are open-closed in $M(p)$ hence closed in X. We may assume $p \in K$. There exists open U in X so $K \subset U$ and $\overline{U} \cap L = \emptyset$. Then $\partial U \cap M(p) = \emptyset$ (since it hits neither K nor L), and each point of ∂U is separated from p. Since ∂U is closed, Lemma 14.1.6 says ∂U can be separated from p by disjoint open-closed neighborhood V of ∂U and $W = U \setminus V = U \setminus \overline{V}$ of p. But W is disjoint from L, so p is separated from points in L, contrary to the definition of $M(p)$. The contradiction shows $M(p)$ is indeed connected. □

Corollary 14.1.8. *For compact sets X conditions (1)-(4) are equivalent. In particular, compact totally disconnected sets have dimension zero.*

Proof. Assume X is totally disconnected, i.e., no connected subset contains more than one point. Then by Lemma 14.1.7 for each $p \in X$ $M(p)$ is connected, hence equals p. Thus (0) implies (1). Lemma 14.1.6 gives (1) implies (2). The implication (2) implies (3) is Lemma 14.1.1 and opposite directions are e all trivial. □

14.2. Subsets, unions and products

Lemma 14.2.1. *A subset Y of a set X of dimension n has dimension $\leq n$.*

Proof. We use induction and note it is trivial for $n = -1$. Suppose $p \in Y \subset X$. By definition, for any $\delta > 0$, there is a neighborhood U of p in X with $U \subset B(p, \delta)$ and $\mathrm{Dim}(\partial U) \leq n - 1$. Let $V = U \cap Y$. Then V is a neighborhood of p in Y and $\partial V \subset \partial U \cap Y$ and this has dimension $\leq n - 1$ by induction. □

Lemma 14.2.2 (Proposition III.2.A). *A subset $Y \subset X$ has dimension $\leq n$, if and only if every point $p \in Y$ has arbitrarily small neighborhoods in X whose boundaries have intersections with Y of dimension $\leq n - 1$.*

Proof. Suppose the condition holds. For any $\delta > 0$ choose a neighborhood $U \subset B(p, \delta)$ of p in X so that $\mathrm{Dim}(\partial U \cap Y) \leq n-1$. Then $V = U\cap$ has $\partial V \subset \partial U \cap Y$ so also has dimension $\leq n - 1$. This proves $\mathrm{Dim}(Y) \leq n$.

Conversely, suppose $\mathrm{Dim}(Y) \leq n$ and let $p \in Y$. For any δ we can choose a neighborhood $V \subset B(p, , \delta)$ of p and $\mathrm{Dim}(\partial V) \leq n - 1$. Since V and $Y \setminus \overline{V}$ are disjoint open subsets of Y, there is an open set W in X so that $V \subset W \subset B(p, \delta)$ and $\overline{W} \cap (Y \setminus \overline{V}) = \emptyset$. It follows that $\partial W \cap Y \subset \partial V$ and hence $\mathrm{Dim}(\partial W \cap Y) \leq n-1$. □

Lemma 14.2.3. *If $A, B \subset X$, then* $\mathrm{Dim}(A \cup B) \leq 1 + \mathrm{Dim}(A) + \mathrm{Dim}(B)$. *Thus a union of n zero dimensional sets has dimension at most $n - 1$.*

Proof. We use induction on both the dimension of A and B, noting that the cases $(m, -1)$ and $(-1, n)$ are all trivial. Assume it is true for the cases $(m, n - 1)$ and $(m - 1, n)$ and we will deduce it for (m, n); this suffices since we can then fill in the whole quadrant $(m, n), m \geq 0, n \geq 0$.

Suppose $p \in A \cup B$; we may assume $p \in A$. Let U be a neighborhood of p in X. By Lemma 14.2.2 there is a neighborhood $V \subset U$ of p with $\dim(\partial V \cap A) \leq m - 1$. Since $\partial V \cap B \subset B$ it also has dimension $\leq n$, so by the induction hypothesis,

$$\mathrm{Dim}(\partial V \cap (A \cup B)) \leq 1 + (m-1) + n = m + n,$$

and this proves $\mathrm{Dim}(A \cup B) \leq m + n + 1$ by Lemma 14.2.2. $\square$

Lemma 14.2.4. *Let $\mathcal{M}_n^m$ be the set of points in $\mathbb{R}^n$ that have at most m rational coordinates. Then $\mathcal{R}_m^n$ has dimension $\leq m$.*

Proof. Since $\mathcal{M}_n^m = \cup_{j=0}^m \mathcal{R}_n^j$, it is a union of $m+1$ sets of dimension 0. The result follows from the final conclusion of Lemma 14.2.3. $\square$

Lemma 14.2.5 (Theorem III.2, Sum Theorem). *A countable union of closed sets of dimension n has dimension $\leq n$.*

Proof. We use induction. The case $n = 0$ is trivial and the case $n = 0$ was proven as Theorem 14.1.2. We claim that the case $n - 1$ implies:

Lemma 14.2.6 (Δ_n). *Any space X of dimension $\leq n$ is a union of a subset of dimension $\leq n - 1$ and a space of dimension ≤ 0.*

Proof. By the definition of dimension, there is a basis of open sets whose boundaries have dimension $\leq n-1$ and since X is separable, this may be taken to be countable, $\{U_k\}$. Then by hypothesis $B = \cup \partial U_k$ has dimension $\leq n - 1$. We claim that $\mathrm{Dim}(X \setminus B) \leq 0$. (PROOF ????) The lemma then follows from Lemma 14.2.2. $\square$

By induction we get a fact we will need later.

Corollary 14.2.7 (Theorem III.3). *A space has dimension $\leq n$ iff it can be written as a union of $n+1$ zero dimensional spaces.*

We now resume the proof of Lemma 14.2.5. Suppose $X = \cup K_j$ where each K_j is a closed set of dimension $\leq n$. Let $X_1 = K_1$ and

$$X_k = K_k \setminus \cup_{j=1}^{k-1} K_j.$$

Then these sets are disjoint, cover X, and has dimension $\leq n$ since $X_k \subset K_k$. Moreover each X_k is a F_σ, i.e., a countable union of closed sets. This holds since

$X \setminus \cup_{j=1}^{k-1} K_j$ is open and hence F_σ; thus X_k is the intersection of a closed set and a F_σ and hence is F_σ.

By Lemma 14.2.6, $X_k M_k \cup N_k$ where $\mathrm{Dim}(M_k) \leq n-1$ and $\mathrm{Dim}(N_k) \leq 0$. Thus $X = M \cup N = (\cup_k M_k) \cup (\cup_k N_k)$. Note that M_k is F_σ inside M since

$$M_k = M_k \cap X_k = (M_1 \cup \dots) \cap X_k = M \cap X_k,$$

is the intersection of a the F_σ set X_k and M (which is closed in itself). Thus by the induction hypothesis, $\mathrm{Dim}(M) = n-1$. A similar argument shows $\mathrm{Dim}(N) = 0$. Since $X = M \cup N$, we have

$$\mathrm{Dim}(X) \leq 1 + \mathrm{Dim}(M) + \mathrm{Dim}(N) \leq 1 + (n-1) + 0 = n.$$

□

Using the same arguments as with Theorem 14.1.2 we obtain:

COROLLARY 14.2.8. *The union of two sets of dimension $\leq n$, one of which is closed, has dimension $\leq n$*

COROLLARY 14.2.9. *Adding a single point to a set does not increase its dimension.*

LEMMA 14.2.10 (Proposition II.2.F). *If K, L are disjoint, closed subsets of X and $Y \subset X$ has dimension ≤ 0 then there is a separating set B for K and L so that $B \cap A = \emptyset$.*

PROOF. There are open sets U, V with disjoint closures that contain K and L respectively. Since $\overline{U} \cap A$ and $\overline{V} \cap A$ are closed in A, they can be separated in A using Lemma 14.1.1, since $\mathrm{Dim}(A) = 0$. Thus $A = Y \cup Z$ where Y, Z are disjoint open-closed sets in A and $\overline{U} \cap A \subset Y$. Then there is an open set W in X such that $K \cup Y \subset W$ and $\overline{W} \cap (L \cup Z) = \emptyset$. Thus $B = \partial W$ separated K from L and B is disjoint from both Y and Z and hence $B \cap A = \emptyset$. □

LEMMA 14.2.11 (Proposition III.5.B). *If K, L are disjoint, closed subsets of X and $Y \subset X$ has dimension $\leq n$ then there is a separating set B for K and L so that* $\mathrm{Dim}(B \cap A) \leq n-1$. *If we take $A = X$, this says that disjoint, closed sets of a n-dimensional space X can always be separated by a $(n-1)$-dimensional set.*

PROOF. We use induction. If $\mathrm{Dim}(A) = -1$, then $A = \emptyset$ and the result is obvious. If $\mathrm{Dim}(A) = 0$ then we proved this in Lemma 14.2.10. Suppose $n > 0$. By Lemma 14.2.6, We can write $A = D \cup E$ as a union of sets of dimension $\leq n-1$ and ≤ 0 respectively. By the case $n = 0$ of the induction, there is a separating set C for K and L that does not intersect E, so $A \cap B \subset D$ has dimension $\leq n-1$. □

LEMMA 14.2.12 (Proposition III.5.C). *Suppose X is a set of dimension $\leq n-1$, and suppose $\{C_j, C_j'\}_{j=1}^n$ be n pairs of closed sets so that $C_j \cap C_j = \emptyset$ for $j = 1, \dots n$. Then there are n closed sets $\{B_j\}$ so that B_j separates C_j from C_j' and $\cap_{j=1}^n B_j = \emptyset$.*

PROOF. By Lemma 14.2.11 C_1, C_1' can be separated by a set B_1 of dimension $\leq n-2$. By Lemma 14.2.11 C_2, C_2' can be separated by a set B_2 so that $\mathrm{Dim}(B_1 \cap B_2) \leq n-3$. Continuing in this way we get separating sets $\{B_k\}$ whose intersection has dimension $n - (n+1) = -1$, i.e., is empty. □

THEOREM 14.2.13 (Theorem III.4, Product Theorem). $\mathrm{Dim}(A \times B) \leq \mathrm{Dim}(A) + \mathrm{Dim}(B)$.

PROOF. We use induction. The result is trivial if either A or B is empty, i.e., for dimensions pairs $(m, -1)$ or $(-1, n)$, so we may assume it for both $(m, n-1)$ and $(m-1, n)$ and deduce it for (m, n).

Each point of $A \times B$ has a neighborhood of the form $U \times V$ where the boundaries of U and V have dimensions $\leq m-1$ and $\leq n-1$ respectively. Since

$$\partial(U \times V) \subset \overline{U} \times \partial V \cup \partial U \times \overline{V},$$

the induction hypothesis and Theorem 14.2.5 imply

$$\mathrm{Dim}(\partial(U \times V)) \leq (m-1) + (n-1) + 1 = m + n - 1,$$

which proves the result. □

Equality holds in Theorem 14.2.13 if $\mathrm{Dim}(B) = 0$, but not in general.

14.3. $\mathrm{Dim}(\mathbb{R}^n) = n$

The direction $\mathrm{Dim}(\mathbb{R}^n) \leq n$ is a rather obvious induction since points in $\mathbb{R}^k$ have small neighborhoods whose boundaries are $k-1$-spheres and one can show

$\mathrm{Dim}(S_k) = \mathrm{Dim}(\mathbb{R}^k)$ since dimension is unchanged by homeomorphisms and adding a single point.

The hard part is to show $\mathrm{Dim}(\mathbb{R}^n) \geq n$. We showed in Lemma 14.2.12 that if X is a set of dimension $\leq n-1$, and $\{C_j, C_j'\}_{j=1}^n$ are n pairs of closed sets so that $C_j \cap C_j = \emptyset$ for $j = 1, \dots n$, then there are n closed sets $\{B_j\}$ so that B_j separates C_j from C_j' and $\cap_{j=1}^n B_j = \emptyset$. We will show that $\mathbb{R}^n$ does not have this property, and hence $\mathrm{Dim}(\mathbb{R}^n) \geq n$.

Lemma 14.3.1 (Proposition IV.1.D). *Let $X = I_n = [-,1]^n \subset \mathbb{R}^n$ and let $\{C_j^-, C_j^+\}$ be the two components of $I_n \cap \{x = (x_1, \dots, x_n) : x_j = \pm 1\}$ (i.e., pairs of opposite faces of the cube). If $\{B_j\}$ are closed subsets of I_n so that B_j separates C_j and C_j', then $\cap_j B_j \neq \emptyset$. In particular, we must have* $\mathrm{Dim}(I_n) \geq n$.

This follows from the famous Brouwer fixed point theorem:

Theorem 14.3.2. *Every continuous map of I_n into itself has a fixed point.*

Proof of Lemma 14.3.1. To see how to deduce the lemma from Brouwer's theorem, let U_j^-, U_j^+ be open subsets of distinct components of $I_n \setminus B_j$ and define $v : I_n \to \mathbb{R}^n$ by setting the jth component to be

$$v(x) = \pm \mathrm{dist}(x, B_j),$$

with sign being chosen > 0 on the component of $I_n \setminus B_j$ containing U_j^+ and < 0 on the component containing U_j^- (and arbitrarily on any other components). Then $f(x) = x + v(x)$ is continuous and maps I_n into itself, because if x is not in U_j^+, then

$$1 - x_j = \mathrm{dist}(x, C_j^+) \geq \mathrm{dist}(x, B_j) = v_j(x),$$

so adding $v(x)$ to x can't make the jth coordinate larger than 1. Thus by Brouwer's theorem f has a fixed point y and so $v(y) = 0$, which means $\mathrm{dist}(y, B_j) = 0$ for $j = 1, \dots, n$, Since each B_j is closed, this means $y \in \cap_j B_j$, so the latter set in non-empty, as claimed. $\square$

The proof of Brouwer's theorem can be found in various places. The proof in [**72**] is based on first proving that S_n is not contractile to a point, basically but an invariance of degree under homotopy argument, but reduced to a parity counting argument for simplicial triangulations of the n-sphere.

Now that we know that $\mathbb{R}^n$ has topological dimension n, we know that different Euclidean spaces are not homeomorphic, indeed, $\mathbb{R}^n$ cannot be homeomorphic to any subset of $\mathbb{R}^k$ for $k < n$. This implies that no continuous map $f : I_n \to \mathbb{R}^k$ can be 1-to-1 (since I_n is compact this implies f would be a homeomorphism)). Thus in the setting of our goal, Theorem 14.0.3, we now know that there must be an image point so that $f^{-1}(y)$ has more than one point. We need to improve this to a preimage of dimension ≥ 1.

14.4. Embedding in $\mathbb{R}^{2n+1}$

A cover of a set X is s collection of open sets whose unions contains X. We say it uses diameter δ if every set in the collection has diameter $\leq \delta$. The cover has order n is at most $n+1$ elements can contain a common point.

LEMMA 14.4.1. *If X is compact and $Tdim(X) = 0$ then X has a cover of order 1 using diameter $\leq \delta$ (i.e., a pairwise disjoint cover by small elements).*

PROOF. By definition, each point has a neighborhood of diameter $\leq \delta$ and empty boundary (hence the set is both open and closed in X) and since X is compact, a finite number of these cover X. Replacing each open set by itself with the other removed gives a pairwise disjoint cover with even smaller diameters. $\square$

LEMMA 14.4.2 (Corollary to Theorem V.1). *If X is compact and* $\mathrm{Dim}(X) \leq n$*, then X has open covers using arbitrarily small diameters and order $\leq n$.*

PROOF. By Theorem 14.2.7 X is the union of $n+1$ dimension zero sets $X_1, \dots, X_{n+1}$, and each of these can be covered by collection of disjoint open sets using diameters $\leq \delta$. We claim the union of these $n+1$ collections has order $n+1$; if $n+2$ of the sets all contained the point p then be the pigeon hole principle, two come from the same collection and they can't both contain p since they are disjoint. $\square$

LEMMA 14.4.3 (Theorem V.2). *If X is compact and* $\mathrm{Dim}(X) \leq n$ *then the set of homeomorphisms from X into I_{2n+1} is a dense G_δ in the set of all continuous maps $X \to I_{2n+1}$. (This is non-empty since constant maps are obviously in it.)*

PROOF. We say g is an ϵ-mapping if $\mathrm{diam}(g^{-1}(y)) < \epsilon$ for every y (the empty set has diameter 0). It is easy to check that if X is compact and a continuous map g on X

is an ϵ-mapping for every $\epsilon > 0$ then g is a homeomorphism. Similarly, compactness implies the ϵ-mappings form an open set: if h is close enough to g in the supremum norm and g is an ϵ-mapping, then so is h. We leave this as an exercise for the reader.

So by Baire's theorem suffices to show that ϵ-maps are dense. Given any continuous $f : X \to I_{2n+1}$ we must approximate it to within $\eta > 0$ in the supremum norm by an ϵ-map g. Choose $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| \leq \eta/2$. Let $\{U_j\}$ be a cover of X of order n using diameters $\leq \delta$ and for each U_j choose a point $p_j \in I_{2n+2}$ so that $\mathrm{dist}(p_j, f(U_j)) \leq \eta/2$ and the p_j's are in general position, i.e., if we take two disjoint sets of the p_j's each with $\leq n+1$ points then the convex hulls in I_{2n+1} do not intersect.

For $x \in X$ let $w_j = \mathrm{dist}(x, X \setminus U_j)$ and define

$$g(x) = \frac{\sum w_j(x) p_j}{\sum w_j(x)}.$$

This is well defined and continuous since $w_j(x) > 0$ holds for at least one j for each x. Moreover, g approximates f at x since at most $n+1$ terms in the sum are non-zero, corresponding to the at most $n+1$ elements of the cover containing x. Since these all have diameter $\leq \delta$, the values of f at these points differs from $f(x)$ by at most $\eta/2$ and hence the same is true for any weighted average.

Finally, associate to each $x \in X$ the linear space spanned by the points p_j where $w_j(x) > 0$. If $g(x) = g(y)$ then the convex hulls of the points p_j corresponding to x and y overlap, so the set of points themselves overlap by out general position condition. Thus x and y are in a common U_j and hence within $\delta < \epsilon$ of each other, as desired. □

14.5. Stable values

If $f : X \to Y$ is continuous and $y \in f(X)$, we call y a stable value of f if $y \in g(X)$ for every continuous $g : X \to Y$ that is sufficiently close to f in the supremum metric. Otherwise, we can make arbitrarily small perturbations of X that omit the value y. In this case, y is called an unstable value of f.

Lemma 14.5.1 (Theorem VI.1). *If X has dimension $\leq n$ and $f : X \to I_n$ is continuous, then f has no stable values.*

PROOF. No value in ∂I_n can be stable since we can approximate f by $(1-\delta)f$. If y is an interior point we may apply a homeomorphism of I_n that maps y to 0 and so assume $y = 0$. Fix a small $\delta > 0$ and let $C_j^+ = \{x : f_j(x) \geq \delta\}$ where f_j is the jth coordinate of f, and similarly define $C_j^- = \{x : f_j(x) \leq -\delta\}$. By Lemma 14.2.12 there are separating sets B_j for these pairs so that $\cap B_j = \emptyset$. Define $g_j = f_j$ on $C_j^+ \cup C_j^-$, $g_j = 0$ on B_j and

$$g_j(x) = \frac{\mathrm{dist}(x, B_j)}{\mathrm{dist}(x, C_j^+) + \mathrm{dist}(x, B_j)},$$

on $U_j^+ \setminus C_j^+$ where U_j^+ is the component of $I_n \setminus B_j$ that contains C_j^+. Define g_j on $U_j^- \setminus C_j^-$ analogously. Then g is continuous, approximates f and never take the value 0, since the B_j's contain no common point. Thus 0 is an unstable value. $\square$

We can now state and prove the converse of Lemma 14.5.1. Note that this gives a characterization of n-dimensional sets in terms of the existence of stable values: X has dimension n if and only if there is a continuous map $f : X \to I_n$ that has a stable value.

LEMMA 14.5.2 (Theorem VI.2). *If $X \subset I_n$, $n < \infty$ has dimension $\geq m$, then there exists a continuous map $f : X \to I_n$ with a stable value.*

PROOF. We will prove the contrapositive: if f has no stable values, then it is homeomorphic to a subset of R_n^m, and hence has dimension $\leq m$.

Consider the identity $f : X \to I_n$. If $X \subset \mathcal{R}_n^m$ then clearly X has dimension $\leq m$. If not, choose an image value y that has $k \leq m$ rational values and let M be the linear subspace of points that agree with y in these coordinates and let N be the k-dimensional orthogonal complement. Let f_N be the f followed by projection onto N. Since y is not a stable value of f_N, then for any $\delta > 0$, there is a continuous map g_N that approximates f_N to within δ. Letting $g = (f_M, g_N)$, we get a map $g : X \to I_n$ that approximates f as closely as we wish and so that $\overline{g(X)} \cap M_y = \emptyset$. Moreover, any map h that is close enough to g will also have this property, i.e., the approximating maps to f form an open set in the sup-norm topology.

There are a finite number of ways of choosing m coordinates out of n and a countable number of ways of assigning rational values to these coordinates. For each such choice the construction above gives an open dense set of maps $X \to I_n$ that do not take those particular values at those particular coordinates. By the Baire category theorem, the intersection of these open dense sets is a dense G_δ set of maps that never takes rational values at m different coordinates. By Lemma 14.5.2 there is a dense G_δ of homeomorphisms $X \to I_{2n+1}$ and applying Baire's theorem again gives a homeomorphism sending X into $\mathcal{M}_n^{m-1}$, that has dimension $\leq m-1$ by Lemma 14.2.4. □

LEMMA 14.5.3 (Proposition VI.1.B). *Suppose $f : X \to I_n$ is continuous and y is an interior point of I_n that is unstable. Fix $\delta > 0$. Then f can be approximated within δ by a map g that omits the value y but agrees with f whenever it takes values more than δ away from y. Thus stability is a local property.*

PROOF. Without loss of generality we assume y is the origin and $U = B(0, \delta)$. Since y is unstable there is a $h : X \to I_n$ that approximates f and omits the value y. Define $g = h$ when $|f(x)| \leq \delta/2$, $g = f$ when $|f(x)| \geq \delta$ and

$$g(x) = (1 - \phi(t))h(x) + \phi(t)f(x),$$

otherwise where $t = |f(x)|$ and ϕ increases linearly from 0 to 1 as t goes from $\delta/2$ to δ. It is easy to check g has the desired properties. □

14.6. Continuous extensions

LEMMA 14.6.1 (Theorem VI.4). *X has dimension $\leq n$ if and only if for each closed set $K \subset X$ and each continuous mapping $f : K \to S_n$, f has a continuous extension $X \to S_n$.*

LEMMA 14.6.1. **Sufficiency:** By Lemma 14.5.2, it is enough to show that that no continuous mapping $f : X \to I_{n+1}$ has stable values. A stable value can't occur on the boundary of I_n, so assume there is a stable interior value y, and let U be a small ball around y. Let $K = f^{-1}(\partial U)$. This set is closed and by assumption there is a map $F : X \to S_n$ that extends $f : K \to \partial U = S_n$. Define g by setting $g = f$ on $f^{-1}(U)$ and $g = F$ otherwise. Then g approximates f uniformly and never equals y, so y is not a stable value of f.

Necessity: Suppose X has dimension $\leq n$, $K \subset X$ is closed and $f : K \to S_n$ is continuous. With loss of generality we may assume f maps into ∂I_{n+1} instead. By the Tietze extension theorem, f can be extended to a map $F : X \to I_{n+1}$. Lemma

14.5.1 implies F has no stable values, so in particular, origin in not stable, so we can approximate F by a map G that never vanishes and agrees with F for values on ∂I_{n+1}. Hence G can be composed with radial projection to give a continuous map onto ∂I_{n+1} that extends f. $\square$

In the next proof we will need

THEOREM 14.6.2 (Tietze Extension Theorem). *If K is a closed subset if a space X and $f : K \to I_1$ is continuous, then f can be extended to a continuous $F : X \to I_1$.*

Clearly I_1 can be replaced by I_n by extending the coordinates separately. Also since S_n is homeomorphic to ∂I_{n+1}, the Tietze theorem implies that a map $f : K \to S_n$ can be extended to an open neighborhood of K, by replacing S_n by ∂I_{n+1}, extending to I_{n+1}, restricting to the open subset where F avoids the origin and composing by radial projection back onto ∂I_{n+1}.

LEMMA 14.6.3 (Corollary to Theorem VI.4). *Suppose K is a closed subset of X. If $\mathrm{Dim}(X \setminus K) \leq n$, then every continuous map $f : K \to S_n$ has a continuous extension to X.*

LEMMA 14.6.3. Suppose $f : K \to S_n$ is continuous. Using Tietze's extension theorem as before, f has a continuous extension F to an open neighborhood U of K. Choose an open V with $K \subset V \subset \overline{V} \subset U$ and note that the restriction maps $\overline{V} \setminus K$ to S_n. Thus by the necessity part of Lemma 14.6.1, this map can be extended to a continuous map G of $X \setminus K \to S_n$ and since this agrees with F on V, setting $G = f$ on K gives a continuous extension of f to all of X. $\square$

Next we need Borsuk's theorem; as with Brouwer's and Tietze's theorem we take this result as given.

THEOREM 14.6.4 (Borsuk's theorem, Theorem VI.5). *Suppose K is closed subset of X and $f, g : KtoS_n$ are homotopic. If there is an extension of f to X, then then there is an extension of g and the extensions are homotopic.*

We say $f, g : X \to S_n$ are homotopic if there is a continuous map $F : X \times [0, 1] \to S_n$ so that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

LEMMA 14.6.5 (Proposition VI.3.B). *Suppose $f, g : X \to S_n$ are continuous and disagree on a set Y of dimension $\leq n-1$. Then f and g are homotopic.*

PROOF. Y is open. Define a closed set $Z \subset X \times [0,1]$ by

$$Z = (X \times \{0\}) \cup (X \times \{1\} \cup (X \setminus Y) \times (0,1).$$

Define the homotopy F by $F(x,t) = f(x) = g(x)$ for $x \notin Y$, $F(x,0) = f(x)$, $F(x,1) = g(x)$. The complement of Z is the product $Y \times (0,1)$ which has dimension $\leq n$ by the product theorem Theorem 14.2.13. By Lemma 14.6.3 we can extend F to all of $X \times [0,1]$, proving f and g are homotopic. □

LEMMA 14.6.6 (Proposition VI.3.C). *Suppose X is the union of two closed subspaces K, L and $F : K \to S_n$ an $G : L \to S_n$ are continuous and they agree on $K \cap L$ except possibly on a set of dimension $\leq n-1$. Then F can be extended to all of X.*

PROOF. The mappings are homotopic by Lemma 14.6.5, so it follows from Borsuk's theorem that F extends from $K \cap L$ to a function H on L. Taking F on K and H on L gives the extension to X. □

LEMMA 14.6.7 (Proposition VI.3.F). *Suppose $K \subset X$ is closed and $\{V_\lambda\}$ is a collection of open sets that cover X and whose boundaries all have dimension $\leq n-1$. If $f : K \to S_n$ can be extended continuously to each V_λ, then it can be extended continuously to all of X.*

LEMMA 14.6.7. Since X is separable we may assume $\{V_\lambda\}$ is a countable collection $\{V_j\}$. Assume we have already extended f to F_k on $X_k = K \cup \overline{V_1} \cup \cdots \cup \overline{V_k}$ and set $Y_k = (K \cup \overline{V_{k+1}}) \setminus (V_1 \cup \cdots \cup V_k)$. By hypothesis f has a continuous extension to both these sets and these extensions can only disagree in

$$(Y_k \cap Z_k) \setminus K \subset \cup_{j=1}^{k} \partial V_k$$

which has dimension $\leq n-1$ by Lemma 14.2.5. Hence we may apply Lemma 14.6.6. □

14.7. Preimages with large dimension

We can now obtain the goal of this chapter.

LEMMA 14.7.1 (Proposition VI.3.G). *Suppose a set X is a union of sets K_λ of dimension $\leq m$ and each K_λ has the property that any open neighborhood U of K_λ contains an open neighborhood V of K_λ whose boundary has dimension $\leq m-1$. Then X has dimension $\leq m$.*

LEMMA 14.7.1. Suppose K is compact and $f : K \to S_n$ is continuous. By Lemma 14.6.3, f can be extended to $K \cup K_\lambda$ and hence to an open neighborhood U_λ of $K \cup K_\lambda$ (by the Tietze extension theorem each coordinate function can be extended to some f_j since it is real-valued and then we restrict to an open neighborhood where $\sum f_j^2 > 0$). The Tietze theorem is Theorem VI.3 of [**72**], and can be found in other texts, such as Munkres' book [**102**].

By hypothesis, each U_λ contains a sub-neighborhood V_λ whose boundary has dimension $\leq m-1$ and hence f extends to $K \cup \overline{V_\lambda} \subset U_\lambda$. By Lemma 14.6.7, f extends to all of X, and by Lemma 14.6.1 this proves Lemma 14.7.1 □

Theorem 14.0.3 clearly follows from

LEMMA 14.7.2. *Suppose X has dimension n, Y has dimension k and $f : X \to Y$ has the property that* $\mathrm{Dim}(f^{-1}(y)) \leq m$ *for all $y \in Y$. Then $n \leq k+m$.*

PROOF. To prove this, we use induction on k, keeping m fixed. If $k = -1$, the set Y is empty and the result is trivially true. Next we assume the result for $k-1$ and deduce it for k.

Consider the family of all preimages $\{K_y\} = \{f^{-1}(y)\}$ for $y \in f(X)$. This is a decomposition of X into disjoint compact sets of dimension $\leq m$. We claim that these sets satisfy the hypotheses of Lemma 14.7.1. To see this, take any neighborhood U of K_y and let $C = f(I_n \setminus U$. This a ball neighborhood V of y that is disjoint from C and has boundary of dimension $k-1$. Then $f^{-1}(V)$ is an open neighborhood of K_y inside U and its boundary has dimension $\leq k-1+m$ by induction on k. Thus Lemma 14.7.1 can be applied to deduce Lemma 14.7.2. □

This completes our "topological" proof of Sullivan's theorem.

APPENDIX A

Background material

A.1. Hausdorff dimension

Given any set K in a metric space we define the *α-dimensional Hausdorff content* as

$$\mathcal{H}^{\alpha}_{\infty}(K) = \inf\{\sum_i |U_i|^{\alpha}\},$$

where $\{U_i\}$ is a countable cover of K by any sets and $|E|$ denotes the diameter of a set E.

The *Hausdorff dimension* of K is defined to be

$$\dim(K) = \inf\{\alpha : \mathcal{H}^{\alpha}_{\infty}(K) = 0\}.$$

This is equivalent to the original [**?**], (reprinted in English translation in [**45**]) definition, using Hausdorff measure, see Proposition A.1.3.

More generally we define

$$\mathcal{H}^{\alpha}_{\epsilon}(K) = \inf\{\sum_i |U_i|^{\alpha} : K \subset \cup_i U_i, |U_i| < \epsilon\},$$

where each U_i is now required to have diameter less than ϵ. The *α-dimensional Hausdorff measure* of K is defined as

$$\mathcal{H}^{\alpha}(K) = \lim_{\epsilon \to 0} \mathcal{H}^{\alpha}_{\epsilon}(K).$$

This is an outer measure; an *outer measure* on a nonempty set X is a function μ^* from the family of subsets of X to $[0, \infty]$ that satisfies

- $\mu^*(\emptyset) = 0$,
- $\mu^*(A) \le \mu^*(B)$ if $A \subset B$,
- $\mu^*(\cup_{j=1}^{\infty} A_j) \le \sum_{j=1}^{\infty} \mu^*(A_j)$.

The α-dimensional Hausdorff measure is even a Borel measure in $\mathbb{R}^d$ if $\alpha < d$ (proved below).

THEOREM A.1.1. *Let μ be a metric outer measure. Then all Borel sets are μ-measurable.*

For a proof see [**?**].

The construction of Hausdorff measure can be made a little more general by considering a positive, increasing function φ on $[0, \infty)$ with $\varphi(0) = 0$. This is called a *gauge function* and we may associate to it the Hausdorff content

$$\mathcal{H}^{\varphi}_{\infty}(K) = \inf\{\sum_i \varphi(|U_i|)\},$$

and $\mathcal{H}^{\varphi}_{\epsilon}(K)$, as well as $\mathcal{H}^{\varphi}(K) = \lim_{\epsilon \to 0} \mathcal{H}^{\varphi}_{\epsilon}(K)$ just as before. The case $\varphi(t) = t^{\alpha}$ is just the case considered above. We will not use other gauge functions in the first few chapters, but they are important in many applications.

LEMMA A.1.2. *If $\mathcal{H}^{\alpha}(K) < \infty$ then $\mathcal{H}^{\beta}(K) = 0$ for any $\beta > \alpha$.*

PROOF. It follows from the definition of $\mathcal{H}^{\alpha}_{\epsilon}$ that

$$\mathcal{H}^{\beta}_{\epsilon}(K) \leq \epsilon^{(\beta-\alpha)} \mathcal{H}^{\alpha}_{\epsilon}(K),$$

which gives the desired result as $\epsilon \to 0$. □

Thus if we think of $\mathcal{H}^{\alpha}(K)$ as a function of α, the graph of $\mathcal{H}^{\alpha}(K)$ versus α shows that there is a critical value of α where $\mathcal{H}^{\alpha}(K)$ jumps from ∞ to 0. This critical value is equal to the Hausdorff dimension of the set. Note that $\mathcal{H}^{\alpha}_{\infty}(K) = 0$ if and only if $\mathcal{H}^{\alpha}(K) = 0$. This gives us the following proposition.

PROPOSITION A.1.3.

$$\dim(K) = \inf\{\alpha : \mathcal{H}^{\alpha}(K) = 0\}.$$

Upper bounds for Hausdorff dimension are computed using explicit covering of the set. Lower bounds are given by constructing measures supported on the set. The simplest version of this idea is:

LEMMA A.1.4 (Mass Distribution Principle). *If E supports a strictly positive Borel measure μ which satisfies*

$$\mu(B(x, r)) \leq C r^{\alpha},$$

for some constant $0 < C < \infty$ and for every ball $B(x, r)$, then $\mathcal{H}^{\alpha}(E) \geq \mathcal{H}^{\alpha}_{\infty}(E) \geq \mu(E)/C$. In particular, $\dim(E) \geq \alpha$.

PROOF. Let $\{U_i\}$ be a cover of E. For $\{r_i\}$, where $r_i > |U_i|$, we look at the following cover: choose x_i in each U_i, and take open balls $B(x_i, r_i)$. By assumption,

$$\mu(U_i) \le \mu(B(x_i, r_i)) \le C{r_i}^{\alpha}.$$

We deduce that $\mu(U_i) \le C|U_i|^\alpha$, i.e.,

$$\sum_i |U_i|^\alpha \ge \sum_i \frac{\mu(U_i)}{C} \ge \frac{\mu(E)}{C}.$$

Thus $\mathcal{H}^\alpha(E) \ge \mathcal{H}^\alpha_\infty(E) \ge \mu(E)/C$. □

A more refined version of the mass distribution principle is

LEMMA A.1.5 (Billingsley's lemma). *Let $A \subset [0,1]$ be Borel and let μ be a finite Borel measure on $[0,1]$. Suppose $\mu(A) > 0$. If*

$$\alpha_1 \le \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \le \beta_1,$$

for all $x \in A$, then $\alpha_1 \le \dim(A) \le \beta_1$.

For a proof see [?]. Billingsley's lemma has a further refinement by Rogers and Taylor which we will not discuss here (see []).

The following result often makes Frostman's lemma easier to apply by allowing us to only check the measure estimate on certain sets coming from an iterative construction, instead of all disks.

LEMMA A.1.6. *Suppose $E = \cap_{n=1}^\infty E_n$ where $E_1 \supset E_2 \supset E_3 \dots$. Suppose each $E_n = \cup_k E_n^k$ is a union of closed sets $\{E_n^k\}$ and that there is a $C < \infty$ so that $\mathrm{diam}(E_n^k) \ge C\mathrm{diam}(E_{n+1}^j)$ if $E_{n+1}^j \subset E_n^k$ and so that at most C sets of diameter $\ge r$ can hit any disk of radius r. If there is a measure μ on E so that $\mu(E_n^j) \le M_1 \mathrm{diam}(E_n^j)^\alpha$ for all elements of $\{E_n^j\}$, then μ is a Frostman measure with the same exponent α, i.e., $\mu(D(x,r)) \le M_2 r^\alpha$ for some M_2 that depends only on C and M_1.*

PROOF. Without loss of generality we may assume $D = D(x,r)$ has diameter smaller than the diameter of E. Let $\mathcal{D}$ be elements of $\{E_n^k\}$ that are maximal with respect to inclusion in $D(x,r)$ (a set is included if it is in D and is not contained in any other element contained in D). Every set in $\mathcal{D}$ has diameter at most $2r$ and is therefore contained in an element of $\mathcal{E} = \cup_n \cup_k E_n^k$ with diameter at least $2r$ and at most $2Cr$ and that hits D (we can pass to a sequence of larger sets multiplying the diameter by at most C each time). By assumption there are at most C such sets. Thus $\mu(D)$ is less than the mass of at most C sets in our collection, each with mass at most $M_1(2Cr)^\alpha$. Hence $\mu(D(x,r) \leq CM_1(2C)^\alpha \cdot r^\alpha$, as desired. □

Suppose $U \subset \overline{U} \subset V$ are Jordan domains and we are given a finite number of Jordan domains $\{V_k\}$ with closures contained in U and a collection $\mathcal{F}$ of conformal maps $f_k : V_k \to V$. Let $U_k = f_k^{-1}(U) \subset V_k$. Let $E_1 = \cup_k \overline{U_k}$, and in general $E_{k+1} = \cup_k f_k^{-1}(E_k) \subset E_k$. Define $E = \cap_k E_k$. We call E a **Cantor repeller** for the **iterated function system** $\{U, V\mathcal{F}\}$.

LEMMA A.1.7. *There is a constant M, depending only on U and V (but not on $\mathcal{F}$) so that if*

$$\sum_k \operatorname{diam}(U_k)^\alpha \geq M \operatorname{diam}(U)^\alpha,$$

then $\dim(E) \geq \alpha$.

PROOF. We want to show that lemma **??** applies to the sets $\{E_k\}$ defined above.

Note that each component of E_k is surrounded by an annulus that is a conformal image of $V \backslash U$, and hence has a modulus that is independent of k and of the particular component. This implies that any two component that both have diameter $\geq r$ are separated by distance $\gtrsim r$. Thus the number of different such components that can hit the same set of diameter $\leq r$ is uniformly bounded. By the distortion theorems for conformal maps, $|f_k'|$ is comparable to $\operatorname{diam}(U)/\operatorname{diam}(U_k)$ on all of U_k with a constant that only depends on U and V (in fact, it only depends on the modulus of the annulus $V \setminus U$). Thus if $\{W_k\}_1^k$ are the components of E_{n+1} contained in some component W of E_n, we have

$$\operatorname{diam}(W_j) \geq \frac{1}{C} \operatorname{diam}(W) \cdot \frac{\operatorname{diam}(U_j)}{\operatorname{diam}(U)},$$

for some C depending only on U and V. Hence

$$\begin{aligned}
\sum_{j=1}^k \operatorname{diam}(W_j)^\alpha &\geq \operatorname{diam}(W)^\alpha C^{-\alpha} \frac{\sum_{j=1}^k \operatorname{diam}(U_k)^\alpha}{\operatorname{diam}(U)^\alpha} \\
&\geq \operatorname{diam}(W)^\alpha C^{-\alpha} M \\
&\geq \operatorname{diam}(W)^\alpha.
\end{aligned}$$

This is the second condition in Lemma A.1.6.

Finally, we define a measure on R by setting inductively setting

$$\mu(W_j) = \frac{\mathrm{diam} {W_j}^s}{\sum_{j=1}^k \mathrm{diam}(W_j)^\alpha}\mu(W).$$

Clearly $\mu(U) \le C\mathrm{diam}(U)$ for some finite constant C, and then we argue by induction,

$$\begin{aligned} \mu(W_j) &\le \frac{\mathrm{diam} {W_j}^s}{\sum_{j=1}^k \mathrm{diam}(W_j)^\alpha} C\mathrm{diam}(W)^\alpha \\ &= C\mathrm{diam} {W_j}^s \frac{\mathrm{diam}(W)^\alpha}{\sum_{j=1}^k \mathrm{diam}(W_j)^\alpha} \\ &\le C\mathrm{diam} {W_j}^s \cdot 1. \end{aligned}$$

Thus Lemma A.1.6 applies and so the set E has Hausdorff dimension at least α. $\square$

A.2. Minkowski dimension

Suppose K is a bounded set in $\mathbb{R}^d$ (or a totally bounded set in any metric space) and let $N(K, \epsilon)$ be the minimal number of open balls of diameter ϵ needed to cover K. We define the **upper Minkowski dimension** as

$$\overline{\mathrm{Mdim}}(K) = \limsup_{\epsilon \to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon},$$

and the *lower Minkowski dimension*

$$\underline{\mathrm{Mdim}}(K) = \liminf_{\epsilon \to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon}.$$

If the two values agree, the common value is simply called the *Minkowski dimension* of K and denoted by $\overline{\mathrm{Mdim}}(K)$. When the Minkowski dimension of a set K exists, the number of balls of diameter ϵ needed to cover K grows like $\epsilon^{-\overline{\mathrm{Mdim}}(K)+o(1)}$ as $\epsilon \to 0$. Minkowski dimension is sometimes called the *box counting dimension.*

In the definitions of $\overline{\mathrm{Mdim}}(K)$ and $\underline{\mathrm{Mdim}}(K)$ it is equivalent to replace $N(K, \epsilon)$ by $N_D(K, \epsilon)$ where the covering sets of diameter $\le \epsilon$ are not required to be balls. This is because $N(K, 2\epsilon) \le N_D(K, \epsilon) \le N(K, \epsilon)$. Also, for a bounded A, $\overline{\mathrm{Mdim}}(A) = \overline{\mathrm{Mdim}}(\bar{A})$ and $\underline{\mathrm{Mdim}}(A) = \underline{\mathrm{Mdim}}(\bar{A})$, where $\bar{A}$ denotes the closure of A. We leave the proofs to the reader. The Minkowski dimension has several drawbacks. For example, it need not exist for a general set (see Example **??** and Exercise **??**).

EXERCISE: Show that $\{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ has Minkowski dimension 1.

EXERCISE: Construct a compact subset of [0.1] that has different upper and lower Minkowski dimensions.

The following relationship between Minkowski and Hausdorff dimension is clear

$$\dim(K) \leq \underline{\mathrm{Mdim}}(K) \leq \overline{\mathrm{Mdim}}(K). \tag{54}$$

Indeed, if $B_i = B(x_i, \epsilon/2)$ are $N(K,\epsilon)$ balls of radius $\epsilon/2$ and centers in x_i that cover K, then consider the sum

$$S_\epsilon = \sum_{i=1}^{N(K,\epsilon)} |B_i|^\alpha = N(K,\epsilon)\epsilon^\alpha = \epsilon^{\alpha - R_\epsilon},$$

where $R_\epsilon = \frac{\log N(K,\epsilon)}{\log(1/\epsilon)}$. For $\alpha > \liminf_{\epsilon\to 0} R_\epsilon = \underline{\mathrm{Mdim}}(K)$ we have $\inf_{\epsilon>0} S_\epsilon = 0$. Strict inequalities in (54) are possible.

Also, for $E_1 \subset E_2 \subset \dots$, it is possible that

$$\overline{\mathrm{Mdim}}(\cup_n E_n) \neq \lim_{n\to\infty} \overline{\mathrm{Mdim}}(E_n)$$

and the Minkowski dimension of a countable set can be non-zero (see Example **??**). Thus the Minkowski dimension of a countable union of sets is not necessarily the supremum of the individual dimensions. We shall see in the next section how to "fix" this by defining packing dimension. First we give some alternate methods for computing the upper Minkowski dimension.

For any compact set $K \subset \mathbb{R}^d$ we can define an **exponent of convergence**

$$\kappa = \kappa(K) = \inf\{\alpha : \sum_{Q\in\mathcal{W}} |Q|^\alpha < \infty\}, \tag{55}$$

where the sum is taken over all cubes in some Whitney decomposition $\mathcal{W}$ of $\Omega = K^c$ that are within distance 1 of K (we have to drop the "far away" cubes or the series might never converge). It is easy to check that κ is independent of the choice of Whitney decomposition (see Exercise **??**).

LEMMA A.2.1. *For any compact set K, $\kappa \leq \overline{\mathrm{Mdim}}(K)$. If K also has zero Lebesgue measure then $\kappa = \overline{\mathrm{Mdim}}(K)$.*

PROOF. Let $D = \overline{\mathrm{Mdim}}(K)$. We start with the easy assertion, $\kappa \leq D$. Choose $\epsilon > 0$ and for for each $n \in \mathbb{N}$, let $\mathcal{Q}_n$ be a covering of K by $O(2^{n(D+\epsilon)})$ dyadic cubes of side length 2^{-n}. Let $\mathcal{W}$ denote the dyadic Whitney cubes that are within distance 1 of K and let $\mathcal{W}_n \subset \mathcal{W}$ be the cubes with side $\ell(Q) = 2^{-n}$. For each $Q \in \mathcal{W}_n$, choose a point $x \in K$ with $\mathrm{dist}(x, Q) \leq 3|Q|$ and let $S(x,Q) \in \mathcal{Q}_n$ be a cube containing

x. Since $|S(x,Q)| = |Q|$ and $\mathrm{dist}(Q, S(x,Q)) \leq 3|Q|$, each $S \in \mathcal{Q}_n$ can only be associated to a uniformly bounded number of Q's in $\mathcal{W}_n$. Hence

$$\#(\mathcal{W}_n) = O(2^{n(D+\epsilon)}),$$

and thus

$$\begin{aligned}
\sum_{Q\in\mathcal{W}} |Q_j|^{D+2\epsilon} &= O(\sum_{n=0}^{\infty} \#(\mathcal{W}_n) 2^{-n(D+2\epsilon)}) \\
&= O(\sum_{n=0}^{\infty} 2^{-\epsilon)}) \\
&< \infty,
\end{aligned}$$

which proves $\kappa \leq D + 2\epsilon$. Taking $\epsilon \to 0$ gives $\kappa \leq D$.

Next we assume K has zero Lebesgue measure and will prove $\kappa \geq D$. Let $\epsilon > 0$. We have

$$N(K, 2^{-n}) \geq 2^{n(D-\epsilon)},$$

for infinitely many n, so suppose n is a value where this occurs and let $\mathcal{S} = \{S_k\}$ be a covering of K with dyadic cubes of side 2^{-n}. Let $\mathcal{U}_n$ be cubes in the dyadic Whitney decomposition of $\Omega = K^c$ with side lengths $< 2^{-n}$. For each $S_k \in \mathcal{S}$ let $\mathcal{U}_{nk} \subset \mathcal{U}_n$ be the subcollection of cubes that intersect S_k. Because of the nesting property of dyadic cubes, every dyadic Whitney cube intersecting the interior of some S_k is contained in that S_k. Since the volume of K is zero, this gives

$$|S_k|^d = \sum_{Q\in\mathcal{U}_{nk}} |Q|^d$$

(The right side $d^{d/2}$ times the Lebesgue measure of $S_k \setminus K$, and the left side is $d^{d/2}$ times the measure of S_k; these are equal by assumption.) Since $-d + D - 2\epsilon < 0$, we get

$$\begin{aligned}
\sum_{Q\in\mathcal{U}_{nk}} |Q|^{D-2\epsilon} &= \sum_{Q\in\mathcal{U}_{nk}} |Q|^d |Q|^{-d+D-2\epsilon} \\
&\geq |S_k|^{-d+D-2\epsilon} \sum_{Q\in\mathcal{U}_{nk}} |Q|^d \\
&= |S_k|^{D-2\epsilon}
\end{aligned}$$

Hence, when we sum over the entire Whitney decomposition,

$$\begin{aligned}\sum_{Q\in\mathcal{U}_0}|Q|^{D-2\epsilon} &\geq \sum_{S_k\in\mathcal{S}}\sum_{Q\in\mathcal{U}_{nk}}|Q|^{D-2\epsilon}\\ &\geq \sum_{S_k\in\mathcal{S}}|S_k|^{D-2\epsilon}\\ &\geq N(K,2^{-n})\cdot 2^{-n(D-2\epsilon)}\\ &= 2^{n\epsilon}.\end{aligned}$$

Taking $n\to\infty$, shows $\kappa \geq D-2\epsilon$ and taking $\epsilon\to 0$ gives $\kappa\leq D$. $\square$

A.3. Packing dimension

Tricot [**137**] introduced packing dimension, which is dual to Hausdorff dimension in several senses and comes with an associated measure.

For any increasing function $\varphi\colon [0,\infty)\to\mathbf{R}$ such that $\varphi(0)=0$ and any set E in a metric space, define first the **packing pre-measure** (in gauge φ) by

$$\tilde{\mathcal{P}}^{\varphi}(E)=\lim_{\epsilon\downarrow 0}\left(\sup\sum_{j=1}^{\infty}\varphi(\text{diam } B_j)\right),$$

where the supremum is over all collections of disjoint closed balls $\{B_j\}_{j=1}^{\infty}$ with centers in E and diameters $\text{diam}(B_j)<\epsilon$. This pre-measure is finitely sub-additive, but not countably sub-additive, see Exercise **??**. Then define the **packing measure** in gauge φ:

$$\mathcal{P}^{\varphi}(E)=\inf\left\{\sum_{i=1}^{\infty}\tilde{\mathcal{P}}^{\varphi}(E_i): E\subset\bigcup_{i=1}^{\infty}E_i\right\}. \tag{56}$$

It is easy to check that $\mathcal{P}^{\varphi}$ is a metric outer measure, hence all Borel sets are $\mathcal{P}^{\varphi}$-measurable, see Theorem A.1.1 in Chapter 1. When $\varphi(t)=t^{\theta}$ we write $\mathcal{P}^{\theta}$ for $\mathcal{P}^{\varphi}$ ($\mathcal{P}^{\theta}$ is called **θ-dimensional packing measure**).

Finally, define the **packing dimension** of E:

$$\text{Pdim}(E)=\inf\left\{\theta:\mathcal{P}^{\theta}(E)=0\right\}. \tag{57}$$

We always have

$$\dim(E)\leq \text{Pdim}(E)\leq \overline{\text{Mdim}}(E). \tag{58}$$

The set $K = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ is of packing dimension 0, since the packing dimension of any countable set is 0. Thus

$$\dim(K) = 0 = \mathrm{Pdim}(K) < 1/2 = \overline{\mathrm{Mdim}}(K).$$

(See Example **??** in Chapter **??**.)

Packing measures are studied in detail in Taylor and Tricot [**135**] and in Saint-Raymond and Tricot [**123**]; here we only mention the general properties we need.

Proposition A.3.1. *The packing dimension of any set A in a metric space may be expressed in terms of upper Minkowski dimensions:*

$$\mathrm{Pdim}(A) = \inf \left\{ \sup_{j \geq 1} \overline{\mathrm{Mdim}}(A_j) : A \subset \bigcup_{j=1}^{\infty} A_j \right\}, \tag{59}$$

where the infimum is over all countable covers of A.

(See Tricot [**137**], Proposition 2, or Falconer [**50**], Proposition 3.8.)

For the proof see [**?**].

Lemma A.3.2. *Let A be a separable metric space.*

(i) *If A is complete and if every non-empty open set V in A satisfies $\overline{\mathrm{Mdim}}(V) \geq \alpha$, then $\mathrm{Pdim}(A) \geq \alpha$.*

(ii) *If $\mathrm{Pdim}(A) > \alpha$, then there is a closed nonempty subset $\widetilde{A}$ of A, such that $\mathrm{Pdim}(\widetilde{A} \cap V) > \alpha$ for any open set V which intersects $\widetilde{A}$.*

Corollary A.3.3. *[Tricot [**137**], Falconer [**50**]] Let K be a compact set in a metric space which satisfies*

$$\overline{\mathrm{Mdim}}(K \cap V) = \overline{\mathrm{Mdim}}(K)$$

for any open set V which intersects K. Then

$$\mathrm{Pdim}(K) = \overline{\mathrm{Mdim}}(K).$$

A gauge function φ is called **doubling** if

$$\sup_{x > 0} \frac{\varphi(2x)}{\varphi(x)} < \infty.$$

THEOREM A.3.4. *Assume* $\{f_1, \ldots, f_\ell\}$ *are contracting self bi-Lipschitz maps of a complete metric space, i.e.*

$$\epsilon_j d(x,y) \leq d(f_j(x), f_j(y)) \leq r_j d(x,y)$$

for all $1 \leq j \leq \ell$ *and any* x, y*, where*

$$0 < \epsilon_j \leq r_j < 1.$$

Denote by K *the compact attractor satisfying (**??**). Then*

(i) $\mathrm{Pdim}(K) = \overline{\mathrm{Mdim}}(K)$.

(ii) *For any doubling gauge function* φ *such that* K *is* σ*-finite for* $\mathcal{P}^\varphi$ *we have* $\tilde{\mathcal{P}}^\varphi(K) < \infty$.

A.4. The Riemann-Hurwitz formula

The Riemann-Hurwitz formula gives a relation between the Euler characteristic of two Riemann surfaces and the number of critical points of holomorphic map between them. We will only deal with the case of plane domains, where the connectivity c (the number of boundary components) replaces the Euler characteristic g (for planar domains $g = 2 - n$).

THEOREM A.4.1 (Riemann-Hurwitz formula for planar domains). *Let* $f : V \to W$ *be a* k*-sheeted, proper, holomorphic map between finitely connected domains on the Riemann sphere and suppose* f *has* r *critical points. Then* $\mathrm{conn}(V) - 2 = k(\mathrm{conn}(W) - 2) + r$.

PROOF. We follow the proof given by Norbert Steinmetz in [].

First suppose f has no critical points and W is simply connected. Then f is a covering map, so V must be simply connected as well, so $k = 1$. Thus the formula holds in this case.

Next suppose that f still has no critical points, but $\mathrm{conn}(W) = n \geq 2$. We use induction on n, the case $n = 1$, having been proven above. Take a cross-cut γ of W (i.e., a Jordan arc in W with endpoints on ∂W) that connects different components of ∂W. Then $W^* = W \setminus \gamma$ is still connected, but $\mathrm{conn}(W^*) = \mathrm{conn}(W) - 1 = n - 1$. The inverse images of γ are k cross-cuts of V and cut V into some number of connected subdomains $V_1, \dots, V_q$ and f is k_j sheeted covering $V_j \to W^*$ where $\sum_j k_j = k$. By induction, the Riemann-Hurwitz formula is true for maps onto W^*, so

$$\mathrm{conn}(V_j) - 2 = k_j((n-1)-2) + 0,$$

and summing over j gives

$$\sum_{j=1}^{q} (\mathrm{conn}(V_j) - 2) = k(n-3) = k(n-2) - k.$$

Thus we need to show

$$\sum_{j=1}^{q} (\mathrm{conn}(V_j) - 2) = \mathrm{conn}(V) - 2 - k. \tag{60}$$

To prove this, we use induction by removing cross-cuts from V one at a time. When no cross-cuts have been removed, (60) is trivial. If the cross-cut has endpoints on the same boundary component, then it does not divide the domain and the connectivity is lowered by one; thus both sides in (60) are lowered by 1 and equality is maintained. If the cross-cut connected points on the same boundary component, then some subdomain of connectivity m is divided into two subdomains whose connectivities add to $m+1$. Thus each side in (60) again decreases by 1 and equality is maintained. This completes the proof of the Riemann-Hurwitz formula in the case when there are no critical points.

If f has r critical points (with multiplicity) and s critical values, let W^* be the domain obtained by removing the critical values from W and let $V^* = f^{-1}(W^*)$ (we remove the critical points plus any other points mapping to critical values). Then $f : V^* \to W^*$ is proper, holomorphic and has no critical points, so the previous case applies to show

$$= \mathrm{conn}(V^*) - 2 = k(\mathrm{conn}(W^*) - 2).$$

Suppose f has r distinct critical points; the orders at these points must sum to k Suppose the jth critical value w_j has k_j distinct preimages with multiplicities $p_{j,i}$, $i = 1, \dots, k_j$. Since w_j has k preimages counted with multiplicity,

$$\sum_i p_{j,i} = k,$$

while

$$\sum_{i,j} (p_{i,j} - 1) = r,$$

since there are r critical points counted with multiplicity. So the number of distinct preimages of the s critical points is

$$\sum_k k_j = \sum_{j,i} 1 = \sum_{j,i}(p_{j,i} + 1 - p_{j,i}) = \sum_j(\sum_j p_{j,i}) - \sum_{j,i}(p_{j,i} - 1) = sk - r.$$

Thus

$$\mathrm{conn}(V) - 2 = \mathrm{conn}(V^*) - sk + r - 2 = k(\mathrm{conn}(W^*) - 2) - sk - r = k(\mathrm{conn}(W) - 2) + r,$$

which completes the proof of the Riemann-Hurwitz formula in the general case. □

A.5. Approximation theorems

Given a compact set $E \subset \mathbb{C}$, the "holes" of E are the bounded complementary components of E. E does not separate the plane iff there are no holes.

THEOREM A.5.1 (Runge's Theorem). *If $K \subset \mathbb{C}$ does not separate the plane and f is holomorphic on a neighborhood U of K, then f can be uniformly approximated on K by holomorphic polynomials.*

PROOF. Surround K by a piecewise smooth curve γ in U (e.g., set $\epsilon = \mathrm{dist}(K, \partial U)/10$ and cover K by ϵ-boxes and take the boundary of the unbounded complementary component). Then use the Cauchy integral formula to write

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}dw.$$

For any $\epsilon > 0$ we can find a finite set of points $\{w_j\}$ on γ so that

$$|\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}dw - \frac{1}{2\pi i}\sum_j \frac{f(w_j)}{w_j - z}| < \epsilon,$$

for all $z \in K$. Thus

$$|f(z) - \frac{1}{2\pi i}\sum_j \frac{f(w_j)}{w_j - z}| < \epsilon,$$

gives an approximation of f by rational functions with poles off k.

Next we use a "pole-pushing" argument to show that each pole can be uniformly approximated on K by a polynomials. Choose a disk $\mathbb{D}_r$ large enough to contain K and choose one of the n poles $w = w_j$ as above. Fix $\epsilon > 0$ Connect w w_j to $\partial\mathbb{D}_r$ by a piecewise linear path and choose a finite collection of points $w = z_0, z_1, \ldots z_n \in \partial\mathbb{D}_r$ so that $D(z_k, 2|z_k - z_{k-1}|) \cap K = \emptyset$. Then any rational function r_{k-1} with poles only at z_{k-1} has a Laurent expansion at z_k that converges on K and by truncating this series, r_{k-1} can be approximated to within $2^{-k}\epsilon/n$ by a rational function with poles only at z_k. When we reach r_k, we can approximate r_n by its Taylor series in $\mathbb{D}_r$ to obtain a polynomial approximation to $(w-z)^{-1}$ that is within ϵ/n. Summing over all the simple poles on γ and letting $\epsilon \to 0$ we obtain a uniform polynomial approximation to f on K. $\square$

Suppose E is a closed set that does not separated the plane. E is called an **Arakelian set** if the holes of $E \cup D$ form a bounded set for every closed disk in $\mathbb{C}$. This can also be stated by saying that E is locally connected at ∞.

THEOREM A.5.2 (Arakelian). *If E is an Arakelian set and f is holomorphic on a neighborhood of E, then for any $\epsilon > 0$ there is an entire function g such that*

$$\sup_{z \in E} |f(z) - g(z)| < \epsilon.$$

PROOF. We follow the proof of Rosay and Rudin [**119**].

Choose nested closed disks $\{D_k\}_1^\infty$ centered at the orgin and filling the plane so that $\frac{1}{4}D_{k+1}$ contains D_k and H_k, the closure of the union of holes of $E \cup D_k$. Let E_k be the $E \cup D_k \cup H_k$. Note that E_k has no holes.

Set $h_0 = f$ and assume by induction that we have a holomorphic function h_{k-1} on neighborhood U of E_{k-1}. Choose a differentiable function ψ such that $\psi \equiv 1$ on $\frac{1}{2}D_{k+1} \subset D_k \cup H_k$ and $\psi \equiv 0$ off D_{k+1}.

Since E_{k-1} has no holes, neither does $E_{k-1} \cap D_{k+1}$, so Runge's theorem gives a polynomial P so that both

$$|h_{k-1} - P| \le q2^{-k-1}\epsilon,$$

on $E_{k-1} \cap D_{k+1}$ and

$$\frac{1}{\pi} \iint_{E_{k-1}} |(h_{k-1} - P)(w)\overline{\partial}\psi(s)| \frac{dxdy}{|z-w|} < 2^{-k-1}\epsilon,$$

for all $z \in \mathbb{C}$. The latter holds because $\overline{\partial}\psi = \psi_x + i\psi_y$ is zero off D_{k+1} and $|z - w|^{-1}$ has a uniformly bounded integral over D_{k+1}, independent of where the pole is. Because the integrands are bounded, the same inequality holds if E_{k-1} is replaced by a sufficiently small open neighborhood $V \subset U$ of E_{k-1}. Thus

$$r(z) = \frac{1}{\pi} \iint_{E_{k-1}} |(h_{k-1} - P)(w)\overline{\partial}\psi(w)| \frac{dxdy}{|z-w|}$$

is holomorphic off V, r is bounded by $2^{-k-1}\epsilon$ everywhere and

$$\overline{\partial} r = (h_{k-1} - P)\overline{\partial}\psi.$$

Set

$$h_k = \psi P + (1-\psi)h_{k-1} + r,$$

in $V \cup \frac{1}{2}D_{k+1}$. Then, since $\overline{\partial}P = 0$ on $\mathbb{C}$ and $\overline{\partial}h_{k-1} = 0$ on V,

$$\overline{\partial}h_k = P\overline{\partial}\psi - h_{k-1}\overline{\partial}\psi - \overline{\partial}r = 0,$$

so h_k is holomorphic on V. Since ψ is constant on $\frac{1}{2}D_{k+1}$, $\overline{\partial}\psi = 0$ there, and hence r is holomorphic there. Since $h_k = r + P$, it is also holomorphic on $\frac{1}{2}D_{k+1}$.

Thus h_k is holomorphic on $\frac{1}{2}D_{k+1} \cup V$, an open neighborhood of E_k and

$$|h_k - h_{k-1}| = |(P - h_k)\psi + r| < 2^-k\epsilon$$

on E_{k-1}. Since $\cup_k E_k = \mathbb{C}$, this implies that h_k converges uniformly on compact sets to an entire function g that satisfies the theorem. $\square$

We can improve Runge's theorem by getting the same conclusion from a weak hypothesis:

THEOREM A.5.3 (Mergelyan's Theorem). *If $K \subset \mathbb{C}$ does not separate the plane and f is continuous on K and holomorphic on the interior of K, then f can be uniformly approximated on K by holomorphic polynomials.*

PROOF. By Runge's it suffices to prove that f can be approximated by a holomorphic function on some neighborhood of K. First extend f to be continuous on compact set E containing K in its interior. We also denote the extension by f. The extension is uniformly continuous on E so given any δ, there is any ϵ so that $|z-w| < 10\epsilon$ implies $|f(z) - f(w)| < \delta$. Convolve f with a smooth, positive bump function h of total mass 1, supported in an ϵ disk and with all first partial bounded by $O(1/epsilon)$. Call the result F. By the mean value property $F = f$ at the points U of the interior of K that are more than distance ϵ from the boundary. Furthermore, $|\overline{\partial}F(z)| = O(\delta/\epsilon)$ everywhere (note that $\overline{\partial}F = \overline{\partial}(F - F(z_0)) = (\overline{\partial}h) * (f - f(z_0))$ which is less than δ on the support of h).

Cover $K \setminus U$ by boxes from an ϵ-grid. For each such box Q_j, $4Q_j \setminus K$ contain a of diameter ϵ and the Riemann map ψ_j from the complement of this arc to the interior of $\mathbb{D}$ satisfies

$$|\psi_j(z)| > a > 0, z \in Q,$$
$$|\psi_j(z)| \leq b\epsilon/\mathrm{dist}(z, Q_j), z \in \mathbb{C} \setminus 8Q,$$

for constants a, b that independent of Q.

Let $\varphi(j)$ be a partition of unity with respect to the doubles of the ϵ-boxes and set

$$H(z) = \sum_j \psi_j^3(z)[\frac{1}{2\pi i} \iint \frac{1}{z-w}(\frac{\varphi_j(w)\overline{\partial}F(w)}{\psi_j^3(w)})dxdy],$$

satisfies $\overline{\partial}H = F$ on the union of the boxes and

$$|H(z)| \leq \sum_j O(b^3 \frac{\epsilon^3}{(1+\mathrm{dist}(z,Q_j))^3} a^{-3} \cdot \epsilon \cdot \frac{\delta}{/} epsilon) = O(\delta),$$

Since the sum $(1+|z|)^{-3}$ over a square lattice is finite. Thus $\overline{\partial}(F-H) = 0$ so $F - H$ is holomorphic and $|f - H| \leq |f - F| + |F - H| = O(\delta)$ is small. This proves Mergelyan's theorem. □

If Runge's theorem is replaced by in the previous proof of Arakelian's theorem, we get:

THEOREM A.5.4. *If E is an Arakelian set and f is continuous on E and holomorphic on the interior of E, then for any $\epsilon > 0$ there is an entire function g such that*

$$\sup_{z \in E} |f(z) - g(z)| < \epsilon.$$

We can easily strengthen this to

COROLLARY A.5.5. *If E is an Arakelian set with empty interior and f is continuous on E and holomorphic on the interior of E, then for any continuous, positive function $\epsilon(t)$ on $[0, \infty)$ there is an entire function F such that*

$$|f(z) - F(z)| < \epsilon(|z|)$$

for all $z \in E$.

PROOF. Apply the second version of Arakelian's theorem to deduce that there is an entire function g so that $\Re g(z) < \log \epsilon(|z|)$ for $z \in E$ and an entire function h so that $|h - fe^{-g}| < 1$. Then $F = he^g$ is entire and

$$|F - f| = |he^g - f| = |e^g| \cdot |h - fe^{-g}| < |e^g| \leq \epsilon(|z|).$$

□

A.6. Logarithmic capacity is a capacity

The title of the section sounds a little odd, but the point is to show that logarithmic capacity as defined in Section **??** satisfies certain properties. A **capacity** is a set function f that satisfies

(1) $E \subset F$ implies $f(E) \leq f(F)$,
(2) $f(E) = \sup\{f(K) : K \subset E, K \text{compact}\}$.
(3) If $E_1 \subset E_2 \subset \dots$, then $f^*(\cup_k E_k) = \lim f^*(E_k)$,

where

$$f^*(E) \equiv \inf\{f(U) : E \subset U, U \text{open}\}.$$

It is clear from the definition that logarithmic capacity satisfies the first two conditions, so we now have to prove the third.

Theorem A.6.1. *If $E_1 \subset E_2 \subset \dots$, then* $\operatorname{cap}^*(\cup_k E_k) = \lim \operatorname{cap}^*(E_k)$*,*

Our proof closely follows that given in Section III.3 of Carleson's book [**40**]. As there, we give a series of preliminary results before giving the proof of the theorem.

Lemma A.6.2. *If $U_\mu \leq 1$, then for any $\epsilon > 0$ there is an open set U with* $\operatorname{cap}(U) <$ *ϵ such that U_μ is continuous off U,*

Proof. Consider

$$U_\mu^\delta(z) = \int_{|w-z|<\delta} \log \frac{1}{|z-w|} dxdy.$$

Since U_μ is bounded, U_μ^δ converges to zero pointwise, and so by Egorov's theorem, for any $\delta > 0$ there is a compact set K with $\mu(K) > \|\mu\| - \delta$ on which it converges uniformly to zero. Let μ_1 be the restriction of μ to this set K and let $\mu_2 = \mu - \mu_1$. Let $U_1 = U_{\mu_1}$ and $U_2 = U_{\mu_2}$. Note that U_1 is continuous.

Let

$$S_n = \{z : U_2(z) > \frac{1}{n}\}.$$

Since potentials are lower semi-continuous, this is an open set and by definition $\operatorname{cap}^*(S_n) = \operatorname{cap}(S_n)$. If $\sigma$is an admissible measure for S_n then

$$\delta > \int U_\sigma d\mu_2 = \int U_2 d\sigma > \frac{1}{n}\|\sigma\||,$$

and so $\operatorname{cap}(S_n) \leq n\delta$.

Choose a sequence δ_n so that $sum_n n\delta n < \epsilon$ and set $S = \cup_n S_n$ to be the union of the corresponding sets defined above. Then outside S_n, U_μ is the sum of a continuous function and a function bounded by $1/n$. Hence it is continuous outside S. By sub-additivity (Lemma **??**) $\operatorname{cap}(S) < \epsilon$, so the lemma is proven. $\square$

LEMMA A.6.3. *If $\mu_n \to \mu$ weak*, and $U_n = U_{\mu_n}$ then*

$$\liminf_{n\to\infty} U_n(z) = U_\mu(z),$$

except on a set of outer capacity zero.

PROOF. We already know that $\liminf_{n\to\infty} U_n(z) \geq U_\mu(z)$ by Lemma **??**, so it suffices to prove the other direction. By the previous lemma there is an open set S with small capacity ϵ so that U_μ is continuous off S Consider pairs of rational numbers $p < q$ and set

$$F_{n,p,q} = \{z : U_\mu(z) \leq p < q \leq U_n(z)\}.$$

Note that these set are closed and

$$\cup_{p,q\in\mathbb{Q}} \cup_k \cap_{n>k} F_{n,p,q}$$

contains the exception set for the lemma. Thus by sub-additivity of outer capacity (Lemma 34), it suffices to prove

$$E_{k,p,q} = \cap_{n>k}\{z : U_\mu(z) \leq p < q \leq U_n(z)\}$$

has outer capacity zero.

Since this set is closed, it suffices to prove this for capacity. If this set had positive capacity, there would be non-zero measure σ supported in it that had a continuous potential bounded by 1. Thus

$$0 = \lim \int U_\sigma d(\mu - \mu_n) = \lim \int (U_\mu - U_n) d\sigma \geq (q-p)\|\sigma\| > 0,$$

which is a contradiction and proves the lemma. $\square$

LEMMA A.6.4. *For any open set S there is a measure μ so that*

(1) *$U_\mu = 1$ on S except for a set of zero outer capacity,*
(2) *$U_\mu \leq 1$,*
(3) *$\|\mu\| = \operatorname{cap}(S)$.*

PROOF. Write S as a nested union of compact sets $F_1 \subset F_2 \subset \dots$, let μ_n be the equilibrium measure on F_n, let U_n be its potential function and assume $\{\mu_n\}$ converges weak* to a measure μ. Fix $\delta > 0$ and note that $E_n = \{z \in F_n : U_n(z) \leq 1 - \delta\}$ is a closed set of capacity zero. By the subadditivity of outer capacity (Lemma **??**) $U_n = 1$ everywhere on F_n except a set of outer capacity zero. Lemma A.6.3 and subadditivity then imply each of the desired properties. □

PROOF OF THEOREM A.6.1. Suppose $E_1 \subset E_2 \subset \dots$ and $E = \cup E_n$. Let V_n be open sets containing E_n with corresponding measure μ_n given by Lemma A.6.4 so that

$$\|\mu_n\| = \mathrm{cap}(V_n) \leq \mathrm{cap}^*(E_n) + \frac{1}{n}.$$

Passing to subsequence if necessary we can assume $\mu_n \to \mu$ weak*, and by Lemma A.6.3 $U_\mu = 1$ on E except possibly on a set S of outer capacity zero. Let $V_\epsilon = \{z : U_\mu > 1 - \epsilon\}$. Then V_ϵ contains $E \setminus S$, so

$$\mathrm{cap}^*(E) \leq \mathrm{cap}(E \setminus S) + \mathrm{cap}^*(S) \leq lc2^*(V_\epsilon) \leq \|\mu\|/(1-\epsilon) \leq \lim_n \mathrm{cap}^*(E_n)/(1-\epsilon).$$

Taking $\epsilon \to 0$ proves then result and shows that logarithmic capacity is, indeed, a capacity. □

A.7. Analytic and Borel sets

In this section we prove that every analytic set is capacitable (due to Choquet **??** for Newtonian capacities and to Kishi **??** in general), following the presentation in Carleson's book [**40**]. We start by recalling the relevant definitions.

The **Borel sets** are the smallest σ-algebra containing the open sets. By a **σ-algebra** we mean a collection of sets that is closed under complements and countable unions. Since

$$\cap A_n = (\cup A_n^c)^c,$$

a σ-algebra is also closed under countable intersections. The Borel sets are also divided into subcollections denoted $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, indexed by countable ordinals α. These classes are defined inductively the the conditions

(1) $\mathbf{\Sigma}^0_1$ are the open sets,
(2) $\mathbf{\Pi}^0_\alpha$ are the complements of sets in $\mathbf{\Sigma}^0_\alpha$,

(3) $E \in \mathbf{\Sigma}^0_\alpha$ iff it is a countable union of sets in $\mathbf{\Pi}^0_{\alpha_k}$ for a collection of ordinals α_k that are strictly less than α.

Details can be found in Kekchris' book [**77**]. The main point for us is that any collection of sets that contains the open sets and the closed sets and that is closed under countable unions and intersections must contain the Borel sets, even if the collection itself is not a σ-algebra. The analytic sets form such a collection.

A set A is called **analytic**, if it can be written as

$$A = \cup_{\mathbf{n}} A_{\mathbf{n}},$$

where the union is over $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$, the (uncountable) collection of sequences of natural numbers $\mathbf{n} = \{n_1, n_2, \dots\}$, and

$$A_{\mathbf{n}} = A_{n_1} \cap A_{n1,n2} \cap \dots,$$

where $\{A_{n_1,\dots,n_k}\}$ is a countable collection of closed sets indexed by finite strings of natural numbers.

Clearly the analytic sets contain any closed set E since we can simply take $A_{n_1,\dots n_k} = E$ for every index set.

The analytic sets are closed under countable unions because if

$$E_k = \cup_{\mathbf{n}} A^k_{\mathbf{n}},$$

then we can define a new collection of closed sets by

$$A_{n_1,\dots n_k} = A^{n_1}_{n_2,\dots n_k},$$

so that

$$\cup_{\mathbf{n}} A_{\mathbf{n}} = \cup_k \cup_{\mathbf{m}} A^k_{\mathbf{m}} = \cup_k E_k,$$

is analytic.

To see that the analytic sets are closed under countable intersections, suppose we are given analytic sets

$$E_k = \cup_{\mathbf{n}} A^k_{\mathbf{n}},$$

and define a new collection of closed sets as follows. Divide the natural numbers into countable many disjoint sequences $\mathbb{N} = \cup_k \cup_k s^k_j$ and choose a sequence of naturals numbers $\{t_k\}$ that takes every value infinitely often. This choice allows us to merge countably many sequences $\{n_j^k\}$ into a single sequence by

$$n_i = n^{t_k}_{s^{t_k}_j},$$

where j is the number of times the value t_k has occurred in the string $t_1, \dots, t_k$, including t_k.

Using this merge operation on sequences we can define an analytic set

$$E = \cup_{\mathbf{b}} A_{\mathbf{n}} i.$$

For each fixed infinite string $\mathbf{n}$, $x \in A_{\mathbf{n}}$ if and only if it is in every set of the form $A^k_{n_{s^k_1},\dots,n_{s^k_j}}$. This occurs if and only if there are sequences $\{\mathbf{n}_k\}$ so that x is in every set $A_{\mathbf{n}_k}$. Thus $E = \cap_k E_k$.

Thus the analytic sets contain the closed sets, are closed under countable unions (hence contain all open sets) and are closed under countable intersections. By our remarks above, then must contain all Borel sets. Analytic sets are also called Suslin sets. Analytic sets can be characterized as the continuous images of Borel sets and Suslin found an error in an argument of Lebesgue purporting to show projections of Borel sets are Borel; it turns out that there are analytic sets that are not Borel. Moreover, the complements of analytic sets need not be analytic; it is a famous theorem of Suslin [] that if an analytic set has an analytic complement, then it is a Borel set. Thus the analytic sets do not form a σ-algebra.

Recall that a capacity is a set function f that satisfies

(1) $E \subset F$ implies $f(E) \le f(F)$,
(2) $f(E) = \sup) f(K) : K \subset E, K\text{compact}\}$.
(3) If $E_1 \subset E_2 \subset \dots$, then $f^*(\cup_k E_k) = \lim f^*(E_k)$,

and that a set E is called capacitable iff

$$f(E) = f^*(E) \equiv \inf\{f(U) : E \subset U, U\text{open}\}.$$

We showed in Section **??**, that logarithmic capacity is a capacity in this sense and we showed in Section **??** that all closed sets are capacitable for logarithmic capacity. Now we generalize that result from closed to all analytic sets (hence all Borel sets).

THEOREM A.7.1. *If all closed sets are capacitable, then all analytic sets are capacitable.*

PROOF. We will not use any particular properties of logarithmic capacity except for the properties of a general capacity listed above.

Suppose E is analytic. Then it can be written as

$$E = \cup_{\mathbf{n}} A_{\mathbf{n}},$$

as described above. We can write this union as a countable union

$$E = \cup_k E_k = \cup_k \cup_{\mathbf{n}:n_1<k} A_{\mathbf{n}},$$

where n_1 denotes the first element of $\mathbf{n}$. By property (3) in the definition of capacity,

$$f^*(E_k) \nearrow f^*(E).$$

Fix $\epsilon > 0$ and choose k_1 so that

$$f^*(S_1) > f^*(E) - \epsilon/2,$$

where $S_1 = E_{k_1}$.

Now repeat the argument with E replaced by E_{k_1}. We can find a k_2 so that

$$S_2 = \cup_{\mathbf{n}:n_1<k_1,n_2<k_2} A_{\mathbf{n}},$$

satisfies $S_2 \subset S_1$ and $f^*(s_2) > f^*(S_1) - \epsilon/4$. Continuing by induction we define analytic sets $S_1 \supset S_2 \supset \dots$ so that $f^*(S_n) > f^*(E) - \epsilon$ for all n.

We claim that $\cap_n \overline{S_n} \subset E$. This is a little tricky since it need not be true that each closed set $\overline{S}_n$ is in E. To prove the claim define

$$F_m = \cup_{n_1<k_1,\dots n_m<k_m} A_{n_1,\dots n_m}.$$

This is a finite union of closed sets, so is closed and hence contains the closure of S_n. Let $F = \cap_m F_m$. It suffices to show $F \subset E$. If $x \in F$, then for every m there is an $A_{n_1,\dots,n_m}$ that contains x. Since each index lies in a finite set, there is at least one index in each position that occurs infinitely often, and thus there is a sequence $\{n_1, n_2, \dots\}$ so that $x \in A_{\mathbf{n}} \subset E$, proving the claim.

Thus,

$$f^*(\overline{S_n}) \geq f^*(\overline{S_n}) \geq f^*(E) - \epsilon.$$

If U is an open set containing $S = \cap \overline{S_n}$ then U contains $\overline{S_n}$ if n is large enough. Thus

$$f(U) \geq f^*(\overline{S_n}) \geq f^*(E) - \epsilon.$$

Taking the infimum over all such U's proves

$$f^*(S) \geq f^*(E) - \epsilon.$$

Since S is closed, it is capacitable, and hence

$$f(E) \geq f(S) = f^*(S) \geq f^*(E) - \epsilon.$$

Taking $\epsilon \to 0$ proves the theorem. $\square$

A.8. Boundary continuity of conformal maps and capacity

Given a compact set $E \subset \mathbb{T}$ we will now define the associated "sawtooth" region W_E and a 2-quasiconformal map between W_E and $\mathbb{D}$ which keeps E fixed pointwise. Suppose $\{I_n\}$ are the connected components of $\mathbb{T} \setminus E$ and for each n let $\gamma_n(\theta)$ be the circular arc in $\mathbb{D}$ with the same endpoints as I_n and which makes angle θ with I_n (so $\gamma_n(0) = I_n$ and $\gamma_n(\pi/2)$ is the hyperbolic geodesic with the same endpoints as I_n). Let $C_n(\theta)$ be the region bounded by I_n and $\gamma_n(\theta)$, and let $W_E(\theta) = \mathbb{D} \setminus \cup_n C_n(\theta)$.

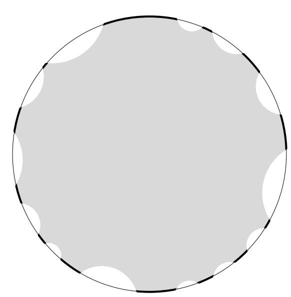


FIGURE 1. The sawtooth domain W_E

Let $W_E = W_E(\pi/8)$ (and let $W_E^* \subset \mathbb{D}^*$ be its reflection across $\mathbb{T}$). We can map $\mathbb{D}$ to W_E by a 2-quasiconformal map f as follows. First let f be the identity on $W_E(\pi/2)$. Then map $U_n = C_n(\pi/2) \setminus C_n(\pi/4)$ (which is a crescent of angle $\pi/4$) to $V_n = C_n(\pi/2) \setminus C_n(3\pi/8)$ (which is a crescent of angle $\pi/8$) as follows: map U_n to the cone $\{z : 0 < \arg(z) < \pi/4\}$ by a Möbius transformation, then to $\{z : 0 < \arg(z) < \pi/8\}$ by halving the angle and then to V_n by another Möbius transformation. Finally, map $C_n(\pi/4)$ to $C_n(3\pi/8) \setminus C_n(\pi/8)$ by a Möbius transformation. See Figure 2. It is easy to check these maps can be chosen to match up along the common boundaries and hence define a 2-quasiconformal map.

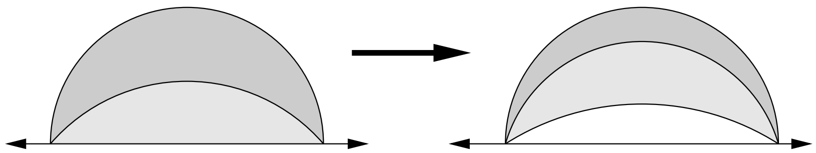


FIGURE 2. Mapping the disk to W_E

If $f : \mathbb{D} \to \Omega$ and $0 < r < 1$, then define

$$d_f(r) = \sup\{|f(z) - f(w)| : |z| = |w| = r \text{ and } |z - w| \leq 1 - r\}.$$

If $\partial\Omega$ is bounded in the plane, then it is easy to see this goes to zero as $r \nearrow 1$, since otherwise any neighborhood of $\partial\Omega$ would contain infinitely many disjoint disks of a fixed, positive size.

LEMMA A.8.1. *Suppose* $f : \mathbb{D} \to \Omega \subset S^2$ *is conformal. Then for any* $\epsilon > 0$ *there is a compact set* $E \subset \mathbb{T}$ *with* $\text{cap}(\mathbb{T} \setminus E) < \epsilon$ *such that* f *is continuous on* $\overline{W_E}$.

PROOF. By applying a square root and a Möbius transformation, we may assume that $\partial\Omega$ is bounded in the plane. Given $r < 1$ let

$$E(\epsilon, r) = \{x \in \mathbb{T} : |f(sx) - f(tx)| > \epsilon \text{ for some } r < s < t < 1\}$$

and note that by Lemmas **??** and 5.5.3

$$\widetilde{\text{cap}}(E(\epsilon, r)) \leq \exp(-\pi\epsilon^2/Ca(r)).$$

Moreover, this set is open since f is continuous at the points sx and tx. So if we take $\epsilon_n = 2^{-n}$, and use the relationship between cap and $\widetilde{\text{cap}}$ we can choose r_n so close to 1 that $\text{cap}(E_n) \equiv \text{cap}(E(\epsilon_n, r_n)) \leq \epsilon 2^{-n}$. If we define $E = \mathbb{T} \setminus \cup_{n>1} E_n$, then E is closed and $\mathbb{T} \setminus E$ has capacity $\leq \epsilon$ by subadditivity .

To show f is continuous at every $x \in \overline{W_E}$, we want to show that $|x - y|$ small implies $|f(x) - f(y)|$ is small. We only have to consider points $x \in \partial W_E \cap \mathbb{T}$. First suppose $y \in \partial W_E \cap \mathbb{T}$. Choose the maximal n so that $s = |x - y| \leq 1 - r_n$. Then $x, y \notin E_n$, so

$$|f(x) - f(y)| \leq |f(x) - f(sx)| + |f(sx) - f(sy)| + |f(sy) - f(y)|.$$

The first and last terms on the right are $\leq \epsilon_{n-1}$ by the definition of E. The middle term is at most $d_f(1-s)$ (which tends to 0 as $s \to 0$). Thus $|f(x) - f(y)|$ is small if $|x - y|$ is.

Now suppose $x \in \partial W_E \cap \mathbb{T}$, $y \in \partial W_E \setminus \mathbb{T}$. From the definition of W_E it is easy to see there is a point $w \in \partial W_E \cap \mathbb{T}$ such that $|w - y| \leq 2(1 - |y|) \leq 2|x - y|$. For the point w we know by the argument above that $|f(x) - f(w)|$ is small. On the other hand, if $t = 1 - |y|$, then

$$|f(y) - f(w)| \leq |f(y) - f(tw)| + |f(tw) - f(w)|.$$

The first term is bounded by $Cd_f(1-t)$ and the second is small since $w \notin E_n$. Thus $|f(x) - f(y)|$ is small depending only on $|x - y|$. Hence f is continuous on $\overline{W_E}$. $\square$

A.9. Borel sets are analytic

A **Polish space** is a topological space that can be equipped with a metric that makes it complete and separable.

If Y is Polish, then a subset $E \subset Y$ is called **analytic** if there exists a Polish space X and a continuous map $f\colon X \to Y$ such that $E = f(X)$.

Analytic sets are also called Souslin sets in honor of Mikhail Yakovlevich Souslin. The analytic subsets if Y are often denoted by $A(Y)$ or $\Sigma_1^1(Y)$. In any uncountable Polish space there exist analytic sets which are not Borel sets, see e.g., Proposition 13.2.5 in [**?**] or Theorem 14.2 in [**?**]. By definition, if $g\colon Y \to Z$ is a continuous mapping between Polish spaces and $E \subset Y$ is analytic, then $g(E)$ is also analytic. In other words, continuous images of analytic sets are themselves analytic, whereas it is known that continuous images of Borel sets may fail to be Borel sets. This fact is the main reason why it can be useful to work with analytic sets instead of Borel sets. The next couple of lemmas prepare for the proof that every Borel set in a Polish space is analytic.

We first show that analytic sets have a nice representation in terms of sequences. Let $\mathbb{N}^\infty$ be the space of all infinite sequences of nonnegative integers equipped with the metric given by $d((a_n),(b_n)) = e^{-m}$, where $m = \max\{n \geq 0\colon a_k = b_k \text{ for all } 1 \leq k \leq n\}$ (this space also sometimes denoted $\mathbb{N}^\mathbb{N}$).

LEMMA A.9.1. *For every Polish space X there exists a continuous mapping $f\colon \mathbb{N}^\infty \to x$ such that $X = f(\mathbb{N}^\infty)$. Moreover, for all $(b_n) \in \mathbb{N}^\infty$, the sequence of diameters of the sets $f(\{(a_n)\colon a_n = b_n$ for $1 \le n \le m\})$ is converging to zero, as $m \uparrow \infty$.*

PROOF. Given a Polish space x we construct a continuous and surjective mapping $f\colon \mathbb{N}^\infty \to X$. Fix a metric ϱ making X complete and separable. By separability, we can cover x by a countable collection of closed balls $B(j), j \in \mathbb{N}$ of radius one. We continue the construction inductively. Given closed sets $X(a_1, \dots, a_k)$, for $(a_1, \dots, a_k) \in \mathbb{N}^k$, we write $X(a_1, \dots, a_k)$ as the union of countably many non-empty closed sets $X(a_1, \dots, a_k, j), j \in \mathbb{N}$ of diameter at most 2^{-k}; we can do this by covering $X(a_1, \dots, a_k)$ by countable many closed balls of diameter $\le 2^{-k}$ with centers in $X(a_1, \dots, a_k)$ and then intersecting these balls with $X(a_1, \dots, a_k)$. Given $(a_n) \in \mathbb{N}^\infty$ the set $\bigcap_{k=1}^\infty X(a_1, \dots, a_k)$ has diameter zero, hence contains at most one point. By construction all the sets are non-empty and nested, so if we choose a point $x_k \in X(a_1, \dots, a_k)$ it is easy to see this forms a Cauchy sequence and by completeness it converges to some point x . Since each $X(a_1, \dots, a_k)$ is closed it must contain x and hence $\bigcap_{k=1}^\infty X(a_1, \dots, a_k)$ contains x. Define $f((a_n)) = x$.

By construction, if $(b_n) \in \mathbb{N}^\infty$, the set $f(\{(a_n)\colon a_n = b_n \text{ for } 1 \le n \le m\})$ has diameter at most $2^{-m+3} \to 0$, which implies continuity of f. Finally, by the covering property of the sets, every point $x \in X$ is contained in a sequence of sets $B(a_1, \dots, a_k)$, $k \in \mathbb{N}$, for some infinite sequence (a_k) which implies, using the nested property, that $f((a_n)) = x$ and hence $f(\mathbb{N}^n) = X$, as required. ☐

LEMMA A.9.2. *If $E \subset X$ is analytic, then there exists a continuous mapping $f\colon \mathbb{N}^\infty \to X$ such that $E = f(\mathbb{N}^\infty)$. Moreover, for any sequence $(k_n) \in \mathbb{N}^\infty$,*

$$\bigcap_{n=1}^\infty \overline{f\big(\{(a_n)\colon a_n \le k_n \text{ for } 1 \le n \le m\}\big)} = f\big(\{(a_n)\colon a_n \le k_n \text{ for all } n \ge 1\}\big).$$

PROOF. If $E \subset X$ is analytic, there exists a Polish space Y and $g_1\colon Y \to X$ continuous with $g_1(Y) = E$. From Lemma A.9.1 we have a continuous mapping $g_2\colon \mathbb{N}^\infty \to Y$ such that $Y = g_2(\mathbb{N}^\infty)$. Letting $f = g_1 \circ g_2\colon \mathbb{N}^\infty \to X$ gives $f(\mathbb{N}^\infty) = E$.

Fix a sequence of positive integers $k_1, k_2, \dots$ and note that the inclusion $\supset$ in the displayed equality holds trivially. If x is a point in the set on the left hand side, then there exist a_n^m with $a_n^m \leq k_n$ for $1 \leq n \leq m$ such that $\varrho(x, f((a_n^m : n \geq 1))) < \frac{1}{m}$. We successively pick integers $a_1, a_2, \ldots$ with the property that, for every n, the integer a_n occurs infinitely often in the collection $\{a_n^m : a_1^m = a_1, \ldots, a_{n-1}^m = a_{n-1}\}$. Then there exists $m_j \uparrow \infty$ such that $a_1^{m_j} = a_1, \ldots, a_j^{m_j} = a_j$ and hence $(a_n^{m_j} : n \geq 1)$ converges, as $j \to \infty$, to (a_n). Since f is continuous, $\varrho(x, f((a_n^{m_j} : n \geq 1)))$ converges to $\varrho(x, f((a_n)))$ and by construction also to zero, whence $x = f((a_n))$. This implies the x is contained in the set on the right hand side in the displayed equation. $\square$

Our next task is to show that every Borel set is analytic. We will use several times the simple fact that the finite or countable product of Polish spaces is Polish, which we leave for the reader to verify.

LEMMA A.9.3. *Open and closed subsets of a Polish space are Polish, hence analytic.*

PROOF. Let X be Polish and ϱ a metric making X a complete and separable metric space. If $C \subset X$ is closed then the metric ϱ makes C complete and separable, hence C is a Polish space. If $O \subset X$ is open, let $Y = \{(x, y) : y = 1/\varrho(x, O^{\mathrm{c}})\} \subset X \times \mathbb{R}$. Then Y is a closed subset of the Polish space $X \times \mathbb{R}$ and hence itself a Polish space. As Y and O are homeomorphic, O is Polish. $\square$

LEMMA A.9.4. *Let $E_1, E_2, \ldots \subset X$ be analytic sets. Then*

(1) $\displaystyle\bigcup_{i=1}^{\infty} E_i$ *is an analytic set;*

(2) $\displaystyle\bigcap_{i=1}^{\infty} E_i$ *is an analytic set.*

PROOF. For every analytic set $E_i \subset X$ there exists a continuous $f_i : \mathbb{N}^\infty \to X$ such that $f_i(\mathbb{N}^\infty) = E_i$. Then $f : \mathbb{N}^\infty \to X$ given by $f((a_n)) = f_{a_1}((a_{n+1}))$ is continuous and satisfies $f(\mathbb{N}^\infty) = \bigcup_{i=1}^\infty E_i$, as required to show (a).

Now look at continuous mappings $f_i : X_i \to X$ with $f_i(X_i) = E_i$. Define a continuous mapping $g : \prod_{i=1}^\infty X_i \to X^\infty$ by $g(x_1, x_2, \ldots) = (f_1(x_1), f_2(x_2), \ldots)$. The diagonal $\Delta \subset X^\infty$ is closed, and so is $g^{-1}(\Delta)$, by continuity of g. In particular, $Y = g^{-1}(\Delta)$ is a Polish space and it is easy to see that $f_1(Y) = \bigcap_{i=1}^\infty E_i$, proving (b). $\square$

LEMMA A.9.5. *If X is Polish, then every Borel set $E \subset X$ is analytic.*

PROOF. The collection $\{S \subset X \colon S \text{ and } S^{\mathrm{c}} \text{ are analytic}\}$ of sets contains the open sets by Lemma A.9.3 and is closed under countable unions by Lemma A.9.4. As it is obviously closed under taking the complement it must contain the Borel sets, which, by definition, is the smallest collection of sets with these properties. $\square$

A.10. Choquet capacitability

The main step in the extension of Frostman's Lemma to Borel sets is a technical device called the *Choquet Capacitability Theorem*, which we now introduce.

Let X be a Polish space. A set function Ψ defined on all subsets of X is called a **Choquet capacity** if

(a) $\Psi(E_1) \le \Psi(E_2)$ whenever $E_1 \subset E_2$;

(b) $\Psi(E) = \inf_{O \supset E \text{open}} \Psi(O)$ for all $E \subset X$;

(c) for all increasing sequences $\{E_n \colon n \in \mathbb{N}\}$ of sets in X,
$$\Psi\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \Psi(E_n).$$

Given Ψ we can define a set function Ψ_* on all sets $E \subset X$ by
$$\Psi_*(E) = \sup_{F \subset E \text{compact}} \Psi(F).$$
A set E is called **capacitable** if $\Psi(E) = \Psi_*(E)$.

THEOREM A.10.1 (Choquet Capacitability Theorem). *If Ψ is a Choquet capacity on a compact metric space X, then all analytic subsets of X are capacitable.*

PROOF. Let $E = f(\mathbb{N}^\infty) \subset X$ be analytic. We define sets
$$S^k = f\big(\{(a_n) \colon a_1 \le k\}\big).$$
The sequence of sets S^k, $k \ge 1$, is increasing and their union is E. Hence, by (c), given $\epsilon > 0$ we can find $k_1 \in \mathbb{N}$ such that $S_1 := S^{k_1}$ satisfies
$$\Psi(S_1) \ge \Psi(E) - \frac{\epsilon}{2}.$$
Having found $S_1, \ldots, S_{m-1}$ and $k_1, \ldots, k_{m-1}$ we continue the sequence by defining
$$S_m^k = f\big(\{(a_n) \colon a_i \le k_i \text{ for } i \le m-1,\ a_m \le k\}\big),$$

and as the sequence of sets S_m^k, $k \geq 1$, is increasing and their union is S_{m-1} we find $k_m \in \mathbb{N}$ with $\Psi(S_m) \geq \Psi(S_{m-1}) - \epsilon 2^{-m}$. We conclude that

$$\Psi(S_m) \geq \Psi(E) - \epsilon \qquad \text{for all } m \in \mathbb{N}.$$

Denoting by $\overline{S_m}$ the closure of S_m we now define a compact set

$$S = \bigcap_{m=1}^{\infty} \overline{S_m}.$$

By Lemma A.9.2 S is a subset of E. Now take an arbitrary open set $O \supset S$. Then there exists m such that $O \supset \overline{S_m}$ and hence $\Psi(O) \geq \Psi(S_m) \geq \Psi(E) - \epsilon$. Using property (b) infer that $\Psi(S) \geq \Psi(E) - \epsilon$ and, as $\epsilon > 0$ was arbitrary, $\Psi_*(E) \geq \Psi(E)$. As $\Psi_*(E) \leq \Psi(E)$ holds trivially, we get that E is capacitable. $\square$

We now look at the boundary ∂T of a tree. Recall from Lemma **??** that ∂T is a compact metric space. For any edge e we denote by $T(e) \subset \partial T$ the set of rays passing through e. Then $T(e)$ is a closed ball of diameter $2^{-|e|}$ (where $|e|$ is tree distance to the root vertex of the endpoint of e further from the root) and also an open ball of diameter r, for $2^{-|e|} < r \leq 2^{-|e|+1}$. Moreover, all closed balls in ∂T are of this form. If $E \subset \partial T$ then a set Π of edges is called a **cut-set of** E if every ray in E contains at least one edge of Π or, equivalently, if the collection $T(e)$, $e \in \Pi$, is a covering of E. Recall from Definition 4.5 that the α-Hausdorff content of a set $E \subset \partial T$ is defined as

$$\begin{aligned}\mathcal{H}_\infty^\alpha(E) &= \inf\Big\{ \textstyle\sum_{i=1}^\infty |E_i|^\alpha \colon E_1, E_2, \dots \text{ is a covering of } E \Big\} \\ &= \inf\Big\{ \textstyle\sum_{e\in\Pi} 2^{-\alpha|e|} \colon \Pi \text{ is a cut-set of } E \Big\},\end{aligned}$$

where the last equality follows from the fact that every closed set in ∂T is contained in a closed ball of the same diameter.

Lemma A.10.2. *The set function Ψ on ∂T given by $\Psi(E) = \mathcal{H}_\infty^\alpha(E)$ is a Choquet capacity.*

Proof. Property (a) holds trivially. For (b) note that given E and $\epsilon > 0$ there exists a cut-set Π such that the collection of sets $T(e)$, $e \in \Pi$, is a covering of E with $\Psi(E) \geq \sum_{e\in\Pi} 2^{-\alpha|e|} - \epsilon$. As $O = \bigcup_{e\in E} T(e)$ is an open set containing E and $\Psi(O) \leq \sum_{e\in\Pi} 2^{-\alpha|e|}$ we infer that $\Psi(O) \leq \Psi(E) + \epsilon$, from which (b) follows. We now prove (c). Suppose $E_1 \subset E_2 \subset \dots$ and let $E = \bigcup_{n=1}^{\infty} E_n$. Fix $\epsilon > 0$ and choose cut-sets Π_n of E_n such that

$$\sum_{e \in \Pi_n} 2^{-\alpha|e|} \leq \Psi(E_n) + \frac{\epsilon}{2^{n+1}}. \tag{61}$$

For each positive integer m we will prove that

$$\sum_{e \in \Pi_1 \cup \dots \cup \Pi_m} 2^{-\alpha|e|} \leq \Psi(E_m) + \epsilon. \tag{62}$$

Taking the limit as $m \to \infty$ gives

$$\Psi(E) \leq \sum_{e \in \Pi_1 \cup \dots} 2^{-\alpha|e|} \leq \lim_{m \to \infty} \Psi(E_m) + \epsilon.$$

Taking $\epsilon \to 0$ gives $\Psi(E) \leq \lim_{m \to \infty} \Psi(E_m)$. Since the opposite inequality is obvious, we see that (62) implies (c).

For every ray $\xi \in E$ we let $e(\xi)$ be the edge of smallest order in $\xi \cap \bigcup_n \Pi_n$. Note that $\Pi = \{e(\xi) \colon \xi \in E\}$ is a cut-set of E and no pair of edges $e_1, e_2 \in \Pi$ lie on the same ray. Fix a positive integer m and let $Q_m^1 \subset E_m$ be the set of rays in E_m that pass through some edge in $\Pi \cap \Pi_1$. Let Π_m^1 be the set of edges in Π_m that intersect a ray in Q_m^1. Then Π_m^1 is a cut-set for Q_m^1 and hence for $Q_m^1 \cap E_1$, and hence $\Pi_m^1 \cup (\Pi_1 \setminus \Pi)$ is a cut-set for E_1. From our choice of Π_1 in (61) and the fact that $\Psi(E_1)$ is a lower bound for any cut-set sum for E_1 we get

$$\sum_{e \in \Pi_1} 2^{-\alpha|e|} \leq \Psi(E_1) + \frac{\epsilon}{4} \leq \sum_{e \in \Pi_m^1 \cup (\Pi_1 \setminus \Pi)} 2^{-\alpha|e|} + \frac{\epsilon}{4}.$$

Now subtract the contribution from edges in $\Pi_1 \setminus \Pi$ on both sides, to get

$$\sum_{e \in \Pi \cap \Pi_1} 2^{-\alpha|e|} \leq \sum_{e \in \Pi_m^1} 2^{-\alpha|e|} + \frac{\epsilon}{4}.$$

Now iterate this construction. Suppose $1 \leq n < m$ and $\Pi_m^1, \dots, \Pi_m^n$ are given. Set $\Pi_{n+1}^* = \Pi_{n+1} \setminus (\Pi_1 \cup \ldots \cup \Pi_n)$ and let Q_m^{n+1} be the set of rays in E_m that pass through some edge in $\Pi \cap \Pi_{n+1}^*$. Let Π_m^{n+1} be the set of edges in Π_m that intersect a ray in Q_m^{n+1}. Then, as above, Π_m^{n+1} is a cut-set for $Q_m^{n+1} \cap E_{n+1}$, and hence $\Pi_m^{n+1} \cup (\Pi_{n+1} \setminus (\Pi \cap \Pi_{n+1}^*))$ is a cut-set for E_{n+1}. Using (61) and subtracting equal terms as before gives

$$\sum_{e \in \Pi \cap \Pi_{n+1}^*} 2^{-\alpha|e|} \leq \sum_{e \in \Pi_m^{n+1}} 2^{-\alpha|e|} + \sum_{n=1}^{m} \frac{\epsilon}{2^{n+2}}.$$

$$\sum_{e\in\Pi\cap(\Pi_1\cup\cdots\cup\Pi_m)} 2^{-\alpha|e|} \le \sum_{n=1}^{m}\Big(\sum_{e\in\Pi_m^n} 2^{-\alpha|e|} + \frac{\epsilon}{2^{n+1}}\Big) \le \sum_{e\in\Pi_m} 2^{-\alpha|e|} + \frac{\epsilon}{4} \le \Psi(E_m) + \epsilon.$$

This is (62) and completes the proof. □

The following is immediate from Lemma A.10.2.

Corollary A.10.3. *If an analytic set $E \subset \partial T$ has $\mathcal{H}^\alpha(E) > 0$, then there exists a compact set $A \subset E$ with $\mathcal{H}^\alpha(A) > 0$.*

Proof. Recall that by Proposition **??** Hausdorff content and Hausdorff measure vanish simultaneously. Hence, if E is analytic and $\mathcal{H}^\alpha(E) > 0$, then $\Psi(E) > 0$. Lemma A.10.2 implies that there exists a compact set $A \subset E$ with $\Psi(A) > 0$, and therefore $\mathcal{H}^\alpha(A) > 0$. □

All that remains to be done now is to transfer this result from the boundary of a suitable tree to Euclidean space.

Theorem A.10.4. *Let $E \subset \mathbb{R}^d$ be a Borel set and assume $\mathcal{H}^\alpha(E) > 0$. Then there exists a closed set $A \subset E$ with $\mathcal{H}^\alpha(A) > 0$.*

Proof. We find a hypercube $Q \subset \mathbb{R}^d$ of unit sidelength such that $\mathcal{H}^\alpha(E\cap Q) > 0$, and a continuous mapping $\Phi\colon \partial T \to Q$ from the boundary of the 2^d-ary tree T to Q, mapping closed balls $T(e)$ onto compact dyadic subcubes of sidelength $2^{-|e|}$. The α-Hausdorff measure of images and inverse images under Φ changes by no more than a constant factor. Indeed, for every $B \subset \partial T$, we have $|\Phi(B)| \le \sqrt{d}|B|$. Conversely, every set $B \subset Q$ of diameter 2^{-k} lies in the interior of the union of no more than 3^d compact dyadic cubes of sidelength 2^{-k}, whence the edges corresponding to these dyadic cubes form a cut-set of $\Phi^{-1}(B)$. Therefore, $\mathcal{H}^\alpha(E\cap Q) > 0$ implies $\mathcal{H}^\alpha(\Phi^{-1}(E\cap Q)) > 0$. As $\Phi^{-1}(E\cap Q)$ is a Borel set and hence analytic we can use Corollary A.10.3 to find a compact subset A with $\mathcal{H}^\alpha(A) > 0$. Now $\Phi(A) \subset E\cap Q$ is a compact subset of E and $\mathcal{H}^\alpha(\Phi(A)) > 0$, as required. □

Given a Borel set $E \subset \mathbb{R}^d$ with $\mathcal{H}^\alpha(E) > 0$ we can now pick a closed set $A \subset E$ with $\mathcal{H}^\alpha(A) > 0$, apply Frostman's lemma to A and obtain a probability measure on A (and, by extension, on E) such that $\mu(D) \le C|D|^\alpha$ for all Borel sets $D \subset \mathbb{R}^d$. The proof of Theorem A.10.4 also holds for Borel sets E with Hausdorff measure $\mathcal{H}^\varphi(E) > 0$ taken with respect to a gauge function φ.

We have adapted the proof of Theorem A.10.4 from [?]. For a brief account of Polish spaces and analytic sets see [?] For a more comprehensive treatment of analytic sets and the general area of descriptive set theory, see the book of [?]. There are several variants of Choquet's Capacitability Theorem, see, for example, [?] for an alternative treatment and for other applications of the result, e.g., the infimum of the hitting times of Brownian motion in a Borel set define a stopping time.

A.11. Wiman-Valiron theory via power series

In this and the following section we outline an alternate method for proving the existence of escaping (and of fast points). This was the method originally used by Eremenko, although in the text we have presented more recent proofs due to Dominguez (Theorem 1.7.1) and to Bergweiler, Rippon and Stallard (Theorem 6.5.2). The later is a generalization of the Wiman-Valiron method to functions defined on tracts; here we discuss the original case of entire functions, with the proof based on power series.

If f is entire, then is has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z_n.$$

Since the radius of convergence is ∞, the coefficients satisfy

$$\lim_{n\to\infty} |a_n|^{1/n} = 0.$$

Thus for any fixed r, $|a_n|r^n \to 0$ and hence the supremum

$$\mu(r) = \mu(r, f) = \sup_n |a_n|r^n,$$

is attained. We let $N(r) = N(r, f)$ denote the largest index n where the maximum is taken. This is an increasing step function and is right continuous at the jump points. Note that the jumps are > 1 when the maximum of $|a_n|r^n$ is attained at several different indices. $N(r)$ is called the central index of f at r. (From now on we will fix f and simply write $M(r), \mu(r), N(r)$ unless we need to stress the dependence on the function f.)

LEMMA A.11.1. $\mu(r) \leq M(r)$.

PROOF. By rescaling we may assume $r = 1$, $\mu(r) = 1$ and $a_N = 1$ where $N = N(1)$. Then $g(z) = f(z) - z^N$ is perpendicular to z^N in L^2 of the unit circle, so

$$0 = \int_{|z|=1} z^N \overline{g(z)} ds = \int_{|z|=1} z^N \overline{f(z) - z^N} ds = \int_{|z|=1} z^N \overline{f(z)} ds - 2\pi.$$

This implies $\int_{|z|=1} |f| ds \geq 2\pi$ and hence $|f| \geq 1$ somewhere on the unit circle. Thus $M(1) \geq \mu(1)$, as desired. □

One of the remarkable consequences of Wiman-Valiron theory is that this inequality can be reversed in a certain sense. There are several ways to make this precise; one says that for any $\delta > 0$, we have

$$M(r) \leq \mu(r)(\log \mu(r))^{1/2}(\log \log \mu(r))^{1+\delta},$$

for all $r > 0$ except a set of finite logarithmic measure. See Theorem 6 of [**65**]. However, we will not pursue this particular line of reasoning. Our goal is to use the Wiman-Valiron method to show that near a point where f attains a circular maximum, i.e., near a point z such that

$$|f(z)| = M(|z|, f),$$

f behaves like the power function $w^{N(|z|)}$. Before making and proving a precise statement, we will deal with a number of more technical details.

As noted above, the sequence $\{|a_n|r^n\}$ attains a maximum value at $n = N = N(r)$. By definition, moving to the left or right of N gives smaller terms, but we would like these terms to decay quickly to zero as we move away from N. An ideal estimate would be

$$|a_n|r^n \leq \mu(r) \exp(-bk^2),$$

where $n = N + k$ and $c > 0$, but this is too much to ask for. However, the estimate is true if we replace the constant b by a function of $N + |k|$ that tends slowly to zero. The reader by take $b(t) = t^{-1-\delta}$ in the proof below.

LEMMA A.11.2. *If $f(z) = \sum a_n z^n$ is an entire function, and $b(x)$ is a positive, strictly decreasing, integrable function on $[0, \infty)$. Then there is a set E of finite logarithmic measure and a constant $K > 0$ so that for $r \in [1, \infty) \setminus E$*

$$|a_n|r^n \leq \mu(r) \exp(-\frac{1}{2} b(\max(N, n)) \cdot (N - n)^2).$$

PROOF. Let

$$r_n = r_n(f) = \inf\{r : N(r) \geq n\}.$$

Note that r_n increases to infinity with n and that for each n there is an $r \in [r_n, r_{n+1}]$ such that

$$|a_n|r_n^n \leq |a_n|r^n = |a_{n+1}|r^{n+1} \leq |a_{n+1}|r_{n_1}^{n+1}.$$

Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{r^n}{r^{n+1}} = \frac{1}{r},$$

so

$$\frac{1}{r_{n+1}} \leq \frac{|a_{n+1}|}{|a_n|} = \frac{r^n}{r^{n+1}} \leq \frac{1}{r_n}.$$

Thus

$$\frac{a_n}{a_0} = \frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdots \frac{a_n}{a_{n-1}} \cdot = \frac{1}{r_0 r_1 \cdots r_{n-1}}. \tag{63}$$

Set $B(t) = \int_0^t b(s)ds$. This is an increasing, concave down function that approaches as finite limit as $t \to \infty$. For integers $n \geq 0$ define

$$\alpha_n = \exp(-\int_0^n B(t)dt), \quad \rho_n = \exp(B(n)).$$

Note that the α's decrease to zero, and the ρ's increase to a finite limit. Since b is strictly decreasing, B is increasing but concave down, so

$$\log \rho_{n+1} = \log B(n+1) \geq \int_n^{n+1} B(t)dt = -\log \frac{\alpha_{n+1}}{\alpha_n} \geq B(n) = \log \rho_n,$$

and hence

$$\log \frac{\alpha_n}{\alpha_{n-1}} > -\log \rho_n > \log \frac{\alpha_{n+1}}{\alpha_n},$$

or equivalently,

$$0 < \rho_0 < \frac{\alpha_0}{\alpha_1} < \rho_1 < \frac{\alpha_1}{\alpha_2} \dots.$$

Using a telescoping product, this implies for $n > N$,

$$\rho_n^{N-n} \leq \frac{1}{\rho_n} \cdots \frac{1}{\rho_{N+1}} \leq \frac{\alpha_n}{\alpha_N} = \frac{\alpha_n}{\alpha_{n-1}} \cdots \frac{\alpha_{N+1}}{\alpha_N} \leq \frac{1}{\rho_{n-1}} \cdots \frac{1}{\rho_N} \leq \rho_N^{N-n} \tag{64}$$

If $m < n$, the definitions above and integration by parts give

$$\begin{aligned}\frac{\alpha_n}{\alpha_m}\rho_m^{n-m} &= \exp(-\int_m^n [B(t)-B(m)]dt)\\ &= \exp(-\int_m^n (n-t)b(t)dt)\\ &\leq \exp(-\int_m^n (n-t)b(n)dt)\\ &\leq \exp(-\frac{1}{2}b(n)(n-m)^2)\end{aligned}$$

Thus for general n and m,

$$\frac{\alpha_n}{\alpha_m}\rho_m^{n-m} = \exp(-\frac{1}{2}b(\max(m,n))\cdot(n-m)^2).$$

To finish the proof we must show that if $N = N(r)$, then for all $n \in \mathbb{N}$,

$$|a_n|r^n \leq \mu(r)\cdot\frac{\alpha_n}{\alpha_N}\rho_N^{n-N} = |a_N|r^N\cdot\frac{\alpha_n}{\alpha_N}\rho_N^{n-N} \tag{65}$$

holds for all r outside some exceptional set E of finite logarithmic measure.

First let us estimate the set of r's we must throw away when $n \leq N$. In this case, we have $\alpha_n > \alpha_N$ and so (65) follows from

$$|a_n|r^n \leq |a_N|r^N\rho_N^{n-N},$$

or equivalently

$$|a_n|(\frac{r}{\rho_N})^n \leq |a_N|(\frac{r}{\rho_N})^N. \tag{66}$$

This holds by definition if $N(r/\rho_N) = N(r)$, i.e., $r \in \rho_N I_N = (\rho_N r_N, \rho_N r_{N+1})$ for some N. The complement of this union of intervals has finite logarithmic measure by Lemma 6.3.1.

Next we estimate the set we must omit due to indices $n > N$. In this case, (66) follows from

$$\frac{|a_n|(r\rho_N)^n}{|a_N|(r\rho_N)^N} \leq \frac{\alpha_n\rho_N^n}{\alpha_N\rho_N^N} \tag{67}$$

since (64) implies the right hand side is ≤ 1. We cancel the ρ_N's and rewrite this as

$$\frac{|a_n|}{|\alpha_n}r^n \leq \frac{|a_N|}{\alpha_N}r^N \tag{68}$$

or

$$c_n r^n \leq c_N r^N \tag{69}$$

where we set $c_n = |a_n|/\alpha_n$. Suppose for the moment that $F(z) = \sum_n c_n z^n$ is entire, so that the central index of F makes sense. Then (66) holds as long as $N(r, f) = N(r, F)$. (67) implies this is true if $r \in (R_n\rho_n, R_{n+1}\rho_n)$, and hence the exception set of r's is the union of complementary intervals

$$\cup_n (R_n\rho_{n-1}, R_{n+1}\rho_{n+1}),$$

which has finite logarithmic measure by Lemma 6.3.1.

Finally, we have to prove that the $\{c_n\}$ are the Taylor coefficients of an entire function, i.e., that

$$\lim_n c_n^{1/n} = 0.$$

Combining (63) and (64) gives

$$(\frac{a_n}{\alpha_n})^{1/n} \leq (\frac{a_0}{\alpha_0}\frac{\rho_1\cdots\rho_n}{r_1\cdots r_n})^{1/n},$$

and this tends to zero since the ρ_n's are bounded above and the r_n's tend to ∞. $\square$

Now that we have the pointwise estimate, we show that the sum of terms far from the central index is small. This is somewhat involved, but is really just a matter of breaking an infinite sum into pieces and estimating each piece separately.

Lemma A.11.3. *Suppose f is a transcendental entire function, $r > 0$, and $N = N(r, f)$. Assume b has been chosen to be positive, decreasing, integrable and satisfy*

$$\frac{1}{Ct^2} \leq b(t) \leq \frac{C}{t|\log t|} \text{ and } b'(t) = O(b(t)/t).$$

If $\gamma > 1$ and

$$k = \left\lfloor \frac{1}{\sqrt{\gamma b(N)|\log b(N)|}} \right\rfloor,$$

$|\log \rho/r| < 2/k$ and $\sigma < (\gamma - 1)/2$, then

$$\sum_{|n-N|\geq k} \frac{|a_n|\rho^n}{|a_N|\rho^N} = o(b(N(r))^\sigma),$$

uniformly as $r \to \infty$ outside a set of finite logarithmic measure.

Proof. One difficulty with the desired estimate is that our pointwise estimates of the terms on the left hand side from Lemma A.11.2 involve the central index of ρ, whereas the right hand side only involves the central index of r. This would be fine if $N(r) = N(\rho)$, but will be a problem if the central index changes rapidly near

r, so our first step will be to exclude the set of finite logarithmic measure where this happens.

The basic strategy is to choose an $\eta > 0$ and cut the sum into pieces

$$\sum_{|n-N|\geq k} = \sum_{n\geq(1+\eta)N} + \sum_{n\leq(1-\eta)N} + \sum_{N+k\leq n<(1+\eta)N} + \sum_{(1-\eta)N<n\leq N-k} = I + II + III + IV,$$

and prove

$$I, II = O(N^{-\beta}),$$

for any $\beta > 0$ and

$$II, IV = O(b(N(r))^{-\sigma}).$$

Given any $\epsilon > 0$ our assumptions on b imply that there is an $0 < \eta < 1$ so that

$$b((1+\eta)t) \geq (1-\epsilon)b(t).$$

Fix ϵ so that $\sigma < (1-\epsilon)^2\gamma$ and choose η so the inequality above is true. Next choose $\alpha \leq \eta/2$, and set $A = \exp(N^{-\alpha})$ and $M = N(Ar, f)$. By Lemma **??** $N \leq M \leq (1+\alpha)N$, outside a set of finite logarithmic measure. Assume that we exclude this set, as well as the exceptional set for Lemma A.11.2. Since M is the central index for Ar, we have

$$|a_n|(Ar)^n \leq |a_M|(Ar)^M,$$

for every n. Using the fact that N and M are central indices for r and Ar we get

$$\begin{aligned}\frac{|a_n|\rho^n}{|a_N|\rho^N} &= \frac{|a_n|(A\rho)^n}{|a_M|(A\rho)^N} \cdot \frac{|a_M|r^M}{|a_N|r^N} \cdot A^{M-n}(\frac{\rho}{r})^{n-M} \\ &\geq 1 \cdot 1 \cdot A^{M-n}(\frac{\rho}{r})^{n-M}.\end{aligned}$$

First assume that $n > (1+\eta)N$. Since $M \leq (1+\alpha)N \leq (1+\eta/2)N$, we have

$$n - M \geq n - (1+\eta/2)N \geq n\eta/2$$

hence

$$A^{M-n} \leq A^{\eta n/2} = \exp(-\frac{1}{2}\eta n N^{-\alpha}).$$

Also,

$$(\frac{\rho}{r})^{n-M} \leq \exp((n-M)2/k).$$

Hence

$$\frac{|a_n|\rho^n}{|a_N|\rho^N} \leq \exp(-\frac{1}{2}\eta n N^{-\alpha} + 2n/k).$$

Our definition of k and assumptions on b imply

$$\frac{1}{k} = O(\sqrt{b(N)|\log b(N)|}) = O((N \log N)^{-1/2})) = o(N^{-\alpha}),$$

since $\alpha \leq \eta/2 < 1$. Thus

$$\frac{|a_n|\rho^n}{|a_N|\rho^N} \leq \exp(-\frac{1}{3}\eta n N^{-\alpha}),$$

if r is large enough (and hence n is large). Thus by the geometric sum formula

$$\begin{aligned}
\sum_{n \geq (1+\eta)N} \frac{|a_n|\rho^n}{|a_N|\rho^N} &\leq \sum_{n \geq (1+\eta)N} \exp(-\frac{1}{3}\eta n N^{-\alpha}) \\
&= O(\frac{\exp(-\frac{1}{3}\eta(1+\eta)N^{1-\alpha})}{1-\exp(-\eta N^{-\alpha}/3)}) \\
&= O(\frac{1}{\eta}N^{\alpha}\exp(-\eta N^{1-\alpha}/3)) \\
&= O(N^{-\beta}) \\
&= O(b(N)^{\beta/2})
\end{aligned}$$

for any $\beta > 0$.

Next assume $n \leq (1-\eta)N$ and repeat the argument above, replacing Ar by r/A. Since $M \geq (1-\alpha)N \geq (1-\eta/2)N$, we have

$$M - n \geq (1-\eta/2)N - n \geq N\eta/2$$

and hence

$$A^{n-M} \geq A^{-\eta N/2} = \exp(-\frac{1}{2}\eta N^{1-\alpha}).$$

We get

$$\frac{|a_n|\rho^n}{|a_N|\rho^N} \leq \exp(-\frac{1}{2}\eta N^{1-\alpha} + 2N/k) \leq \exp(-\frac{1}{3}\eta N^{1-\alpha}),$$

using the definition of k as before. Now we are summing $\leq N$ terms of the same size, so

$$\begin{aligned}
\sum_{n \leq (1-\eta)N} \frac{|a_n|\rho^n}{|a_N|\rho^N} &\leq N\exp(-\frac{1}{3}\eta N^{1-\alpha}) \\
&= O(N^{-\beta})
\end{aligned}$$

for any $\beta > 0$.

Finally we consider $(1-\eta)N < n \leq N-k$ and $N+k \leq n < (1+\eta)N$. Recall that $b((1+\eta)N) > (1-\epsilon)b(N)$ and $\rho = re^t$ where $t \leq 2/k$. Lemma A.11.2 implies that for $N-p| \leq \eta n$ (except for a set of finite logarithmic measure),

$$\begin{aligned}
\frac{|a_n|\rho^n}{|a_N|\rho^N} &\leq \frac{|a_n|(re^t)^n}{|a_N|(re^t)^N} \\
&\leq \frac{|a_n|r^n}{|a_N|r^N}e^{t(n-N)} \\
&\leq \frac{|a_n|r^n}{|a_N|r^N}e^{t|p|} \\
&\leq \exp(-\frac{1}{2}b(N+\eta N)p^2 + t|p|) \\
&\leq \exp(t|p| - \frac{1}{2}(1-\epsilon)b(N)p^2) \\
&\leq \exp(|p|(t - b|p|))
\end{aligned}$$

where $b = \frac{1}{2}(1-\epsilon)b(N)$. Recall that $1/k = O(\sqrt{b(N)/|\log b(N)|}) = o(\sqrt{b})$, so

$$t = \frac{2}{o(k\sqrt{b})} = o(b/\sqrt{b}) = o(bk) \leq \epsilon b|p|,$$

if r is large enough. Thus

$$\frac{|a_n|\rho^n}{|a_N|\rho^N} \leq \exp(-(1-\epsilon)bkp)$$

and thus by the geometric formula

$$\begin{aligned}
\sum_{k\leq|p|\leq\eta N}\frac{|a_n|\rho^n}{|a_N|\rho^N} \leq \sum_{p=k}^{\infty}\exp(-(1-\epsilon)bp^2/2) \\
= O(\frac{1}{bk}\exp(-bk^2/2)) \\
= O(\frac{1}{\sqrt{b}}\exp(-\frac{1}{2}(1-\epsilon)^2 b(N)\gamma b(N)^{-1}|\log b(N)|)) \\
= O(b(N)^{-1/2}\exp(-\frac{\gamma}{2}(1-\epsilon)^2|\log b(N)|)) \\
= O(b(N)^{-\sigma})
\end{aligned}$$

$\square$

Most of the hard work is finished now. All that remains is a simple lemma about polynomials and then we can state and prove the main consequence of the Wiman-Valiron method that we will need for dynamics.

LEMMA A.11.4. *Suppose P is polynomial of degree m and $|P| \le M$ on $\mathbb{D}_r$. Then*

$$|P'(z)| \le eMn|z|^{m-1}r^{-m}.$$

Moreover, if $|P(z)| = M$, then

$$\frac{1}{2}|P(z)| \le |P(w)| \le \frac{3}{2}|P(z)|,$$

for $|w - z| < r/8m$.

PROOF. Since $P(z)/z^m$ is holomorphic on $\overline{\mathbb{D}}_r^c$ with a removable singularity at ∞, the maximum principle implies $p(Z) \le M|z|^m$. The Cauchy estimate on a circle of radius $t = |z|/m$ around z gives

$$|P'(z)| \le \frac{m}{|z|} \max_{|w-z|=t} |P(z)| \le \frac{m}{|z|} Mr^{-m}|z|^m(1 + \frac{1}{m})^m \le emMr^{-m}|z|^{m-1}.$$

If $|z| = r$ and $|z - w| \le r/8m$, then on the line segment between z and w we have

$$|P'| \le emMr^{-m}(r(1 + \frac{1}{8m}))^{m-1} \le \frac{4mM}{r}$$

so

$$|P(z) - P(w)| \le |z - w|\frac{4mM}{r} \le \frac{1}{2}M = \frac{1}{2}|P(z)|,$$

which implies the desired estimate. □

THEOREM A.11.5. *Suppose f is entire, $r > 0$, $N = N(r, f)$, $|z| = r$, and $|f(z)| = M(r, f)$. Suppose that b and k satisfy the conditions of Lemma A.11.3. Then there is a set E of finite logarithmic measure so that for $r \notin E$,*

$$f(w) = f(z)(\frac{w}{z})^N(1 + O(\delta) + o(1))),$$

whenever $|z - w| \le \delta|z|/k$ and $r \nearrow \infty$.

PROOF. Taking $\sigma = 4$ in Lemma A.11.3 we get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{N-k}^{N+k} a_n z_n + o(|a_N||z|^N b(N)^4),$$

for all large r outside a set of finite logarithmic measure. Using Lemma A.11.1 and our assumption $b(N) = O(1/N)$,

$$f(z) = \sum_{N-k}^{N+k} a_n z_n + o(|a_N||z|^N N^{-4}) = z^{N-k}P(z) + o(M(|z|, f)N^{-4}),$$

where P is a polynomial of degree $\leq 2k$. On the circle of radius r,

$$r^{N-k}|P(z)| = |f(z) - o(M(^f)N^{-4})| \leq |f(z)| + o(M(r, f)) = (1 + o(1))M(r, f).$$

Fix an $\epsilon > 0$ and set $M = (1 + \epsilon)r^{k-N}N(r, f)$. Then if r is large enough, we have $|P| \leq M$ on the circle of radius r. Suppose z is the point on this circle where $|f|$ attains its maximum, $m(r, f)$. Then

$$r^{N-k}|P(z)| = |f(z)| - o(M(r, f)) \geq (1 - o(1))M(r, f),$$

so

$$|P(z)| \geq (1 - o(1)M_0 \geq \frac{1 - \epsilon}{1 + \epsilon}M_0 \geq (1 - 3\epsilon)M_0,$$

if r is large enough (and not in E) and ϵ is small enough. Thus by Lemma A.11.4

$$\frac{1}{2}|P(z)| \leq |P(w)| \leq \frac{3}{2}|P(z)|,$$

for $|z - w| < (1 - 3\epsilon)r/16k$. For these same w, we have for $\rho = |w|$

$$\begin{aligned}
w^{-N}f(w) &= w^{-k}P(w) + o(|a_N|N^{-4}) \\
&= w^{-k}P(w) + o(r^{-N}\mu(r, f)N^{-4}) \\
&= w^{-k}P(w) + o(r^{-N}M(r, f)N^{-4}) \\
&= w^{-k}P(w) + o(r^{-k}MN^{-4}) \\
&= w^{-k}P(w) + o(\rho^{-k}MN^{-4}) \\
&= w^{-k}P(w) + o(\rho^{-k}P(z))
\end{aligned}$$

In the last line we replaced r^k by ρ^k. This is justified since $|r - \rho| \leq \frac{r}{k}$ and hence

$$(1 - \frac{1}{k})^k \leq (\frac{r}{\rho})^k \leq (1 + \frac{1}{k})^k,$$

which is uniformly bounded above and below. Hence

$$f(w) = w^{N-k}P(w) + o(\rho^{N-k}|P(z)|).$$

Setting $w = z$ gives

$$f(z) = z^{N-k}P(z) + o(r^{N-k}P(z))$$

or

$$1 = \frac{f(z)}{z^{N-k}P(z)}(1+o(1)),$$

and multiplying equations gives

$$\begin{aligned} f(w) &= [f(z)(\frac{w}{z})^{N-k}\frac{P(w)}{P(z)} + o(|\frac{\rho}{r}|^{N-k}||f(z)|)](1+o(1)) \\ &= f(z)(\frac{w}{z})^{N-k}(1+O(\frac{|P(w)-P(z)|}{|P(z)|})(1+o(1) \\ &= f(z)(\frac{w}{z})^{N-k}(1+O(2k|z-w|)+o(1) \\ ) &= f(z)(\frac{w}{z})^{N-k}(1+O(\delta)+o(1)). \end{aligned}$$

□

A.12. Extremal length, symmetry and Koebe's $\frac{1}{4}$-theorem

Here we given another proof of Koebe' theorem using extremal length.

If γ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let $\gamma^+ = (\gamma \cap \mathbb{H}) \cup \overline{\gamma \cap \mathbb{L}}$, where $\mathbb{H}, \mathbb{L}$ denote the upper and lower half-planes. If Γ is a path family in the plane then $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma^+ = \{\gamma^+ : \gamma \in \Gamma\}$.

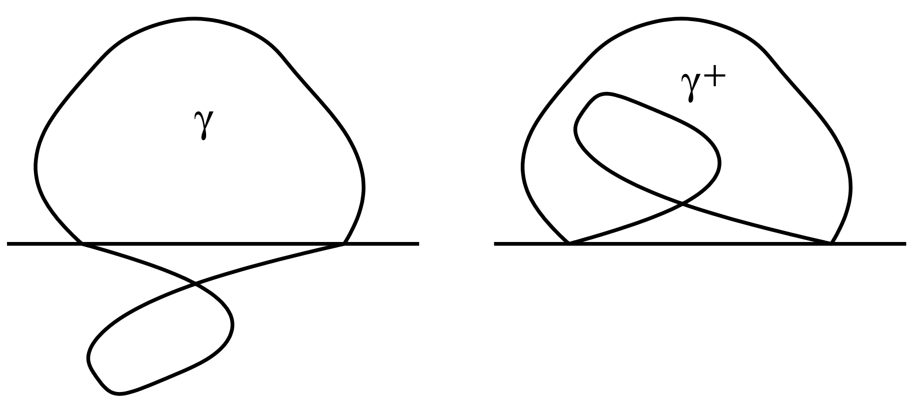


FIGURE 3. The curves γ and γ^+

LEMMA A.12.1. *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma^+)$.

PROOF. We start by proving $M(\Gamma) \leq 2M(\Gamma^+)$. Given a metric ρ, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$. Then for any $\gamma \in \Gamma$,

$$\int +\gamma^+ \sigma ds \geq \int_{\gamma^+} \rho ds \geq \inf_{\gamma \in \Gamma} \int_\gamma \rho ds.$$

Thus if ρ admissible for Γ^+, then σ is admissible for Γ Thus, since $\max(a, b)^2 \leq a^2+b^2$,

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z)dxdy + \int \rho^2(\bar{z})dxdy \leq 2\int \rho^2(z)dxdy.$$

Taking the infimum over admissible ρ's for Γ^+ makes the right hand side equal to $2M(\Gamma^+)$, proving the claim.

For the other direction, given ρ define $\sigma(z) = \rho(z) + \rho(\bar{z})$ for $z \in \mathbb{H}$ and $\sigma = 0$ if $z \in lhp$. Then

$$\begin{aligned} \int_{\gamma^+} \sigma ds &= \int_{\gamma^+} \rho(z) + \rho(\bar{z}) ds \\ &= \int_{\gamma\cap\mathbb{H}} \rho(z)ds + \int_{\gamma\cap\mathbb{H}} \rho(\bar{z})ds + \int_{\gamma\cap\mathbb{L}} \rho(z) + \int_{\gamma\cap\mathbb{L}} \rho(\bar{z})ds \\ &= \int_\gamma \rho(z)ds + \int_{\bar{\gamma}} \rho(z)ds \\ &\geq 2\inf_r ho \int_\gamma \rho ds. \end{aligned}$$

Thus if ρ is admissible for Γ, $\frac{1}{2}\sigma$ is admissible for Γ^+. Hence, since $(a+b)^2 \leq 2(a^2+b^2)$,

$$\begin{aligned} M(\Gamma^+) &\leq \int (\frac{1}{2}\sigma)^2 dxdy \\ &= \frac{1}{4}\int_{\mathbb{H}} (\rho(z) + \rho(\bar{z}))^2 dxdy \\ &\leq \tfrac{1}{2} \int_{\mathbb{H}} \rho^2(z)dxdy + \int_{\mathbb{H}} \rho^2(\bar{z})dxdy \\ &= \frac{1}{2}\int \rho^2 dxdy. \end{aligned}$$

Taking the infimum over all admissible ρ's for Γ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. $\square$

LEMMA A.12.2. *Let $\mathbb{D}^* = \{z : |z| > 1\}$ and $\Omega_0 = \mathbb{D}^* \setminus [R, \infty)$ for some $R > 1$. Let $\Omega = \mathbb{D}^* \setminus K$, where K is a closed, unbounded, connected set in $\mathbb{D}^*$ which contains the point $\{R\}$. Let Γ_0, Γ denote the path families in these domains with separate the two boundary components. Then $M(\Gamma_0) \leq M(\Gamma)$.*

Proof. We use the symmetry principle we just proved. The family Γ_0 is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma^+) = \frac{1}{2}M(\Gamma_0)$. The family Γ may not be symmetric, but we can replace it by a larger family that is. Let Γ_R be the collection of rectifiable curves in $\mathbb{D}^*\setminus\{R\}$ which have zero winding number around $\{R\}$, but non-zero winding number around 0. Clearly $\Gamma \subset \Gamma_R$ and Γ_R is symmetric so$M(\Gamma) \geq M(\Gamma_R) = 2M(\Gamma_R^+)$. Thus all we have to do is show $M(\Gamma_R^+) = M(\Gamma_0^+)$. We will actually show $\Gamma_R^+ = \Gamma_0^+$. Since $\Gamma_0 \subset \Gamma_R$ is obvious, we need only show $\Gamma_R^+ \subset \Gamma_0^+$.

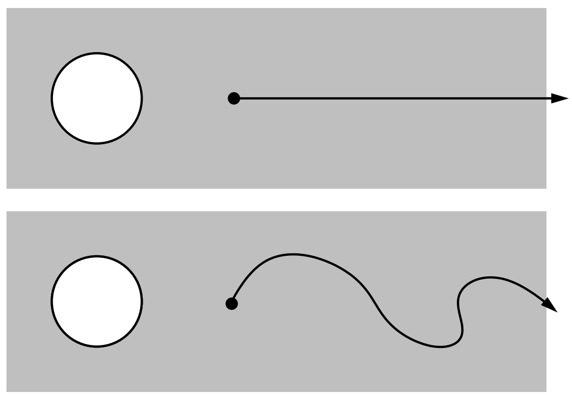


Figure 4. The annulus on top has smaller modulus than any other annulus formed by connecting R to ∞.

Suppose $\gamma \in \Gamma_R$. Since γ has non-zero winding around 0 it must cross both the negative and positive real axes. If it never crossed $(0, R)$ then the winding around 0 and R would be the same, which false, so γ must cross$(0, R)$ as well. Choose points $z_- \in \gamma \cap (-\infty, 0)$ and $z_+ \in \gamma \cap (0, R)$. These points divide γ into two subarcs γ_1 and γ_2.Then $\gamma^+ = \gamma_1^+ \cup \gamma_2^+$. But if we reflect γ_2^+ into the lower half-plane and join it to γ_1^+ it forms a closed curve γ_0 that is in Γ_0 and $\gamma_0^+ = \gamma^+$. Thus $\gamma^+ \in \Gamma_0^+$, as desired. □

Let $\Omega_{\epsilon,R} = \{z : |z| > \epsilon\} \setminus [R, \infty)$. Thus $\Omega_{1,R}$ is the domain considered in the previous lemma. We can estimate the moduli of these domains using the Koebe map

$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \dots,$$

which conformal maps the unit disk to $\mathbb{R}^2 \setminus [\frac{1}{4}, \infty)$ and satisfies $k(0) = 0$, $k'(0) = 1$. Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\epsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius$\frac{\epsilon}{4R}(1 \pm O(\frac{\epsilon}{R})$. Thus the modulus of $\Omega_{\epsilon,R}$ is $2\pi \log \frac{4R}{\epsilon} + O(\frac{\epsilon}{R})$.

LEMMA A.12.3. *Suppose $z, w \in \mathbb{D}$ and K is a compact connected set in $\mathbb{D}$ which contains both these points. Let Γ be the path family that separates K and $\mathbb{T}$. Then the modulus of this family is maximized when K is the hyperbolic geodesic between z and w in which case the modulus is $2\pi \log \frac{4}{\rho}(z, w) + O(\rho(z, w))$, where ρ denotes the hyperbolic distance.*

PROOF. By conformal invariance we may use a Möbius transformation to move z to 0and w onto the positive axis. Applying an inversion, the path family is mapped to one as in Lemma A.12.2, showing that the radial line from z to w maximizes the modulus. The estimate of the modulus follows from our previous remarks. □

We now give an elegant second proof of the Koebe $\frac{1}{4}$-theorem due to Mateljevic [**92**].

THEOREM A.12.4 (The Koebe $\frac{1}{4}$ Theorem). *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then$D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

PROOF. Recall that the modulus of a doubly connected domain is the modulus of the path family that separates the two boundary components (and is equal to the extremal distance between the boundary components). Let $R = \text{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\epsilon,r} = \{z : \epsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\epsilon} = M(A_{\epsilon,1}) = M(f(A_{\epsilon,1}).$$

Let $\delta = \min_{|z|=\epsilon} |f(z)|$.Since $f'(0) = 1$, $\delta = \epsilon + O(\epsilon^2)$. Note that $f(\mathbb{D}) \setminus D(0, \delta) \supset f(A_{\epsilon,1})$, so

$$M(f(\mathbb{D}) \setminus D(0, \epsilon_2)) \geq M(f(A_{\epsilon,1})).$$

By Lemma A.12.2

$$M(f(\mathbb{D}) \setminus D(0, \epsilon_2)) \leq M(\Omega_{\epsilon_2,R}) = 2\pi \log \frac{4R}{\epsilon_2} + O(\frac{\epsilon_2}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\epsilon},$$

or

$$\log 4R - \log(\epsilon + O(\epsilon^2)) + O(\frac{\epsilon}{R}) \geq -\log \epsilon.$$

Taking $\epsilon \to 0$ shows $\log 4R \geq 0$, or $R \geq \frac{1}{4}$. □

A.13. The product formula

LEMMA A.13.1 (Jensen's' formula). *If f is holomorphic on a neighborhood of $\overline{\mathbb{D}}$, non-zero on $\partial\mathbb{D}$ and $\{z_k\}$ are its zeros in $\mathbb{D}$ listed according to multiplicity, then*

$$|f(0)| = \prod_k |z_k| \sup_{\partial\mathbb{D}} |f(z)|.$$

PROOF. If $f(0) = 0$, there is nothing to do, so assume otherwise. Let

$$B(z) = \prod_k \frac{z - z_k}{1 - \overline{z_k} z},$$

be the finite Blaschke product with the same zeros as f. Then $g = f/B$ has no zeros and $|f| = |g|$ on $\partial\mathbb{D}$. So by the maximum principle,

$$|f(0)| = |g(0)| \cdot |B(0)| \leq \max_{\partial\mathbb{D}} |g(z)| \cdot \prod_k |z_k| = \max_{\partial\mathbb{D}} |f(z)| \cdot \prod_k |z_k|.$$

□

We often use this in the following form after taking logarithms

$$\log|f(0)| = \sum_k \log|z_k| + \sup_{\partial\mathbb{D}} \log|f|. \tag{70}$$

THEOREM A.13.2 (Hadamard product formula). *Let f be an entire function of order α and let $a = \lfloor \alpha \rfloor$ be the greatest integer less or equal to α. Let $\{z_k\}$ be the zeros of f listed according to multiplicity. Then*

$$f(z) = \exp(g(z)) z^r \prod_k \left((1 - \frac{z}{z_k}) \exp[\sum_{j=1}^{a} \frac{1}{j} (\frac{z}{z_k})^j] \right),$$

where g is a polynomial of degree at most a. The product converges uniformly on compact subsets of $\mathbb{C}$ and

$$n(r) = \#\{k : |z_k| < r\} = o(n^{\alpha+\epsilon})\},$$

for every $\epsilon > 0$. Conversely, if $\{z_k\}$ is a sequence that satisfies (71) then the product converges uniformly on compact sets to an entire function of order at most α.

PROOF. Using Jensen's formula and the fact that f is order α, for $R \geq 1$ we get

$$\begin{aligned} n(R) &= n(R) \int_R^{eR} \frac{dt}{t} \\ &\leq \int_0^{eR} n(t) \frac{dt}{t} \\ &= \int_0^{eR} \frac{1}{|z_k|} \frac{dt}{t} \\ &= -Rn(R) - \log |f(0)| + \log \max_{|z|=R} |f| \\ &= O(R^{\alpha+\epsilon}), \end{aligned}$$

for every $\epsilon > 0$. Thus for $\beta > \alpha + \epsilon$,

$$\sum_{|z_k|>1} |z_k|^{-\beta} \leq \sum_{j=1}^{\infty} \sum_{2^j < |z_k| \leq 2^{j+1}} 2^{-j\beta} \leq \sum_{j=1}^{\infty} 2^{(j+1)(\alpha+\epsilon)} 2^{-j\beta}$$

converges and hence, taking $\beta = 1$, the product

$$P(z) = z^r \prod_k (1 - \frac{z}{z_k}),$$

converges on compact sets to an entire function with the same zeros (counted with multiplicities) as f. Thus f/P is a non-vanishing entire function and hence is of the form e^g for some entire g. We claim that e^g is also order α. This requires a lower bound on $|P|$; because of the zeros, such a bound can't hold everywhere, but we will show that it holds on "most" circles centered at the origin.

Fix $R > 0$ and let $A = \{z : R \leq |z| \leq 2R\}$. Factor

$$P(z) = P_1(z) P_2(z) P_3(z),$$

with zeros in $\{|z| < R\}$, $\{R < |z| < 4R\}$ and $\{|z| \geq 4R\}$ respectively.

If $z \in A$ and $|z_k| \leq R$ then $|z - z_k| \geq |z_k|$ so $|P_1| \geq 1$ on A. Similarly, $z \in A$, then

$$\log |P_3(z)| \leq \sum_{|z_k| \geq 4R} \log |1 - \frac{z}{|z_k|}| \leq O(|z| \sum_{|z_k| \geq 4R} \frac{1}{|z_k|}| \leq O(R R^{\alpha+\beta-1}) \leq O(R^{\alpha+\beta})$$

Finally, for $|P_2|$ we give an average, rather than uniform bound:

$$\begin{aligned}
\frac{1}{R}\int_R^{2R} \log\max_{|z|=r}|P_3|dr &\leq \frac{1}{R}\int_R^{2R} \sum_{R\leq|z_k|\leq 2R} \log|1-\frac{r}{|z_k|}|dr \\
&\leq \frac{1}{R}\sum_{R\leq|z_k|\leq 2R}\int_R^{2R} \log|1-\frac{r}{|z_k|}|dr \\
&\leq \frac{1}{R}\sum_{R\leq|z_k|\leq 2R}\int_{2R/|z_k|}^{R/|z_k|} \log|1-t||z_k|dt \\
&\leq \frac{1}{R}\cdot O(R^{\alpha+\beta})\cdot 1R\cdot(\int_{1/4}^{2}\log|1-t|dt) \\
&= O(R^{\alpha+\beta})
\end{aligned}$$

Thus there is some $r \in [R, 2R]$ where $\max_{|z|=r}|1/P_2| = O(R^{\alpha+\beta}$. Combined with our previous estimates, this shows e^g is order at most α.

Let $g = u + iv$ be its real and imaginary parts. Since $|e^g| = \Re g$, u is a harmonic function so that $|u(z)| = O(|z|^{\alpha+\epsilon}$ and hence $|\nabla u(z)| = O(|z|^{\alpha+\epsilon-1}$. Since $|\nabla u| = |\nabla v|$, we deduce that $|\nabla v(z)| = O(|z|^{\alpha+\epsilon-1})$ and integrating along radial segments gives $|v(z)| = O(|z|^{\alpha+\epsilon})$. Thus $|g(z)| = O(|z|^{\alpha+\epsilon}$ and the usual Cauchy estimates show g is a polynomial of order at most $a = \lfloor\alpha\rfloor$. Since we assumed $\alpha < 1$, g must be constant. □

COROLLARY A.13.3. *If f is an entire function with order of growth < 1 then*

$$f(z) = Cz^r\prod_k(1-\frac{z}{z_k}),$$

where $\{z_k\}$ are the zeros of f, counted with multiplicity.

EXERCISE: Prove the product formula for the general case $\alpha > 0$. The only new observation is that $\sum_{j=1}^a \frac{1}{j}(\frac{z}{z_k})^j$ are the first a terms of the Taylor expansion of $\log 1 - \frac{z}{z_k}$ and hence

$$|\log(1-\frac{z}{z_k}) + \sum_{j=1}^{a}\frac{1}{j}(\frac{z}{z_k})^j = O(|\frac{z}{z_k}|).$$

A.14. Beurling's $\cos \rho\pi$ theorem

LEMMA A.14.1. *Suppose u is subharmonic on $\mathbb{H}_r$, $u \le 0$ on $\partial\mathbb{H}_r$ and for all $\epsilon > 0$, $u(z) \le \epsilon|z|$ when $|z| = r_n$ for some sequence $r_n \nearrow \infty$. Then $u \le 0$ on $\mathbb{H}_r$.*

The following is a Phragmén-Lindelöf type theorem that we will use later.

THEOREM A.14.2 (Beurling's Theorem). *Suppose f is holomorphic in $\mathbb{H}_r$ and continuous on its closure. Suppose $o < \rho < 1$ and $|f(iy)| \le \phi(|y|)$ where ϕ is unbounded and*

$$\limsup_{r\to\infty} \frac{\log \phi(r)}{r^\rho} = 0.$$

Assume that for every $\epsilon > 0$, and $|f(z)| = o(\exp(-\epsilon|z|))$ on a sequence of circles $\{|z| = r_n\}$ tending to ∞. Then

$$M(r, f) < \frac{\log \phi(r)}{\cos \rho\pi/2},$$

on some sequence $r_n \to \infty$.

PROOF. Choose $a = a(\epsilon)$ so that

$$\log \phi(r) \le \epsilon r^\rho \cos \rho\pi/2 + a(\epsilon),$$

holds for all r with equality at some $r = r(\epsilon)$. Note that $r(\epsilon) \to \infty$ as $\epsilon \searrow 0$ and that $a(\epsilon) > 0$ when ϵ is large enough.

Let $F(z) = f(z)\exp(-\epsilon z^\rho - a)$ where z^ρ is the branch that is positive on $\mathbb{R}^+$ and a will be chosen below. Then

$$\begin{aligned} \log|F(iy)| &\le \log|f(iy)| + \Re(-\epsilon z^\rho - a) \\ &\le \log|\phi(|y|) - a - \epsilon y^\rho \cos \rho\pi/2 \\ &\le 0, \end{aligned}$$

on the imaginary axis and by assumption

$$\log|F(z)| \le \log|f(z)| \le C + \epsilon r/2),$$

on an infinite set of radii tending to ∞, and hence $|F| \le 1$ on $\mathbb{H}_r$ by Lemma A.14.1. For $r = r(\epsilon)$, we have

$$M(r, f) \le \epsilon r^\rho + a(\epsilon),$$

and

$$\log \phi(r) = \epsilon r^\rho \cos \rho\pi/2 + a(\epsilon),$$

so

$$\log \phi(r) \geq (M(r,f) - a(\epsilon)) \cos \rho\pi/2 + a(\epsilon) = M(r,f) \cos \rho\pi/2 + a(\epsilon)(1 - \cos \rho\pi/2).$$

Since $a(\epsilon) > 0$ and $\rho < 1$, the final term is positive and we get the desired inequality. $\square$

Suppose f is an entire function with order of growth < 1 with product expansion

$$f(z) = Cz^r \prod_k (1 - \frac{z}{z_k}),$$

and define a new function

$$g(z) = Cz^r \prod_k (1 + \frac{z}{|z_k|}).$$

Since $|z_k| = O(R^{1-\epsilon})$ for some $\epsilon > 0$, we see that this product converges uniformly on compact sets and that g has the same order as f. Moreover, for $r > 0$,

$$m(r,g) = |g(-r)| \leq m(r,f) \leq M(r,f) \leq g(r) = M(r,g).$$

Thus various questions about $m(r,f)$ and $M(r,f)$ for general functions of order < 1 can be answered by looking at the special case when the zeros all lie on the negative real axis. One result where this works is:

THEOREM A.14.3. *If f has growth at most order $1/2$, minimal type, then*

$$\limsup_{r \to \infty} m(r,f) = \infty.$$

PROOF. By our remarks above, it suffices to prove this when f attains its minimum modulus on the negative real axis, i.e., $|f(z)| \geq f(-|z|)$ for all z. If $m(r,f)$ is bounded by M for all r, then f is bounded by M on the negative real axis, so

Replace $g(z)$ by $g(z^2)$

Apply Beurling's theorem $\square$

REMARK: Wiman proved in [**142**] that for any ϵ and any non-vanishing entire function f

$$m(r) > M(r)^{-1-\epsilon},$$

for some sequence of r's tending to ∞. He conjectured this was true in general and this was verified by Beurling [**25**] in the special case $|f(r)| = m(r)$ (i.e., the minimal values are attained along $\mathbb{R}^+$), but was disproved by Hayman in general [**64**]. Later we will show how to construct an entire functions so that $m(r, f) < M(r, f)^{-C}$ for every C and r large enough.

Bibliography

[1] Stephen Agard. Distortion theorems for quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 413:12, 1968.

[2] Lars V. Ahlfors. *Complex analysis*. McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[3] Lars V. Ahlfors. *Lectures on quasiconformal mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 2006. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard.

[4] Lars V. Ahlfors and Leo Sario. *Riemann surfaces*. Princeton Mathematical Series, No. 26. Princeton University Press, Princeton, N.J., 1960.

[5] Glen D. Anderson, Toshiyuki Sugawa, Mavina K. Vamanamurthy, and Matti Vuorinen. Hypergeometric functions and hyperbolic metric. *Comput. Methods Funct. Theory*, 9(1):269–284, 2009.

[6] I. N. Baker. The iteration of polynomials and transcendental entire functions. *J. Austral. Math. Soc. Ser. A*, 30(4):483–495, 1980/81.

[7] I. N. Baker. Wandering domains in the iteration of entire functions. *Proc. London Math. Soc. (3)*, 49(3):563–576, 1984.

[8] I. N. Baker. Wandering domains for maps of the punctured plane. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 12(2):191–198, 1987.

[9] I. N. Baker and P. Domínguez. Analytic self-maps of the punctured plane. *Complex Variables Theory Appl.*, 37(1-4):67–91, 1998.

[10] I. N. Baker and P. Domínguez. Boundaries of unbounded Fatou components of entire functions. *Ann. Acad. Sci. Fenn. Math.*, 24(2):437–464, 1999.

[11] Zoltan Balogh and Mario Bonk. Lengths of radii under conformal maps of the unit disc. *Proc. Amer. Math. Soc.*, 127(3):801–804, 1999.

[12] Zoltán M. Balogh and Stephen M. Buckley. Geometric characterizations of Gromov hyperbolicity. *Invent. Math.*, 153(2):261–301, 2003.

[13] Detlef Bargmann. Simple proofs of some fundamental properties of the Julia set. *Ergodic Theory Dynam. Systems*, 19(3):553–558, 1999.

[14] W. Bergweiler. The size of Wiman-Valiron discs. *Complex Var. Elliptic Equ.*, 56(1-4):13–33, 2011.

[15] Walter Bergweiler. On the Julia set of analytic self-maps of the punctured plane. *Analysis*, 15(3):251–256, 1995.

[16] Walter Bergweiler. Bloch's principle. *Comput. Methods Funct. Theory*, 6(1):77–108, 2006.

[17] Walter Bergweiler and A. Hinkkanen. On semiconjugation of entire functions. *Math. Proc. Cambridge Philos. Soc.*, 126(3):565–574, 1999.

[18] Walter Bergweiler, Philip Rippon, and Gwyneth Stallard. Multiply connected wandering domains of entire functions. *Proc. London Math. Soc.*, 107:1261–1301, 2013.

[19] Walter Bergweiler, Philip J. Rippon, and Gwyneth M. Stallard. Dynamics of meromorphic functions with direct or logarithmic singularities. *Proc. Lond. Math. Soc. (3)*, 97(2):368–400, 2008.

[20] Walter Bergweiler and Steffen Rohde. Omitted values in domains of normality. *Proc. Amer. Math. Soc.*, 123(6):1857–1858, 1995.

[21] L. Bers. The moduli of Kleinian groups. *Uspehi Mat. Nauk*, 29(2(176)):86–102, 1974. Translated from the English by M. E. Novodvorskiĭ, Collection of articles dedicated to the memory of Ivan Georgievic Petrovskiĭ (1901–1973), I.

[22] Daniel Bertilsson. *On Brennan's conjecture in conformal mapping*. Royal Institute of Technology, Stockholm, 1999. Dissertation, Royal Institute of Technology, Stockholm, 1999.

[23] Dimitrios Betsakos. Estimation of the hyperbolic metric by using the punctured plane. *Math. Z.*, 259(1):187–196, 2008.

[24] A. Beurling and L. Ahlfors. The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96:125–142, 1956.

[25] Arne Beurling. Some theorems on boundedness of analytic functions. *Duke Math. J.*, 16:355–359, 1949.

[26] P. Bhattacharyya. An extension of a theorem of Baker. *Publ. Math. Debrecen*, 27(3-4):273–277, 1980.

[27] Christopher J. Bishop. A transcendental Julia set of dimension 1. preprint, 2015.

[28] Christopher J. Bishop. Some homeomorphisms of the sphere conformal off a curve. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 19(2):323–338, 1994.

[29] Christopher J. Bishop. Conformal welding and Koebe's theorem. *Ann. of Math. (2)*, 166(3):613–656, 2007.

[30] Christopher J. Bishop. Bounds for the CRDT conformal mapping algorithm. *Comput. Methods Funct. Theory*, 10(1):325–366, 2010.

[31] Christopher J. Bishop. Constructing entire functions by quasiconformal folding. *Acta Math.*, 214(1):1–60, 2015.

[32] A. Bloch. Les fonctions holomorphes and méronorphes dans le cercle-unité. *Gauthiers-Villars*, 1926. Paris.

[33] A. Bolsch. *Iteration of meromorphic functions with countably many essential singularities.* PhD thesis, 1997. Zbl 901.3002.

[34] A. Bolsch. Periodic Fatou components of meromorphic functions. *Bull. London Math. Soc.*, 31(5):543–555, 1999.

[35] Mario Bonk, Pekka Koskela, and Steffen Rohde. Conformal metrics on the unit ball in Euclidean space. *Proc. London Math. Soc. (3)*, 77(3):635–664, 1998.

[36] Melkana A. Brakalova and James A. Jenkins. On solutions of the Beltrami equation. *J. Anal. Math.*, 76:67–92, 1998.

[37] James E. Brennan. The integrability of the derivative in conformal mapping. *J. London Math. Soc. (2)*, 18(2):261–272, 1978.

[38] Xavier Buff and Arnaud Chéritat. Quadratic Julia sets with positive area. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1701–1713, New Delhi, 2010. Hindustan Book Agency.

[39] Xavier Buff and Arnaud Chéritat. Quadratic Julia sets with positive area. *Ann. of Math. (2)*, 176(2):673–746, 2012. ArKiv prerprint at arxiv.org/abs/math/0605514.

[40] Lennart Carleson. *Selected problems on exceptional sets*. Van Nostrand Mathematical Studies, No. 13. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[41] Guy David. Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 13(1):25–70, 1988.

[42] P. Domínguez. Dynamics of transcendental meromorphic functions. *Ann. Acad. Sci. Fenn. Math.*, 23(1):225–250, 1998.

[43] Adrien Douady and Clifford J. Earle. Conformally natural extension of homeomorphisms of the circle. *Acta Math.*, 157(1-2):23–48, 1986.

[44] Adrien Douady and John Hamal Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. École Norm. Sup. (4)*, 18(2):287–343, 1985.

[45] Gerald A. Edgar, editor. *Classics on fractals*. Studies in Nonlinearity. Westview Press. Advanced Book Program, Boulder, CO, 2004.

[46] D. Th. Egoroff. Sur les suites de fonctions mesurables. *C. R. Acad. Sci., Paris*, 152:244–246, 1911.

[47] A. È. Erëmenko. On the iteration of entire functions. In *Dynamical systems and ergodic theory (Warsaw, 1986)*, volume 23 of *Banach Center Publ.*, pages 339–345. PWN, Warsaw, 1989.

[48] A. È. Erëmenko and M. Ju. Ljubich. Examples of entire functions with pathological dynamics. *J. London Math. Soc. (2)*, 36(3):458–468, 1987.

[49] A. È. Erëmenko and M. Yu. Lyubich. Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier (Grenoble)*, 42(4):989–1020, 1992.

[50] Kenneth Falconer. *Fractal geometry*. John Wiley & Sons Ltd., Chichester, 1990. Mathematical foundations and applications.

[51] Li Ping Fang. Complex dynamical systems on $\mathbf{C}^*$. *Acta Math. Sinica*, 34(5):611–621, 1991.

[52] Li Ping Fang. Area of Julia sets of holomorphic self-maps on $\mathbf{C}^*$. *Acta Math. Sinica (N.S.)*, 9(2):160–165, 1993. A Chinese summary appears in Acta Math. Sinica **37** (1994), no. 3, 431.

[53] Liping Fang. Completely invariant domains of holomorphic self-maps on $\mathbf{C}^*$. *J. Beijing Inst. Tech.*, 6(3):187–191, 1997.

[54] Liping Fang. On the iteration of holomorphic self-maps of $\mathbf{C}^*$. *Acta Math. Sinica (N.S.)*, 14(1):139–144, 1998.

[55] P. Fatou. Sur l'itération des fonctions transcendantes Entières. *Acta Math.*, 47(4):337–370, 1926.

[56] Dominique S. Fleischmann. A family of entire functions with Baker domains. *Ergodic Theory Dynam. Systems*, 29(2):495–514, 2009.

[57] Frederick P. Gardiner and Nikola Lakic. Comparing Poincaré densities. *Ann. of Math. (2)*, 154(2):245–267, 2001.

[58] John B. Garnett. *Bounded analytic functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, first edition, 2007.

[59] John B. Garnett and Donald E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008. Reprint of the 2005 original.

[60] F. W. Gehring and W. K. Hayman. An inequality in the theory of conformal mapping. *J. Math. Pures Appl. (9)*, 41:353–361, 1962.

[61] F. W. Gehring and Olli Lehto. On the total differentiability of functions of a complex variable. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 272:9, 1959.

[62] Lisa R. Goldberg and Linda Keen. A finiteness theorem for a dynamical class of entire functions. *Ergodic Theory Dynam. Systems*, 6(2):183–192, 1986.

[63] W. Hayman. Symmetrization in the theory of functions. 1950. Tech. Rep. no. 11, Navy Contract N 6-ori-106 Task order 5 (Stanford University).

[64] W. K. Hayman. The minimum modulus of large integral functions. *Proc. London Math. Soc. (3)*, 2:469–512, 1952.

[65] W. K. Hayman. The local growth of power series: a survey of the Wiman-Valiron method. *Canad. Math. Bull.*, 17(3):317–358, 1974.

[66] Håkan Hedenmalm and Serguei Shimorin. Weighted Bergman spaces and the integral means spectrum of conformal mappings. *Duke Math. J.*, 127(2):341–393, 2005.
[67] Juha Heinonen and Raimo Näkki. Quasiconformal distortion on arcs. *J. Anal. Math.*, 63:19–53, 1994.
[68] Juha Heinonen and Steffen Rohde. The Gehring-Hayman inequality for quasihyperbolic geodesics. *Math. Proc. Cambridge Philos. Soc.*, 114(3):393–405, 1993.
[69] Joachim A. Hempel. The Poincaré metric on the twice punctured plane and the theorems of Landau and Schottky. *J. London Math. Soc. (2)*, 20(3):435–445, 1979.
[70] M. E. Herring. Mapping properties of Fatou components. *Ann. Acad. Sci. Fenn. Math.*, 23(2):263–274, 1998.
[71] David A. Herron. Conformal deformations of uniform Loewner spaces. *Math. Proc. Cambridge Philos. Soc.*, 136(2):325–360, 2004.
[72] Witold Hurewicz and Henry Wallman. *Dimension Theory.* Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941.
[73] Xavier Jarque. On the connectivity of the escaping set for complex exponential Misiurewicz parameters. *Proc. Amer. Math. Soc.*, 139(6):2057–2065, 2011.
[74] Peter W. Jones. On removable sets for Sobolev spaces in the plane. In *Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991)*, volume 42 of *Princeton Math. Ser.*, pages 250–267. Princeton Univ. Press, Princeton, NJ, 1995.
[75] Peter W. Jones and Stanislav K. Smirnov. Removability theorems for Sobolev functions and quasiconformal maps. *Ark. Mat.*, 38(2):263–279, 2000.
[76] R. Kaufman. Fourier-Stieltjes coefficients and continuation of functions. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 9:27–31, 1984.
[77] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
[78] Linda Keen. Dynamics of holomorphic self-maps of $\mathbf{C}^*$. In *Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986)*, volume 10 of *Math. Sci. Res. Inst. Publ.*, pages 9–30. Springer, New York, 1988.
[79] Linda Keen. Topology and growth of a special class of holomorphic self-maps of $\mathbf{C}^*$. *Ergodic Theory Dynam. Systems*, 9(2):321–328, 1989.
[80] Masashi Kisaka and Mitsuhiro Shishikura. On multiply connected wandering domains of entire functions. In *Transcendental dynamics and complex analysis*, volume 348 of *London Math. Soc. Lecture Note Ser.*, pages 217–250. Cambridge Univ. Press, Cambridge, 2008.
[81] B. Knaster and C. Kuratowski. Sure les ensembles conexes. *Fund. Math.*, 2:206–255, 1921.
[82] Pekka Koskela and Päivi Lammi. Gehring-Hayman theorem for conformal deformations. *Comment. Math. Helv.*, 88(1):185–203, 2013.
[83] J. Kotus. Iterated holomorphic maps on the punctured plane. In *Dynamical systems (Sopron, 1985)*, volume 287 of *Lecture Notes in Econom. and Math. Systems*, pages 10–28. Springer, Berlin, 1987.
[84] Janina Kotus. The domains of normality of holomorphic self-maps of $\mathbf{C}^*$. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 15(2):329–340, 1990.
[85] Irwin Kra. On Teichmüller's theorem on the quasi-invariance of cross ratios. *Israel J. Math.*, 30(1-2):152–158, 1978.
[86] Olli Lehto. Distribution of values and singularities of analytic functions. *Ann. Acad. Sci. Fenn. Ser. A.I. no.*, 249/3:16, 1957.
[87] Olli Lehto. Homeomorphisms with a given dilatation. In *Proceedings of the Fifteenth Scandinavian Congress (Oslo, 1968)*, Lecture Notes in Mathematics, Vol. 118, pages 58–73, Berlin, 1970. Springer.

[88] N. G. Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc. (3)*, 51(2):369–384, 1985.

[89] P. M. Makienko. Iterations of analytic functions in $\mathbf{C}^*$. *Dokl. Akad. Nauk SSSR*, 297(1):35–37, 1987.

[90] P. M. Makienko. Herman rings in the theory of iterations of endomorphisms of $\mathbf{C}^*$. *Sibirsk. Mat. Zh.*, 32(3):97–103, 1991.

[91] D. Marshall. The uniformization theorem. preprint.

[92] Miodrag Mateljević. Quasiconformal and quasiregular harmonic analogues of Koebe's theorem and applications. *Ann. Acad. Sci. Fenn. Math.*, 32(2):301–315, 2007.

[93] John C. Mayer. An explosion point for the set of endpoints of the Julia set of $\lambda \exp(z)$. *Ergodic Theory Dynam. Systems*, 10(1):177–183, 1990.

[94] John C. Mayer. An explosion point for the set of endpoints of the Julia set of $\lambda \exp(z)$. *Ergodic Theory Dynam. Systems*, 10(1):177–183, 1990.

[95] Curt McMullen. Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.*, 300(1):329–342, 1987.

[96] Helena Mihaljević-Brandt. Semiconjugacies, pinched Cantor bouquets and hyperbolic orbifolds. *Trans. Amer. Math. Soc.*, 364(8):4053–4083, 2012.

[97] Helena Mihaljević-Brandt and Lasse Rempe-Gillen. Absence of wandering domains for some real entire functions with bounded singular sets. *Math. Ann.*, 357(4):1577–1604, 2013.

[98] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.

[99] Michał Misiurewicz. On iterates of e^z. *Ergodic Theory Dynamical Systems*, 1(1):103–106, 1981.

[100] Shunsuke Morosawa. An example of cyclic Baker domains. *Mem. Fac. Sci. Kochi Univ. Ser. A Math.*, 20:123–126, 1999.

[101] A. N. Mukhamedshin. Mapping of a punctured plane with wandering domains. *Sibirsk. Mat. Zh.*, 32(2):184–187, 214, 1991.

[102] James R. Munkres. *Topology: a first course*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[103] J. H. Neuwirth. Ergodicity of some mappings of the circle and the line. *Israel J. Math.*, 31(3-4):359–367, 1978.

[104] Rolf Nevanlinna. *Eindeutige analytische Funktionen*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd XLVI. Springer-Verlag, Berlin, 1953. 2te Aufl.

[105] Rolf Nevanlinna. *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*. Chelsea Publishing Co., New York, 1974. Reprinting of the 1929 original.

[106] G. Polya. On an integral function of an integral function. *J. London Math. Soc.*, 1:12–15, 1926.

[107] Ch. Pommerenke. On ergodic properties of inner functions. *Math. Ann.*, 256(1):43–50, 1981.

[108] Ch. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[109] Hans Rådström. On the iteration of analytic functions. *Math. Scand.*, 1:85–92, 1953.

[110] Lasse Rempe. Rigidity of escaping dynamics for transcendental entire functions. *Acta Math.*, 203(2):235–267, 2009.

[111] Lasse Rempe. The escaping set of the exponential. *Ergodic Theory Dynam. Systems*, 30(2):595–599, 2010.

[112] Lasse Rempe. Connected escaping sets of exponential maps. *Ann. Acad. Sci. Fenn. Math.*, 36(1):71–80, 2011.

[113] Lasse Rempe. Hyperbolic entire functions with full hyperbolic dimension and approximation by Eremneko-Lyubich functions. 2011. preprint, arXiv:1106.349v2.

[114] Seppo Rickman. Removability theorems for quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 449:8, 1969.

[115] P. J. Rippon and G. M. Stallard. Families of Baker domains. II. *Conform. Geom. Dyn.*, 3:67–78 (electronic), 1999.

[116] P. J. Rippon and G. M. Stallard. On questions of Fatou and Eremenko. *Proc. Amer. Math. Soc.*, 133(4):1119–1126 (electronic), 2005.

[117] P. J. Rippon and G. M. Stallard. Boundaries of escaping Fatou components. *Proc. Amer. Math. Soc.*, 139(8):2807–2820, 2011.

[118] P. J. Rippon and G. M. Stallard. Fast escaping points of entire functions. *Proc. Lond. Math. Soc. (3)*, 105(4):787–820, 2012.

[119] Jean-Pierre Rosay and Walter Rudin. Arakelian's approximation theorem. *Amer. Math. Monthly*, 96(5):432–434, 1989.

[120] Günter Rottenfusser, Johannes Rückert, Lasse Rempe, and Dierk Schleicher. Dynamic rays of bounded-type entire functions. *Ann. of Math. (2)*, 173(1):77–125, 2011.

[121] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill Book Co., New York, third edition, 1976. International Series in Pure and Applied Mathematics.

[122] S.

[123] Xavier Saint Raymond and Claude Tricot. Packing regularity of sets in n-space. *Math. Proc. Cambridge Philos. Soc.*, 103(1):133–145, 1988.

[124] Dragomir Šarić. Bounded earthquakes. *Proc. Amer. Math. Soc.*, 136(3):889–897 (electronic), 2008.

[125] Dierk Schleicher. Dynamics of entire functions. In *Holomorphic dynamical systems*, volume 1998 of *Lecture Notes in Math.*, pages 295–339. Springer, Berlin, 2010.

[126] C. Severini. Sulle successioni di funzioni ortogonali. *Atti Acc. Gioenia (5)*, 3:10, 1910.

[127] Serguei Shimorin. A multiplier estimate of the Schwarzian derivative of univalent functions. *Int. Math. Res. Not.*, (30):1623–1633, 2003.

[128] Mitsuhiro Shishikura. The boundary of the Mandelbrot set has Hausdorff dimension two. *Astérisque*, (222):7, 389–405, 1994. Complex analytic methods in dynamical systems (Rio de Janeiro, 1992).

[129] Mitsuhiro Shishikura. The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. *Ann. of Math. (2)*, 147(2):225–267, 1998.

[130] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of entire functions. III. *Math. Proc. Cambridge Philos. Soc.*, 122(2):223–244, 1997.

[131] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions. *Math. Proc. Cambridge Philos. Soc.*, 127(2):271–288, 1999.

[132] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of entire functions. IV. *J. London Math. Soc. (2)*, 61(2):471–488, 2000.

[133] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in topology*. Dover Publications Inc., Mineola, NY, 1995. Reprint of the second (1978) edition.

[134] Toshiyuki Sugawa and Matti Vuorinen. Some inequalities for the Poincaré metric of plane domains. *Math. Z.*, 250(4):885–906, 2005.

[135] S. James Taylor and Claude Tricot. Packing measure, and its evaluation for a Brownian path. *Trans. Amer. Math. Soc.*, 288(2):679–699, 1985.

[136] William P. Thurston. Earthquakes in two-dimensional hyperbolic geometry. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume 112 of *London Math. Soc. Lecture Note Ser.*, pages 91–112. Cambridge Univ. Press, Cambridge, 1986.

[137] Claude Tricot, Jr. Two definitions of fractional dimension. *Math. Proc. Cambridge Philos. Soc.*, 91(1):57–74, 1982.

[138] Pekka Tukia. Compactness properties of μ-homeomorphisms. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 16(1):47–69, 1991.

[139] Jussi Väisälä. Quasihyperbolic geodesics in convex domains. *Results Math.*, 48(1-2):184–195, 2005.

[140] G. Valiron. Sur les theoremes des mm. bloch, landau, montel et schottky. *C. R. Acad. Sci. Paris*, 183:728–730, 1926.

[141] Allen Weitsman. Symmetrization and the Poincaré metric. *Ann. of Math. (2)*, 124(1):159–169, 1986.

[142] A. Wiman. Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Funktion und dem grössten Betrage bei gegebenem Argumente der Funktion. *Acta Math.*, 41(1):1–28, 1916.

[143] Jian-Hua Zheng. On multiply-connected Fatou components in iteration of meromorphic functions. *J. Math. Anal. Appl.*, 313(1):24–37, 2006.