Introduction to Transcendental Dynamics

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Intentions

Despite many previous opportunities and a brush with conformal dynamics in the form of Kleinian groups, I had never worked on the iteration of holomorphic functions until April 2011 when I met Misha Lyubich on the stairwell of the Stony Brook math building he told me that Alex Eremenko had a question for me. Alex was visiting for a few days and his question was the following: given any compact, connected, planar set K and an $\epsilon > 0$, is there a polynomial $p(z)$ with only two critical values, whose critical points approximate K to within ϵ in the Hausdorff metric?

It turns out that we can assume the critical values are ±1 and then $T = p^{-1}([-1, 1])$ is a finite tree in the plane and Alex's question is equivalent to asking if such trees are dense in all compact sets. Such a tree is conformally balanced in the sense that every edge gets equal harmonic measure from ∞ and every subset of an edge gets equal harmonic measure from either side. This makes the existence of such trees seem unlikely, or at least requiring very special geometry, but I was able to show that Alex's conjecture was correct by building an approximating tree that is balanced in a quasiconformal sense and then "fixing" it with the measurable Riemann mapping theorem.

When I asked for the motivation behind the questionm, Alex explained that entire functions with a finite number of critical points play an important role in transcendental dynamics because they mimic certain properties of polynomials, e.g., Dennis Sullivan's "no wandering domains" theorem can be extended to such functions. He wanted to know if the geometry of polynomials of high degree but with a low number of critical points was somehow "special". At least in the case of this particular question, the answer is no.

This discussion led to an analogous question for entire functions. If f is entire and has only two critical points, say ±1, then $T = f^{-1}([-1, 1])$ is an unbounded tree in the plane. Conversely, given an unbounded tree in the plane can we approximate it by a tree of this form? We need to impose a few conditions, but the basic answer is yes, any "reasonable" tree T can be approximated. The "reasonable" conditions include that the tree has uniformly bounded degrees, the edges are uniformly C^2 arcs, adjacent edges have comparable Euclidean lengths and one more technical looking condition. Since T is connected and unbounded, its complementary components are simply connected and can be conformally mapped to a half-plane with ∞ mapping to infinity. Pulling back Lebesgue measure on the boundary of the half-plane to T defines a conformal length for each side of each edge. We require that each side of each edge in the tree has conformal length that is uniformly bounded away from zero. Given this, we can approximate the tree in a precise sense by a tree T' of the form $f^{-1}([-1, 1])$ where f is entire and has exactly two critical values.

With this construction in hand, one can create a large number of new entire functions with finite singular set $S(f)$ (critical values and finite asymptotic values). This collection is called the Speiser class. With a small variation we can also build functions in the larger Eremenko-Lyubich class (bounded singular set). One of the first applications was to build an entire function in the Eremenko-Lyubich that has a wandering domain; as noted above, this is impossible in the Speiser class, but was previously known for more general entire functions. Another application of the ideas (though not of the precise method) led to the construction of an entire function whose Julia set has Hausdorff dimension 1. Noel Baker had proven in 1975 [] that the Julia set of an entire function always contains an non-trivial continuum, so its dimension is always ≥ 1, and Gwyneth Stallard [] had given examples with dimension $1 + \epsilon$ for any $\epsilon > 0$. Furthermore, the new example has packing dimension 1 and is the first entire function known to have packing dimension < 2. At this writing, no known examples have packing dimension strictly between 1 and 2 (but I hope this will change shortly).

In the course of thinking about my examples, I looked at many papers in the field and benefited greatly from a number of well written surveys, especially by Bergweiler and Schleicher. However, I felt that I needed to learn some of the basic theory in more detail and the standard way to do this quickly is to offer to each a graduate course on the topic, which I volunteered to due during Spring 2013. These notes are my attempt to record and organize some of the topics I hope to cover.

First, there are topics from complex analysis that are not usually covered in the standard first year graduate course, such as the hyperbolic and spherical metrics, the uniformization theorem, Koebe's $\frac{1}{4}$-theorem, normal families, the Ahlfors 5 islands theorem, Arakelian's approximation theorem (or is it Arakeljan?), Hausdorff and packing dimensions, and Wiman-Valiron theory describing the behavior of holomorphic function near a boundary point of maximum modulus. We will also need a certain amount of quasiconformal theory, especially the measurable Riemann mapping theorem, and some extensions of to more general homeomorphisms, but at this stage I am only planning to quote the results, since developing the theory seems too time consuming for a one semester course.

Second, there are the topics from transcendental dynamics that are the real goal of the course. I will interpret "transcendental dynamics" as the iteration of entire functions. One can also develop parallel theories for holomorphic functions mapping $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ to itself, for meromorphic functions on the plane, or even for maps between domains on a Riemann surface, but I will limit myself to entire functions (except in a few cases If f is entire let $\{f^\}$ denote the iterates of f. The basic objects of study are the Fatou set

$$\mathcal{F}(f) = \{z : \{f^\} \text{ is a normal family on a neighborhood of } z\},$$

its complement the Julia set, $\mathcal{J}(f)$. However, the escaping set

$$I(f) = \{z : f^n(z) \to \infty\},$$

plays a crucial role in the theory and we will spend a great deal of effort trying to understand its geometry. Indeed, the first "dynamical" result we prove (after some background material) is that $I(f) \neq \emptyset$, a fundamental result of Eremenko. Some of the ogther results we will cover include:

- The Julia set is non-empty, the closure of the repelling fixed points and the boundary of the escaping set. It is either the whole plane or nowhere dense.
- The Fatou set has no unbounded multiply connected components. Thus the Julia set can't be totally disconnected and must have dimension ≥ 1.
- Multiply connected components of the Fatou set must be wandering domains. Examples exists with all connectivities $2, 3, \ldots, \infty$.

• Properties of various subsets of the escaping set defined in terms of rates of escape, especially the "fast escaping set".

• Special results for the Speiser and Eremenko-Lyubich classes, e.g., the Julia set contains the escaping set, the Hausdorff dimension of the Julia set is strictly bigger than 1, but the escaping set can have dimension 1. The Julia set has packing dimension 2. How to construct such functions with prescribed geometry near infinity.

• The Julia set of $\exp(z)$ is the whole plane. Examples where the Julia set has positive area, but is not the whole plane, where it has zero area but Hausdorff dimension 2, examples of dimension < 2.

• Periodic components of the Fatou set of an entire functions inclue all the possibilities for polynomial plus one other. There can be Baker domains, i.e., unbounded periodic domains that iterate to ∞, a sort of analog to a the petal associated to a rationally neutral fixed point. We will give examples, estimate the escape rate in such a domain and show that they force the presence of nearby singular points (although the Baker domain itself need not contain any singular points).

• Progress on Eremenko's question: is every component of the escaping set is unbounded? Among other partial results we know this is false for path components, true for the closure of the escaping set, and there is always at least one unbounded component.

• Progress on Baker's conjecture: an entire function that grows slower than $\exp(\sqrt{z})$ has only bounded Fatou components. This is known to be true under various extra conditions such as very slow growth or some regularity on growth (e.g., the maximum modulus on the circle of radius r is "nice" as a function of r).

Well, this should be enough to get started. In general, I am most interested in topics where the behavior for entire functions contrasts with that for polynomials. For example, the "simplest" examples of transcendental Julia sets have dimension 2; it requires work to build one of dimension < 2 and more work to reach the minimum value 1. For polynomials, the situation is reversed; it is easy to build Julia sets that are small Cantor sets but required sophisticated methods to construct examples with dimension 2 [**?**], [**?**] or positive area [**?**], [**?**].

The prerequisite for reading these notes is to be Lars Ahlfors, or failing this, to have read Ahlfors' books "Complex Variables" and "Lectures on Quasiconformal Mappings", or at the very least, to believe results quoted from these books.

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Contents

CHAPTER 1

The hyperbolic metric

1. Schwarz's lemma

The hyperbolic metric on $\mathbb{D}$ is given by $d\rho(z) = 2|dz|/(1-|z|^2)$. This means that the hyperbolic length of a rectifiable curve γ in $\mathbb{D}$ is defined as

$$\ell_\rho(\gamma) = \int_\gamma \frac{2|dz|}{1-|z|^2},$$

and the hyperbolic distance between two points $z, w \in \mathbb{D}$ is the infimum of the lengths of paths connecting them (we shall see shortly that there is an explicit formula for this distance in terms of z and w).

We define the hyperbolic gradient of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ as

$$|\nabla_H^H f(z)| = |f'(z)| \frac{1-|z|^2}{1-|f^2(z)|^2}.$$

More generally, given a map f between metric spaces (X, d) and (Y, ρ) we define the gradient at a point z as

$$|\nabla_d^\rho f(z)| = \limsup_{x \to z} \frac{\rho(f(z), f(x))}{d(x, z)}.$$

In these notes, the most common metrics we will use are the usual Euclidean metric on $\mathbb{C}$, the spherical metric

$$\frac{ds}{1+|z|^2},$$

on the Riemann Sphere, $\widehat{\mathbb{C}}$ and the hyperbolic metric on the disk or on some other hyperbolic planar domain (these will be defined in Section **??**). To simplify (?) notation, we use E, S and H to denote whether we are taking a gradient with respect to Euclidean, Spherical or Hyperbolic metrics. For example if $f : U \to V$, the symbol $\nabla_H^H f$ means that we are taking a gradient from the hyperbolic metric on U to the hyperbolic metric on V (assuming the domains are clear from context; otherwise we write ∇_U^V or $\nabla_{\rho_U}^{\rho_v}$ it we need to be very precise).

In this notation, the spherical derivative of a function, usually denoted

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2},$$

is be written $|\nabla_E^S f(z)|$ since it is limit of quotients where the numerator is measured in the spherical metric and the denominator is measured in the Euclidean metric. Similarly ∇_H^S denotes a gradient measuring expansion from a hyperbolic to the spherical metric. This particular gradient will be important in Chapter **??** when we characterize conformally invariant normal families.

EXERCISE: Show that the linear fractional transformations that map $\mathbb{D}$ 1-to-1, onto itself are exactly those of the form $z \to \lambda(z-a)/(1-\bar{a}z)$ where $|a| < 1$ and $|\lambda| = 1$.

LEMMA 1. *Möbius transformations of $\mathbb{D}$ to itself are isometries of the hyperbolic metric.*

PROOF. When f is a Möbius transformation of the disk we have

$$f(z) = \frac{z-a}{1-\bar{a}z},$$

$$f'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

Thus

$$\begin{aligned}
|\nabla_H^H f(z)| &= \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|f(z)|^2} \\
&= \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|\frac{z-a}{1-\bar{a}z}|^2} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2-|z-a|^2} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z)(1-a\bar{z})-(z-a)(\bar{z}-\bar{a})} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z-a\bar{z}+|az|^2)-(|z|^2-a\bar{z}-z\bar{a}+|a|^2)} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1+|az|^2-|z|^2-|a|^2)} \\
&= 1.
\end{aligned}$$

Note that

$$\ell_\rho(f(\gamma)) \leq \int_\gamma |\nabla_H f|(z) \frac{|dz|}{1-|z|^2}.$$

Thus Möbius transformations multiply hyperbolic length by at most one. Since the inverse also has this property, we see that Möbius transformation preserve hyperbolic length. □

The segment $(-1, 1)$ is clearly a geodesic for the hyperbolic metric and since isometries take geodesics to geodesics, we see that geodesics for the hyperbolic metric are circles orthogonal to the boundary.

On the disk it is convenient to define the pseudo-hyperbolic metric

$$T(z, w) = |\frac{z-w}{1-\bar{w}z}|.$$

The hyperbolic metric between two points can then be expressed as

$$\rho(w, z) = \log \frac{1+T(w,z)}{1-T(w,z)}.$$

On the upper half-plane the corresponding function is

$$T(z, w) = |\frac{z-w}{w-\bar{z}}|,$$

and ρ is related as before.

EXERCISE: Show a hyperbolic ball in the disk is also a Euclidean ball, but the hyperbolic and Euclidean centers are different (unless they are both the origin). Compute the Euclidean center and radius of a hyperbolic ball of radius r centered at z in $\mathbb{D}$.

LEMMA 2 (Schwarz's Lemma). *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ then $|f'(0)| \leq 1$ with equality iff f is a rotation. Moreover, $|f(z)| \leq |z|$ for all $|z| < 1$, with equality for $z \neq 0$ iff f is a rotation.*

PROOF. Define $g(z) = f(z)/z$ for $z \neq 0$ and $g(0) = f'(0)$. This is a holomorphic function since if $f(z) = \sum a_n z^n$ then $a_0 = 0$ and so $g(z) = \sum a_n z^{n-1}$ has a convergent power series expansion. Since $\max_{|z|=r} |g(z)| \leq \frac{1}{r} \max_{|z|=r} |f| \leq \frac{1}{r}$. By the maximum principle $|g| \leq \frac{1}{r}$ on $\{|z| < r\}$. Taking $r \nearrow 1$ shows $|g| \leq 1$ on $\mathbb{D}$ and equality anywhere implies g is constant. Thus $|f(z)| \leq |z|$ and $|f'(0)| = |g(0)| \leq 1$ and equality implies f is a rotation. □

In terms of the hyperbolic metric this says that

$$\rho(f(0), f(z)) = \rho(0, f(z)) \leq \mathbb{H}_r(0, z),$$

which shows the hyperbolic distance from 0 to any point is non-increasing. For an arbitrary holomorphic self-map of the disk f and any point $w \in \mathbb{D}$ we can always choose Möbius transformations τ, σ so that $\tau(0) = w$ and $\sigma(f(w)) = 0$, so that $\sigma \circ f \circ \tau(0) = 0$. Since Möbius transformations are hyperbolic isometries, this shows

COROLLARY 3. *If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic then $\rho(f(w), f(z)) \leq \rho(w, z)$.*

Another formulation is

COROLLARY 4. *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic then $|\nabla_H^H f(z)| \leq 1$.*

EXERCISE: Show that the only isometries of the hyperbolic disk are Möbius transformations and their reflections across $\mathbb{R}$.

It is a useful curiosity that the hyperbolic gradient of f can be well defined even if f itself is not well defined. Consider $f(z) = z^{1/2}$ on $\mathbb{D}$. There is a not a well defined branch of f on the whole unit dist, but if $z \neq 0$, and we take a small disk D around z, then $f(z)$ does have two well defined branches in this disk. If we use either branch to compute the hyperbolic gradient at z we get the same answer since we take absolute values in the definition and the branches differ by ± 1.

Since z^2 maps the disk to itself, it strictly contracts the hyperbolic metric; a more explicit computation shows we know that $|\nabla_H^H f|(z) > 1$.

$$\nabla_H(z^2) = |2z| \frac{1 - |z|^2}{1 - |z|^4} = \frac{2|z|}{1 + |z|^2} < 1.$$

Thus $f(z) = \sqrt{z}$ is locally an expansion of the hyperbolic metric and for $z \neq 0$,

$$|\nabla_H f|(z) = |\frac{1}{2\sqrt{z}}| \frac{1 - |z|^2}{1 - |z|} \geq \frac{1 + |z|}{2\sqrt{z}}.$$

For $z = 0$

$$|\nabla_H f|(0) = \limsup_{z \to 0} \frac{\rho(0, \sqrt{z})}{\rho(0, z} = \infty.$$

Similarly, if $\alpha > 0$, then the map $p_\alpha(z) = z^\alpha$ sends $\mathbb{D}$ to $\mathbb{D}$, and satisfies

$$|\nabla_H^H p_\alpha|(z) = \alpha |z|^{\alpha - 1} \frac{1 - |z|^2}{1 - |z|^{2\alpha}} < 1,$$

hence is a hyperbolic contraction if $\alpha > 1$ and an expansion if $\alpha < 1$.

EXERCISE: If $a > 1$ and $0 < r < 1$ show that

$$\alpha r^{a-1}\frac{1-r^2}{1-r^{2a}} < 1, \tag{1}$$

if $a > 1$ and $0 < r < 1$.

LEMMA 5. *[Product rule] If F, G are multivalued maps $\mathbb{D} \to \mathbb{D}$, but $|F|, |G|$ are single valued, then*

$$|\nabla_H^H FG|(z) \leq \max(|\nabla_H^H F|(z), |\nabla_H^H G|(z)) \cdot \frac{|F(z)| + |G(z)|}{1 + |F(z)G(z)|}, \tag{2}$$

and if they are both holomorphic,

$$|\nabla_H^H FG|(z) \leq \frac{|F(z)| + |G(z)|}{1 + |F(z)G(z)|} = T(|F(z)|, -|G(z)|) < 1. \tag{3}$$

PROOF. We copy the usual proof of the product rule to get

$$|\nabla_H^H FG|(z) \quad = \quad |\nabla_H^H F|(z) \cdot \frac{|G(z)|(1-|z|^2)}{1-|F(z)G(z)|^2} + |\nabla_H^H G|(z) \cdot \frac{|F(z)|(1-|z|^2)}{1-|F(z)G(z)|^2}$$

The first claim follows from the equality

$$x\frac{1-y^2}{1-x^2y^2} + y\frac{1-x^2}{1-x^2y^2} = \frac{x+y}{1+xy},$$

which can be proved by simple algebra. The second claim is then immediate from Schwarz's lemma. □

LEMMA 6. *If $F : \mathbb{D} \to \mathbb{D}$ is holomorphic and every zero has order at least m, then $|\nabla_H^H F^{1/m}|(z) \leq 1$.*

PROOF. We may assume that F is holomorphic on a neighborhood of $\overline{\mathbb{D}}$; otherwise we replace $F(z)$ by $F(rz)$ and let $r \nearrow 1$ at the end of the proof. We can write $F = GB$ where $G = H^m$ has a holomorphic m-root and B is a finite Blaschke product

$$B(z) = \prod_k B_k(z) = \prod_k (\frac{z - z_n}{1 - \overline{z_k} z})^{a_k}$$

where $a_k \geq m$ for all k. Then by (2)

$$|\nabla_H^H F^{1/m}|(z) \quad \leq \quad |\nabla_H^H H B^{1/m}|(z) \leq |\nabla_H^H B^{1/m}|(z).$$

Using (3) and induction gives

$$|\nabla_H^H F^{1/m}|(z) \quad \leq \quad \max_k |\nabla_H^H B_k^{1/m}|(z).$$

However, B_k is the composition of two maps: $z \to \frac{z-z-k}{1-\overline{z}_k z}$, and $z \to z^{a_k/m}$ and $z \to z^{1/m}$. The first is an isometry and second is a contraction by (**??**). ☐

Corollary 7 (Schwarz's lemma for multiple roots). *If $F : \mathbb{D} \to \mathbb{D}$ is holomorphic and every zero of F has order at least m, then*

$$|F'(0)|^m \leq m^m |F(0)|^{m-1}.$$

Proof. The stated inequality follows by setting $z = 0$ in

$$|\nabla_H^H F^{1/m}|(z) = \frac{|F(z)|^{\frac{1}{m}-1}|F'(z)|(1-|z|^2)}{m(1-|F(z)|^{2/m})} \leq 1.$$

☐

2. Uniformization for planar domains

let $p : E \to B$ be continuous and surjective. An open set $U \subset B$ is evenly covered if the inverse image $p^{-1}(U)$ can be written as a disjoint union of sets V_α so that p restricted to each V_α is a homeomorphism onto U. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a covering map. A space X is simply connected if it is path connected and if its fundamental group is trivial, i.e., every closed loop in X can be homotoped to a point.

Lemma 8 ([**?**], Lemma 8.4.1). *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Any path $f[0,1] \to B$ beginning at b_0 has a unique lift to a path $\tilde{f}$ in E beginning at e_0.*

Lemma 9 ([**?**], Exercise 8.4.12(a)). *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $f : Y \to B$ be continuous with $f(y_0) = b_0$. If Y is locally path connected and simply connected then f can be lifted uniquely to a continuous map $\tilde{f} : (Y, y_0), \to (E, e_0)$.*

The theory of covering spaces says that every Riemann surface has a universal covering surface that is also a Riemann surface. Koebe's uniformization theorem says that there are only three simply connected Riemann surfaces (up to conformal isomorphism): $\mathbb{D}$, $\mathbb{C}$ and $\widehat{\mathbb{C}}$. Any other Riemann surface (and there are many) is the quotient of one of these by a discrete group of Möbius transformations. An element of such group can't have a fixed point, and this implies that the sphere covers only itself and the plane covers only genus 1 tori and the once punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Every other Riemann surface is the quotient of the disk by a Fuchsian group (i.e., a discrete group of Möbius transformations acting on $\mathbb{D}$).

We will not prove the complete uniformization theorem here, although there are proofs using potential theory that take only about a dozen pages (e.g. see Don Marshall's paper []). However, we will give the proof for hyperbolic planar domains, a case that we will occasion to use throughout these notes. A planar domain Ω is called **hyperbolic** if $\mathbb{C} \setminus \Omega$ has at least two points.

THEOREM 10. *Every hyperbolic plane domain Ω is holomorphically covered by $\mathbb{D}$ (i.e., there is a locally 1-to-1, holomorphic covering map from $\mathbb{D}$ to Ω).*

We will first prove this for bounded domains, then for general simply connected domains, and then finally general hyperbolic domains. The proof will use Montel's theorem that a sequence of uniformly bounded holomoprhic functions on $\mathbb{D}$ has a subsequence that converges uniformly on compact sets. We will also need the following.

LEMMA 11. *If $\{f_n\}$ are holomorphic functions on a domain Ω that converge uniformly on compact sets to f and if $z_n \to z \in \Omega$, then $f_n(z_n) \to f(z)$.*

PROOF. We may ssume $\{z_n\}$ are contained in some disk $D \subset \Omega$ around z. Let $E = \{z_n\} \cup \{z\}$. This is a compact set so it has a positive distance d from $\partial\Omega$. The points within distance $d/2$ of E form a compact set F on which the functions $\{f_n\}$ are uniformly bounded on E, say by M. By the Cauchy estimate the derivatives are bounded by a constant M' on E. Thus

$$|f(z) - f_n(z_n) \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_n)| \leq |f(z) - f_n(z)| + M'|z - z_n|,$$

and both terms on the right tend to zero by hypothesis.

□

PROOF OF THEOREM 10 FOR BOUNDED DOMAINS. If Ω is bounded, then by a translation and rescaling, we may assume $\Omega \subset \mathbb{D}$ and $0 \in \Omega$. We will define a sequence of domains $\{\Omega_n\}$ with $\Omega_0 = \Omega$ and covering maps $p_n : \Omega_n \to \Omega_{n-1}$ such that $p(0) = 0$. We will show that Ω_n contains hyperbolic disks centered at 0 of arbitrarily large radius and that the covering map $q_n = p_1 \circ \cdots \circ p_n : \Omega_n \to \Omega_0 = \Omega$ converges uniformly on compacta to a covering map $q : \mathbb{D} \to \Omega$.

If $\Omega_0 = \mathbb{D}$ we are done, since the identity map will work. In general assume that we have $q_n : \Omega_n \to \Omega_0$ and that there is a point $w \in \mathbb{D} \setminus \Omega_n$. Let τ and σ be Möbius transformations of the disk to itself so that $\tau(w) = 0$, choose a square root α of $\tau(0)$ and choose σ so $\sigma(\alpha) = 0$. Then $p_{n+1}(z) = \sigma(\sqrt{\tau(z)})$ and let Ω_{n+1} be the component of $U = p_{n+1}^{-1}(\Omega_n)$ that contains the origin (the set U will have one or two components; two if w is in a connected component of $\mathbb{D} \setminus \Omega_n$ that is compact in $\mathbb{D}$, and one otherwise). Since σ and τ are hyperbolic isometries and $\sqrt{z}$ expands the hyperbolic metric, we see that Ω_{n+1} contains a larger hyperbolic ball around 0 than Ω_n did. Suppose $\operatorname{dist}(\partial\Omega_n, 0) < r < 1$ for all n. Then (**??**) says that

$$|\nabla_H^H p_n|(0) > \frac{1+r}{2\sqrt{r}} > 1,$$

and hence $|\nabla_H^H q_n|(0)$ increases by this much at every step. But $|\nabla_H^H q_n|(0) \leq 1$, which is a contradiction. Thus $d_n \to 1$.

Thus $\{q_n\}$ is a sequence of uniformly bounded holomorphic functions on the disk. By Montel's theorem (we shall dicuss this in the next chapter), there a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic map $q : \mathbb{D} \to \Omega$. It is non-constant since it has non-zero gradient at the origin; moreover, by Hurwicz's theorem, q' never vanishes on $\mathbb{D}$ since it is the locally uniform limit of the q_n' which never vanish (q_n is a covering map). Next we show that q is a covering map $\mathbb{D} \to \Omega$.

Fix $a \in \Omega$ and let $d = \operatorname{dist}(a, \partial\Omega)$. Since Ω is bounded, this is finite. Let $D = D(a, d) \subset \Omega$. Since q_n is a covering map, every branch of q_n^{-1} is 1-to-1 holomorphic map of D into $\mathbb{D}$ and hence each q_n is a contractio from the hyperbolic metric on D to the hyperbolic metric on $\mathbb{D}$. Thus every preimage of $\frac{1}{2}D$ has uniformly bounded hyperbolic diameter.

Now fix a point $b \in q^{-1}(a)$. Since $q_n(b) \to q(b) = 1$, $q_n(b) \in \frac{1}{2}D$ for n large enough, so there is branch of q_n^{-1} that contains b. Since these branches are uniformly bounded holomorphic functions, by Montel's theorem we can pass to a subsequence so that they converge to a holomorphic function g from $\frac{1}{2}D$ into $\mathbb{D}$. Moreover,

$$q(g(z)) = \lim_n q_n(q_k^{-1}(z)) = z,$$

by Lemma **??**. $\square$

This proves the existence of a covering map for bounded domains Ω. If Ω is bounded and simply connected, then we have proved the Riemann mapping theorem for Ω. To deduce Riemann's theorem for all proper simply connected plane domains, we only need:

LEMMA 12. *Any simply connected planar domain, except for the plane itself, can be conformally mapped to a bounded domain.*

PROOF. If the domain Ω is bounded, there is nothing to do. If Ω. omits a disk $D(x, r)$ then the map $z \to 1/(z - x)$ conformal maps Ω to a bounded domain. Otherwise, translate the domain so that 0 is on the boundary and consider a continuous branch of $\sqrt{z}$. The image is a 1-1, holomorphic image of Ω, but does not contain both a point and its negative. Since the image does contain some open ball, it also omits an open ball and hence can be mapped to a bounded domain by the previous case. $\square$

Next we want to deduce the uniformization theorem for all hyperbolic plane domains (we have only proved it for bounded domains so far). It suffices to show that any hyperbolic plane domain has a covering map from some bounded domain W, for then we can compose the covering maps $\mathbb{D} \to W$ and $W \to \Omega$.

To do this, it suffices to consider the case $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$. $q : \mathbb{D} \to W$ is a covering map and if $\{a, b\} \in \mathbb{C} \setminus \Omega$ then $h(z) = bq(z) + a$ is a covering map from $U = h^{-1}(\Omega) \subset \mathbb{D}$ to Ω. Choose a connected component of U shows that Ω has a covering from a bounded plane domain, finishing the proof.

Thus we are reduced to proving:

THEOREM 13. *There is a holomorphic covering map from $\mathbb{D}$ to $W = \mathbb{C} \setminus \{0, 1\}$*

PROOF. Let

$$\Omega = \{z = x + iy : y > 0, 0 < x < 1, |z - \frac{1}{2}| > \frac{1}{2}\} \subset \mathbb{H}.$$

This is simply connected and hence can be conformally mapped to $\mathbb{H}$ with $0, 1, \infty$ each fixed. We can then use Schwarz reflection to extend the map across the sides of Ω. Every such reflection of Ω stays in $\mathbb{H}$ maps to either the lower or upper half-planes. Continuing this forever gives a covering map from a simply connected subdomain U

of $\mathbb{H}$ to W. Since U is simply connected an not the whole plane (it is a subset of $\mathbb{H}$) it is conformally equivalent to $\mathbb{D}$ and hence a covering $q : \mathbb{D} \to W$ exists. $\square$

EXERCISE: Show that the domain U formed by repeated reflections across $\partial\Omega$ is, in fact, all of $\mathbb{H}$.

EXERCISE: The conformal map $\lambda : \Omega \to \mathbb{H}$ is called the modular functions and has a explicit formula. Here we state give the formula and outline how to prove it is correct. see Section 7.3.4 of [**?**].

There is an explicit formula for the covering map from the upper half-plane to $\mathbb{C} \setminus \{0, 1\}$. in terms of the Weiserstrass P-function. See [].

3. Estimates for hyperbolic metric

If $\Omega \subset \mathbb{C}$ has at least two boundary points then we can define a hyperbolic metric on Ω by $\rho_\Omega(z)ds$ where

$$\rho_\Omega(z) = |p'(w)|\rho_{\mathbb{D}}(w) = \frac{|p'(w)|}{1 - |w|^2}.$$

where $p : \mathbb{D} \to \Omega$ is a holomorhpic covering map and $p(w) = z$. Different choices of p and w give the same value for $\rho_\Omega(z)$ since they differ by an isometry of $\mathbb{D}$. Thus every hyperbolic planar domain has a hyperbolic metric. In this section we want to give some useful estimates for ρ_Ω in terms of more geometric quantitities, such as the quasi-hyperbolic metric, defined as

$$\tilde{\rho}_\Omega(z)ds = \frac{ds}{\operatorname{dist}(z, \partial\Omega}.$$

For simply connected domains, ρ and $\tilde{\rho}$ are boundedly equivalent; for more general domains this can fail, but some useful estimates are still available.

Recall Green's theorem,

$$\iint_\Omega u\Delta v + v\Delta u dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n} ds, \tag{4}$$

where n denotes the inward pointing normal vector of $\partial\Omega$. We will also use Green's theorem in the following form:

$$\int_{\partial\Omega} f(x,y)dx + g(x,y)dy = \iint_\Omega \frac{\partial g}{\partial x} - \frac{\partial f}{\partial g} dxdy \tag{5}$$

and its simple consequence that the area of a region Ω is given by

$$\text{area}(\Omega) = \frac{1}{2}\int_{\partial\Omega} xdy - ydx = \frac{1}{2i}\int \partial\Omega\bar{z}dz. \tag{6}$$

We now come to some well known estimates for univalent mappings. The first step is to prove:

THEOREM 14 (Area theorem). *Suppose* $g(z) = \frac{1}{z} + b_0 + b_1 z + \dots$ *is univalent in* $\mathbb{D}$. *Then* $\sum_{n=0}^{\infty} n|b_n|^2 \leq 1$. *In particular,* $|b_1| \leq 1$.

PROOF. For $0 < r < 1$ let $D_r = \mathbb{C} \setminus g(D(0,r))$. If $z = g(w)$ and $w = e^{i\theta}$ then $dw = iwd\theta$, so by (19),

$$\text{area}(D_r) = \iint_{D_r} dxdy = \frac{1}{2i}\int_{\partial D_r} \bar{z}dz = \frac{-1}{2i}\int_{\partial D(0,r)} \bar{g}(w)g'(w)dw.$$

To evaluate the right hand side note that

$$\begin{aligned} g(z) &= \frac{1}{z} + b_0 + b_1 z + \dots, \\ g'(z) &= 1\frac{1}{z^2} + 0 + b_1 + 2b_2 z + \dots, \end{aligned}$$

so that

$$\begin{aligned} \int_{|w|=r} \bar{g}(w)g'(w)dw &= i\int \bar{g}(w)g'(w)wd\theta \\ &= i\int (\frac{1}{\bar{w}} + \bar{b}_0 + \bar{b}_1\bar{w} + \dots)(-\frac{1}{w} + b_1 w + 2b_2 w + \dots)d\theta \\ &= 2\pi i(-\frac{1}{r^2} + |b_1|^2 r^2 + 2|b_2|r^4 + \dots 0 \end{aligned}$$

Thus,

$$0 \leq \text{area}(D_r) = \pi(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}).$$

Taking $r \to 1$ gives the result. □

COROLLARY 15. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is univalent on the unit disk, then* $|a_2| \leq 2$.

PROOF. Let $g(z) = (f(z^2))^{-1/2} = 1/z - a_2 z/2 + \dots$. We claim g is one-to-one. To see this suppose $g(z) = g(w)$. Then $f(z^2) = f(w^2)$, so $z = \pm w$. Note that g is odd, so $z = w$. Since $b_1 = a_2/2$, the previous result implies $|a_2| \leq 2$. □

THEOREM 16 (Koebe 1/4 theorem). *If f is univalent on $\mathbb{D}$, then*

$$\frac{1}{4}|f'(z)|(1-|z|^2) \leq \mathrm{dist}(f(z), \partial\Omega) \leq |f'(z)|(1-|z|^2).$$

PROOF. By pre-composing with a Möbius transformation and post-composing by a linear map, we may assume $z=0$, $f(0)=0$ and $f'(0)=1$. Then the right hand inequality is just Schwarz's lemma applied to f^{-1}. □

COROLLARY 17. *If f is univalent on $\mathbb{D}$, then*

$$\frac{1}{4}|f'(z)|(1-|z|^2) \leq \mathrm{dist}(f(z), \partial\Omega) \leq |f'(z)|(1-|z|^2).$$

PROOF. By pre-composing with a Möbius transformation and post-composing by a linear map, we may assume $z=0$, $f(0)=0$ and $f'(0)=1$. Then the right hand inequality is just Schwarz's lemma applied to f^{-1}. The right side is the previous result. □

Because of Koebe's theorem we have

COROLLARY 18. *For simply connected domains, the hyperbolic and quasi-hyperbolic metrics are bi-Lipschitz equivalent, i.e.,*

$$d\rho_\Omega \leq d\tilde{\rho}_\Omega \leq 4 d\rho_\Omega. \tag{7}$$

COROLLARY 19. *Suppose Ω is simply connected, $z, w \in \Omega$. Then*

$$\rho(z,w) \geq |\log \frac{\mathrm{dist}(z,\partial\Omega)}{\mathrm{dist}(w,\partial\Omega)}|.$$

PROOF. Suppose γ is a curve in Ω connecting the two points. Then the quasi-hyperbolic length of γ is at least

$$|\int_{\mathrm{dist}(z,\partial\Omega)}^{\mathrm{dist}(w,\partial\Omega} \frac{dt}{t}| = |\log \frac{\mathrm{dist}(z,\partial\Omega}{\mathrm{dist}(w,\partial\Omega)}|.$$

By our previous remarks, the hyperbolic distance is at least $\frac{1}{4}$ of this. □

COROLLARY 20. *If $U \subset V$ are both hyperbolic, then $\rho_U \geq \rho_V$.*

This is immediate from Schwarz's lemma.

COROLLARY 21. *If U is hyperbolic and $V = \mathbb{D}^* \subset U$, then*

$$d\rho_U(z) = O(\frac{ds}{|z|\log|z|},$$

for $|z|>1$. If $\{1\} \in \partial U$, then the two sides are comparable.

Proof. Compare to a hyperbolic metric on the punctured dist V, which can be explicitly computed via the covering map $\exp : \mathbb{H}_r \to V$. This gives the first claim. Then lift U and v into $\mathbb{H}$ by the covering map on $\mathbb{C} \setminus \{0,1\}$. V lifts to a simply connected subdomain $\tilde{V}$ of $\mathbb{H}$ lying above a curve that is 1-periodic and lies below some horizontal line $\{y = C\}$. U lifts to a domain $\tilde{U}$ and we have the containments

$$\mathbb{H} + iC \subset V\tilde{\subset}\tilde{U} \subset \mathbb{H}.$$

Above the line $\{y = 2C\}$ the hyperbolic metrics for $\mathbb{H}$ and $\mathbb{H} + iC$ have ratio $1 - \frac{C}{y}$ which is between 1 and $\frac{1}{2}$. Thus the hyperbolic metrics on $\tilde{V}$ and $\tilde{U}$ have the same ratio bounds, and hence so do the hyperbolic metrics on U and V. $\square$

The twice punctured plane is one of the few domains for which there is an explicit expression for the hyperbolic metric. Assuming the punctures are $0, 1$, the metric is $\rho(z)ds$ where

$$\frac{1}{\rho(z)} = \frac{1}{\pi}\iint_{\mathbb{C}} |\frac{w)(w-1)}{(z(z-1)(z-w)} dudv.$$

See [?], [?]. For $\widehat{\mathbb{C}} \setminus \{a, b, c\}$ it has the more symmetric looking form

$$\frac{1}{\rho(z)} = \frac{1}{\pi}\iint_{\mathbb{C}} |\frac{(w-a))(w-b)(w-c)}{((z-a)(z-b)(z-c)(z-w)} dudv.$$

We will not prove these formaulas since the estimate Corollary 21 will be good enough for our purposes. For example, it suffice to prove:

Lemma 22 (Schottky's Lemma). *Suppose f is a holomorphic function on a hyperbolic domain Ω and that f omits the values $0, 1$. If $K \subset \Omega$ is compact, then there is are constants B, C, depending only on K, so that*

$$\max_K |f(z)| \leq B(1 + \min_K |f(z)|)^C.$$

Proof. f is a holomorphic map from Ω to $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$ and hence is a contraction of the hyperbolic metrics. Since $K \subset \Omega$ is compact, it has finite hyperbolic diameter in Ω and hence $f(K)$ has finite hyperbolic diameter d in $\mathbb{C}^{**}$. We may also assume $f(K)$ is connected (if not, replace it by a covering hyperbolic ball of at most twice the diameter).

Let $r_0 = 1$, $r_1 = 2$ and, in general, $r_n = (r_{n-1})^2 = 2^{2^n}$. By Corollary 21 the hyperbolic distance between these circles is at least

$$\int_{2^{2^n}}^{2^{2^{n+1}}} \frac{dr}{r\log r} \geq \log log 2^{2^{n+1}} - \log log 2^{2^n} \geq (n+1)\log 2 + \log 2 - n\log 2 - \log 2 = \log 2.$$

Thus the circles are a uniformly positive hyperbolic distance apart in $\mathbb{C}^{**}$ and hence $f(K)$ hits at most a bounded number of them and hence is trapped between r_k and r_{k+M} some k (depending on f) and some M (depending only on d). This proves the result with $B = 2 = C = 2^{M+1}$. □

4. Maximum modulus

If f is entire we define its minimum and maximum modulus on the circle of radius r as

$$m(r, f) = \min\{|f(z)| : |z| = r\},$$

$$M(r, f) = \max\{|f(z)| : |z| = r\}.$$

The iterates of M are defined iteratively as

$$M^n(r, f) = M(M^{n-1}(r, f), f),$$

where $M^1(r, f) = M(r, f)$. Note that

$$M(r, f^n) \leq M^n(r, f),$$

and strict inequality is possible, but we shall see later (Theorem **??**) that the two quantities are always close in a certain sense. Also note that if $|z| \leq M^n(rmf)$ then $|f^k(z)| \leq M^{n+k}(r, f)$.

The most important properties of $M(r, f)$ are described in the following results.

LEMMA 23. *For $r > 0$, $M(r, f)$ is an increasing, convex function of* $\log r$.

PROOF. Clearly $M(r, f)$ increases with r by the maximum principle. Since $g(z) = f(e^z)$ is entire, $\log|g(z)|$ is subharmonic on the plane. Hence

$$M(e^x, f) = \sup\{\log|g(x + iy))| :\in \mathbb{R}\},$$

is a subharmonic function of x alone and hence convex (a supremum of subharmonic functions is subharmonic). □

As a consequence of convexity we see that

$$\frac{\log M(r,f) - \log M(1,f)}{\log r}$$

increases with $r \geq 0$ and hence has a limit in $[0,\infty]$ as $r \to \infty$. If f is a polynomial of degree d then it is easy to check the limit is d. Conversely, if the limit is $\leq d$, then for $|z|$ large we have

$$\log M(r,f) \leq \log M(1,f) + d \log r, r \geq 1$$

or

$$|f(z)| \leq M(1,f)|z|^d, |z| \geq 1$$

which implies f is a polynomial of degree $\leq d$ by a well known argument (use the Cauchy estimates to show all derivatives of order $> d$ vanish at the origin). Thus

LEMMA 24. *If f is a transcendental entire function,*

$$\lim_{r\to\infty} \frac{\log M(r,f)}{\log r} = \infty.$$

LEMMA 25. *If f is a transcendental entire function and $\lambda > 1$, then*

$$\lim_{r\to\infty} \frac{M(\lambda r,f)}{M(r,f)} = \infty.$$

PROOF. Since $\log M(r,f)$ is an increasing convex function of $\log r$, $\log(M(\lambda r,f) -$ $\log M(r,f)$ is increasing in r if $\lambda > 1$ is fixed. If this difference is bounded above by $A < \infty$ for all $r \geq 0$, then setting $r_n = \lambda^n$ and summing a telescoping series shows

$$M(r_n,f) \leq (n+1)A = (1+\frac{1}{n})\frac{A}{\log \lambda}\log r_n,$$

which implies f is a polynomial of degree $d \leq A\log\lambda$ by Lemma 24. □

LEMMA 26. *If f is a transcendental entire function and $\lambda > 1$, then there is an $R > 0$ so that for all $r \geq R$ and all $c \geq 1$,*

$$\log M(r^c,f) \geq \lambda(c-1)\log M(r,f).$$

PROOF. Since $\log M(r,f)$ is an increasing, convex function of $\log r$ it is the integral of some positive, increasing function, say

$$\log M(r,f) = \int_{-\infty}^{\log r} a(t)dt.$$

Choose R so that $a(t) \geq \lambda$ if $t \geq \log R$. Then if $r \geq R$,

$$\log M(r^c, f) - \log M(r, f) = \int_{\log r}^{c \log r} a(t)dt \geq \lambda(c-1)\log r.$$

$\square$

Theorem 27. *For any increasing function* $\phi : [0, \infty) \to [0, \infty)$ *there is an entire function* f *so that* $M(r, f) > \phi(r)$ *for all sufficiently large* r.

NOTE: For an entire function f, we let

$$m(r) = \min_{|z|=r} |f(z)|, \qquad M(r) = \max_{|z|=r} |f(z)|.$$

Wiman proved in [?] that for any ϵ and any non-vanishing entire function f

$$m(r) > M(r)^{-1-\epsilon},$$

for some sequence of r's tending to ∞. He conjectured this was true in general and this was verified by Beurling [?] in the special case $|f(r)| = m(r)$ (i.e., the minimal values are attained along $\mathbb{R}^+$), but was disproved by Hayman in general [?]. We can show Hayman's counterexample can be taken in $\mathcal{S}_3$.

Corollary 28. *There are* $A > 0, r_0 < \infty$ *and an entire function* $f \in \mathcal{S}_3$ *so that*

$$m(r) < M(r)^{-A \log\log\log M(r)} \tag{8}$$

for all $r > r_0$. *Hence* $m(r) < M(r)^{-C}$ *for every* C *and* r *large enough.*

Lemma 29 (Bohr's lemma). *If* $0 < r < 1$ *there is a number* $s = s(r) > 0$ *so that whenever* f *is holomorphic on* $\mathbb{D}$ *and* $|f| < 1$ *on* $r\mathbb{D}$, *then* $t\partial\mathbb{D} \subset f(\mathbb{D})$ *for some* $t \geq s$.

Proof. Replacing f by a slight dilation and passing to a limit we may assume f is holomorphic on $\overline{\mathbb{D}}$. Note that $U = f(r\mathbb{D})$ has a single unbounded complementary component W and this component is bounded by piecewise smooth curve. Therefore ∂W contains a point z closest to the origin. Let $s = |z|$ and let $\Omega = \mathbb{C} \setminus W$ (this is the polynomial hull of U). By the Schwarz lemma, the f-image of $r\mathbb{D}$ lies within a ρ-hyperbolic neighborhood of 0. If $w \in \Omega$, then

$$\rho_\Omega(w, 0) \geq \frac{1}{4}\int_0^{|} w|\frac{dt}{s+t} = \frac{1}{4}\log 1 + \frac{|w|}{s},$$

so any point $w \in f(r\mathbb{D})$ satisfies

$$|w| \leq s(e^{4\rho} - 1).$$

Since $f(r\mathbb{D})$ contains a point of modulus 1, we deduce

$$s \geq \frac{1}{e^{4\rho} - 1}.$$

□

Taking $r = \frac{1}{2}$ gives $\rho = \frac{1}{2} \log 1 + r1 - r = \frac{1}{2} \log 3$ and $s = 1/8$.

LEMMA 30 (Polya's lemma []). *Suppose* g, h *are entire and define an entire function* $f = g \circ h$. *Then*

$$M(f, r) \geq M(g, \frac{1}{8} M(h, \frac{1}{2} r).$$

PROOF. Apply Bohr's lemma to $h(rz)/M(h, \frac{1}{2}r)$ to deduce that this function covers a circle centered at the origin of radius $R \geq \frac{1}{8} M(h, \frac{1}{2}r)$. So if w is a point on $R\partial\mathbb{D}$ where $|g|$ takes a maximum value then there is a $z \in r\mathbb{D}$ so that $h(z) = w$ and

$$M(\frac{1}{8} M(\frac{1}{2} h)) \leq M(g, R) = |g(w)| = |g(h(z))| \leq M(f, r).$$

□

EXERCISE Use Polya's lemma to prove: if g and h are entire functions such that $f = g \circ h$ has finite order then either

1. h is polynomial and g has finite order, or
2. h transcendental of finite order and g is zero order.

5. $I(f) \neq \emptyset$

If f is holomoprhic on $\mathbb{C}$,

$$I(f) = \{z : f^n(z) \to \infty\}.$$

Our first step is to show this set is non-empty. This was first proven by Eremenko in [], but here we follow a later proof by Dominguez []. We shall give Eremenko's proof late whenn we study the rate of escape of iterates of f.

THEOREM 31. *If* f *is a transcendental entire function then* $I(f) \neq \emptyset$. *In fact,* $I(f)$ *intersects every circle* $\{|z| = r\}$ *for all sufficiently large* r.

Proof. Let $\gamma_1 = \partial\mathbb{D}_r$ where r is chosen so large that $M(\frac{1}{2}s, f) \geq 16s$ for $s \geq r$. It follows from Bohr's lemma (Lemma **??**) that $f(\mathbb{D}_r)$ contains a circle $\partial\mathbb{D}_s$ for some $s \geq \frac{1}{8}M(\frac{1}{2}r, f) \geq 2r$. Let γ_2 be the boundary of the unbounded component of $\mathbb{C} \setminus f(\mathbb{D}_r)$. It is a subset of $f(\partial\mathbb{D}_r)$, so $K_2 = f^{-1}(\gamma_2)$ is a compact subset of γ_1 and $|f| > 2r$ on K_2.

Now repeat the argument with $\mathbb{D}_r$ replaced by the region D_2 bounded by γ_2. Bohr's lemma implies $f(D_2)$ contains a circle $\mathbb{D}_s$ for some $s \geq \frac{1}{8}M(\frac{1}{2}(2r), f) \geq 4r$. Let γ_3 be the boundary of the unbounded component of $\mathbb{C} \setminus f(D_2)$ and $K_3 = f^{-2}(\gamma_3)$. Thus $K_3 \subset K_2 \subset \gamma_1$ and $|f^2| \geq 4r$ on K_3.

Continuing in this way we build a sequence of nested sets $\gamma_1 \supset K_2 \supset K_3 \supset \dots$ so that $|f^n| \geq 2^{n-1}$ on K_n. Clearly $\cap_n K_n$ is non-empty and contained in $I(f)$. $\square$

The lower bound on the rate of escape we have given is rather slow. Later we will give another proof, due to Eremenko and based on Wiman-Valiron theory, that constructs points that escape as fast as possible.

CHAPTER 2

Normal families

1. Definitions

A family $\mathcal{F}$ of functions from one metric space (X, d) to another (Y, ρ) is called **equicontinuous** if for each $\epsilon >$ there is a $\delta > 0$ so that $d(x, y) < \delta \Rightarrow \rho(f(x) - f(y)) < \epsilon$ for every $f \in \mathcal{F}$. This is the same as the definition of continuity at a point, except that δ can be chosen independent of the point and of the function f.

A family $\mathcal{F}$ of functions is **normal** if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on every compact set or converges uniformly to ∞ on every compact set.

Lemma 32. *If a sequence of meromorphic functions converges uniformly on compact sets in the sense of spherical distance, then the limit is meromorphic or identically ∞. If a sequence of homomorphic functions converges in the same sense, then the limit is either holomorphic or identically ∞.*

Proof. $\square$

If a sequence of homomorphic functions converges uniformly on compacta to a holomorphic limit, then the derivatives also converge uniformly on compacta. However, if $\mathcal{F}$ is a normal family, $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ need not be normal. i.e., $\{n(z^2 - n)\}$.

Theorem 33 (Arzela-Ascoli). *A family $\mathcal{F}$ of continuous functions from a planar domain Ω to a metric space (X, d) is normal if and only if*

(1) *$\mathcal{F}$ is equicontinuous on every compact $E \subset \Omega$.*
(2) *For any $z \in \Omega$, $\{f(z) : f \in \mathcal{F}\}$ is pre-compact (lies in a compact subset).*

In the case that (X, d) is $\mathbb{C}$ with the Euclidean metric we get

Theorem 34 (Montel's theorem). *A family $\mathcal{F}$ of holomorphic functions on Ω is normal iff the functions are uniformly bounded on every compact subset of Ω.*

THEOREM 35. *There is a conformal covering map from $\mathbb{D}$ to $\mathbb{C} \setminus \{0,1\}$.*

PROOF. $\square$

THEOREM 36 (Montel's theorem). *If Ω is a planar domain and $\mathcal{F}$ is a family of holomorphic functions on Ω that all omit the values $\{0,1\}$ then $\mathcal{F}$ is a normal family.*

PROOF. $\square$

THEOREM 37 (Marty's theorem). *A family $\mathcal{F}$ of meromorphic functions is normal iff*

$$\sup_{f \in \mathcal{F}} \sup_{z \in E} |\nabla_E^S f(z)| < \infty,$$

for every compact $E \subset \Omega$.

PROOF. One direction is easy; if the spherical gradient is bounded at each point then $\mathcal{F}$ is equicontinuous, hence normal. Conversely, suppose $\mathcal{F}$ is normal but there is a sequence $\{f_n\} \subset \mathcal{F}$ such that $|\nabla^S f(z)| \to \infty$. By passing to a subsequence if necessary, we may assume f_n converges uniformly to a meromorphic function f on an open disk around z ($f \equiv \infty$ is allowed).

If the limit function f is finite at z, then it is bounded on some disk around z and hence f' is bounded on a smaller disk. Since $f_n' \to f$ on this disk, f_n' is uniformly bounded. Since $|\nabla^S f| \leq |f'|$, the spherical gradient is uniformly bounded on a neighborhood of z.

If $f(z) = \infty$, then consider $1/f$. Since $z \to 1/z$ corresponds to a rotation of the sphere by 180 degrees, its spherical gradient is 1 everywhere and hence

$$|\nabla^S \frac{1}{f}(z)| = |\nabla^S f(z)|.$$

Thus the argument above, applied to $1/f$, again shows that the spherical gradient of $\{f_n\}$ is uniformly bounded on some neighborhood of E. Since E is compact, it can be covered by a finite number of these neighborhoods and we deduce that $|\nabla^S f|$ is uniformly bounded over $z \in E$ and $f \in \mathcal{F}$. $\square$

The following lemma, due to Zalcman, is extremely helpful. It turns the failure of normality into a useful property. We shall use it again and again in our study of Julia sets.

LEMMA 38 (Zalcman's lemma). *Suppose Ω is a planar domain and $\mathcal{F}$ is a family of meromorphic functions on Ω. If $\mathcal{F}$ is not normal, then there is a sequence of points $\{z_k\}$ in Ω converging to a point $z_0 \in \Omega$, a sequence $\{\rho_k\}$ of positive real numbers converging to 0 and a sequence $\{f_k\} \subset \mathcal{F}$ so that $f_k(z_k + \rho_k z)$ converges uniformly on compact sets to a meromorphic function f on $\mathbb{C}$. Moreover, f has satisfies $|\nabla_S f(z)| \leq 1 = |\nabla_S f(0)|$ for all $z \in \mathbb{C}$.*

PROOF. By Marty's criterion, there is a sequence $\{f_n\} \in \mathcal{F}$ and $\{w_n\} \in \Omega$ so that $w_n \to w_0 \in \Omega$ and $|\nabla_S f_n(w_n)| \nearrow 0$. Without loss of generality we assume $z_0 = 0$ and $\overline{\mathbb{D}} \subset \Omega$. Since $(1-|z|)|\nabla_S f_n(z)|$ is continuous on $\overline{\mathbb{D}}$ and vanishes on its boundary, it attains a maximum M_n at some point $z_n \in \mathbb{D}$. Note that $M_k \to \infty$ since $M_k \geq (1-|w_n|)|\nabla_S f_n(w_n)| \to \infty$. Define

$$\rho_n = \frac{1}{|\nabla_S f_n(z_n)} = \frac{1-|z_n|}{M_n},$$

and

$$g_n(z) = f_n(z_n + \rho_n z).$$

Note that $0 < \rho_n \leq 1/M_n \to 0$ and $|z_n + \rho_n z| < 1$ if $|z| < (1-|z_n|)/rho_n = M_n$. Thus g_n is defined on $\mathbb{D}_{M_n}$ and for $z \in \mathbb{D}_{M_n}$,

$$\begin{aligned}
|\nabla_S g_n(z)| &= \rho_n |\nabla_S f_n(z_n + \rho_n z)| \\
&= \frac{(1-|z_n|}{M_n} |\nabla_S f_n(z_n + \rho_n z)| \\
&\leq \frac{1-|z_n|}{1-|z_n + \rho_n z|} \\
&\leq \frac{1-|z_n|}{1-|z_n| - |\rho_n z|} \\
&\leq \frac{1}{1 - z/M_n|}.
\end{aligned}$$

By Marty's theorem, $\{g_n\}$ has a subsequence that converges uniformly on compact subsets of $\mathbb{C}$ to a function f. Passing to another subsequence, if necessary, we may assume z_k converges to some point $z_0 \in \overline{\mathbb{D}} \subset \Omega$. Since $|\nabla_S g_n(0)| = 1$ for all n, $|\nabla_S f(0)| = 1$, hence f is non constant and $|\nabla_S f| \leq 1$ everywhere on $\mathbb{C}$. □

A meromorphic function f on $\mathbb{D}$ is called a **normal function** if the family

$$\mathcal{F} = \{f \circ \sigma : \sigma \in \mathcal{A}(\mathbb{D}), \},$$

is a normal family. A family of functions on $\mathbb{D}$ is called a conformally invariant normal family if

$$\mathcal{G} = \{f \circ \sigma : f \in \mathcal{F}, \sigma \in \mathcal{A}(\mathbb{D})\},$$

is a normal family.

THEOREM 39. *f is normal on $\mathbb{D}$ iff $|\nabla_H^S f|$ is bounded on $\mathbb{D}$. $\mathcal{F}$ is a conformally invariant normal family iff $|\nabla_H^S f|$ is uniformly bounded over $\mathbb{D}$ and $\mathcal{F}$.*

PROOF. Let

$$\sigma(z) = \frac{z+a}{1+\bar{a}z}.$$

This is a Möbius transformation of $\mathbb{D}$ to itself that maps 0 to a. Thus by Marty's theorem, $|\nabla_E^S f \circ \sigma|(0)$ is uniformly bounded, say by M. Thus

$$\begin{aligned} M &\geq |\nabla_H^S f \circ \sigma|(0) \\ &= |\nabla_H^S f|(\sigma(0)) \cdot |\nabla_H^H \sigma|(a) \cdot |\nabla_E^H \mathrm{Id}|(0) \\ &= |\nabla_H^S f|(a) \cdot 1 \cdot 1, \end{aligned}$$

which proves the first claim. The second follows by essentially the same argument. □

A function f on a hyperbolic plane domain is called **normal** if $f \circ p$ is normal on the disk, where $p : \mathbb{D} \to \Omega$ is the covering map. A family of meromorphic functions on Ω is called a **conformally invariant normal family** on Ω if $\mathcal{G} = \{f \circ p : f \in \mathcal{F}\}$ is conformally invariant normal family on the disk.

THEOREM 40. *Suppose Ω is a hyperbolic planar domain. f is normal on Ω iff $|\nabla_H^S f|$ is bounded on Ω. $\mathcal{F}$ is a conformally invariant normal family iff $|\nabla_H^S f|$ is uniformly bounded over Ω and $\mathcal{F}$.*

LEMMA 41. *[Lehto, Lehto-Virtanen] If f is meromorphic in $\mathbb{D}_r^*$ and has an essential singularity at ∞, then*

$$\limsup_{|z|\to\infty} |z| \cdot |\nabla_E^S f^{|}(z) \geq 1/2.$$

PROOF. Recall that points w, z are antipodal on $\widehat{\mathbb{C}}$ iff $z\overline{w} = 1$. For any $|\lambda| = 1$, $F(z) = f(\lambda z)\overline{f(\overline{z})}$ is meromorphic in $\mathbb{D}_r^*$ and for some choice of λ, F has an essential singularity at ∞ (if F meromorphic at ∞, the vanishing of its Laurent series for large negative indices implies λ is in some countable set, so for a.e. λ).

If $F(z) = -1$ let r =—z— and note that f maps z and λz to antipodal points on $\widehat{\mathbb{C}}$ and hence $\gamma = f(\partial \mathbb{D}_r)$ has spherical length at least π. This means that $\nabla^S_E f| \geq \frac{1}{2}$ somewhere on $\partial \mathbb{D}_r$. This if F takes the value -1 infinitely often near ∞ the result is proved. If not, f takes values $1 - \epsilon$ arbitrarily close to ∞ and the argument above then shows $|\nabla^S_E f| \to \frac{1}{2}$ along some sequence tending to ∞. ☐

This immediately gives its contrapositive:

Corollary 42. *[Lehto, Lehto-Virtanen] If f is meromorphic in $\mathbb{D}^*_r$ and*

$$\limsup_{|z| \to \infty} |z| \cdot |\nabla^S_E f|(z) < 1/2,$$

then f is meromorphic at ∞.

Theorem 43. *Suppose f is a normal function on a hyperbolic planar domain Ω. Then f has a meromorphic extension to any isolated boundary point of Ω.*

Proof. Without loss of generality, we may assume the boundary point is ∞. Since f is normal

$$\begin{aligned} |\nabla^S_E f|(z) &= |\nabla^H_E \mathrm{Id}|(\mathrm{z}) \cdot |\nabla^{\mathrm{S}}_{\mathrm{H}} \mathrm{f}|(\mathrm{z}) \\ &= O(\frac{1}{|z| \log |z|}) \cdot O(1) \\ &= = o(\frac{1}{|z|}), \end{aligned}$$

so by Corollary **??**, f has a meromorphic extension at ∞. ☐

2. Picard's theorem and beyond

Theorem 44 (Picard's little theorem). *If f is a non-constant entire function, then $E = \mathbb{C} \setminus f(\mathbb{C})$ contains at most one point.*

Proof. If E contains two points $\{a, b\}$, then using the covering map $p : \mathbb{D} \to \mathbb{C} \setminus \{a, b\}$, f can be lifted to a holomoprhic map $f : \mathbb{C} \to \mathbb{D}$. By Liouville's theorem, the lift must be constant and hence so must f. ☐

Theorem 45 (Picard's great theorem). *If f is a non-constant entire function, then for every $r > 0$, $E = \mathbb{C} \setminus f(\mathbb{D}^*_r)$ contains at most one point.*

Proof. Suppose for r sufficiently large, $f(\mathbb{D}^*)$ omits two points. Then f is a normal function on $\mathbb{D}_r$ and hence has a meronorphic extension to ∞ by Lemma 43, a contradiction. $\square$

Theorem 46 (Montel's theorem). *If $\mathcal{F}$ is a family of holomoprhic functions on a planar domain Ω all taking values in $W = \mathbb{C} \setminus \{a, b\}$ for some $a \neq b$, then $\mathcal{F}$ is a normal family.*

First proof. If not, then by Zalcman's lemma we can form a sequence

$$g_n(z) = f_n(\rho_n + z_n)$$

so that $f_n \in \mathcal{F}$, $\rho_n \searrow 0$ and $z_n \to z \in \Omega$ and g_n converges unifromly on compact sets to a non-constant entire function g. If g ever took the value a, then so would f_n for n sufficiently large (apply Rouche's theorem to a small disk around some g-preimage of a). Thus g omits a, b and is constant, a contradiction. $\square$

Second proof. One can also prove this by using the covering map $p : \mathbb{D} \to W$ to lift $\mathcal{G}$ be the family of bounded holomorphic functions obtained by lifting elements of $\mathcal{F}$ via p. Then

$$|\nabla^S_H f|(z) = |\nabla^H_H \tilde{f}|(z) |\nabla^p_H|(z) |\nabla^S_H \mathrm{Id}|(\mathrm{z}) \leq 1 \cdot 1 \cdot \mathrm{O}(1),$$

because the hyperbolic distance between points in W is smaller that a fixed constant times the spherical distance. On Ω, $|\nabla^H_E \mathrm{Id}|$ is bounded on compact sets, so this implies $|\nabla^S_E|$ is locally bounded, which implies normality by Marty's theorem. $\square$

Thus omitting two values has two consequences: it implies normality when applied to functions on a hyperbolic domain and it implies constantcy when applied to functions on $\mathbb{C}$. This is a common phenonmenon and **Bloch's principle** say that a proprety P implies one of these conclusions iff it implies the other. This is not always true, but it does hold for a number of interesting cases and can be made into a precise mathematical statement. See Bergweiler's paper [**?**].

Our next goal is a result that contain both Picard's theorem and Montel's theorem as speical cases, as well as various results of Ahlfors, Nevanlinna and others. We will state the general result first, then a number of corollaries and then prove the general statement.

THEOREM 47. *Suppose* $\mathcal{E} = \{E_1, \dots, E_q\}$ *are connected sets that are contained in Jordan domains* $\{\Omega_1, \dots, \Omega_q\}$ *on* $\widehat{\mathbb{C}}$ *that have pairwise disjoint closures. Suppose* $\mathbf{m} = \{m_1, \dots, m_q\} \subset \mathbb{N}$ *satisfy*

$$\sum_{k=1}^{q}(1 - \frac{1}{m_k}) > 2. \tag{9}$$

If $\Omega \subset \widehat{\mathbb{C}}$ *is a domain let* $\mathcal{F} = \mathcal{F}(\mathcal{E}, \mathbf{m})$ *be the collection of meromorphic functions* f *on* Ω *so that every compact connected component of* $f^{-1}(E_j)$ *contains critical points whose degrees sum to at least* m_j*. Then* $\mathcal{F}$ *is a conformally invariant normal family.*

Recall from Theorem 40 that the conclusion holds iff

$$\sup_{f \in \mathcal{F}} |\nabla_H^S f(z)| < \infty.$$

and that if Ω is not hyperbolic (i.e., $\widehat{\mathbb{C}} \setminus \Omega$ has two or fewer points), then $\mathcal{F}$ contains only constants. Also recall that every function in an conformally invariant normal family is normal and has a meromorphic extension to any isolated boundary point of Ω.

If $a \in E_j$ is an omitted value of f then $f^{-1}(E)$ has no pre-compact components. Thus the hypothesis is trivially fulfilled for any m_j. In this case we may take $m_j = \infty$ in (9). Thus a meromorphic function on Ω with 3 omitted values is normal and if Ω is not hyperbolic, then it is constant. Similarly, holomorphic function with two normal values is normal and an entire functions with two omitted values is constant; thus this result contains Picard's little theorem. It also contains Picard's great theorem, for if an entire function f omits two values on some neighborhood $\mathbb{D}_r^* = \{|z| > r\}$ of ∞, then it has a meromorphic extension to ∞ and hence is a polynomial. Thus Theorem **??** also contains Picard's great theorem.

An a-point of f is any point $z \in f^{-1}(a)$. It is called simple if $f'(z) \neq 0$ and a is called perfectly branched if there are no simple a points (hence $f(z) = a \Rightarrow f'(z) = 0$).

A map $\Omega \to \widehat{\mathbb{C}}$ is said to have a simple island over an Jordan domain D if there is a component W of $f^{-1}(D)$ that has compact closure in Ω and so that $f : W \to D$ is conformal (1-to-1). If $\overline{W}$ is compact in Ω, then this occurs iff f has no critical points in W.

In some sources, the requirement that an island have compact closure is dropped. Saying there are no islands over D then becomes a stronger conditions, since more components of $f^{-1}(D)$ have to be checked for critical points. However, the weaker version that checks only pre-compact components is sufficient to prove normality. Thus we get:

Corollary 48. *Suppose $\Omega \subset \widehat{\mathbb{C}}$ and $\mathcal{F}$ is one of the following families:*

- **H1:** *Meromorphic functions on Ω with no simple islands over five Jordan domains with disjoint closures.*
- **H2:** *Meromorphic f with five perfectly branched values.*
- **H3:** *Holomorphic functions on Ω with no simple islands over three Jordan domains with disjoint closures.*
- **H4:** *Holomorphic functions with three perfectly branched values.*

Then all the following hold

- **C1:** *$\mathcal{F}$ is a conformally invariant normal family.*
- **C2:** *$\mathcal{F}$ contains only constants if Ω is not hyperbolic.*
- **C3:** *Each element of $\mathcal{F}$ extends to be meromorphic at any isolated boundary points of Ω.*

The implication "H1 ⇒ C1" is called the Ahlfors 5-Island theorem. "H1 ⇒ C2" is due to Nevanlinna [**?**, **?**, **?**], [**?**, **?**, **?**, **?**], while "H1 ⇒ C1" appears in the work of Boch [**?**, **?**] and Valiron [**?**, **?**].

Proof of Theorem 47. It is clear that $\mathcal{F}$ is conformally invariant; if we replace $f : \Omega \to \widehat{\mathbb{C}}$ by pre-composing with a covering map $\varphi : \mathbb{D} \to \Omega$ then a component W $(f \circ \varphi)^{-1}(\{E_j\})$ is compact iff $\varphi(W)$ is and the critical points W contains correspond 1-to-1 with the critical points in $\varphi(W)$ and have the same degrees. Thus we only have to show $\mathcal{F}$ is a normal family.

If $\mathcal{F}$ is not a normal family, then we can use Zalcman's lemma (Lemma 38) to build a sequence

$$f(\rho_n z + z_n),$$

that converges to a non-constant, meromorphic f on the whole plane that has bounded spherical gradient and still satisfies (9). If we can prove any such f must be constant, we get a contradiction and can deduce $\mathcal{F}$ is indeed a normal family.

We first do the case when each set E_j is a single point $\{a_j\}$. Our hypothesis now means that every preimage of a_j is a critical point of degree at least m_j. If $\{a_j\}$ is an omitted value of f, then we can choose any $m_j \in \mathbb{N}$ and the proof below will work.

So assume that f is a non-constant and meromorphic on $\mathbb{C}$, that it has bounded spherical gradient, and that every preimage of a_j is a critical point of degree at least m_j, where the $\{m_j\}_1^q$ satisfy (9). We will prove f is constant.

Let m be the least common multiple of $\{m_1, \dots, m_q\}$ and define

$$g(z) = \frac{(f'(z))^M}{\prod_{j=1}^q (f(z) - a_j)^{(m_j-1)M/m_j}}.$$

This function is entire; near a preimage w of a_j we have

$$|f(z) - a_j|^{(m_j-1)M/m_j} \lesssim |z-w|^{(m_j-1)M},$$

$$|f'(z)| \simeq |z-w|^{(m_j-1)M},$$

so g has removable singularities at these points. If $m = \sum(1 - \frac{1}{m_j}) > 2$ then

$$|g(z)|^{1/M} \leq \frac{|f'(z)|}{|f(z)|^m} = O(|f(z)|^{2-m}).$$

Thus g is small whenever f is large. Since g is entire and not constant, it is unbounded along some sequence $\{z_n\}$. Because f has bounded spherical gradient, Marty's theorem says we can find a convergent subsequence of $h_n(z) f(z+z_n)$ that converges to a meromorphic h. This h must be constant and equal to one of the a_j for otherwise

$$g(z+z_n) \to \frac{(h'(z))^M}{\prod_{j=1}^q (h(z) - a_j)^{(m_j-1)M/m_j}},$$

contradicting the fact that $g(z_n) \to \infty$.

The function $F(z) = f(z+z_n) - a_k$ only has zeros of order at least m_k so we can apply the Schwarz Lemma for m_kth roots (Lemma 7). This gives

$$|f'(z_n)|^{m_k} = |F'(0)|^{m_k} \leq m_k^{m_k} |F(0)|^{m_k - 1} = m_k^{m_k} |f(z_n) - a_k|^{m_k-1},$$

or

$$|f'(z_n)|^M \leq m_k^M |f(z_n) - a_k|^{(m_k-1)M/m_k}.$$

Thus

$$|g(z)| \leq \frac{m^{mM}}{\prod_{j \neq k} (f(z) - a_j)^{(m_j-1)M/m_j}}.$$

However, this is impossible since $f(z_n) \to h(0) = a_k$ and $g(z_n) \to \infty$. The contradiction proves that

$$\sum_{j=1}^{q}(1 - \frac{1}{m_j}) > 2$$

is impossible.

We have now completed the proof of Theorem **??** in the case when the sets $\{E_j\}$ are all points. Next we consider all E_j as in the theorem. Suppose $\mathcal{F}(\mathbb{C}, \{E_j\}_1^q)$ contains a non-constant function f.

Choose any five distinct points $\{a_1, \dots a_5\}$ and choose ϵ so the disks $D_j = D(a_j, \epsilon)$ have pairwise disjoint closures. There is a smooth quasiconformal map ψ_ϵ defined on $\mathbb{C}$ that maps each Ω_j into D_j for $j = 1, \dots, 5$ (and hence the set E_j is also mapped into D_j). By the measurable Riemann mapping theorem (Theorem **??**), there is another quasiconformal map ϕ_ϵ so that

$$g = \psi_\epsilon^{-1} \circ f \circ \phi_\epsilon,$$

is meromorphic and in $\mathcal{F}(\mathbb{C}, \{D_j\}_1^5)$.

Fix any sequence of $\epsilon_k \searrow 0$.We may suppose that $\{g_{\epsilon_k}\}$ is not normal, for otherwise replace $g_\epsilon(z)$ by $g_\epsilon(Mz)$ for m large. Then by Zalcman's lemma, we can form a sequence

$$g_{\epsilon_k}(\rho_k z + z_k)$$

that converges uniformly on compact sets to a, non-constant meromorphic function g that is perfectly branched over each a_j with degree at least m_j. By our previous arguments, such a g must be constant, and the contradiction shows that f must have been constant after all. This completes the proof of the theorem. □

3. The Julia set

The Julia set is completely invariant

$\mathcal{J}(f) = \mathcal{J}(f^n)$

Julia set has no isolated points (assuming non-empty)

Julia set is in accumulation set of every backwards orbit with at most one exception.

The Julia set is either the whole plane or has empty interior.

The Julia set is the minimal closed completely invariant set.

$\mathcal{J}(f) = \mathbb{C}$ is possible.

LEMMA 49 (Rosenbloom, 1952). *If g is entire and $h(z) = (g^2(z) - z)/(g(z) - z)$ is constant then g is constant or linear.*

PROOF. If $h \equiv 0$, then $g^2(z) = z$ implying g is i-to-1, hence linear. If $h \equiv 1$, then $g^2 = g$ so g is constant or $g(z) = z$. So assume h is a constant $c \neq 0, 1$, i.e,

$$g^2(z) - z = c(g(z) - z),$$

and differentiate to get

$$g'(g(z))g'(z) - 1 = c(g'(z) - 1),$$

or

$$g'(z)(g'(g(z)) - c) = 1 - c.$$

Since $c \neq 1$, the left side is never zero, hence both factors are never zero. Thus g' omits both 0 and c and hence is constant by Picard's theorem. □

EXERCISE: Show that if h is rational, then g must be rational too.

THEOREM 50. *If g is entire and not constant or linear then it has at least two pre-periodic points. If g is transcendental, there are infinitely many pre-periodic points.*

PROOF. Out assumption im[plies that h in Lemma is a non-constant meromorphic function. If $h(z) = \infty$ then $g(z) = z$, so z is a fixed point of g. If $h(z) = 1$, then $g^2(z) = g(z)$ so $g(z)$ is a fixed point of g. If $h(z) = 0$ then $g^2(z) = z$ so z is a fixed point of g^2.

If h is a rational of degree $d \geq 1$, then each point has d preimages counted with multiplicity and there are at most $2d$ critical points counted with multiplicity, so the number of distinct preimages of three points is at least $3d - 2(d - 1) = d + 1 \geq 2$.

If h is transcendental, then Picard's great theorem says that it takes on at least one of the values $\{0, 1, \infty\}$ infinitely often. If 0 or ∞ are taken infinitely often then g or g^2 has infinitely many fixed points and we are done. If $h(z) = 1$, then $g(z)$ is a fixed point of g. If $g(z) = z$ for infinitely many such $z's$ then there are infinitely many fixed points of g. If $g(z) \neq z$ then at least one of z or $g(z)$ has infinitely many preimages, all of which are pre-periodic. □

THEOREM 51. *If f is entire, $\mathcal{J}(f) = \partial I(f)$.*

Proof. Suppose $z \in \mathcal{J}(f)$ and a neighborhood V of z. Then $\{f^n\}$ is not normal on V, so takes every complex value except possibly one (Theorem **??**). For any $w \in I(f)$ we have $f(w) \neq w$ (since fixed points don't escape), so $f^n(V)$ eventually contains either w or $f(w)$. Hence V contains escaping points. Thus the Julia set is contained in the closure of the escaping set.

On the other hand, if D is a disk in $I(f)$, then $f^n(D)$ never hits a non-escaping point. We know that there are at least two pre-periodic points, so $\{f^n\}$ is normal on D, hence $D \subset \mathcal{F}(f)$. Thus the Julia set is disjoint from the interior of the escaping set. □

The same proof shows

Theorem 52. *If f is entire and V is any neighborhood of any point $z \in \mathcal{J}(f)$ then $\cup_n f^n(V)$ covers the whole plane with at most one exception.*

Since we know that there are at least two pre-periodic points and since preimages of such points are again pre-periodic, we get:

Corollary 53. *If f is entire, $\mathcal{J}(f)$ is the closure of the pre-periodic points.*

A stronger result, that the Julia set is the closure of repelling fixed points will be proved later, Theorem **??**. Another strengthening is that a finite number of iterates of V suffice to cover any bounded part of the Julia set (?)

Corollary 54. *If g is entire, and not constant or linear, then $\mathcal{J}(g) \neq \emptyset$. In fact, the Julia set is infinite.*

Proof. Pre-periodic points don't escape, so we now know that both escaping set $I(g)$ and its complement are non-empty Thus $\mathcal{J}(g) = \partial I(g) \neq \emptyset$. In fact, we have shown that $I(g)$ is infinite and its complement has at least two pre-periodic points. Thus $\mathcal{J}(g) = \partial I(g)$ has at least two points. By Picard's great theorem, one of the points in $\mathcal{J}(g)$ has infinitely many preimages, and all are in $\mathcal{J}(g)$ since it is an invariant set. □

Alternate proof that $\mathcal{J}(f)$ is infinite. Suppose $\mathcal{J}(g)$ contained only one point. Conjugating by a translation we can assume this point is 0 and hence that $g^{-1}(0) = 0$. Thus $f = \log(g(e^z))$ is a well defined entire function and $f^n =$

$\log(g^n(e^z)))$. Thus $\mathcal{F}(g) \subset \exp(\mathcal{F}(f))$ and hence $\mathcal{J}(g) \supset \exp(\mathcal{J}(f))$. Since f has a non-empty Julia set, g has a non-zero point in its Julia set, i.e., its Julia has at least two points and hence infinitely many by Picard's theorem. $\square$

The opposite inclusion $\mathcal{F}(g) \supset \exp(\mathcal{F}(f))$ (and hence equality) is also true, but more difficult to prove. See [**?**], [**?**], [**?**]. Bergweiler's proof is based on the density of repelling fixed points.

4. Repelling fixed points are dense

$\mathcal{M}$ are functions meromorphic off a compact countable set.

LEMMA 55. *Let $a_1, \dots, a_5$ be be distinct complex numbers. Then there is an $\epsilon > 0$ so that for any meromorphic f and any domain D hitting $\mathcal{J}(f)$, there is a $j \in \{1, \dots, 5\}$ and a simple island over D_j contained inside D_j.*

PROOF. For a disk centered on the Julia set of f, the iterates of f form a non-normal family and the Ahlfors island theorem may be applied. Inside each of the five disks D_k, $k = 1, \dots 5$, choose 5 disks centered on $\mathcal{J}$ and with disjoint closures. By Theorem **??** each of these 25 disks contains a simple island for some iterate of f over one of the D_j's. Thus one of the D_j's occurs at least 5 times. Choose a D_j and five smaller disks containing an island over D_j. Then D_j contains a simple island over one of these disks and hence over itself. $\square$

LEMMA 56. *Let $a_1, \dots, a_5$ be be distinct complex numbers and let ϵ be as in Lemma* **??**. *Let $0 < \delta < \epsilon$ and let $V_1, \dots, V_5$ be be domains that hit $\mathcal{J}(f)$ and $V_j \subset D_j$ for all $j \in \{1, \dots, 5\}$. Then there is a U in one of the V_j that is mapped by some f^n univalently to D_j.*

PROOF. For a disk centered on the Julia set of f, the iterates of f form a non-normal family and the Ahlfors island theorem may be applied. Inside each of the five domains v_k, $k = 1, \dots 5$, choose 5 disks centered on $\mathcal{J}$ and with disjoint closures. By Theorem **??** each of these 25 disks contains a simple island for some iterate of f over one of the D_j's. Thus one of the D_j's occurs at least 5 times. Choose a D_j and five smaller disks containing an island over D_j. Then D_j contains a simple island over one of these disks and composition gives an island over itself. $\square$

Theorem 57. *Suppose f is a transcendental entire function. Then $\mathcal{J}(f)$ is the closure of its repelling periodic points.*

Proof. Repelling periodic points are clearly in $\mathcal{J}$ so we only need show every disk D hitting $\mathcal{J}$ contains such a point. Since $\mathcal{J}$ is perfect, $D \cap \mathcal{J}$ contains at least five points $a_1, \dots, a_5$. Choose $\epsilon > 0$ so small that the disks $D_j = D(a_j, \epsilon)$ have disjoint closures and Lemma **??** applies. By Lemma 56 there is a domain U inside one of the D_j that is mapped univalently to D_j by some f^n. The inverse branch is a mapping of D_j into itself and must have an attracting fixed point (U must be a proper subset of D_j, otherwise f^n would be a Möbius transformation, which is impossible). This fixed point is clearly a repelling periodic point for f. □

Lemma 58. *Let $a_1, \dots, a_5$ be be distinct complex numbers in $\mathcal{J}(f)$ and let ϵ be as in Lemma 55. Let $0 < \gamma < \delta < \epsilon$. Then there exists a disk of the form $D_k = D(a_k, \gamma)$ and domains $U_1, U_2 \subset D_k$ so that some iterate f^m of f maps each domain univalently to D_k.*

Proof. □

5. $\dim(\mathcal{J}(f)) > 0$

Theorem 59. *If f is an entire function and not a linear polynomial, then the set of points in $\mathcal{J}(f)$ with bounded orbit has positive Hausdorff dimension.*

Proof. □

It is easy to construct polynomials whose Julia set is a Cantor set with dimension close to zero. However, we shall see later that a transcendental Julia set always has dimension at least 1. The result above is still sharp since it it possible to construct examples where the set of points with bounded orbits has dimension close to zero.

CHAPTER 3

Rates of escape

1. Slow escaping points

Theorem 60 (Rippon Stallard). *Let f be a transcendental entire function. Then given any sequence $\{a_n\} \subset (0, \infty)$ such that $a_n \nearrow \infty$ there is a point $z \in \mathcal{J}(f) \cap I(f)$ such that $|f^n(z)| \leq a_n$.*

Theorem 61 (Bergweiler-Peter). *Let $f \in \mathcal{B}$. Then given any sequence $\{a_n\} \subset (0, \infty)$ such that $a_n \nearrow \infty$ the set of points $z \in \mathcal{J}(f) \cap I(f)$ such that $|f^n(z)| \leq a_n$ has Hausdorff dimension ≥ 1.*

Equality is possible since Rempe-Stallard have shown $\dim(I(f)) = 1$ is possible.

Def slow escaping set

Def moderately slow escaping set

2. Logarithmic measure

If $E \subset [1, \infty)$ is a measurable set, we let *log*(E) denote the image of E under the map and define $|\log E|$, the logarithmic measure of E, as the Lebesgue measure of $\log E$, i.e.,

$$|\log E| = \int_E \frac{dt}{t}.$$

Lemma 62. *Suppose $0 < r_n \nearrow \infty$ and $0 < \rho_n \nearrow L < \infty$. The the complement of $\cup_n (\rho_n r_n, \rho_n, r_{n+1})$ has finite logarithmic measure.*

We introduce this because many interesting properties of $M(r, f)$ fail to hold for all r, but do hold except on a set of finite logarithmic measure. Here are two results that we will use later.

Proof. The complement is covered by intervals of the form $[r_n \rho n - 1, r_n, \rho_n$ that has logarithmic length $\log \rho_{n_1} - \log \rho_n$. Summing the telescoping series shows the complement has total logarithmic length $\leq L - \rho_0$. $\square$

LEMMA 63. *Suppose $N(r)$ is an increasing function on $I = [r_0, \infty)$ and $\alpha > 0$. Then there is a set E of finite logarithmic measure so that $|h| < N(r)^{-\alpha}$ implies*

$$|N(r) - N(re^h)| \leq \alpha N(r), r \in I \setminus E.$$

PROOF. Let $X \subset I$ be a compact set where the inequality fails. Each point of $\log X$ is contained in an interval $(a, b) = (\log r - |h|, \log r + |h|)$ so that $N(e^b) > (1+\alpha)N(e^a)$. By the easy form of Vitali's covering lemma, there is a finite, disjoint subcollection of these intervals that covers a fixed fraction of the length of $\log X$. Ordering the intervals as $\{I_k\} = \{(a_k, b_k)\}_1^K$ from left to right, we get $N(e^{b_k}) > (1+\alpha)^k N(e^{a_1})$ and hence

$$|I_k| = 2|h| \leq N(e^{a_k})^{-\alpha} \leq (1+\alpha)^{-\alpha k},$$

and so if $A = (1+\alpha)^{-\alpha} < 1$,

$$|\log X| = O(\sum |I_k|) = O(\sum_{k=1}^{\infty} A^n) == O(\frac{1}{A-1}).$$

$\square$

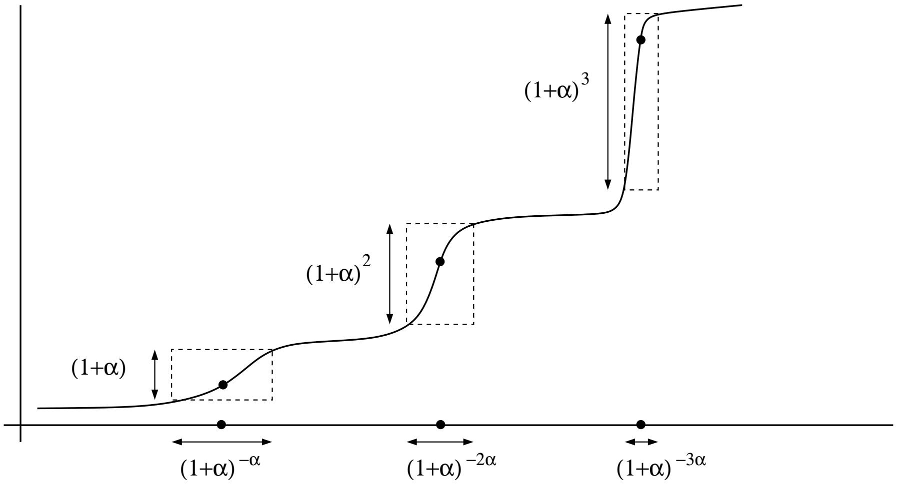

FIGURE 1. Proof of Lemma **??**; an increasing function can have extreme growth only a set of finite total length, hence finite logarithmic length.

3. The Wiman-Valiron method

4. The central index

If f is entire, then is has a Taylor series expansion

$$f(z)=\sum_{n=0}^{\infty}a_n z_n.$$

Since the radius of convergence is ∞, the coefficients satisfy

$$\lim_{n\to\infty}|a_n|^{1/n}=0.$$

Thus for any fixed r, $|a_n|r^n\to 0$ and hence the supremum

$$\mu(r)=\mu(r,f)=\sup_n |a_n|r^n,$$

is attained. We let $N(r)=N(r,f)$ denote the largest index n where the maximum is taken. This is an increasing step function and is right continuous at the jump points. Note that the jumps are >1 when the maximum of $|a_n|r^n$ is attained at several different indices. This is called the central index of f at r. (From now on we will fix f and simply write $M(r),\mu(r),N(r)$ unless we need to stress the dependence on the function f.)

Let

$$r_n=r_n(f)=\inf\{r:N(r)\geq n\}.$$

Note that $r_n\nearrow\infty$. Also note that if $r_n=r_m$ then

$$|a_n|r_n^n=|a_{n+1}|r_{n+1}^{n+1}.$$

Hence

$$\frac{a_{n+1}}{a_n}=\frac{r_n^n}{r_{n+1}^{n+1}}\leq\frac{1}{r_{n+1}},$$

since $r_n\leq r_{n+1}$. Thus

$$\frac{a_n}{a_0}=\frac{a_n}{a_0}\cdot\frac{a_1}{a_0}\cdot\frac{a_2}{a_2}\cdots\frac{a_n}{a_{n-1}}\cdot=\frac{1}{r_1r_2\cdots r_n}. \tag{10}$$

Lemma 64. *$\mu(r)\leq M(r)$.*

Proof. By rescaling we may assume $r=1$, $\mu(r)=1$ and $a_N=1$ where $N=N(1)$. Then $g(z)=f(z)-z^N$ is perpendicular to z^N in L^2 of the unit circle, so

$$0=\int_{|z|=1}z^N\overline{g(z)}ds=\int_{|z|=1}z^N\overline{f(z)-z^N}ds=\int_{|z|=1}z^N\overline{f(z)}ds-2\pi.$$

This implies $\int_{|z|=1} |f| ds \geq 2\pi$ and hence $|f| \geq 1$ somewhere on the unit circle. Thus $M(1) \geq \mu(1)$, as desired. □

One of the remarkable consequences of Wiman-Valiron theory is that this inequality can be reversed a in certain sense. There are several ways to make this precise; one says that for any $\delta > 0$, we have

$$M(r) \leq \mu(r)(\log \mu(r))^{1/2}(\log\log \mu(r))^{1+\delta},$$

for all $r > 0$ except a set of finite logarithmic measure. See Theorem 6 of [**?**]. We will not pursue this particular line of reasoning. Our goal is to use the Wiman-Valiron method to show that near a point where f attains a circular maximum, i.e., a point z such that

$$|f(z)| = M(|z|, f),$$

it behaves like the power function $w^{N(z)}$. Before making and proving a precise statement, we will deal with a number of more technical details.

As noted above, the sequence $\{|a_n|r^n\}$ attains a maximum value at $n = N = N(r)$. By definition, moving to the left or right of N gives smaller terms, but we would like these terms to decay quickly to zero as we move away from N. An ideal estimate would be

$$|a_n|r^n \leq \mu(r) \exp(-bk^2),$$

where $n = N + k$ and $c > 0$, but this is too much to ask for. However, the estimate is true if we replace the constant b by a function of $N + |k|$ that tends slowly to zero.

LEMMA 65. *If $f(z) = \sum a_n z^n$ is an entire function, and $b(x)0$ is a positive, strictly decreasing, integrable function on $[0, \infty)$. Then there is a set E of finite logarithmic measure and a constant $K > 0$ so that for $r \in [1, \infty) \setminus E$*

$$|a_n|r^n \leq \mu(r) \exp(-\frac{1}{2} b(\max(N, n))(N - n)^2).$$

PROOF. Set $B(t) = \int_0^t b(s)ds$. This is an increasing, concave down function that approaches as finite limit as $t \to \infty$. For integers $n \geq 0$ define

$$\alpha_n = \exp(-\int_0^n B(t)dt), \quad \rho_n = \exp(B(n)).$$

Note that the α's are decreasing to zero, and the ρ's are increasing with a finite limit. Since b is strictly decreasing,

$$\log \frac{\alpha_n}{\alpha_{n-1}} > -\log \rho_n > \log \frac{\alpha_{n+1}}{\alpha_n},$$

or equivalently,

$$0 < \rho_0 < \frac{\alpha_0}{\alpha_1} < \rho_1 < \frac{\alpha_1}{\alpha_2} \dots .$$

Using a telescoping product, this implies for $n > N$,

$$\frac{\alpha_n}{\alpha_N} = \frac{\alpha_n}{\alpha_{n-1}} \cdots \frac{\alpha_{N+1}}{\alpha_N} \le \frac{1}{\rho_{n-1}} \cdots \frac{1}{\rho_N} \le \rho_N^{N-n} \tag{11}$$

Similarly,

$$\frac{\alpha_n}{\alpha_N} \ge \rho_n^{N-n}$$

$$\frac{\alpha_{n+1}}{\alpha_n} \ge \frac{1}{\rho_n},$$

a telescoping product gives be the smallest r such that

$$\frac{\alpha_n}{\alpha_0} \ge \frac{1}{\rho_1 \cdots \rho_n}.$$

If $m < n$, the definitions above and integration by parts give

$$\begin{aligned}\frac{\alpha_n}{\alpha_m}\rho_m^{n-m} &= \exp(-\int_m^n B(t) - B(m) dt)\\ &= \exp(-\int_m^n (t-m)b(t)dt)\\ &\le \exp(-\int_m^n (t-m)b(n)dt)\\ &\le \exp(-\frac{1}{2}b(m)(n-m)^2)\end{aligned}$$

Thus

$$\frac{\alpha_n}{\alpha_m}\rho_m^{n-m} = \exp(-\frac{1}{2}b(\max(m,n))(n-m)^2).$$

To finish the proof we must show that for all $r \notin E$ and for all $n \in \mathbb{N}$,

$$|a_n|r^n \le mu(r) \cdot \alpha_n \alpha_N \rho_N^{n-N} \tag{12}$$

First let us estimate the set of r's we must throw away when $n = 1, \dots, N$. In this case $\alpha_n > \alpha_N$, so (12) holds if

$$|a_n|r^n \le |a_N| r^N \rho_N^{n-N},$$

or equivalently

$$|a_n|(\frac{r}{\rho_N})^n \leq |a_N|(\frac{r}{\rho_N})^N.$$

This holds by definition if $N(r/\rho_N) = N$, i.e., $r \in \rho_N I_N = (\rho_N r_N, \rho_N r_{N+1})$ for some N. The complement of this union of intervals has finite logarithmic measure by Lemma 62.

Next we estimate the set we must omit due to indices $n > N$. Note that (12) follows from

$$\frac{|a_n|(r\rho_N)^n}{|a_N|(r\rho_N)^N} \leq \frac{\alpha_n \rho_N^n}{\alpha_N \rho_N^N} \tag{13}$$

since (11) implies the right hand side is ≤ 1. We cancel the ρ_M's and rewrite this as

$$\frac{|a_n|}{|\alpha_n} r^n \leq \frac{|a_N|}{\alpha_N} r^N \tag{14}$$

or

$$c_n r^n \leq c_N r^N \tag{15}$$

where we set $c_n = |a_n|/\alpha_n$. Suppose for the moment that $F(z) = \sum_n c_n z^n$ is entire, so that the central index makes sense. Then (15) holds as long as $N(r, f) = N(r, F)$. (13) implies this is true if $r \in (R_n\rho_n, R_{n+1}\rho_n)$, and hence the exception set of r's is the union of complementary intervals

$$\cup_n (R_n \rho_{n-1}, R_{n+1}\rho_{n+1}$$

, which has finite logarithmic measure by Lemma 62.

Finally, we have to prove that the $\{c_n\}$ are the Taylor coefficients of an entire function, i.e., that

$$\lim_n c_n^{1/n} = 0.$$

Combining (10) and (11) gives

$$(\frac{a_n}{\alpha_n})^{1/n} \leq (\frac{a_0}{\alpha_0} \frac{\rho_1 \cdots \rho_n}{r_1 \cdots r_n})^{1/n},$$

and this tends to zero since the ρ_n's are bounded above and the r_n's tend to ∞. $\square$

Now that we have the pointwise estimate, we show that the sum of terms far from the central index is small.

LEMMA 66. *Suppose f is a transcendental entire function, $r > 0$, and $N = N(r, f)$. Assume b has been chosen to be positive, decreasing, integrable and satisfy*

$$\frac{1}{Ct^2} \leq b(t) \leq \frac{C}{t|\log t|}),$$

and

$$b'(t) = O(b(t)/t).$$

If

$$k = \lfloor \sqrt{\gamma b(N) |\log b(N)|} \rfloor,$$

$|\log \rho/r| < 2/k$ *and* $\sigma < (\gamma - 1)/2$ *, then*

$$\sum_{|n-N| \geq k} \frac{|a_n|\rho^n}{|a_N|\rho^N} = o(b(N)^\sigma),$$

uniformly as $r \to \infty$ outside a set of finite logarithmic measure.

PROOF. On difficulty with the desired estimate is that our estimates of the terms on the left hand side from Lemma **??** involve the central index of ρ, whereas the right hand side only involves the central index of r. This will be a problem if the central index changes rapidly near r, so our first step is to exclude the set of finite logarithmic measure where this happens.

Given any $\epsilon > 0$ our assumptions on b imply that there is an $0 < \eta < 1$ so that

$$b((1 + \eta)t) \geq (1 - \epsilon)b(t).$$

Fix ϵ so that $\sigma < (1 - \epsilon)^2\gamma$ and choose η so the inequality above is true.

Next choose $\alpha \leq \eta/2$, and set $A = \exp(N^{-\alpha})$ and $M = N(Ar, f)$. By Lemma 63 $N \leq M \leq (1 + \alpha)N$, outside a set of finite logarithmic measure. Assume that we exclude this set, as well as the exceptional set for Lemma **??**. Since M is the central index for Ar, we have

$$|a_n|((Ar)^n \leq |a_M|(Ar)^M,$$

for every n. Using the fact that N and M are central indices for r and Ar we get

$$\begin{aligned} \frac{|a_n|\rho^n}{|a_N|\rho^N} &= \frac{|a_n|(A\rho)^n}{|a_M|(A\rho)^N} \cdot \frac{|a_M|r^M}{|a_N|r^N} \cdot A^{M-n}(\frac{\rho}{r})^n - M \\ &\geq 1 \cdot 1 \cdot A^{M-n}(\frac{\rho}{r})^{n-M}. \end{aligned}$$

First assume that $n > (1+\eta)N$. Since $M \leq (1+\alpha)N \leq (1+\eta/2)N$, we have

$$n - M \geq n - (1+\eta/2)N \geq n\eta/2$$

hence

$$A^{M-n} \leq A^{\eta n/2} = \exp(-\frac{1}{2}\eta n N^{-\alpha}).$$

Also,

$$(\frac{\rho}{r})^{n-M} \leq \exp((n-M)2/k).$$

Hence

$$\frac{|a_n|\rho^n}{|a_N|\rho^N} \quad \leq \quad \exp(-\frac{1}{2}\eta n N^{-\alpha} + 2n/k).$$

Our definition of k and assumptions on b imply

$$\frac{1}{k} = O(\sqrt{b(N)|\log b(N)|}) = O((N\log N)^{-1/2})) = o(N^{-\alpha}),$$

since $\alpha \leq \eta/2 < 1$. Thus

$$\frac{|a_n|\rho^n}{|a_N|\rho^N} \quad \leq \quad \exp(-\frac{1}{3}\eta n N^{-\alpha}),$$

if r is large enough (and hence n is large). Thus by the geometric sum formula

$$\begin{aligned}
\sum_{n\geq(1+\eta)N} \frac{|a_n|\rho^n}{|a_N|\rho^N} \quad &\leq \quad \sum_{n\geq(1+\eta)N} \exp(-\frac{1}{3}\eta n N^{-\alpha}) \\
&= \quad O(\frac{\exp(-\frac{1}{3}\eta(1+\eta)N^{1-\alpha})}{1-\exp(-\eta N^{-\alpha}/3)} \\
&= \quad O(\frac{1}{\eta}N^{\alpha}\exp(-\eta N^{1-\alpha}/3) \\
&= \quad O(N^{-\beta}) \\
&= \quad O(b(N)^{\beta/2})
\end{aligned}$$

for any $\beta > 0$.

Next assume $n \leq (1-\eta)N$ and repeat the argument above, replacing Ar by r/A. Since $M \geq (1-\alpha)N \geq (1-\eta/2)N$, we have

$$M - n \geq (1-\eta/2)N - n \geq N\eta/2$$

and hence

$$A^{n-M} \geq A^{-\eta N/2} = \exp(-\frac{1}{2}\eta N^{1-\alpha}).$$

We get

$$\frac{|a_n|\rho^n}{|a_N|\rho^N} \leq \exp(-\frac{1}{2}\eta N^{1-\alpha} + 2N/k) \leq \exp(-\frac{1}{3}\eta N^{1-\alpha}),$$

using the definition of k as before. Now we are summing $\leq N$ terms of the same size, so

$$\begin{aligned} \sum_{n\leq(1-\eta)N} \frac{|a_n|\rho^n}{|a_N|\rho^N} &\leq N\exp(-\frac{1}{3}\eta N^{1-\alpha}) \\ &= O(N^{-\beta}) \end{aligned}$$

for any $\beta > 0$.

Finally we consider $(1-\eta)N < n < N-k$ and $N+k \geq< n < (1+\eta)N$. Recall that $b((1+\eta)N) > (1-\epsilon)b(N)$ and $\rho = re^t$ where $t \leq 2/k$. Lemma **??** implies that for $N-p| \leq \eta n$ (except for a set of finite logarithmic measure),

$$\begin{aligned} \frac{|a_n|\rho^n}{|a_N|\rho^N} &\leq \frac{|a_n|(re^t)^n}{|a_N|(re^t)^N} \\ &\leq \frac{|a_n|r^n}{|a_N|r^N}e^{t(n-N)} \\ &\leq \frac{|a_n|r^n}{|a_N|r^N}e^{t|p|} \\ &\leq \exp(-\frac{1}{2}b(N+\eta N)p^2 + t|p|) \\ &\leq \exp(t|p| - \frac{1}{2}(1-\epsilon)b(N)p^2) \\ &\leq \exp(|p|(t-b|p|)) \end{aligned}$$

where $b = \frac{1}{2}(1-\epsilon)b(N)$. Recall that $1/k = O(\sqrt{b(N)/|\log b(N)|}) = o(\sqrt{b})$, so

$$t = 2/ko(\sqrt{b}) = o(b/\sqrt{b}) = o(bk) \leq \epsilon b|p|,$$

if r is large enough. Thus

$$\frac{|a_n|\rho^n}{|a_N|\rho^N} \leq \exp(-(1-\epsilon)bkp)$$

and thus by the geometric formula

$$\begin{aligned}\sum_{k\le|p|\le\eta N}\frac{|a_n|\rho^n}{|a_N|\rho^N} &\le \sum_{p=k}^{\infty}\exp(-(1-\epsilon)bp^2/2)\\ &= O(\frac{1}{bk}\exp(-bk^2/2))\\ = O(\frac{1}{\sqrt{b}}\exp(-\frac{1}{2}(1-\epsilon)^2 b(N)\gamma b(N)^{-1}&|\log b(N)|))\\ = O(b(N)^{-1/2}\exp(-\frac{\gamma}{2}(1-\epsilon)^2&|\log b(N)|))\\ &= O(b(N)^{-\sigma})\end{aligned}$$

□

Most of the hard work is finished now. All that remains is a simple lemma about polynomials and then we can state and prove the main consequence of the Wiman-Valiron method that we will need for dynamics.

LEMMA 67. *Suppose P is polynomial of degree m and $|P|\le M$ on $\mathbb{D}_r$. Then*

$$|P'(z)|\le eMn|z|^{m-1}r^{-m}.$$

Moreover, if $|P(z)|=M$, then

$$\frac{1}{2}|P(z)|\le|P(w)|\le\frac{3}{2}|P(z)|,$$

for $|w-z|<r/8m$.

PROOF. Since $P(z)/z^m$ is holomorphic on $\mathbb{D}_r^*$ with a removable singularity at ∞, the maximum principle implies $p(Z)\le M|z|^m$. The Cauchy estimate on a circle of radius $t=|z|/m$ around z gives

$$|P'(z)|\le\frac{m}{|z|}\max_{|w-z|=t}|P(z)|\le\frac{m}{|z|}Mr^{-m}|z|^m(1+\frac{1}{m})^m\le emMr^{-m}|z|^{m-1}.$$

If $|z|=r$ and $|z-w|\le r/8m$, then on the line segment between z and w we have

$$|P'|\le emMr^{-m}(r(1+\frac{1}{8m}))^{m-1}\le\frac{4mM}{r}$$

so

$$|P(z)-P(w)|\le|z-w|\frac{4mM}{r}\le\frac{1}{2}M=\frac{1}{2}|P(z)|,$$

which implies the desired estimate. □

Theorem 68. *Suppose f is entire, $r > 0$, $N = N(r, f)$, $|z| = r$, and $|f(z)| = M(r, f)$. Suppose that b and k satisfy the conditions of Lemma 66. Then there is a set E of finite logarithmic measure so that for $r \notin E$,*

$$f(w) = f(z)(\frac{w}{z})^N(1 + O(\delta) + o(1))),$$

whenever $|z - w| \leq \delta/k$ and $r \nearrow \infty$.

Proof. Taking $\sigma = 4$ in Lemma 66 we get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{N-k}^{N+k} a_n z_n + o(|a_N||z|^N b(N)^4,$$

for all large r outside a set of finite logarithmic measure. Using Lemma 64 and our assumption $b(N) = O(1/N)$,

$$f(z) = \sum_{N-k}^{N+k} a_n z_n + o(|a_N||z|^N N^{-4}) = z^{N-k}P(z) + o(M(|z|, f)N^{-4}),$$

where P is a polynomial of degree $\leq 2k$. On the circle of radius r,

$$r^{N-k}|P(z)| = |f(z) - o(M(^f)N^{-4})| \leq |f(z)| + o(M(r, f)) = (1 + o(1))M(r, f).$$

Fix an $\epsilon > 0$ and set $M = (1 + \epsilon)r^{k-N}N(r, f)$. Then if r is large enough, we have $|P| \leq M$ on the circle of radius r. Suppose z is the point on this circle where $|f|$ attains its maximum, $m(r, f)$. Then

$$r^{N-k}|P(z)| = |f(z)| - o(M(r, f)) \geq (1 - o(1))M(r, f),$$

so

$$|P(z)| \geq (1 - o(1)M_0 \geq \frac{1 - \epsilon}{1 + \epsilon}M_0 \geq (1 - 3\epsilon)M_0,$$

if r is large enough (and not in E) and ϵ is small enough. Thus by Lemma 67

$$\frac{1}{2}|P(z)| \leq |P(w)| \leq \frac{3}{2}|P(z)|,$$

for $|z-w| < (1-3\epsilon)r/16k$. For these same w, we have for $\rho = |w|$

$$\begin{aligned} w^{-N}f(w) &= w^{-k}P(w) + o(|a_N|N^{-4}) \\ &= w^{-k}P(w) + o(r^{-N}\mu(r,f)N^{-4}) \\ &= w^{-k}P(w) + o(r^{-N}M(r,f)N^{-4}) \\ &= w^{-k}P(w) + o(r^{-k}MN^{-4}) \\ &= w^{-k}P(w) + o(\rho^{-k}MN^{-4}) \\ &= w^{-k}P(w) + o(\rho^{-k}P(z)) \end{aligned}$$

In the last line we replaced r^k by ρ^k. This is justified since $|r-\rho| \leq \frac{r}{k}$ and hence

$$(1-\frac{1}{k})^k \leq (\frac{r}{\rho})^k \leq (1+\frac{1}{k})^k,$$

which is uniformly bounded above and below. Setting $w = z$ gives

$$\begin{aligned} f(z) &= z^{N-k}P(z) + o(r^{N-k}P(z)) \\ &= \frac{f(z)}{z^{N-k}P(z)} + o(r^{N-k}) \end{aligned}$$

and multiplying equations gives

$$\begin{aligned} f(w) &= f(z)(\frac{w}{z})^{N-k}\frac{P(w)}{P(z)} + o(\frac{\rho^{N-k}P(w)}{P(z)^2r^{2(N-k)}}) + o(r^{N-k}P(z)) \\ f(w) &= f(z)(\frac{w}{z})^{N-k}(1+O(\frac{|P(w)-P(z)|}{|P(z)|}) + o(f(z)) \\ f(w) &= f(z)(\frac{w}{z})^{N-k}(1+O(2k|z-w|) + o(f(z)) \\ f(w) &= f(z)(\frac{w}{z})^{N-k}(1+O(\delta)+o(1)). \end{aligned}$$

□

5. The fast escaping set

Define the escaping set of f as $I(f) = \{z : f^n(z) \to \infty\}$. Out first goal is to show there always are such points.

THEOREM 69. *If f is entire, then $I(f) \neq \emptyset$. In fact, if R is sufficiently large, there is a point z_0 so that $|z_0| > R$ and $f^n(z_0) > M^n(R,f)$ for all $n \geq 1$.*

PROOF. This is easy for polynomials, so we assume f is transcendental. Suppose E is the exceptional set in Theorem 68 and fix R so large that the logarithmic measure of $X = E \cap [R, \infty)$ is less than $1/2$. For $r_1 \in X$, let z_1 be the point where $|z_1| = r_1$ and where $|f(z_1)| = M(r_1, f)$. Let $N = N(r_1, f)$ be the central index for r_1. Let

$$S_5(z) = \{w : |\log \frac{w}{z}_1| \leq 5N, |\arg \frac{w}{z}_1| \leq \frac{5}{N},$$

and note that under the map $w \to f(z)(\frac{w}{z})^N$ the image of S covers the annulus $A_s(z) = \{w : e^{-s}|f(z_1)| < |w| < e^s|f(z_1)|\}$ for $s = 5$. Since by Theorem 68,

$$|f(w) - f(z_1)(\frac{w}{z}_1)^N| \leq e^{-5}|f(z)|,$$

if R is large enough, we can deduce that $f(S_5(z))$ covers $A_4(z)$.

Since $[ef(z), e^2f(z)) \subset A_1(z) \cap [R, \infty)$ has logarithmic length 1, it contains a point r_2 not in E. Let z_2 satisfy $|z_2| = r_2$ and $|f(z_2)| = M(r_2, f)$. We now repeat the construction above, obtaining a sector $S_5(z_2) \subset A_4(z_1)$. Thus $f^{-1}(S_5(z_2))$ has a component $U_2 \subset S_5(z-1)$ which maps univalently to $S_5(z_2)$ via F. Note that $|z_2| \geq |f(z_1)| = M(r_1, f)$.

Continuing in this we way we obtain a sequence of sectors $S_5(z_n)$, each of which contains a preimage under f of the next sector. Thus $S_5(z_1)$ contains a nested sequence of closed sets $\{U_n\}$ mapping univalently to ${}_n$ under f. The intersection of these sets is non-empty and contains only escaping points.

The final claim follows because we always choose $r_n > M(r_{n-1}, f)e > M(r_{n-1}, f)e^{5/N(r_n)}$, so the sector $S_5(z_n)$ lies outside the circle of radius $M(r_{n-1}, f)$. □

It is easy to show the sets $\{U_n\}$ have diameters shrinking to zero. This will be helpful in some later arguments, so we record it as a corollary.

The power map covers the annulus at least three times, so sector has at least two univalent preimages inside the previous sector. Thus the nth sector has 2^{n-1} closed preimages in the first sector. Thus we obtain an uncountable set of escaping points.

THEOREM 70. *If f is transcendental entire then* $\dim_P(A(f)) = 2$.

COROLLARY 71. *If $f \in \mathcal{B}$ then* $\dim_P(J(f)) = 2$.

Alternate definitions of $A(f)$.

THEOREM 72. *The different definitions are equivalent.*

CHAPTER 4

Multiply connected components of the Fatou set

1. Multiply connected components are bounded

LEMMA 73. *If Ω is multiply connected Fatou component of an entire function f, then $f^n \nearrow \infty$ uniformly on compact subsets of Ω.*

PROOF. Suppose γ is closed curve in Ω. By normality any subsequence of $\{f^n\}$ contains a subsequence that converges uniformly, either to ∞ or to a bounded function. In the latter case, the maximum principle implies it converges uniformly to a bounded on the whole interior or γ, and therefore the interior of γ is in the Fatou set. Thus if the Fatou set contains a curve γ surrounding a point in the Julia set, every subsequence of f^n must converge uniformly to ∞ on γ. Normality then implies it converges to ∞ on every compact subset of component containing γ. □

LEMMA 74. *If Ω is multiply connected Fatou component of an entire function f, and $\gamma \subset \Omega$ surrounds a point of the Julia set, then $f^n(\gamma)$ has positive index with respect to 0 for all sufficiently large n.*

PROOF. Suppose the index is zero for an infinite subsequence of iterates $\{f^{n_k}\}$. Thus f^{n_k} never vanishes inside γ, and tends to ∞ uniformly on γ. The minimum principle implies f^{n_k} tends to ∞ inside γ. This contradicts that pre-periodic points are dense in the Julia set, Corollary 53 . □

LEMMA 75 (Töpfer). *If the Fatou set of a transcendental entire function has an unbounded component, then all other components are simply connected.*

PROOF. □

THEOREM 76 (Baker []). *If f is a transcendental entire function, then multiply connected component of the Fatou set is bounded.*

Proof. Suppose not, i.e., suppose Ω is an unbounded multiply connected Fatou component. By replacing f by f^n if necessary, and normalizing by linear maps, we may assume 0 and 1 are fixed points (repelling?) and that 0 has infinitely many pre-images under f.

By Lemma **??**, Ω contains a closed curve γ on which f^n converges uniformly to ∞ and surrounds 0. Thus for any large r we can choose n so that $\gamma_1 = f^n(\gamma)$ has distance at least r from the origin. We choose r so large that $t \geq r$ implies

$$\frac{1}{8}M(\frac{1}{4}t, f) > t.$$

Let R be the greatest distance of $f^n(\gamma)$ from the origin and choose a second closed curve $\gamma_2 = f^m(\gamma)$ so that its distance to 0 is $s > M(2R, f^2))$. Now join γ_1 to γ_2 by a arc γ_3 in Ω. Let $K = \gamma_i \cup \gamma_2 \cup \gamma_3$.

The iterates of f on Ω omit all values in the Julia set and hence omit at least two distinct values. By Lemma **??** there are constants B, C so that

$$\max_K |f^n(z)| \leq B(\min_K |f^n(z)|)^C. \tag{16}$$

However, by the choice of s, if $z_2 \in \gamma_2$ is the point where $|f^n|$ is maximized on γ_2, then

$$\begin{aligned} |f^n(z_2)| &\geq M(s, f^n) \\ &\geq M(M(2R, f^2), f^n) \\ &\geq M(2R, f^{n+2}) \\ &\geq M(\frac{1}{8}M(R, f^{n+1}), f), \end{aligned}$$

by Polya's lemma (Lemma **??**). Since f is transcendental, $M(r, f)$ grows faster than any polynomial. In particular $M(r, f) \geq Br^C$ for the constants B and C chosen above, if r is large enough. Since $M(R, f^{n+1}) \nearrow \infty$ as $n \nearrow \infty$, we can therefore choose n so large that

$$\begin{aligned} M(\frac{1}{8}M(R, f^{n+1}), f), &\geq B(M(R, f^{n+1}))^C \\ &\geq B(M(\frac{1}{8}M(R/2, f), f^n))^C \\ &\geq B(M(R, f^n))^C \\ &\geq B(\min_K |f(z)|)^C \end{aligned}$$

since K contains γ_1 which is contained inside $\mathbb{D}_R$. This is a contradiction to (16) and proves the result. $\square$

Corollary 77. *The Julia set of a transcendental entire function contains a non-trivial continuum.*

Corollary 78. *If f is a transcendental entire functions then every multiply connected component of the Fatou set is a wandering domain.*

Proof. We already know that multiply connected components are bounded and iterate to infinity uniformly on compact sets, so they can't be periodic. If they were pre-periodic they would have to land on a periodic domain that iterates to infinity, i.e., a Baker domain. However such a domain is unbounded, whereas $f(U)$ must be bounded. Thus there is a point $w \in V \setminus f(U)$ and a sequence $\{z_n\}$ in U so that $f(z_n) \to w$ Since U is bounded we can pass to a subsequence such $z_n \to z \in \overline{U}$. If $z \in U$, then $w = f(z) \in f(U)$, a contradiction. If $z \in \partial U$ then z is in the Julia set, so w is in the Julia too, also a contradiction. Thus there are no pre-periodic, multiply connected Fatou components. $\square$

We just proved that a bounded Fatou component cannot map into an unbounded component. In fact, if f is a transcendental entire function and U, V are components of $\mathcal{F}(f)$ such that $f(U) \subset V$, then $E = V \setminus f(U)$ has had most one point. This is due to M. Herring [], with another proof give by Bergweiler and Rohde [].

Corollary 79. *If f is transcendental, any completely invariant Fatou component is simply connected.*

Proof. A completely invariant Fatou component Ω must be unbounded (since by Picard's great theorem $f^{-1}(\Omega)$ contains unbounded sets) and hence is simply connected by Baker's theorem. $\square$

Corollary 80. *If f is a transcendental entire function with an unbounded Fatou component, then all Fatou components are simply connected.*

Proof. By Baker's theorem the unbounded component Ω is simply connected. If U is a multiply connected component, then by Lemma **??**, it contains a curve γ whose iterates $f^n(\gamma)$ intersect Ω. Thus Ω contains a closed curve that winds around $\mathbb{D}_r$ for any $r > 0$. Since the Julia set is non-empty this means Ω is not simply connected, contradicting Baker's theorem. Therefore there are no multiply connected Fatou components. $\square$

COROLLARY 81. *If f is a transcendental entire function that is bounded along a curve σ tending to ∞, then all Fatou components are simply connected. In particular, this happens if f has a finite asymptotic value.*

PROOF. As in the previous proof, if there were a multiply connected Fatou component, then there would also be a closed curve γ in it whose iterates tend to ∞ and wind around zero. All iterates of γ eventually hit σ and f at the intersections tends to ∞, contradicting that f is bounded on σ. Thus there are no multiply connected Fatou components. $\square$

THEOREM 82 (Baker, 1970, EL). *There is at most one completely invariant Fatou component.*

THEOREM 83. *If f is a transcendental entire function with an completely invariant Fatou component, then f is univalent on all other Fatou components.*

THEOREM 84. *If f is a transcendental entire function then $\mathcal{J}(f) \cap I(f) \neq \emptyset$.*

PROOF. There are two cases depending on whether there are multiply connected Fatou components or not.

If there is a multiply connected component Ω, then by Lemma **??** there is a closed curve γ in Ω that eventually surrounds every point in the plane. Since Ω wanders and iterates to ∞, $\Omega_m = f^m(\Omega)$ is eventually outside $\gamma_n = f^n(\gamma)$ and hence $\partial\Omega_m$ is outside γ_n. Thus $\partial\Omega$ escaping points in the Julia set.

If there are no multiply connected Fatou components, consider the escaping point z constructed in the proof of Theorem **??**. It is the intersection of nested sets $B_1 \supset B_2 \supset \dots$ whose diameters shrink to zero and with the property that $f^{n+1}(B_n)$ covers an annulus A_n that eventually surrounds every point. If z is in the Fatou set, so if B_n for large n and hence so are the annuli A_n. Thus the Fatou set has multiply connected components, a contradiction. Hence $z \in \mathcal{J}(f) \cap I(f)$. $\square$

THEOREM 85. *The closure of the escaping set has no bounded components.*

PROOF. Since $\mathcal{J}(f) \subset \overline{I(f)}$, if there is a bounded component of $\overline{I(f)}$, it is surrounded by a curve in a non-escaping component of the Fatou set. But this component is then multiply connected component, hence it does escape. □

Eremenko conjectured that all the components of $I(f)$ are unbounded. This is still open, but the so called "Strong Eremenko Conjecture" that all path components of $I(f)$ are unbounded has been disproven by ??????. See Theorem **??**. Partial progress towards Eremenko's conjecture has been given by Rippon and Stallard who showed that $I(f)$ always contains at least one unbounded component.

2. Multiply connected components are fast escaping

DEFN of fast escaping:
equivalence of different definitions

THEOREM 86. *Multiply connected Fatou components are in $A(f)$.*

THEOREM 87 (Zhang). *If f is a transcendental entire function with multiply connected Fatou component U and $\gamma \subset A \subset U$ is closed curve surrounding a Julia point and $\gamma \subset A \subset U$ is any domain then*

$$\{z : r_n < |z| < R_n\} \subset f^b(A) \subset f^n(U)\}$$

where $R_n/r_n \to \infty$.

Thus iterates of multiply connected Fatou components contain large round annuli centered at the origin. In particular, the Julia set is not uniformly perfect in this case.

Stronger version:

THEOREM 88 (Bergweiler, Rippon, Stallard). *If f is a transcendental entire function with multiply connected Fatou component U and $D = D(z_0, r) \subset U$, then*

$$\{z : r_n < |z| < R_n\} \subset f^b(D) \subset f^n(U)\}$$

where $\liminf_n \log R_n / \log r_n > 1$.

There are examples where the liminf is as close to 1 as desired.

Every point in the multiply connected component U eventually iterates into a maximal round annulus in $f^n(U)$ (actually into a smaller annulus).

3. Eventual connectivity

THEOREM 89 (Kisaka and Shishikura). *The eventual connectivity of a multiply connected Fatou component is* 2 *or* ∞.

THEOREM 90. *If* f *is a transcendental entire function with multiply connected Fatou component* U.

1. *If* $c(U_n) = 2$ *then* $\cup_{m=n}^{\infty} U_n$ *contains no critical point of* f.
2. *If* $c(U_n) < \infty$, *then* $c(U_m)$, $m \geq n$ *is non-increasing and eventually equals* 2.
3. *If* $c(U_n) = \infty$ *then* $c(U_m) = m$ *for all* $m \geq n$.

THEOREM 91 (Kisaka and Shishikura). *There exists a transcendental entire function with a component with eventual connectivity* 2.

THEOREM 92 (Bergweiler, Rippon and Stallard). *If* f *is a transcendental entire function with multiply connected Fatou component* U.

1. $c(U_n) = 2$ *iff* $\cup_{m=n}^{\infty} U_n$ *contains no critical point of* f.
2. $2 < c(U_n) < \infty$ *iff* $\cup_{m=n}^{\infty} U_n$ *contains a finite number of critical points.*
3. *If* $c(U_n) = \infty$ *iff* $\cup_{m=n}^{\infty} U_n$ *contains infinitely many critical points.*

4. Buried points of the Julia set

THEOREM 93. *If* f *is a transcendental entire function that has multiply connected Fatou components, then* $\mathcal{J}(f)$ *has connected components that are single points and not on the boundary of any Fatou component. Moreover, such points are dense in* $\mathcal{J}(f)$.

CHAPTER 5

Baker domains

1. Rate of escape

Theorem 94. *If f is a transcendental entire function and $z \in \Omega$, a Baker domain for f, then there is a $1 \leq C < \infty$ so that*

$$\frac{C}{|z|} \leq |f(z)| \leq C|z|$$

. Thus

$$\log|f^n(z)| = O(n).$$

2. Classification of Baker domains

Baker domains are simply connected.

Invariant domain; f corresponds to iteration of inner function converging to boundary.

Denjoy-Wolff theorem

Classification based on conjugacy.

Theorem 95 (Baker-Domínguez). *Let Ω be a Baker domain on which f is not univalent. Then Ω has uncountably many different ends to ∞ and $\partial\Omega$ has infinitely many components.*

3. Singular points

We have already seen (Theorem **??**) that if $S(f)$ is bounded, then f has no Baker domains. In fact, if there are Baker domains, then there large singular points must be fairly common in the following sense.

Theorem 96 (Bargmann). *If f is a transcendental entire function that has a Baker domain, then there is a $1 \leq C < \infty$ so that $S(f) \cap \{z : r < |z| < Cr\} \neq \emptyset$ for all sufficiently large $r > 0$.*

LEMMA 97. *There is a f transcendental entire function that has a Baker domain Ω that contains no singular values.*

THEOREM 98 (Bergweiler). *If f is a transcendental entire function that has a Baker domain Ω such that $\Omega \cap S(f) = \emptyset$, then there is a sequence of complex numbers $p_n \to \infty$ such that $p_n \in P(f)$ (the post-singular set of f), $|P_{n+1}/p_n| \to 1$ and $\operatorname{dist}(p_n, \Omega) = o(|p_n|)$.*

THEOREM 99 (Bergweiler). *Let f be transcendental entire of finite order such that*

$$f(z) = z + a + o(1),$$

as $z \to \infty$, $|\arg z| \leq \eta$ for some $a, \eta > 0$. Then f has an invariant Baker domain Ω which contains

$$\{z = x + iy : |\arg z| \leq \eta, > x > R\},$$

for some $R > 0$ and $\Omega \cap S(f)$ is unbounded.

4. Examples

CHAPTER 6

The Eremenko-Lyubich class

1. The singular set

Suppose f is a transcendental entire function. A critical point of f is a zero of f' and a critical value is $f(z)$ where z is a critical point. A asymptotic value is a $w \in \widehat{\mathbb{C}}$ so that $\lim f(z) = w$ along a curve $\gamma : [0, \infty)$ that tends to ∞.

The assumption that γ tends to ∞ is stronger than we need. If f has a limit along a path γ that is unbounded, but returns to some compact set infinitely often, then there is some annulus $A = \{z : r < |z| < 2r\}$ that gamma crosses infinitely often and hence subarcs $\gamma_n \subset A$ with endpoints on different boundary components such that $|f - a| < 1/n$ on γ_n. This implies $f = a$ at some point on every circle $\{|z| = t\}$ for $r < t < 2t$ and hence f is constant. Thus to check that a is an asymptotic value of f, we only need verity f has limit a along some unbounded path (the fact that γ tends to∞ is then automatically satisfied).

The singular values of f, $S(f)$ is defined as the closure or the union of critical values and finite asymptotic values. This is not completely standard; some authors do not take the closure in this definition. The primary importance of the singular set is that f acts as a covering map off $S(f)$., i.e.,

Lemma 100. *If $a \in \mathbb{C} \setminus S(f)$, then a has a neighborhood U so that $f : f^{-1}(U) \to U$ is conformal map on each component of $f^{-1}(U)$.*

Proof. Choose a disk $D = D(a, \epsilon)$ around a that misses $S(f)$. If $f^{-1}(\frac{1}{2}D)$ has an unbounded component W, then cover $\frac{1}{2}D$ by a finite number of disks of half the radius; one of these must have an unbounded preimage as well. Continuing in this way we can can construct a point $b \in D$ and a sequence of nested disks $D_1 \subset D_2 \supset D_3 \supset \dots$ shrinking down to b so that each has an unbounded preimage $W_1 \supset W_2 \supset W_3 \supset \dots$. Choose a point $z_n \in W_n$ with $|z_n| > n$ and let γ_n be a arc in W_n that joins z_n to z_{n+1}. Then joining these arcs end-to-end creates an unbounded path γ and since

$f(\gamma_n) \subset D_n$, it is clear f approaches $b \in D$ along γ. Thus f has a finite asymptotic values in D, contrary to assumption.

Therefore there is an ϵ so that every preimage of D is bounded. If W is such a component, then each point of ∂W must map to ∂D. Thus $f : W \to D$ is a proper map and so must be a finite-to-1 branched covering. Since D contains no critical values, W contains no critical points, so the covering must be 1-to-1, $\square$

Singular values are further classified by the behavior of f. For $a \in \widehat{\mathbb{C}}$ let D_r be a disk of radius r around a in the spherical metric and let $\{U_r\}$ be a component of $f^{-1}(D_r)$ chosen so that $U_r \subset U_s$ if $r < s$. Then $E = \cap r > 0U(r)$ is a connected set and $f = a$ on E, so either $E = \emptyset$ or E is a single point (otherwise f would be constant on a non-trivial connected set, hence constant everywhere). If E is a single point z and $f'(z) \neq 0$ then z is called a regular a-point. If $E = \{z\}$ and $f'(z) = 0$ then z is a critical a-point. If $E = \emptyset$ the a is an asymptotic value of f (argue as in Lemma 100). In this case, the tract $\{U_r\}$ is called an asymptotic tract of f.

LEMMA 101. *Suppose f attains the value $a \in \widehat{\mathbb{C}}$ only finitely often. Then a is a direct asymptotic value.*

2. Class $\mathcal{B}$

Suppose f is a transcendental entire function. If $S(f)$ is finite, we say f is finite type or in the Speiser class, denoted $\mathcal{S}$. If $S(f)$ is bounded, we say f is bounded type or in the Eremenko-Lyubich class, denoted $\mathcal{B}$. A little care needs to be taken with the terms "finite type" and "bounded type" since these are also used to mean something different in Nevanlinna theory.

Suppose $f \in \mathcal{B}$ and choose $R > 0$ so that $S(f) \subset \mathbb{D}_R$. Let $\Omega = f^{-1}(\mathbb{D}_R^*)$. By Lemma 100, f is a covering map from each component of Ω to $\mathbb{D}_R^*$. Since $\mathbb{D}_R^*$ is unbounded, each component of Ω is unbounded, but $|f| = R$ on the boundary. Thus by Corollary 81 we get

LEMMA 102. *If $f \in \mathcal{B}$, then every component of $\mathcal{F}(f)$ is simply connected.*

The right half-plane $\mathbb{H}_r = \{x + iy : x > 0\}$ is a simply connected covering space of $\mathbb{D}_R^*$ with the map $z \to R\exp(z) = \exp(z + \log R)$. Thus if W is a component of Ω

we can lift $f : W \to \mathbb{D}_R^*$ to a conformal map $\tau : W \to \mathbb{H}_r$, so that $f(z) = R\exp(\tau(z))$ on W.

Choose a point $a \not{i}nW$ and let $U = \log(W - a)$. Since W is simply connected, this is well defined and single valued and the domain U intersects each vertical line in length $\leq 2\Pi$. In particular, no point of U is more than distance π from some boundary point. If $R > |f(0)|$ then we can take $a = 0$; we shall make this assumption from here on.

Define $F : U \to \mathbb{H}_r$ by $F(z) = \tau(a + \exp(z))$. Then F is conjugate to f in the sense that

$$\exp(F(z)) = f(a + \exp(z)),$$

and

$$\exp(F^n(z)) = f^n(\exp(z)),$$

if the orbit of z stays inside Ω. We refer to F as "f in logarithmic coordinates".

Lemma 103. *$|F'(z)| \geq \frac{1}{4\pi}\Re(\tau(z))$.*

Proof. If we apply the Koebe $\frac{1}{4}$-theorem at z, we get

$$\pi|F'(z)| \geq \text{dist}(z, \partial U)|F'(z)| \geq \frac{1}{4}\text{dist}(F(z), \partial\mathbb{H}_r) = \frac{1}{4}\Re F(z).$$

□

Theorem 104. *If $f \in \mathcal{B}$ then $I(f) \subset \mathcal{J}(f)$.*

Proof. Suppose not, i.e., suppose there is a point $z \in \mathcal{F}(f) \cap I(f)$ and let D be a closed disk centered at z and inside $\mathcal{F}(f)$. Then f^n converges uniformly to ∞ on D and hence $f^{n+1}(D) \subset \mathbb{D}_R^*$ for n large enough. This means $f^n(D) \subset \Omega = f^{-1}(\mathbb{D}_R^*)$ for all $n \geq m$, for some m. By replacing D by a disk centered at z_m, we may assume $m = 0$.

Let $w = \log(z) \in U$ and let $V = \log D \subset U$. Then the iterates of V under F stay in U forever, Then $w_n = F^n(w) = \log(f^n(z))$. Since $|f^n(z)| \to \infty$, we have $\Re F(w_n) \to +\infty$, and hence

$$(F^n(w))' = \prod_{k=1}^{n} F'(w_k) \geq \prod_{k=1}^{n} \frac{\Re(w_{k+1})}{4\pi}.$$

Since w_k is never a critical point of F and since $\Re w_k \to \infty$, the product on the right tends to ∞. However, by Koebe's $\frac{1}{4}$-theorem

$$|(F^n(w))'| \leq 4\text{dist}(w_n, U)/\text{dist}(w, V) \leq 4\pi/\text{dist}(w, V),$$

is bounded independent of n. $\square$

Note that although Fatou points don't escape, they can have unbounded orbits (but these orbits return to some compact set infinitely often).

Corollary 105. *If $f \in \mathcal{B}$ then $\mathcal{J}(f) = \overline{(}I(f))$.*

Proof. For any transcendental entire function we have $J(f) = \partial I(f) \subset \overline{I(f)}$. We have just proved $I(f) \subset \mathcal{J}(f)$, hence $\overline{I(f)} \subset \mathcal{J}(f)$. Thus equality holds. $\square$

3. Geometry of bounded type level sets

4. The strong Eremenko conjecture fails in $\mathcal{B}$

5. The Speiser class

Cosh coordinates

Finite dimensional

Non-wandering

QC folding with two critical values

QC folding in general

6. Completely invariant domains

Lemma 106. *If f is a transcendental entire function and Ω is a completely invariant Fatou component, then all critical values and logarithmic singularities of f are contained in Ω*

Theorem 107. *If $f \in \mathcal{S}$ has a completely invariant Fatou component then it is the only Fatou component.*

7. $\mathcal{J}(e^z) = \mathbb{C}$

.

The Julia set can be the whole plane. The classification of Fatou components implies this is true whenever all the singular values are strictly pre-periodic, but it is fairly easy to see that it is also true for other examples such $f(z) = \exp(z)$. This was first verified by Misiurewicz [], but we will give a proof due to Rempe and Mihaljević-Brandt and Rempe.

Theorem 108. $\mathcal{J}(e^z) = \mathbb{C}$.

Proof. Since $f(z) = e^z$ has no critical points and one finite asymptotic value, it is in the Speiser class and hence the Julia set is the closure of the escaping set. Suppose the Fatou set is non-empty and let W be a component. Let $z \in W$. Since $\mathbb{R} \subset I(f)$, W must avoid $f^{-1}(\mathbb{R})$; the latter set consists of horizontal lines at heights πn, $n \in \mathbb{Z}$ and f is 1-to-1 on the strips

$$S_n = \{x + iy : n\pi < y < (n+1)\pi\}.$$

is univalent on W and hence $|\nabla_H(z)| = 1$, where we take the hyperbolic metric on W and $f(W)$. Since f is conformal from S to the upper or lower half-plane $\mathbb{H}$ it is an isometry from the hyperbolic metric on S_n to the hyperbolic metric on the half-plane. However inclusion map S_n into its corresponding half-plane is a contraction of hyperbolic metrics; a strict contraction if $|y| > \epsilon$. Thus $|\nabla_H f(x+iy)| \leq \eta(|y|)$, where η is strictly decreasing and $\eta(0) = 1$ and we consider the hyperbolic metrics on $U = \{|y| > 0]\}$. Since

$$\begin{aligned}
1 = |\nabla_W^W f^n(z)| &= |\nabla_W^U \mathrm{Id(z)}| \cdot |\nabla_{\mathrm{U}}^{\mathrm{U}} \mathrm{f^n(z)}| \cdot |\nabla_{\mathrm{U}}^{\mathrm{f^n(W)}} \mathrm{Id(f^n(z))}| \\
&= |\nabla_W^U \mathrm{Id(z)}| \cdot \prod_{\mathrm{k=0}}^{\mathrm{n-1}} |\nabla_{\mathrm{U}}^{\mathrm{U}} \mathrm{f(f^k(z))}| \cdot 1 \\
&\geq |\nabla_W^U \mathrm{Id(z)}| \cdot \prod_{\mathrm{k=0}}^{\mathrm{n-1}} \eta(\Im((\mathrm{f^k(z)})),
\end{aligned}$$

we see that $\lim_k \Im(f^k(z)) = 0$. However, $|\Im(z)| \leq 1/3$ implies $\Re(f(z)) \geq \Re(z) + (1 - \ln 2)$, so that $\Re f^n(z) \nearrow +\infty$. However, $x > 1$, $|y| < \pi/4$ implies $\Im(f(x+iy) > x$. Thus the sequence $\{|\Im(f^n(z))|\}$ is both eventually increasing and converging to zero. This contradiction proves that $\mathcal{F}(f) = \emptyset$. □

8. $I(e^z)$ is connected

Theorem 109. *Let* $f_a(z) = \exp(z) + a$. *Then one of the following holds*

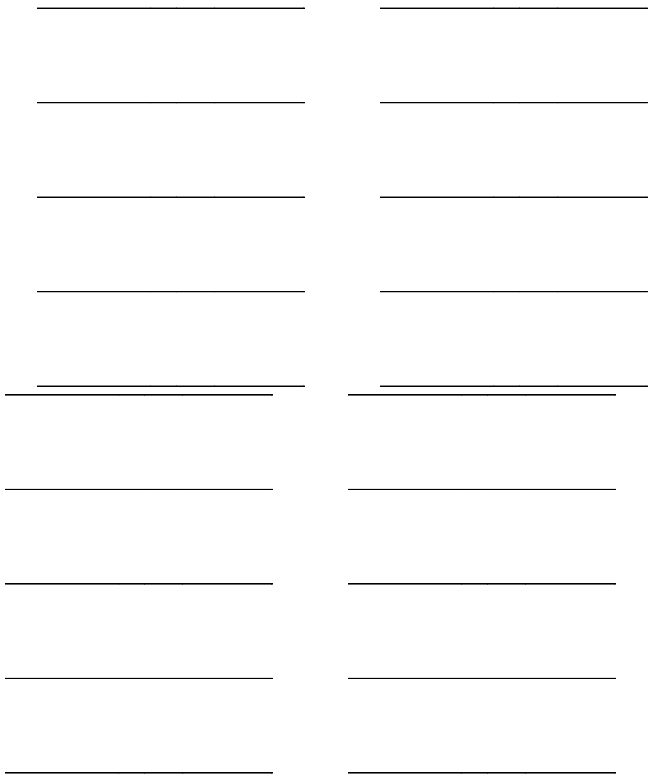

FIGURE 1. The inverse image of $\mathbb{R}$ under e^z is $\mathbb{R} + \frac{\pi}{2}i\mathbb{Z}$; the collection of parallel lines shown at upper left. A preimage of one of these lines is added at upper right. Every other image is a horizontal translate of this by $\log(\frac{\pi}{2}n)$ or a vertical translate of one of these curves. We wish to show that if we continue taking inverses, the resulting system of curves is dense in the plane.

(1) *f_a has a unique attracting or neutral rational cycle and the Fatou set is the attractive region for this cycle. The area of the Julia set is zero and the singular point a belongs to the immediate basin of the periodic cycle but is not pre-periodic.*

(2) *f_a has a cycle of Siegel disks.*

(3) $\mathcal{J}(f_a) = \mathbb{C}$.

CHAPTER 7

Dimension of the Julia set

1. Some basics

THEOREM 110. *Suppose f is a transcendental entire function. Then* $\dim(\mathcal{J}(f)) > 0$.

2. Explosions

3. Positive area

4. Zero area, dimension 2

5. Any dimension between 1 and 2

6. Packing dimension 2

7. Growth rates and dimension

8. Dimension 1

CHAPTER 8

Dimension results for class $\mathcal{B}$.

1. EL implies dimension > 1

2. Zero area for EL class

3. $\dim(I(f) = 1$ is possible

Rempe-Stallard
Any dimension in $[1, 2]$ is possible for $I(f)$ in $\mathcal{B}$.

4. Affine invariance of $\dim(I(f))$.

5. Packing dimension is always 2 in $\mathcal{B}$

6. Eventual dimension

APPENDIX A

Background material

1. A first course in complex analysis

Cauchy integral therorem

Cauchy estimates

Uniform convergece

Harnack's inequality

Theorem 111 (Weierstrass). *If $\{f_n\}$ are holomorphic on Ω and converge uniformly on compact sets of Ω to f, then f is holomorphic and $f_n' \to f'$ uniformly on compact sets.*

2. Distortion of conformal maps

In this section we give the "usual" proof of Koebe's $\frac{1}{4}$ theorem, via the area theorem and deduce the sharp version of the distortion estimates. An alternate derivation using symmetry properties of extremal length is given in Section **??** of the Appendices.

Recall Green's theorem,

$$\iint_\Omega u\Delta v + v\Delta u dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n} ds, \tag{17}$$

where n denotes the inward pointing normal vector of $\partial\Omega$.

$$COMPLEXVERSION$$

We will also use Green's theorem in the following form:

$$\int_{\partial\Omega} f(x,y)dx + g(x,y)dy = \iint_\Omega \frac{\partial g}{\partial x} - \frac{\partial f}{\partial g} dxdy \tag{18}$$

and its simple consequence that the area of a region Ω is given by

$$\text{area}(\Omega) = \frac{1}{2}\int_{\partial\Omega} xdy - ydx = \frac{1}{2i}\int \partial\Omega\bar{z}dz. \tag{19}$$

We now come to some well known (but perhaps not as well known as the results above) estimates for univalent mappings. The basic idea is to show that a univalent map f on $\mathbb{D}$ is well approximated by its linear Taylor approximation $f(z_0)+f'(z_0)(z-z_0)$ in a hyperbolic neighborhood of z_0, with estimates that do not depend on f or z. These so called "distortion estimates" are fundamental to most arguments in geometric function theory. The first step is to prove:

THEOREM 112 (Area theorem). *Suppose* $g(z) = \frac{1}{z} + b_0 + b_1 z + \dots$ *is univalent in* $\mathbb{D}$. *Then* $\sum_{n=0}^{\infty} n|b_n|^2 \le 1$. *In particular,* $|b_1| \le 1$.

PROOF. For $0 < r < 1$ let $D_r = \mathbb{C} \setminus g(D(0,r))$. If $z = g(w)$ and $w = e^{i\theta}$ then $dw = iwd\theta$, so by (19),

$$\text{area}(D_r) = \iint_{D_r} dxdy = \frac{1}{2i}\int_{\partial D_r} \bar{z}dz = \frac{-1}{2i}\int_{\partial D(0,r)} \bar{g}(w)g'(w)dw.$$

To evaluate the right hand side note that

$$\begin{aligned} g(z) &= \frac{1}{z} + b_0 + b_1 z + \dots, \\ g'(z) &= 1\frac{1}{z^2} + 0 + b_1 + 2b_2 z + \dots, \end{aligned}$$

so that

$$\begin{aligned} \int_{|w|=r} \bar{g}(w)g'(w)dw &= i\int \bar{g}(w)g'(w)wd\theta \\ &= i\int (\frac{1}{\bar{w}} + \bar{b}_0 + \bar{b}_1\bar{w} + \dots)(-\frac{1}{w} + b_1 w + 2b_2 w + \dots)d\theta \\ &= 2\pi i(-\frac{1}{r^2} + |b_1|^2 r^2 + 2|b_2|r^4 + \dots 0 \end{aligned}$$

Thus,

$$0 \le \text{area}(D_r) = \pi(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}).$$

Taking $r \to 1$ gives the result. $\square$

COROLLARY 113. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is univalent on the unit disk, then* $|a_2| \le 2$.

PROOF. Let $g(z) = (f(z^2))^{-1/2} = 1/z - a_2 z/2 + \dots$. We claim g is one-to-one. To see this suppose $g(z) = g(w)$. Then $f(z^2) = f(w^2)$, so $z = \pm w$. Note that g is odd, so $z = w$. Since $b_1 = a_2/2$, the previous result implies $|a_2| \le 2$. $\square$

Theorem 114 (Koebe 1/4 theorem). *If f is univalent on $\mathbb{D}$, then*

$$\frac{1}{4}|f'(z)|(1-|z|^2) \leq \operatorname{dist}(f(z), \partial\Omega) \leq |f'(z)|(1-|z|^2).$$

Proof. By pre-composing with a Möbius transformation and post-composing by a linear map, we may assume $z = 0$, $f(0) = 0$ and $f'(0) = 1$. Then the right hand inequality is just Schwarz's lemma applied to f^{-1}. To prove the left hand inequality, suppose f never equals w in $\mathbb{D}$. Then

$$g(z) = \frac{wf(z)}{w - f(z)} = z + (a_2 + \frac{1}{w})z^2 + \dots,$$

is univalent with $f(0) = 0$ and $f'(0) = 1$. Applying Corollary 113 to both f and g gives

$$\frac{1}{|w|} \leq |a_2| + |a_2 + \frac{1}{w}| \leq 2 + 2 = 4.$$

Thus the omitted point w lies outside $D(0, 1/4)$, as desired. □

3. The distortion theorems for conformal maps

Lemma 115. *Suppose f is univalent on $\mathbb{D}$, $f(0) = 0$ and $f'(0) = 1$. Then*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3},$$

Proof. Fix a point $w \in \mathbb{D}$ and write the Koebe transform of f,

$$F(z) = \frac{f(\tau(z)) - f(w)}{(1-|w|^2)f'(w)},$$

where

$$\tau(z) = \frac{z+w}{1-\bar{w}z}.$$

This is univalent, so by Corollary 113, $|a_2(w)| \leq 2$. Differentiation and setting $z = 0$ shows

$$F'(z) = \frac{f'(\tau(z))\tau'(z)}{(1-|w|^2)f'(w)},$$

$$F''(z) = \frac{f''(\tau(z))\tau'(z)^2 + f'(\tau(z))\tau''(z)}{(1-|w|^2)f'(w)},$$

$$\tau'(0) = 1-|w|^2, \tau''(0) = -2(1-|w|^2),$$

$$F''(0) = \frac{f''(w)}{f(w)}(1-|w|^2) - 2\bar{w}.$$

This implies that the coefficient of z^2 (as a function of w) in the power series of F is

$$a_2(w) = \frac{1}{2}((1-|w|^2)\frac{f''(w)}{f'(w)} - 2\bar{w}).$$

Using $|a_2| \leq 2$ and multiplying by $w/(1-|w|^2)$, we get

$$|\frac{wf''(w)}{f'(w)} - \frac{2|w|^2}{1-|w|^2}| \leq \frac{4|w|}{1-|w|^2}.$$

Thus

$$\frac{2|w|^2 - 4|w|}{1-|w|^2} \leq \frac{wf''(w)}{f'(w)} \leq \frac{4|w| + 2|w|^2}{1-|w|^2}.$$

Now divide by $|w|$ and use partial fractions,

$$\frac{-1}{1-|w|} + \frac{-3}{1+|w|} \leq \frac{1}{|w|}\frac{wf''(w)}{f'(w)} \leq \frac{3}{1-|w|} + \frac{1}{1+|w|}$$

Note that

$$\begin{aligned} \frac{\partial}{\partial r}\log|f'(re^{i\theta})| &= \frac{\partial}{\partial r}\mathrm{Re}\log f'(z) \\ &= \mathrm{Re}\frac{z}{|z|}\frac{\partial}{\partial z}\log f'(z) \\ &= \frac{1}{|z|}\mathrm{Re}(\frac{zf''(z)}{f'(z)}) \end{aligned}$$

Since $w = re^{i\theta}$ and $f'(0) = 1$, we can integrate to get

$$\log(1-r) - 3\log(1+r) \leq \log|f'(re^{i\theta})| \leq -3\log(1-r) + \log(1+r).$$

Exponentiating gives the result. $\square$

4. Quasiconformal maps

Three definitions of QC maps

Quasisymmetry

Hölder continuity

Compactness of K-QC maps

3-point condition

Measurable Riemann mapping theorem

Removable sets

5. Approximation theorems

Given a compact set $E \subset \mathbb{C}$, the "holes" of E are the bounded complementary components of E. E does not separate the plane iff there are no holes.

Theorem 116 (Runge's Theorem). *If $K \subset \mathbb{C}$ does not separate the plane and f is holomorphic on a neighborhood U of K, then f can be uniformly approximated on K by holomorphic polynomials.*

Proof. Surround K by a piecewise smooth curve γ in U (e.g., set $\epsilon = \text{dist}(K, \partial U)/10$ and cover K by ϵ-boxes and take the boundary of the unbounded complementary component). Then use the Cauchy integral formula to write

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw.$$

For any $\epsilon > 0$ we can find a finite set of points $\{w_j\}$ on γ so that

$$|\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \sum_j \frac{f(w_j)}{w_j - z}| < \epsilon,$$

for all $z \in K$. Thus

$$|f(z) - \frac{1}{2\pi i} \sum_j \frac{f(w_j)}{w_j - z}| < \epsilon,$$

gives an approximation of f by rational functions with poles off k.

Next we use a "pole-pushing" argument to show that each pole can be uniformly approximated on K by a polynomials. Choose a disk $\mathbb{D}_r$ large enough to contain K and choose one of the n poles $w = w_j$ as above. Fix $\epsilon > 0$ Connect w w_j to $\partial \mathbb{D}_r$ by a piecewise linear path and choose a finite collection of points $w = z_0, z_1, \dots z_n \in \partial \mathbb{D}_r$ so that $D(z_k, 2|z_k - z_{k-1}|) \cap K = \emptyset$. Then any rational function r_{k-1} with poles only at z_{k-1} has a Laurant expansion at z_k that converges on K and ny truncating this series, r_{k-1} can be approximated to within $2^{-k}\epsilon/n$ by a rational function with poles only at z_k. When we reach r_k, we can approximate r_n by its Taylor series in $\mathbb{D}_r$ to obtain a polynomial approximation to $(w-z)^{-1}$ that is within ϵ/n. Summing over all the simple poles on γ and letting $\epsilon \to 0$ we obtain a uniform polynomial approximation to f on K. $\square$

Suppose E is a closed set that does not separated the plane. E is called an **Arkelian set** if the holes of $E \cup D$ form a bounded set for every closed disk in $\mathbb{C}$. This can also be stated by saying that E is locally connected at ∞.

Theorem 117 (Arakelian). *If E is an Arakelian set and f is holomoprhic on a neighborhood of E, then for any $\epsilon > 0$ there is an entire function g such that*

$$\sup_{z \in E} |f(z) - g(z)| < \epsilon.$$

Proof. We follow the proof of Rosay and Rudin [**?**].

Choose nested closed disks $\{D_k\}_1^\infty$ centered at the orgin and filling the plane so that $\frac{1}{4}D_{k+1}$ contains D_k and H_k, the closure of the union of holes of $E \cup D_k$. Let E_k be the $E \cup D_k \cup H_k$. Note that E_k has no holes.

Set $h_0 = f$ and assume by induction that we have a holomorphic function h_{k-1} on neighborhood U of E_{k-1}. Choose a differentiable function ψ such that $\psi \equiv 1$ on $\frac{1}{2}D_{k+1} \subset D_k \cup H_k$ and $\psi \equiv 0$ off D_{k+1}.

Since E_{k-1} has no holes, neither does $E_{k-1} \cap D_{k+1}$, so Runge's theorem gives a polynomial P so that both

$$|h_{k-1} - P| \leq q2^{-k-1}\epsilon,$$

on $E_{k-1} \cap D_{k+1}$ and

$$\frac{1}{\pi} \iint_{E_{k-1}} |(h_{k-1} - P)(w)\overline{\partial}\psi(s)| \frac{dxdy}{|z-w|} < 2^{-k-1}\epsilon,$$

for all $z \in \mathbb{C}$. The latter holds because $\overline{\partial}\psi = \psi_x + i\psi_y$ is zero off D_{k+1} and $|z - w|^{-1}$ has a uniformly bounded integral over D_{k+1}, independent of where the pole is. Because the integrands are bounded, the same inequality holds if E_{k-1} is replaced by a sufficently small open neighborhood $V \subset U$ of E_{k-1}. Thus

$$r(z) = \frac{1}{\pi} \iint_{E_{k-1}} |(h_{k-1} - P)(w)\overline{\partial}\psi(w)| \frac{dxdy}{|z-w|}$$

is holomophic off V, r is bounded by $2^{-k-1}\epsilon$ everywhere and

$$\overline{\partial}r = (h_{k-1} - P)\overline{\partial}\psi.$$

Set

$$h_k = \psi P + (1-\psi)h_{k-1} + r,$$

in $V \cup \frac{1}{2}D_{k+1}$. Then, since $\overline{\partial}P = 0$ on $\mathbb{C}$ and $\overline{\partial}h_{k-1} = 0$ on V,

$$\overline{\partial}h_k = P\overline{\partial}\psi - h_{k-1}\overline{\partial}\psi - \overline{\partial}r = 0,$$

so h_k is holomorphic on V. Since ψ is constant on $\frac{1}{2}D_{k+1}$, $\overline{\partial}\psi = 0$ there, and hence r is holomoprhic there. Since $h_k = r + P$, it is also holomoprhic on $\frac{1}{2}D_{k+1}$.

Thus h_k is holomorphic on $\frac{1}{2}D_{k+1} \cup V$, an open neighborhood of E_k and

$$|h_k - h_{k-1}| = |(P - h_k)\psi + r| < 2^-k\epsilon$$

on E_{k-1}. Since $\cup_k E_k = \mathbb{C}$, this implies that h_k converges uniformly on compact sets to an entire function g that satisfies the theorem. □

We can improve Runge's theorem by getting the same conclusion from a weak hypothesis:

THEOREM 118 (Mergelyan's Theorem). *If $K \subset \mathbb{C}$ does not separate the plane and f is continuous on K and holomorphic on the interior of K, then f can be unifromly approximated on K by holomophic polynomials.*

PROOF. By Runge's it suffices to prove that f can be approximated by a holomoprhic function on some neighborhood of K. First extend f to be continuous on compact set E containing K in its interior. We also denote the extension by f. The extension is uniformly continuous on E so given any δ, there is any ϵ so that $|z - w| < 10\epsilon$ implies $|f(z) - f(w)| < \delta$. Convolve f with a smooth, positive bump function h of total mass 1, supported in an ϵ disk and with all first partial bounded by $O(1/epsilon)$. Call the result F. By the mean value property $F = f$ at the points U of the interior of K that are more than distance ϵ from the boundary. Furthermore, $|\overline{\partial}F(z)| = O(\delta/\epsilon)$ everywhere (note that $\overline{\partial}F = \overline{\partial}(F - F(z_0)) = (\overline{\partial}h) * (f - f(z_0))$ which is less than δ on the support of h).

Cover $K \setminus U$ by boxes from an ϵ-grid. For each such box Q_j, $4Q_j \setminus K$ contain a of diamter ϵ and the Riemann map ψ_j from the complment of this arc to the interior of $\mathbb{D}$ satisfies

$$|\psi_j(z)| > a > 0, z \in Q,$$

$$|\psi_j(z)| \leq b\epsilon/\mathrm{dist}(z, Q_j), z \in \mathbb{C} \setminus 8Q,$$

for constants a, b that independent of Q.

Let $\varphi(j)$ be a partition of unity with repsect to the doubles of the ϵ-boxes and set

$$H(z) = \sum_j \psi_j^3(z)[\frac{1}{2\pi i} \iint \frac{1}{z-w} (\frac{\varphi_j(w)\overline{\partial} F(w)}{\psi_j^3(w)}) dxdy],$$

satisfies $\overline{\partial} H = F$ on the union of the boxes and

$$|H(z)| \leq \sum_j O(b^3 \frac{\epsilon^3}{(1+\operatorname{dist}(z, Q_j))^3} a^{-3} \cdot \epsilon \cdot \frac{\delta}{/} epsilon) = O(\delta),$$

Since the sum $(1+|z|)^{-3}$ over a square lattice is finite. Thus $\overline{\partial}(F-H) = 0$ so $F-H$ is holomorphic and $|f-H| \leq |f-F| + |F-H| = O(\delta)$ is small. This proves Mergelyan's theorem. $\square$

If Runge's theorem is replaced by in the previous proof of Arakelian's theorem, we get:

THEOREM 119. *If E is an Arakelian set and f is continuous on E and holomoprhic on the interior of E, then for any $\epsilon > 0$ there is an entire function g such that*

$$\sup_{z \in E} |f(z) - g(z)| < \epsilon.$$

We can easily strenghen this to

COROLLARY 120. *If E is an Arakelian set with empty interior and f is continuous on E and holomoprhic on the interior of E, then for any continuous, positive function $\epsilon(t)$ on $[0, \infty)$ there is an entire function F such that*

$$|f(z) - F(z)| < \epsilon(|z|)$$

for all $z \in E$.

PROOF. Apply the second version of Arakelian's theorem to deduce that there is an entire function g so that $\Re g(z) < \log \epsilon(|z|)$ for $z \in E$ and an entire function h so that $|h - fe^{-g}| < 1$. Then $F = he^g$ is entire and

$$|F - f| = |he^g - f| = |e^g| \cdot |h - fe^{-g}| < |e^g| \leq \epsilon(|z|).$$

$\square$

6. Dimension

Suppose K is a bounded set in $\mathbb{R}^d$ (or a totally bounded set in any metric space) and let $N(K, \epsilon)$ be the minimal number of open balls of diameter ϵ needed to cover K. We define the **upper Minkowski dimension** as

$$\overline{\dim}_M(K) = \limsup_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon},$$

and the *lower Minkowski dimension*

$$\underline{\dim}_M(K) = \liminf_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon}.$$

If the two values agree, the common value is simply called the *Minkowski dimension* of K and denoted by $\overline{\mathrm{Mdim}}(K)$. When the Minkowski dimension of a set K exists, the number of balls of diameter ϵ needed to cover K grows like $\epsilon^{-\overline{\mathrm{Mdim}}(K)+o(1)}$ as $\epsilon \to 0$. Minkowski dimension is sometimes called the *box counting dimension.*

In the definitions of $\overline{\dim}_M(K)$ and $\underline{\dim}_M(K)$ it is equivalent to replace $N(K, \epsilon)$ by $N_D(K, \epsilon)$ where the covering sets of diameter $\le \epsilon$ are not required to be balls. This is because $N(K, 2\epsilon) \le N_D(K, \epsilon) \le N(K, \epsilon)$. Also, for a bounded A, $\overline{\dim}_M(A) = \overline{\dim}_M(\bar{A})$ and $\underline{\dim}_M(A) = \underline{\dim}_M(\bar{A})$, where $\bar{A}$ denotes the closure of A. We leave the proofs to the reader. The Minkowski dimension has several drawbacks. For example, it need not exist for a general set (see Example **??** and Exercise **??**).

EXERCISE: Show that $\{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ has Minkowski dimension 1.

EXERCISE: Construct a compact subset of [0.1] that has different upper and lower Minkowski dimensions.

Also, for $E_1 \subset E_2 \subset \dots$, it is possible that

$$\overline{\mathrm{Mdim}}(\cup_n E_n) \neq \lim_{n \to \infty} \overline{\mathrm{Mdim}}(E_n)$$

and the Minkowski dimension of a countable set can be non-zero (see Example **??**). Thus the Minkowski dimension of a countable union of sets is not necessarily the supremum of the individual dimensions. There are (at least) two ways to “fix” these problems by modifying the definition of Minkowski dimension. The first is to break the set into pieces and cover each piece separately; this gives packing dimension. The other is to allow coverings by balls of different sizes; this gives Hausdorff dimension. We will investigate Hausdorff dimension first and return to packing dimension later.

Given any set K in a metric space we define the *α-dimensional Hausdorff content* as

$$\mathcal{H}^{\alpha}_{\infty}(K) = \inf\{\sum_i |U_i|^{\alpha}\},$$

where $\{U_i\}$ is a countable cover of K by any sets and $|E|$ denotes the diameter of a set E.

The *Hausdorff dimension* of K is defined to be

$$\dim(K) = \inf\{\alpha : \mathcal{H}^{\alpha}_{\infty}(K) = 0\}.$$

This is equivalent to the original [**?**] definition, using Hausdorff measure, see Proposition 123.

More generally we define

$$\mathcal{H}^{\alpha}_{\epsilon}(K) = \inf\{\sum_i |U_i|^{\alpha} : K \subset \cup_i U_i, |U_i| < \epsilon\},$$

where each U_i is now required to have diameter less than ϵ. The *α-dimensional Hausdorff measure* of K is defined as

$$\mathcal{H}^{\alpha}(K) = \lim_{\epsilon\to 0} \mathcal{H}^{\alpha}_{\epsilon}(K).$$

This is an outer measure; an *outer measure* on a nonempty set X is a function μ^* from the family of subsets of X to $[0,\infty]$ that satisfies

- $\mu^*(\emptyset) = 0$,
- $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$,
- $\mu^*(\cup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$.

(For background on real analysis see [**?**].) The α-dimensional Hausdorff measure is even a Borel measure in $\mathbb{R}^d$ if $\alpha < d$ (proved below).

THEOREM 121. *Let μ be a metric outer measure. Then all Borel sets are μ-measurable.*

For a proof see [**?**].

The construction of Hausdorff measure can be made a little more general by considering a positive, increasing function φ on $[0,\infty)$ with $\varphi(0) = 0$. This is called a *gauge function* and we may associate to it the Hausdorff content

$$\mathcal{H}^{\varphi}_{\infty}(K) = \inf\{\sum_i \varphi(|U_i|)\},$$

and $\mathcal{H}^\varphi_\epsilon(K)$, as well as $\mathcal{H}^\varphi(K) = \lim_{\epsilon\to 0} \mathcal{H}^\varphi_\epsilon(K)$ just as before. The case $\varphi(t) = t^\alpha$ is just the case considered above. We will not use other gauge functions in the first few chapters, but they are important in many applications.

Lemma 122. *If* $\mathcal{H}^\alpha(K) < \infty$ *then* $\mathcal{H}^\beta(K) = 0$ *for any* $\beta > \alpha$.

Proof. It follows from the definition of $\mathcal{H}^\alpha_\epsilon$ that

$$\mathcal{H}^\beta_\epsilon(K) \le \epsilon^{(\beta-\alpha)}\mathcal{H}^\alpha_\epsilon(K),$$

which gives the desired result as $\epsilon \to 0$. □

Thus if we think of $\mathcal{H}^\alpha(K)$ as a function of α, the graph of $\mathcal{H}^\alpha(K)$ versus α shows that there is a critical value of α where $\mathcal{H}^\alpha(K)$ jumps from ∞ to 0. This critical value is equal to the Hausdorff dimension of the set. Note that $\mathcal{H}^\alpha_\infty(K) = 0$ if and only if $\mathcal{H}^\alpha(K) = 0$. This gives us the following proposition.

Proposition 123.

$$\dim(K) = \inf\{\alpha : \mathcal{H}^\alpha(K) = 0\}.$$

The following relationship to Minkowski dimension is clear

$$\dim(K) \le \underline{\dim}_M(K) \le \overline{\dim}_M(K). \tag{20}$$

Indeed, if $B_i = B(x_i, \epsilon/2)$ are $N(K,\epsilon)$ balls of radius $\epsilon/2$ and centers in x_i that cover K, then consider the sum

$$S_\epsilon = \sum_{i=1}^{N(K,\epsilon)} |B_i|^\alpha = N(K,\epsilon)\epsilon^\alpha = \epsilon^{\alpha - R_\epsilon},$$

where $R_\epsilon = \frac{\log N(K,\epsilon)}{\log(1/\epsilon)}$. For $\alpha > \liminf_{\epsilon\to 0} R_\epsilon = \underline{\dim}_M(K)$ we have $\inf_{\epsilon>0} S_\epsilon = 0$. Strict inequalities in (20) are possible.

Upper bounds for Hausdorff dimension are computed using explicit covering of the set. Lower bounds are given by constructing measures supported on the set. The simplest version of this idea is:

Lemma 124 (Mass Distribution Principle). *If* E *supports a strictly positive Borel measure* μ *which satisfies*

$$\mu(B(x,r)) \le Cr^\alpha,$$

for some constant $0 < C < \infty$ *and for every ball* $B(x, r)$*, then* $\mathcal{H}^\alpha(E) \geq \mathcal{H}^\alpha_\infty(E) \geq \mu(E)/C$*. In particular,* $\dim(E) \geq \alpha$*.*

Proof. Let $\{U_i\}$ be a cover of E. For $\{r_i\}$, where $r_i > |U_i|$, we look at the following cover: choose x_i in each U_i, and take open balls $B(x_i, r_i)$. By assumption,

$$\mu(U_i) \leq \mu(B(x_i, r_i)) \leq C{r_i}^\alpha.$$

We deduce that $\mu(U_i) \leq C|U_i|^\alpha$, i.e.,

$$\sum_i |U_i|^\alpha \geq \sum_i \frac{\mu(U_i)}{C} \geq \frac{\mu(E)}{C}.$$

Thus $\mathcal{H}^\alpha(E) \geq \mathcal{H}^\alpha_\infty(E) \geq \mu(E)/C$. □

A more refined version of the mass distribution principle is

Lemma 125 (Billingsley's lemma). *Let* $A \subset [0, 1]$ *be Borel and let* μ *be a finite Borel measure on* $[0, 1]$*. Suppose* $\mu(A) > 0$*. If*

$$\alpha_1 \leq \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \leq \beta_1,$$

for all $x \in A$*, then* $\alpha_1 \leq \dim(A) \leq \beta_1$*.*

For a proof see [**?**]. Billingsley's lemma has a further refinement by Rogers and Taylor which we will not discuss here (see []).

For any compact set $K \subset \mathbb{R}^d$ we can define an **exponent of convergence**

$$\kappa = \kappa(K) = \inf\{\alpha : \sum_{Q\in\mathcal{W}} |Q|^\alpha < \infty\}, \tag{21}$$

where the sum is taken over all cubes in some Whitney decomposition $\mathcal{W}$ of $\Omega = K^c$ that are within distance 1 of K (we have to drop the "far away" cubes or the series might never converge). It is easy to check that κ is independent of the choice of Whitney decomposition (see Exercise **??**).

Lemma 126. *For any compact set* K*,* $\kappa \leq \overline{\dim}_M(K)$*. If* K *also has zero Lebesgue measure then* $\kappa = \overline{\dim}_M(K)$*.*

Proof. Let $D = \overline{\dim}_M(K)$. We start with the easy assertion, $\kappa \leq D$. Choose $\epsilon > 0$ and for for each $n \in \mathbb{N}$, let $\mathcal{Q}_n$ be a covering of K by $O(2^{n(D+\epsilon)})$ dyadic cubes of side length 2^{-n}. Let $\mathcal{W}$ denote the dyadic Whitney cubes that are within distance 1 of K and let $\mathcal{W}_n \subset \mathcal{W}$ be the cubes with side $\ell(Q) = 2^{-n}$. For each $Q \in \mathcal{W}_n$, choose a point $x \in K$ with $\mathrm{dist}(x, Q) \leq 3|Q|$ and let $S(x, Q) \in \mathcal{Q}_n$ be a cube containing x. Since $|S(x, Q)| = |Q|$ and $\mathrm{dist}(Q, S(x, Q)) \leq 3|Q|$, each $S \in \mathcal{Q}_n$ can only be associated to a uniformly bounded number of Q's in $\mathcal{W}_n$. Hence

$$\#(\mathcal{W}_n) = O(2^{n(D+\epsilon)}),$$

and thus

$$\begin{aligned}\sum_{Q \in \mathcal{W}} |Q_j|^{D+2\epsilon} &= O(\sum_{n=0}^{\infty} \#(\mathcal{W}_n) 2^{-n(D+2\epsilon)}) \\ &= O(\sum_{n=0}^{\infty} 2^{-\epsilon)}) \\ &< \infty,\end{aligned}$$

which proves $\kappa \leq D + 2\epsilon$. Taking $\epsilon \to 0$ gives $\kappa \leq D$.

Next we assume K has zero Lebesgue measure and will prove $\kappa \geq D$. Let $\epsilon > 0$. We have

$$N(K, 2^{-n}) \geq 2^{n(D-\epsilon)},$$

for infinitely many n, so suppose n is a value where this occurs and let $\mathcal{S} = \{S_k\}$ be a covering of K with dyadic cubes of side 2^{-n}. Let $\mathcal{U}_n$ be cubes in the dyadic Whitney decomposition of $\Omega = K^c$ with side lengths $< 2^{-n}$. For each $S_k \in \mathcal{S}$ let $\mathcal{U}_{nk} \subset \mathcal{U}_n$ be the subcollection of cubes that intersect S_k. Because of the nesting property of dyadic cubes, every dyadic Whitney cube intersecting the interior of some S_k is contained in that S_k. Since the volume of K is zero, this gives

$$|S_k|^d = \sum_{Q \in \mathcal{U}_{nk}} |Q|^d$$

(The right side $d^{d/2}$ times the Lebesgue measure of $S_k \setminus K$, and the left side is $d^{d/2}$ times the measure of S_k; these are equal by assumption.) Since $-d + D - 2\epsilon < 0$, we get

$$\begin{aligned}\sum_{Q\in\mathcal{U}_{nk}} |Q|^{D-2\epsilon} &= \sum_{Q\in\mathcal{U}_{nk}} |Q|^d |Q|^{-d+D-2\epsilon}\\ &\geq |S_k|^{-d+D-2\epsilon} \sum_{Q\in\mathcal{U}_{nk}} |Q|^d\\ &= |S_k|^{D-2\epsilon}\end{aligned}$$

Hence, when we sum over the entire Whitney decomposition,

$$\begin{aligned}\sum_{Q\in\mathcal{U}_0} |Q|^{D-2\epsilon} &\geq \sum_{S_k\in\mathcal{S}} \sum_{Q\in\mathcal{U}_{nk}} |Q|^{D-2\epsilon}\\ &\geq \sum_{S_k\in\mathcal{S}} |S_k|^{D-2\epsilon}\\ &\geq N(K,2^{-n})\cdot 2^{-n(D-2\epsilon)}\\ &= 2^{n\epsilon}.\end{aligned}$$

Taking $n \to \infty$, shows $\kappa \geq D - 2\epsilon$ and taking $\epsilon \to 0$ gives $\kappa \leq D$. ☐

Tricot [**?**] introduced packing dimension, which is dual to Hausdorff dimension in several senses and comes with an associated measure.

For any increasing function $\varphi\colon [0,\infty) \to \mathbf{R}$ such that $\varphi(0) = 0$ and any set E in a metric space, define first the **packing pre-measure** (in gauge φ) by

$$\tilde{\mathcal{P}}^{\varphi}(E) = \lim_{\epsilon\downarrow 0}\left(\sup\sum_{j=1}^{\infty}\varphi(\text{diam } B_j)\right),$$

where the supremum is over all collections of disjoint closed balls $\{B_j\}_{j=1}^{\infty}$ with centers in E and diameters $\text{diam}(B_j) < \epsilon$. This pre-measure is finitely sub-additive, but not countably sub-additive, see Exercise **??**. Then define the **packing measure** in gauge φ:

$$\mathcal{P}^{\varphi}(E) = \inf\left\{\sum_{i=1}^{\infty}\tilde{\mathcal{P}}^{\varphi}(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i\right\}. \tag{22}$$

It is easy to check that $\mathcal{P}^{\varphi}$ is a metric outer measure, hence all Borel sets are $\mathcal{P}^{\varphi}$-measurable, see Theorem 121 in Chapter 1. When $\varphi(t) = t^{\theta}$ we write $\mathcal{P}^{\theta}$ for $\mathcal{P}^{\varphi}$ ($\mathcal{P}^{\theta}$ is called **θ-dimensional packing measure**).

Finally, define the **packing dimension** of E:

$$\dim_P(E) = \inf\left\{\theta : \mathcal{P}^\theta(E) = 0\right\}. \tag{23}$$

We always have

$$\dim(E) \le \dim_P(E) \le \overline{\dim}_M(E). \tag{24}$$

The set$K = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ is of packing dimension 0, since the packing dimension of any countable set is 0. Thus

$$\dim(K) = 0 = \dim_P(K) < 1/2 = \overline{\dim}_M(K).$$

(See Example **??** in Chapter **??**.)

Packing measures are studied in detail in Taylor and Tricot [**?**] and in Saint-Raymond and Tricot [**?**]; here we only mention the general properties we need.

Proposition 127. *The packing dimension of any set A in a metric space may be expressed in terms of upper Minkowski dimensions:*

$$\dim_P(A) = \inf\left\{\sup_{j\ge 1} \overline{\dim}_M(A_j) : A \subset \bigcup_{j=1}^{\infty} A_j\right\}, \tag{25}$$

where the infimum is over all countable covers of A.

(See Tricot [**?**], Proposition 2, or Falconer [**?**], Proposition 3.8.)

For the proof see [**?**].

Lemma 128. *Let A be a separable metric space.*

(i) *If A is complete and if every non-empty open set V in A satisfies $\overline{\dim}_M(V) \ge \alpha$, then $\dim_P(A) \ge \alpha$.*

(ii) *If $\dim_P(A) > \alpha$, then there is a closed nonempty subset $\widetilde{A}$ of A, such that $\dim_P(\widetilde{A} \cap V) > \alpha$ for any open set V which intersects $\widetilde{A}$.*

Corollary 129. *[Tricot [**?**], Falconer [**?**]] Let K be a compact set in a metric space which satisfies*

$$\overline{\dim}_M(K \cap V) = \overline{\dim}_M(K)$$

for any open set V which intersects K. Then

$$\dim_P(K) = \overline{\dim}_M(K).$$

A gauge function φ is called **doubling** if

$$\sup_{x>0} \frac{\varphi(2x)}{\varphi(x)} < \infty.$$

THEOREM 130. *Assume* $\{f_1, \dots, f_\ell\}$ *are contracting self bi-Lipschitz maps of a complete metric space, i.e.*

$$\epsilon_j d(x,y) \leq d(f_j(x), f_j(y)) \leq r_j d(x,y)$$

for all $1 \leq j \leq \ell$ *and any* x, y*, where*

$$0 < \epsilon_j \leq r_j < 1.$$

Denote by K *the compact attractor satisfying (**??**). Then*

(i) $\dim_P(K) = \overline{\dim}_M(K)$.

(ii) *For any doubling gauge function* φ *such that* K *is* σ*-finite for* $\mathcal{P}^\varphi$ *we have* $\tilde{\mathcal{P}}^\varphi(K) < \infty$.

7. Extremal length and harmonic measure

A conformal invariant is a number which is invariant under conformal mappings. We are often in the situation where we wish to know the value of some conformal invariant (e.g., that harmonic measure of the edge of a polygon) and are able to estimate some other conformal invariant (e.g., the modulus of some path family in the polygon). Using a known relation between the invariants, we can turn an estimate for one into an estimate for the other.

Probably the most important example of a conformal invariant is the (conformal) modulus.

Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω. We say ρ is admissible for Γ if

$$\ell(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho ds \geq 1,$$

and define the modulus of Γ as

$$\mathrm{Mod}(\Gamma) = \inf_{\Omega} \int_M \rho^2 dxdy,$$

where the infimum is over all admissible ρ for Γ. This is a well known conformal invariant whose basic properties are discussed in many sources such as Ahlfors' book [**?**].

It reciprocal is called the extremal length of the path family. Modulus and extremal length satisfy several properties that are helpful in estimating these quantities.

LEMMA 131 (Conformal invariance). *If $\mathcal{F}$ is a family of curves in a domain Ω and f is a one-to-one analytic mapping from Ω to Ω' then $M(\mathcal{F}) = M(f(\mathcal{F}))$.*

PROOF. This is just the change of variables formulas

$$\int_{f(\gamma)} \rho \circ f ds = \int_\gamma \rho ds,$$

$$\int_{f(\Omega)} (\rho \circ f)^2 dxdy = \int_\Omega \rho dxdy.$$

These imply that if $\rho \in \mathcal{A}(\mathcal{F})$ then $\rho \circ f^{-1} \in \mathcal{A}(f(\mathcal{F}))$, and thus $M(f(\mathcal{F})) \leq M(\mathcal{F})$. We get the other direction by considering f^{-1}. □

LEMMA 132 (Monotonicity). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are collections such that every $\gamma \in \mathcal{F}_1$ contains some curve in $\mathcal{F}_2$ then $M(\mathcal{F}_1) \leq M(\mathcal{F}_2)$ and $\lambda(\mathcal{F}_1) \geq \lambda(\mathcal{F}_2)$.*

The proof is immediate since $\mathcal{A}(\mathcal{F}_1) \supset \mathcal{A}(\mathcal{F}_2)$.

LEMMA 133 (Grötsch Principle). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains then $M(\mathcal{F}_1 \cup \mathcal{F}_2) = M(\mathcal{F}_1) + M(\mathcal{F}_2)$.*

LEMMA 134. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains and every curve of $\mathcal{F}$ contains both a curve from $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\lambda(\mathcal{F}) \geq \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2)$.*

PROOF. If $\rho_1 \in \mathcal{A}(\mathcal{F}_i)$ for $i = 1, 2$, then $\rho = t\rho_1 + (1-t)\rho_2$ is admissible for $\mathcal{F}$. Since the domains are disjoint we may assume $\rho_1\rho_2 = 0$ everywhere so taking

$$t = t^2 M(\mathcal{F}_1) + (1-t^2) M(\mathcal{F}_2),$$

gives

$$m(\mathcal{F}) \leq t^2 M(\mathcal{F}_1) + (1-t^2) M(\mathcal{F}_2) = (M(\mathcal{F}_1)^{-1} + M(\mathcal{F}_2)^{-1})^{-1},$$

as required. □

The fundamental example is to compute the modulus of the path family connecting opposite sides of a $a \times b$ rectangle; this serves as the model of almost all modulus estimates. So suppose $R = [0, b] \times [0, a]$ is a a long and b high rectangle and Γ consists of all rectifiable curves in R with one endpoint on each of the sides of length a. Then each such curve has length at least b, so if we let ρ be the constant $1/b$ function on R we have

$$\int_\gamma \rho ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\text{Mod}(\Gamma) \leq \iint_T \rho^2 dxdy = \frac{1}{b^2}ab = \frac{a}{b}.$$

To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$(\int fgdx) \leq (\int f^2dx)(\int g^2dx).$$

To apply this, suppose ρ is an admissible metric on R for γ. Every horizontal segment in R connecting the two sides of length a is in Γ, so since γ is admissible,

$$\int_0^b \rho(x,y)dx \geq 1,$$

and so by Cauchy-Schwarz

$$1 \leq \int_0^b (1 \cdot \rho(x,y))dx \leq \int_0^b 1^2 dx \cdot \int_0^b \rho^2(x,y)dx.$$

Now integrate with respect to y to get

$$\int_0^a 1dy \leq b \int_0^a \int_0^b \rho^2(x,y)dxdy,$$

or

$$\frac{a}{b} \leq \iint_R \rho^2 dxdy,$$

which implies $\text{Mod}(\Gamma) \geq \frac{b}{a}$. Thus we must have equality.

Another useful computation is the modulus of the family of path connecting the inner and out boundaries of the annulus $A = \{z : r < |z| < R\}$. An argument similar to the one above shows that the modulus of this family is $\frac{1}{2\pi} \log \frac{R}{r}$.

The usefulness of extremal length is its ability to estimate a conformal invariant in terms of geometry (length and area). Our main application of this idea is the following special case of a theorem of Pfluger:

THEOREM 135. *Suppose $K \subset \mathbb{D}$ is compact with a smooth boundary and contains 0 in its interior and and $E \subset \partial\mathbb{D}$ is compact and let $\mathcal{F}$ be the family of curves in $\mathbb{D} \setminus K$ separating 0 from E. Then there is a C (depending only on K) such that $\mathcal{H}^1(E) \leq C \exp(-\pi M(\mathcal{F}))$.*

This is not sharp since the right hand side can be positive for sets of zero length. The sharp version uses logarithmic capacity in place of $\mathcal{H}^1$ measure on the left hand side. See Exercise **??** for the sharp version of Pfluger's theorem.

PROOF. We assume E has positive length, since there is nothing to prove otherwise. Furthermore, we may assume E is actually a finite union of closed intervals. Let μ be Lebesgue measure restricted to E, normalized to have mass 1. Define the potential function

$$U_\mu(z) = \log \frac{1}{|z|} * \mu,$$

and set

$$v(z) = U_\mu(z) + U_\mu(\frac{1}{z}).$$

Then v is symmetric with respect to the unit circle, has negative logarithmic poles at 0 and ∞ and bounded above by $2|\log \mathcal{H}^1(E)| + O(1)$ (to prove this note that since $-\log|z|$ is decreasing with $|z|$, the integral defining U is maximized when E is an interval and z is its midpoint). Since v is symmetric with respect to $\mathbb{T}$ we have $\partial v/\partial n = 0$ on $\partial\mathbb{D} \setminus E$. Now suppose $\gamma \in \mathcal{F}$ and let Ω be the component of $\mathbb{D} - \gamma$ containing 0. Since v is harmonic in $\mathbb{D}$ except for a logarithmic pole at 0 we can apply Green's theorem to get

$$\int_\gamma |\nabla v| ds \quad \geq \quad -\int_\gamma \frac{\partial v}{\partial n} ds = \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{\partial v}{\partial n} ds = 2\pi.$$

Thus $|\nabla v|/2\pi \in \mathcal{A}(\mathcal{F})$, so

$$M(\mathcal{F}) \leq \int_{\mathbb{D}\setminus K} (\frac{|\nabla v|}{2\pi})^2 dxdy.$$

Note that $|\nabla v|^2 = \frac{1}{2}\Delta(v^2)$, and use Green's theorem

$$\begin{aligned} \iint_{\mathbb{D}\setminus K} |\nabla v|^2 dxdy &= \frac{1}{2} \iint_{\mathbb{D}\setminus K} \Delta(v^2) dxdy \\ &= -\frac{1}{2} \int_{\partial\mathbb{D}} \frac{\partial v^2}{\partial n} ds - \frac{1}{2} \int_{\partial K} \frac{\partial v^2}{\partial n} ds \\ &\leq -(\max_{\mathbb{T}} v) \int_{\partial\mathbb{D}} \frac{\partial v}{\partial n} ds + O(1) \end{aligned}$$

Using Green's theorem gives

$$-\int_{\partial\mathbb{D}} \frac{\partial v}{\partial n} ds = 2\pi$$

Hence,

$$M(\mathcal{F}) \leq \frac{1}{(2\pi)^2}|2\log \mathcal{H}^1(E)|2\pi + O(1),$$

as desired. $\square$

COROLLARY 136. *Suppose Ω is a Jordan domain, $z_0 \in \Omega$ with $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ and $E \subset \partial\Omega$. Let $\mathcal{F}$ be the family of curves in Ω which separates $D(z_0, 1/2)$ from E. Then $\omega(z_0, E, \Omega) \leq C\exp(-\pi M(\mathcal{F}))$.*

PROOF. Since both harmonic measure and modulus are conformally invariant we need only verify this when $\Omega = \mathbb{D}$. But this is the previous theorem. $\square$

If $E \subset \partial\Omega$ is an arc then the inequality is actually a similarity.

COROLLARY 137 (Ahlfors distortion theorem). *Suppose Ω is a Jordan domain, $z_0 \in \Omega$ with $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ and $x \in \partial\Omega$. For each $0 < t < 1$ let $\theta(t)$ be the length of $\Omega \cap \{|w - x| = t\}$. Then there is an absolute $C < \infty$, so that*

$$\omega(z_0, D(x, r), \Omega) \leq C\exp(-\pi\int_r^1 \frac{dt}{\theta(t)}).$$

PROOF. Let K be the disk of radius $1/2$ around z_0 and let $\mathcal{F}$ be the family of curves in Ω which separate $D(x, r) \cap \partial\Omega$ from K. Let $\mathcal{F}_1 \subset \mathcal{F}$ be the collection of curves of the form

$$L_t = \Omega \cap \{|w - x| = t\}.$$

if ρ is admissible for $\mathcal{F}$ then it is admissible for $\mathcal{F}_1$ and hence

$$1 \leq \int_{L_t} \rho ds \leq (\int_{L_t} \rho^2 ds)\theta(t),$$

so

$$\int_r^1 \int_{L_t} \rho^2 ds dt \geq \int_r^1 \frac{dt}{\theta(t)}.$$

This proves

$$M(\mathcal{F}) \geq \int_r^1 \frac{dt}{\theta(t)},$$

which proves the result by the previous corollary. $\square$

For an alternate version of this using line segments instead of circular arcs, see Exercise **??**.

COROLLARY 138 (Beurling's estimate). *There is a* $C < \infty$ *so that if* Ω *is simply connected,* $z \in \Omega$ *and* $d = \operatorname{dist}(z, \partial\Omega)$ *then for any* $0 < r < 1$ *and any* $x \in \partial\Omega$,

$$\omega(z, D(x, rd), \Omega) \leq Cr^{1/2}$$

PROOF. Apply Corollary 137 at x and use $\theta(t) \leq 2\pi t$ to get

$$\exp(-\pi \int_{rd}^{d} \frac{dt}{\theta(t)t}) \leq C \exp(-\frac{1}{2} \log r) \leq C\sqrt{r}.$$

□

LEMMA 139. *If* Ω *is simply connected then*

$$\omega(z_0, D(x, r), \Omega) \leq C[\frac{r}{\operatorname{dist}(z, \partial\Omega)}]^{1/2}.$$

PROOF. Use extremal length. □

COROLLARY 140. *If* Ω *is simply connected and* $w \in \partial\Omega$, *then*

$$\int_{D(w,r)\cap\partial\Omega} |\log \frac{1}{|z-w|} d\omega_{z_0}(z) \leq C[\frac{r}{\operatorname{dist}(z, \partial\Omega)}]^{1/2}.$$

PROOF. Cut the disk into concentric annuli $a_n = \{z : 2^{-n}r \leq |z - w| \leq 2^{-n+1}r\}$. By Beurling estimate the singleton $\{w\}$ has zero harmonic measure, so $\int_D = \sum \int_{A_n}$. However, the integral over A_n is bounded by $n2^{-n/2} \log \frac{1}{r} \log 2$ which sums to $O(|\log r/\operatorname{dist}(z_0, \partial\Omega)|)$. □

If γ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let $\gamma^+ = (\gamma \cap \mathbb{H}) \cup \overline{\gamma \cap \mathbb{L}}$, where $\mathbb{H}, \mathbb{L}$ denote the upper and lower half-planes. If Γ is a path family in the plane then $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma^+ = \{\gamma^+ : \gamma \in \Gamma\}$.

LEMMA 141. *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma^+)$.

PROOF. We start by proving $M(\Gamma) \leq 2M(\Gamma^+)$. Given a metric ρ, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$. Then for any $\gamma \in \Gamma$,

$$\int +\gamma^+ \sigma ds \geq \int_{\gamma^+} \rho ds \geq \inf_{\gamma\in\Gamma} \int_{\gamma} \rho ds.$$

Thus if ρ admissible for Γ^+, then σ is admissible for Γ Thus, since $\max(a, b)^2 \leq a^2+b^2$,

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z)dxdy + \int \rho^2(\bar{z})dxdy \leq 2 \int \rho^2(z)dxdy.$$

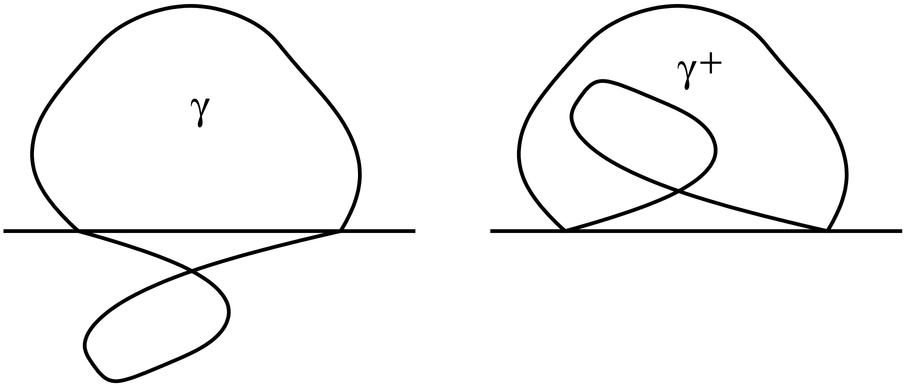

FIGURE 1. The curves γ and γ^+

Taking the infimum over admissible ρ's for Γ^+ makes the right hand side equal to $2M(\Gamma^+)$, proving the claim.

For the other direction, given ρ define $\sigma(z) = \rho(z) + \rho(\bar{z})$ for $z \in \mathbb{H}$ and $\sigma = 0$ if $z \in lhp$. Then

$$\begin{aligned}
\int_{\gamma^+} \sigma ds &= \int_{\gamma^+} \rho(z) + \rho(\bar{z}) ds \\
&= \int_{\gamma \cap \mathbb{H}} \rho(z) ds + \int_{\gamma \cap \mathbb{H}} \rho(\bar{z}) ds + \int_{\gamma \cap \mathbb{L}} \rho(z) + \int_{\gamma \cap \mathbb{L}} \rho(\bar{z}) ds \\
&= \int_{\gamma} \rho(z) ds + \int_{\bar{\gamma}} \rho(z) ds \\
&\geq 2 \inf_r ho \int_\gamma \rho ds.
\end{aligned}$$

Thus if ρ is admissible for Γ, $\frac{1}{2}\sigma$ is admissible for Γ^+. Hence, since $(a+b)^2 \leq 2(a^2+b^2)$,

$$\begin{aligned}
M(\Gamma^+) &\leq \int (\frac{1}{2}\sigma)^2 dxdy \\
&= \frac{1}{4} \int_{\mathbb{H}} (\rho(z) + \rho(\bar{z}))^2 dxdy \\
&\leq \tfrac{1}{2} \int_{\mathbb{H}} \rho^2(z) dxdy + \int_{\mathbb{H}} \rho^2(\bar{z}) dxdy \\
&= \frac{1}{2} \int \rho^2 dxdy.
\end{aligned}$$

Taking the infimum over all admissible ρ's for Γ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. $\square$

Lemma 142. *Let* $\mathbb{D}^* = \{z : |z| > 1\}$ *and* $\Omega_0 = \mathbb{D}^* \setminus [R, \infty)$ *for some* $R > 1$. *Let* $\Omega = \mathbb{D}^* \setminus K$, *where* K *is a closed, unbounded, connected set in* $\mathbb{D}^*$ *which contains the point* $\{R\}$. *Let* Γ_0, Γ *denote the path families in these domains with separate the two boundary components. Then* $M(\Gamma_0) \leq M(\Gamma)$.

Proof. We use the symmetry principle we just proved. The family Γ_0 is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma^+) = \frac{1}{2}M(\Gamma_0)$.The family Γ may not be symmetric, but we can replace it by a larger family that is. Let Γ_R be the collection of rectifiable curves in $\mathbb{D}^* \setminus \{R\}$which have zero winding number around $\{R\}$, but non-zero winding number around 0.Clearly $\Gamma \subset \Gamma_R$ and Γ_R is symmetric so $M(\Gamma) \geq M(\Gamma_R) = 2M(\Gamma_R^+)$. Thus all we have to do is show$M(\Gamma_R^+) = M(\Gamma_0^+)$. We will actually show $\Gamma_R^+ = \Gamma_0^+$. Since $\Gamma_0 \subset \Gamma_R$ is obvious, we need only show $\Gamma_R^+ \subset \Gamma_0^+$.

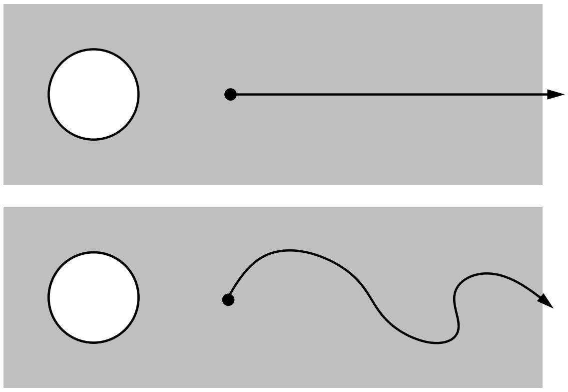

Figure 2. The annulus on top has smaller modulus than any other annulus formed by connecting R to ∞.

Suppose $\gamma \in \Gamma_R$. Since γ has non-zero winding around 0 it must cross both the negative and positive real axes. If it never crossed $(0, R)$ then the winding around 0 and R would be the same, which false, so γ must cross$(0, R)$ as well. Choose points $z_- \in \gamma \cap (-\infty, 0)$ and $z_+ \in \gamma \cap (0, R)$. These points divide γ into two subarcs γ_1 and γ_2.Then $\gamma^+ = \gamma_1^+ \cup \gamma_2^+$. But if we reflect γ_2^+ into the lower half-plane and join it to γ_1^+ it forms a closed curve γ_0 that is in Γ_0 and $\gamma_0^+ = \gamma^+$. Thus $\gamma^+ \in \Gamma_0^+$, as desired. $\square$

Let $\Omega_{\epsilon,R} = \{z : |z| > \epsilon\} \setminus [R, \infty)$. Thus$\Omega_{1,R}$ is the domain considered in the previous lemma. We can estimate the moduli of these domains using the Koebe map

$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \dots,$$

which conformal maps the unit disk to $\mathbb{R}^2 \setminus [\frac{1}{4}, \infty)$ and satisfies $k(0) = 0$, $k'(0) = 1$. Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\epsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius$\frac{\epsilon}{4R}(1 \pm O(\frac{\epsilon}{R})$. Thus the modulus of $\Omega_{\epsilon,R}$is $2\pi \log \frac{4R}{\epsilon} + O(\frac{\epsilon}{R})$.

Lemma 143. *Suppose $z, w \in \mathbb{D}$ and K is a compact connected set in $\mathbb{D}$ which contains both these points. Let Γ be the path family that separates$K$ and $\mathbb{T}$. Then the modulus of this family is maximized when K is the hyperbolic geodesic between z and w in which case the modulus is$2\pi \log \frac{4}{\rho}(z, w) + O(\rho(z, w))$, where ρ denotes the hyperbolic distance.*

Proof. By conformal invariance we may use a Möbius transformation to move z to 0and w onto the positive axis. Applying an inversion, the path family is mapped to one as in Lemma 142, showing that the radial line from z to w maximizes the modulus. The estimate of the modulus follows from our previous remarks. ☐

Theorem 144 (The Koebe $\frac{1}{4}$ Theorem). *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then$D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Proof. This proof is from [**?**]. Recall that the modulus of a doubly connected domain is the modulus of the path family that separates the two boundary components (and is equal to the extremal distance between the boundary components). Let $R = \mathrm{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\epsilon,r} = \{z : \epsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\epsilon} = M(A_{\epsilon,1}) = M(f(A_{\epsilon,1}).$$

Let $\delta = \min_{|z|=\epsilon} |f(z)|$.Since $f'(0) = 1$, $\delta = \epsilon + O(\epsilon^2)$. Note that $f(\mathbb{D}) \setminus D(0, delta) \supset f(A_{\epsilon,1})$, so

$$M(f(\mathbb{D}) \setminus D(0, \epsilon_2)) \geq M(f(A_{\epsilon,1})).$$

By Lemma 142

$$M(f(\mathbb{D}) \setminus D(0, \epsilon_2)) \leq M(\Omega_{\epsilon_2,R}) = 2\pi \log \frac{4R}{\epsilon\ _2} + O(\frac{\epsilon_2}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\epsilon}.,$$

or

$$\log 4R - \log(\epsilon + O(\epsilon^2)) + O(\frac{\epsilon}{R}) \geq -\log \epsilon.$$

Taking $\epsilon \to 0$ shows $\log 4R \geq 0$, or $R \geq \frac{1}{4}$. $\square$