Bull. London Math. Soc. 36 (2004) 263–270 © 2004 London Mathematical Society
DOI: 10.1112/S0024609303002698

DIMENSIONS OF JULIA SETS OF HYPERBOLIC ENTIRE FUNCTIONS

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Abstract

It is known that, if f is a hyperbolic rational function, then the Hausdorff, packing and box dimensions of the Julia set $J(f)$ are equal. It is also known that there is a family of hyperbolic transcendental meromorphic functions with infinitely many poles for which this result fails to be true. In this paper, new methods are used to show that there is a family of hyperbolic transcendental entire functions f_K, $K \in \mathbb{N}$, such that the box and packing dimensions of $J(f_K)$ are equal to two, even though as $K \to \infty$ the Hausdorff dimension of $J(f_K)$ tends to one, the lowest possible value for the Hausdorff dimension of the Julia set of a transcendental entire function.

1. *Introduction*

Let f be a meromorphic function which is not rational of degree one, and denote by f^n, $n \in \mathbb{N}$, the nth iterate of f. The set of normality $N(f)$ is defined to be the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ is well defined and meromorphic, and forms a normal family in some neighbourhood of z. The complement $J(f)$ of $N(f)$ is called the *Julia set* of f. An introduction to the properties of these sets can be found in, for example, [**1**] for rational functions and in [**2**] for transcendental meromorphic functions.

We denote the Hausdorff dimension of a set A by $\dim_{\mathrm{H}} A$, the packing dimension of A by $\dim_{\mathrm{P}} A$ and, if it exists, the box dimension of A by $\dim_{\mathrm{B}} A$. The upper box dimension $\overline{\dim}_{\mathrm{B}} A$ and the lower box dimension $\underline{\dim}_{\mathrm{B}} A$ are defined for all sets A; if they agree, then their common value is the box dimension. These dimensions are related by the inequalities

$$\dim_{\mathrm{H}} A \leqslant \dim_{\mathrm{P}} A \leqslant \overline{\dim}_{\mathrm{B}} A.$$

We use the following notation concerning singularities:

$$S(f) = \{z : z \text{ is a finite singularity of } f^{-1}\},$$
$$P(f) = \{z : z \text{ is a finite singularity of } f^{-n} \text{ for some } n \in \mathbb{N}\}.$$

For a rational function f, it follows fairly easily from results of Sullivan [**11**, Theorems 3 and 4] that $\dim_{\mathrm{B}} J(f) = \dim_{\mathrm{H}} J(f)$ if f is *hyperbolic*; that is, if f satisfies the following equivalent conditions (see [**1**, Section 9.7] and [**3**, Section 5.2]):

(1) $\overline{P}(f) \cap J(f) = \emptyset$;
(2) $\overline{P}(f)$ is a compact subset of $N(f)$;
(3) f is expanding, in the sense that there exist $K > 1$ and $c > 0$ such that $|(f^n)'(z)| > cK^n$ for each $z \in J(f)$, $n \in \mathbb{N}$.

Received 16 July 2002; revised 16 April 2003.

2000 *Mathematics Subject Classification* 30D05, 37F10, 37F15, 37F35, 37F50.

These conditions are not in general equivalent for a transcendental meromorphic function f, but in [**7**] we showed that, if f is in the class

$$\hat{B} = \{f : f \text{ is a transcendental meromorphic function with } S(f) \text{ bounded and } \overline{P}(f) \cap J(f) = \emptyset\},$$

then

(1) $P(f)$ is bounded;
(2) f is expanding in the sense that there exist $K > 1$ and $c > 0$ such that $|(f^n)'(z)| > cK^n(|f^n(z)| + 1)/(|z| + 1)$ for each $n \in \mathbb{N}$ and each $z \in J(f)$ such that f^n is analytic at z.

Thus it makes sense to say that functions in the class $\hat{B}$ are hyperbolic.

In [**10**, Theorem 3] we showed that, if f is a meromorphic function and $\overline{P}(f) \cap J(f) = \emptyset$, then $\dim_{\mathrm{P}} J(f) = \overline{\dim}_{\mathrm{B}} J(f)$. We also gave examples in [**10**] of meromorphic functions in $\hat{B}$ for which the upper box and packing dimensions of the Julia set are strictly greater than the Hausdorff dimension of the Julia set. The arguments used in [**10**] can be applied only to meromorphic functions with poles, and more subtle arguments are needed in order to show that there is a hyperbolic transcendental entire function for which the Hausdorff and box dimensions of the Julia set are different.

In this paper we consider the function defined by

$$E(z) = \frac{1}{2\pi i}\int_L \frac{\exp(e^t)}{t - z}\,dt,$$

where L is the boundary of the half-strip

$$G = \{x + iy : x > 0, -\pi < y < \pi\}$$

described in a clockwise direction, and $z \in \mathbb{C} \setminus \overline{G}$. As shown in [**6**], E can be analytically continued to a transcendental entire function, which we will also denote by E. In [**9**] we showed that, if f_K is the function defined by

$$f_K(z) = E(z) - K,$$

then, as $K \to \infty$, $\dim_{\mathrm{H}} J(f_K)$ tends to one, which is the lowest possible value for the Hausdorff dimension of the Julia set of a transcendental entire function. In this paper we prove the following result.

THEOREM 1.1. *If K is sufficiently large, then $f_K \in \hat{B}$ and is therefore hyperbolic, and* $\dim_{\mathrm{B}} J(f_K) = \dim_{\mathrm{P}} J(f_K) = 2$.

REMARK 1.1. The Julia set of a transcendental meromorphic function is unbounded and so, strictly speaking, the box dimension of $J(f_K)$ is not well defined. In [**10**], however, we showed that the Julia set of a hyperbolic meromorphic function has constant local upper and lower box dimensions; so, if we can find a subset of $J(f_K)$ that has box dimension two, then it makes sense to say that the whole Julia set has box dimension two.

REMARK 1.2. In [**10**, Theorem 3] we showed that, if f is a meromorphic function and $\overline{P}(f) \cap J(f) = \emptyset$, then $\dim_{\mathrm{P}} J(f) = \overline{\dim}_{\mathrm{B}} J(f)$. Since $\overline{\dim}_{\mathrm{B}} J(f) \leqslant 2$ for any meromorphic function f, it follows that, in order to prove Theorem 1.1, it is sufficient to show that $\underline{\dim}_{\mathrm{B}} J(f_K) = 2$.

2. *Preliminary results*

In this section we give some results that will be useful in the proof of Theorem 1.1. We begin with the following result, which is known as Koebe's distortion theorem; see, for example [**4**]. (Note that $B(z,r)$ denotes the disk of radius r, centred at z.)

Lemma 2.1. *If f is univalent in $B(z,r)$, then, for $0 \leqslant s \leqslant r$,*

$$\sup_{v,w\in B(z,s)} \left|\frac{f'(v)}{f'(w)}\right| \leqslant \left(\frac{r+s}{r-s}\right)^4 = L(s/r).$$

In particular, we will use the fact that $L(1/2) = 81 < 100$.

The next result follows from the discussion in [**9**, Section 2].

Lemma 2.2. *If K is sufficiently large, then $P(f_K)$ is contained in a bounded subset of $\{x+iy : x<0\}$ and $J(f_K) \subset G$.*

We note that it follows from Lemma 2.2 that, for large K, f_K belongs to $\hat{B}$ and is therefore hyperbolic.

We now consider the function h defined by $h(z) = \exp(e^z) - K$. The following result follows from [**9**, Equation (3.5)].

Lemma 2.3. *There exists a constant $C_1 > 0$ such that, for $z \in G$,*

$$|f_K(z) - h(z)| < C_1 \quad \text{and} \quad |f_K'(z) - h'(z)| < C_1.$$

We now show that $J(f_K)$ contains all large real values of z. To do this, we use the following two results proved in [**8**, Theorems 3(a) and 4(a)].

Lemma 2.4. *If D is a closed disk in $N(f)$, then there exists $0 < C < \infty$ such that*

$$\frac{\ln(|f^n(z_2)|+1)}{\ln(|f^n(z_1)|+2)} \leqslant C \qquad \text{for } z_1, z_2 \in D \text{ and } n \in \mathbb{N}.$$

Lemma 2.5. *Let $\phi : (t_0,\infty) \longrightarrow \mathbb{R}$ be an increasing function satisfying $\phi(t) > t$. If there exists $C > 1$ and $t_0' \geqslant t_0$ such that*

$$\phi'(t) \geqslant C\frac{\phi(t)\ln\phi(t)}{t\ln t} \qquad \text{for } t \geqslant t_0',$$

then, for $t_0 < t_1 < t_2$,

$$\frac{\ln\phi^n(t_2)}{\ln\phi^n(t_1)} \to \infty \qquad \text{as } n \to \infty.$$

Using these two results, we are able to prove the following.

Lemma 2.6. *Let $K \geqslant 2C_1$. There exists $K_1 > 5C_1$ such that*

$$f_K([K_1,\infty)) \subset [K_1,\infty) \subset J(f_K).$$

Proof. We begin by noting that $f_K(\mathbb{R}) \subset \mathbb{R}$ (see [**6**]). We now put $\phi(t) = f_K(t)$, and we note that it follows from Lemma 2.3 that if K_1 is sufficiently large, then $\phi : [K_1, \infty) \longrightarrow \mathbb{R}$ is an increasing function satisfying $\phi(t) > t$. Also, from Lemma 2.3,

$$\phi'(t) \geqslant e^t \exp(e^t) - C_1$$

and, since $K \geqslant 2C_1$,

$$\phi(t) \leqslant \exp(e^t) - K + C_1 \leqslant \exp(e^t) - C_1$$

so that, for t sufficiently large,

$$\begin{aligned}\phi(t) \ln \phi(t) &\leqslant e^t(\exp(e^t) - C_1)\\ &< e^t \exp(e^t) - C_1 \leqslant \phi'(t).\end{aligned}$$

It now follows from Lemma 2.5 that, for $K_1 < t_1 < t_2$,

$$\frac{\ln \phi^n(t_2)}{\ln \phi^n(t_1)} \to \infty \qquad \text{as } n \to \infty.$$

The result now follows from Lemma 2.4.

We end this section with the following result.

LEMMA 2.7. *There exist* $C_2 > 200$ *and* $K_2 \geqslant K_1$ *such that, for all* $x \geqslant K_2$, $|Y| \leqslant \pi/4$,

$$\{z = x + iy : Y \leqslant y \leqslant Y + C_2/e^x\} \cap h^{-1}([K_1, \infty)) \neq \emptyset.$$

Proof. This follows from the fact that, if $z = x + iy$, then $\exp(e^z)$ is real and positive if

$$e^x \sin y = 2\pi r, \qquad \text{where } r \in \mathbb{Z}.$$

3. *Proof of Theorem* 1.1

We begin this section with a formal definition of the box dimension of a bounded set $A \subset \mathbb{C}$. Suppose that we have a grid composed of lines parallel to the axes, so that each box in the grid has sides of length d. We count the number $N_d(A)$ of boxes in this grid that meet the set A. The box dimension of A is defined to be

$$\dim_{\mathrm{B}} A = \lim_{d \to 0} \frac{\log N_d(A)}{-\log d},$$

if this limit exists. There are several equivalent definitions of the box dimension of a set; for more details see, for example, [**5**].

Now let f be a function f_K as defined in Section 1, with K sufficiently large for Lemma 2.2 and Lemma 2.6 to apply. In order to prove Theorem 1.1, we take a point $x \in \mathbb{R}$ and let $b_D(x, n)$ denote the box centred at x with sides parallel to the axes and of length

$$D(x, n) = \frac{\pi}{400(f^n)'(x)}.$$

We then let $b_d(x,n)$ denote a box within $b_D(x,n)$ centred at a point $z_d(x,n)$ with sides parallel to the axes and of length

$$d(x,n) = \frac{600C_2}{\exp(f^n(x))(f^n)'(x)}.$$

The key step in the proof of Theorem 1.1 is to show that, if x is sufficiently large, then any box of the form $b_d(x,n)$ meets $J(f)$.

LEMMA 3.1. *If x is sufficiently large and $n \in \mathbb{N}$, then*

$$b_d(x,n) \cap J(f) \neq \emptyset.$$

Proof. For simplicity, we will let $b_D = b_D(x,n)$, $b_d = b_d(x,n)$, $z_d = z_d(x,n)$ and $d = d(x,n)$ throughout the proof. We will show that $f^n(b_d)$ meets a curve in $f^{-1}(\mathbb{R})$, and then we deduce from Lemma 2.6 that, if x is sufficiently large, then b_d meets $J(f)$. We first note that, if x is sufficiently large, then it follows from Lemma 2.1 and Lemma 2.2 that

$$f^{-n}\left(B\left(f^n(x), \frac{\pi}{4}\right)\right) \supset B\left(x, \frac{\pi}{400(f^n)'(x)}\right) \supset b_D \supset b_d$$

and so

$$f^n(b_d) \subset B(f^n(x), \pi/4). \tag{3.1}$$

It also follows from Lemma 2.1 and Lemma 2.2 that, if x is sufficiently large, then

$$f^{-n}\left(B\left(f^n(z_d), \frac{3C_2}{\exp(f^n(x))}\right)\right) \subset B\left(z_d, \frac{300C_2}{\exp(f^n(x))(f^n)'(x)}\right) \subset b_d,$$

and so

$$f^n(b_d) \supset B\left(f^n(z_d), \frac{3C_2}{\exp(f^n(x))}\right). \tag{3.2}$$

It follows from Lemma 2.7 and (3.1) that, if x is sufficiently large, then there exists a point z' such that

$$z' \in h^{-1}([K_1,\infty)) \cap B\left(f^n(z_d), \frac{2C_2}{\exp(f^n(x))}\right). \tag{3.3}$$

It follows from Lemma 2.3 that $|f(z') - h(z')| < C_1$ and so $B(f(z'), C_1) \cap [K_1, \infty) \neq \emptyset$. It follows from Lemma 2.6 that

$$B(f(z'), C_1) \cap J(f) \neq \emptyset. \tag{3.4}$$

Since $\Re(f(z')) \geqslant h(z') - C_1 \geqslant K_1 - C_1$ and $K_1 > 5C_1$ it follows from Lemmas 2.1, 2.2 and 2.3 that

$$f^{-1}(B(f(z'), C_1)) \subset B\left(z', \frac{100C_1}{|f'(z')|}\right) \subset B\left(z', \frac{200C_1}{|e^{z'}f(z')|}\right). \tag{3.5}$$

It follows from equations (3.1), (3.2) and (3.3) that $z' \in B(f^n(x), \pi/4)$. We have also just seen that $|f(z')| > 4C_1$, and so

$$\frac{200C_1}{|e^{z'}f(z')|} \leqslant \frac{200C_1 e^{\pi/4}}{\exp(f^n(x))4C_1} \leqslant \frac{200}{\exp(f^n(x))}.$$

Since $C_2 > 200$, it now follows from equation (3.5) that

$$f^{-1}(B(f(z'), C_1)) \subset B\left(z', \frac{C_2}{\exp(f^n(x))}\right). \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$B\left(z', \frac{C_2}{\exp(f^n(x))}\right) \cap J(f) \neq \emptyset.$$

We also know from (3.2) and (3.3) that

$$B\left(z', \frac{C_2}{\exp(f^n(x))}\right) \subset B\left(f^n(z_d), \frac{3C_2}{\exp(f^n(x))}\right) \subset f^n(b_d).$$

Thus $f^n(b_d) \cap J(f) \neq \emptyset$, and hence $b_d \cap J(f) \neq \emptyset$ as claimed.

Lemma 3.2. *If $x \in \mathbb{R}$ is sufficiently large, then, for each $n \in \mathbb{N}$,*

$$d(f^2(x), n) \leqslant d(x, n+1) \leqslant d(x, n)/2.$$

Proof. We have

$$\frac{d(x,n+1)}{d(f^2(x),n)} = \frac{\exp(f^{n+2}(x))(f^n)'(f^2(x))}{\exp(f^{n+1}(x))(f^{n+1})'(x)} = \frac{\exp(f^{n+2}(x))f'(f^{n+1}(x))}{\exp(f^{n+1}(x))f'(x)f'(f(x))}.$$

It follows from Lemma 2.3 that, if x is sufficiently large, then, for each $n \in \mathbb{N}$,

$$\frac{\exp(f^{n+2}(x))f'(f^{n+1}(x))}{\exp(f^{n+1}(x))f'(x)f'(f(x))} \geqslant 1,$$

and hence $d(x, n+1) \geqslant d(f^2(x), n)$ as claimed.

Similarly, if x is sufficiently large, then

$$\frac{d(x,n)}{d(x,n+1)} = \frac{\exp(f^{n+1}(x))(f^{n+1})'(x)}{\exp(f^n(x))(f^n)'(x)} = \frac{\exp(f^{n+1}(x))f'(f^n(x))}{\exp(f^n(x))} \geqslant 2,$$

and hence $d(x, n) \geqslant 2d(x, n+1)$ as claimed.

We now put

$$R(X) = \{x + iy : X - 1 \leqslant x \leqslant f^2(X) + 1,\ |y| \leqslant 1\}$$

and put $A(X) = J(f) \cap R(X)$. Using Lemmas 3.1 and 3.2, we show that, for large values of X, the set $A(X)$ has box dimension equal to two. This is sufficient to prove Theorem 1.1.

Lemma 3.3. *If X is sufficiently large, then* $\dim_{\mathrm{B}} A(X) = 2$.

Proof. Take $d > 0$. If d is sufficiently small and X is sufficiently large, then it follows from Lemma 3.2 that there exist $x \in (X, f^2(X))$ and $n \in \mathbb{N}$ such that $d(x, n) = d$. It now follows from Lemma 3.1 that, if X is sufficiently large, then $N_d(A(X))$ is greater than or equal to the largest number of disjoint boxes of the form $b_d(x, n)$ which can be fitted inside $b_D(x, n)$. Thus, if X is sufficiently large, then

$$\begin{aligned} N_d(A(X)) &\geqslant \inf_{x\in(X,f^2(X))}\left(\frac{D(x,n)}{d(x,n)}-1\right)^2 \\ &= \inf_{x\in(X,f^2(X))}\left(\frac{\pi\exp(f^n(x))(f^n)'(x)}{400(f^n)'(x)\times 600C_2}-1\right)^2 \\ &= \inf_{x\in(X,f^2(X))}\left(\frac{\pi\exp(f^n(x))}{2.4\times 10^5 C_2}-1\right)^2 \\ &\geqslant \inf_{x\in(X,f^2(X))}\frac{\exp(2f^n(x))}{10^{11}C_2^2}. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\log N_d(A(X))}{-\log d} &\geqslant \inf_{x\in(X,f^2(X))}\frac{\log\left(\exp(2f^n(x))/\left(10^{11}C_2^2\right)\right)}{-\log(600C_2/(\exp(f^n(x)(f^n)'(x))))} \\ &= \inf_{x\in(X,f^2(X))}\frac{2f^n(x)-\log\left(10^{11}C_2^2\right)}{f^n(x)+\log(f^n)'(x)-\log(600C_2)} \\ &\geqslant \inf_{x\in(X,f^2(X))}\frac{2-\log\left(10^{11}C_2^2\right)/f^n(X)}{1+\log(f^n)'(x)/f^n(x)-\log(600C_2)/f^{n+2}(X)}. \end{aligned}$$

If X is sufficiently large, then it follows from Lemma 2.3 that for $x\in(X,f^2(X))$,

$$\begin{aligned} (f^n)'(x) &= \prod_{j=0}^{n-1} f'(f^j(x)) \\ &\leqslant \prod_{j=0}^{n-1} 2\exp(f^j(x))f^{j+1}(x) \\ &\leqslant \prod_{j=0}^{n-1} (f^{j+1}(x))^2 \\ &\leqslant (f^n(x))^{2n}. \end{aligned}$$

It follows that, if X is sufficiently large, then for $x\in(X,f^2(X))$,

$$\begin{aligned} \frac{\log(f^n)'(x)}{f^n(x)} &\leqslant \frac{2n\log f^n(x)}{f^n(x)} \\ &\leqslant \frac{2n\log f^n(X)}{f^n(X)}. \end{aligned}$$

Thus, if X is sufficiently large, then

$$\begin{aligned} \liminf_{d\to 0}\frac{\log N_d(A(X))}{-\log d} &\geqslant \lim_{n\to\infty}\frac{2-\log\left(10^{11}C_2^2\right)/f^n(X)}{1+2n\log f^n(X)/f^n(X)-\log(600C_2)/f^{n+2}(X)} \\ &= 2. \end{aligned}$$

This completes the proof of Theorem 1.1.

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