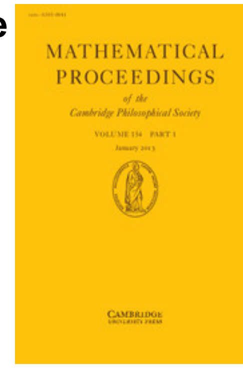

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The Hausdorff dimension of Julia sets of entire functions II

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Abstract

Let f be a transcendental entire function such that the finite singularities of f^{-1} lie in a bounded set. We show that the Hausdorff dimension of the Julia set of such a function is strictly greater than one.

1. *Introduction*

Let $f\colon \mathbb{C} \to \mathbb{C}$ denote a nonlinear rational or entire function and f^n, $n \in \mathbb{N}$, the nth iterate of f. The set of normality, $N(f)$, is defined to be the set of points, $z \in \mathbb{C}$, such that the sequence (f^n) forms a normal family in some neighbourhood of z. It is easy to see that $N(f)$ is open and has the property of complete invariance under f, that is $z \in N(f)$ if and only if $f(z) \in N(f)$. The complement, $J(f)$, of $N(f)$ is known as the Julia set. This set is clearly closed and completely invariant under f. More details of these and other basic properties of the sets $N(f)$ and $J(f)$ can be found in [**7**] and [**8**].

It was shown by Baker [**3**, corollary to theorem 1] that if f is a transcendental entire function then $J(f)$ must contain continua and so the Hausdorff dimension of $J(f)$, $\dim J(f)$, lies in the range $1 \leqslant \dim J(f) \leqslant 2$. In [**10**] we constructed a family of transcendental entire functions, f_K, which satisfied the following result.

THEOREM 1. *Given $\delta > 0$, there exists $K_0(\delta)$ such that*

$$\dim J(f_K) < 1 + \delta$$

for all $K \geqslant K_0$.

It remains open whether there exists a transcendental entire function whose Julia set has dimension equal to one. We put

$$S(f) = \{z : z \text{ is a finite singularity of } f^{-1}\},$$

$$B = \{f : f \text{ is transcendental entire and } S(f) \text{ is bounded}\}.$$

In this paper we study the class B and obtain the following result.

THEOREM 2. *If f is a function in B then* $\dim J(f) > 1$.

When $J(f) = \mathbb{C}$ the result of Theorem 2 is obvious. If $f \in B$ and $J(f) \neq \mathbb{C}$ then, in order to prove Theorem 2, we construct a measure μ_s on a subset of $J(f)$. We do this by taking a point $z_0 \in J(f)$ and taking sets

$$I_n \subset \{z : f^n(z) = z_0\}.$$

For each $z \in I_n$ we define

$$d(z) = (1+|z_0|^2)/[|(f^n)'(z)|\,(1+|z|^2)]$$

and then put

$$s = \inf\left\{t: \sum_{n=1}^{\infty} \sum_{z \in I_n} d(z)^t < \infty\right\}. \tag{1·1}$$

For $s < t \leqslant 2$ we define measures μ_t by

$$\mu_t(B) = c_t \sum_{n=1}^{\infty} \sum_{z \in I_n \cap B} d(z)^t,$$

for each set $B \subset \mathbb{C}$, where c_t is chosen so as to ensure that $\mu_t(\mathbb{C}) = 1$. We then take μ_s to be a weak limit of the measures μ_t as $t \searrow s$. By choosing the point z_0 and the sets I_n carefully we can use the properties of the measure μ_s to prove the following two results, which together imply Theorem 2.

THEOREM 3. *The value s defined by* (1·1) *lies in the range*

$$1 < s \leqslant 2.$$

THEOREM 4. *The value s defined by* (1·1) *satisfies*

$$\dim J(f) \geqslant s.$$

The motivation for looking at measures of this type comes from Sullivan's paper [**11**] where similar measures are used to obtain results on the Hausdorff dimension of Julia sets of rational functions.

In the last section of the paper we give an example of a function in B whose Julia set lies in a domain of finite Lebesgue measure.

2. *Preliminary results*

We begin by giving a formal definition of the Hausdorff dimension of a compact set E. If, for each $\mu > 0$, we put

$$H_\mu(E) = \lim_{\epsilon \to 0} \inf \sum_i (r_i)^\mu,$$

where the inf is taken over all possible covers of E with sets of diameter $r_i < \epsilon$, then the Hausdorff dimension d of the set E is defined to be the unique value satisfying

$$H_\mu(E) = \begin{cases} \infty & \text{for } \mu < d \\ 0 & \text{for } \mu > d. \end{cases}$$

For more details see, for example, [**9**, p. 220].

We now give a list of the notation that will be used throughout this paper.

(i) $d(z,w) = |z-w|/[(1+|z|^2)^{\frac{1}{2}}(1+|w|^2)^{\frac{1}{2}}]$,

(ii) $B(w,r) = \{z: |w-z| < r\}$,

(iii) $\operatorname{diam} B = \sup_{z,w \in B} |z-w|$,

(iv) $\operatorname{Diam} B = \sup_{z,w \in B} d(z,w)$,

(v) $f^\times(z) = |f'(z)|\,(1+|z|^2)/(1+|f(z)|^2)$,

(vi) $C(r) = \{z: |z| = r\}$,

(vii) $D(r) = \{z: |z| \geqslant r\}$.

The next two results are due to Koebe. The first is known as Koebe's quarter theorem and the second as Koebe's distortion theorem.

LEMMA 2·1 (See, for example, [**5**, theorem 2·33]). *If f is univalent in $B(w,r)$ then* $f(B(w,r)) \supset B(f(w), |f'(w)|\, r/4)$.

LEMMA 2·2 (See, for example, [**5**, p. 32]). *If f is univalent in $B(w,r)$ then, for* $0 < s < r$,

$$\sup_{z_1, z_2 \in B(w,s)} |f'(z_1)|/|f'(z_2)| \leqslant L(s/r) = [(r+s)/(r-s)]^4.$$

We put $L(1/2) = L$ and apply this result to functions in the class B.

LEMMA 2·3. *If $S(f) \subset B(0, r_0/2)$, $|w| = r \geqslant r_0$ and $4|w|/5 < |z| < 5|w|/4$ and if we take a branch g of f^{-1} that is defined at w and continue g analytically along a curve γ which lies within $\{t: 4|w|/5 < |t| < 5|w|/4\}$ such that γ winds at most once around 0, then*

$$|g'(w)|/L^{26} \leqslant |g'(z)| \leqslant L^{26}\, |g'(w)|.$$

Proof. If $z = R\exp(i\phi)$ then the circle $C(r)$ can be covered by the discs $B_m = B(r\exp[i(\theta_m + \phi)], r/4)$, $m = 0, 1, 2, \ldots, 24$, where $\theta_m = 2\pi m/25$. We take a point $z_{24} \in B_{24} \cap B_0$ and, for each $0 \leqslant m \leqslant 23$, we take a point $z_m \in B_m \cap B_{m+1}$. As $r \geqslant r_0$, each branch g of f^{-1} is univalent in $B(r\exp(i(\theta_m + \phi)), r/2)$. As $z \in B_0$ and $w \in B_M$ for some $0 \leqslant M \leqslant 24$, it follows from Lemma 2·2 that

$$|g'(z)| \leqslant L\,|g'(z_0)| \leqslant L^2\,|g'(z_1)| \leqslant \ldots \leqslant L^{M+1}\,|g'(z_M)| \leqslant L^{M+2}\,|g'(w)| \leqslant L^{26}\,|g'(w)|.$$

Similarly, we find

$$|g'(z)| \geqslant |g'(w)|\,/L^{26}.$$

LEMMA 2·4. *If f is a transcendental entire function with $S(f) \subset B(0, r_0/2)$ and $|f(0)| < r_0$ then, if $|f(z)| > r_0$, we have*

$$|f'(z)| \geqslant |f(z)|\,[\log|f(z)| - \log(r_0)]/[4\pi\,|z|].$$

Proof. In [**6**, lemma 1], Eremenko and Lyubich show that, if $A = \mathbb{C} \setminus \overline{B(0, r_0)}$, $G = f^{-1}(A)$ and $U = \ln(G)$, then there exists a map F such that, for all $w \in U$, $f(e^w) = e^{F(w)}$ and $|F'(z)| \geqslant [\mathrm{Re}(F(z)) - \log(r_0)]/(4\pi)$. Lemma 2·4 now follows.

As a direct consequence of this result we have

LEMMA 2·5. *If f is in B then there exists $R_1(f) > 0$ such that*

(i) *if $|f(z)| > R_1(f)$ then $|f'(z)| > |f(z)| \log|f(z)|/[8\pi\,|z|]$,*

(ii) *if $|z| > R_1(f)$ then, for each branch g of f^{-1}, $|g'(z)| < 8\pi\,|g(z)|/[|z|\log|z|]$.*

Using Lemma 2·2 together with Lemma 2·5 we are able to prove:

LEMMA 2·6. *If f is in B then there exists $R_2(f) \geqslant R_1(f)$ such that, if $|f^k(z)| \geqslant R_2(f)$ for each $0 \leqslant k \leqslant n$, the branch g of f^{-n} that maps $f^n(z)$ to z satisfies:*

(i) *for each $0 \leqslant k \leqslant n-1$ and each $K \geqslant 4$, $f^k g(B(f^n(z), |f^n(z)|/K))$*
$\subset B(f^k(z), |f^k(z)|/(4K))$,

(ii) *g is univalent in $B(f^n(z), |f^n(z)|/4)$,*

(iii) *if* $w \in B(f^n(z), |f^n(z)|/8)$ *then* $|g'(f^n(z))|/L \leqslant |g'(w)| \leqslant L|g'(f^n(z))|$,
(iv) *if* $w \in B(f^n(z), |f^n(z)|/8)$ *then* $g^{\times}(f^n(z))/(3L) \leqslant g^{\times}(w) \leqslant 3Lg^{\times}(f^n(z))$.

Proof. In what follows we write R_1 for $R_1(f)$ and R_2 for $R_2(f)$. If $S(f) \subset B(0, R_2/2)$ then, for each $1 \leqslant k \leqslant n$, the branch g_k of f^{-1} that maps $f^k(z)$ to $f^{k-1}(z)$ is univalent in $B(f^k(z), |f^k(z)|/2)$ and so, from Lemma 2·2,

$$g_k(B(f^k(z), |f^k(z)|/K)) \subset B(f^{k-1}(z), Lg_k'(f^k(z))\,|f^k(z)|/K) \tag{2·1}$$

for each $K \geqslant 4$. As $R_2 \geqslant R_1$, it follows from Lemma 2·5 that

$$|g_k'(f^k(z))| \leqslant 8\pi |f^{k-1}(z)|/[|f^k(z)| \log |f^k(z)|].$$

If $R_2 > \exp(32\pi L)$ then

$$|g_k'(f^k(z))| \leqslant |f^{k-1}(z)|/(4L\,|f^k(z)|)$$

and so, from (2·1),

$$g_k(B(f^k(z), |f^k(z)|/K)) \subset B(f^{k-1}(z), |f^{k-1}(z)|/(4K)).$$

This proves part (i).

If $S(f) \subset B(0, R_2/2)$ then g_k is univalent in $B(f^k(z), |f^k(z)|/4)$ and so, from part (i), $g = g_1 g_2, \dots, g_{n-1} g_n$ is univalent in $B(f^n(z), |f^n(z)|/4)$. This proves part (ii), and part (iii) then follows from Lemma 2·2.

If $w \in B(f^n(z), |f^n(z)|/4)$ then, from (i), $g(w) \in B(z, |z|/4)$ and so

$$\frac{[1+(3|f^n(z)|/4)^2]}{[1+(5|g(f^n(z))|/4)^2]}|g'(w)| < g^{\times}(w) < \frac{[1+(5|f^n(z)|/4)^2]}{[1+(3|g(f^n(z))|/4)^2]}|g'(w)|.$$

Combining this with part (iii) gives, for $w \in B(f^n(z), |f^n(z)|/8)$,

$$g^{\times}(f^n(z))/(3L) < \frac{9(1+|f^n(z)|^2)\,|g'(f^n(z))|}{25(1+|g(f^n(z))|^2)\,L} < g^{\times}(w)$$

$$< \frac{25(1+|f^n(z)|^2)\,|g'(f^n(z))|\,L}{9(1+|g(f^n(z))|^2)} < 3Lg^{\times}(f^n(z)).$$

The next two results were proved by Eremenko and Lyubich [**6**, pp. 5, 6].

LEMMA 2·7. *If f is in B then there exists $R_3(f) \geqslant R_2(f)$ such that, for each $R \geqslant R_3$, there exists an analytic curve Γ joining a point z_R to ∞ such that $|f(z)| = R$ for each $z \in \Gamma$.*

LEMMA 2·8. *If f is in B then $J(f) = \overline{I(f)}$, where $I(f) = \{z : f^n(z) \to \infty\}$.*

As a direct consequence of Lemma 2·8 we have:

LEMMA 2·9. *If f is in B then, for each $r > 0$, there exists a point $z \in J(f) \cap D(r)$ for which*

$$|f^n(z)| \geqslant r, \quad \textit{for} \quad n \in \mathbb{N}.$$

We now give a couple of results concerning the basic properties of Julia sets. Put

$$O^{-}(w) = \{z : f^n(z) = w \quad \text{for some} \quad n \in \mathbb{N}\},$$

$$E(f) = \{w : O^{-}(w) \quad \text{is finite}\}.$$

If f is nonlinear entire then $E(f)$ contains at most two points [**8**, p. 338].

LEMMA 2·10 ([**8**, p. 356]). *If U is compact, $U \cap E(f) = \varnothing$, $z \in J(f)$ and V is an open neighbourhood of z then there exists $N \in \mathbb{N}$ such that, for all $n \geqslant N$, we have*

$$f^n(V) \supset U.$$

As a simple consequence of Lemma 2·10 we have:

LEMMA 2·11. *If $z \in \mathbb{C} \setminus E(f)$ then $J(f) \subset O^-(z)'$.*

Using Lemma 2·10 we are able to prove:

LEMMA 2·12. *If f is in B, $z \in J(f)$ and $|f^n(z)| \geqslant R_2$ for each $n \in \mathbb{N}$ then, given $K > 0$, there exists $n \in \mathbb{N}$ such that*

$$|(f^n)'(z)| \geqslant K|f^n(z)|.$$

Proof. Suppose that $|(f^n)'(z)| < K|f^n(z)|$ for each $n \in \mathbb{N}$. From Lemma 2·6 we know that the branch g of f^{-n} that maps $f^n(z)$ to z is univalent in $B(f^n(z), |f^n(z)|/4)$ and so, from Lemma 2·1,

$$g(B(f^n(z), |f^n(z)|/4) \supset B(z, |f^n(z)|/[16|(f^n)'(z)|]) \supset B(z, 1/(16K)).$$

As $B(f^n(z), |f^n(z)|/4) \cap B(0, R_2/2) = \varnothing$, it follows that

$$f^n(B(z, 1/(16K))) \cap B(0, R_2/2) = \varnothing$$

for each $n \in \mathbb{N}$, which contradicts Lemma 2·10.

We conclude this section with a result concerning the weak convergence of measures.

LEMMA 2·13 (See, for example, [1, theorem 4·5·1]). *Let $\{\mu_n\}$ be a sequence of finite measures on the Borel sets of a metric space Ω. If the measures μ_n converge weakly to a measure μ as $n \to \infty$ then, for every closed set $A \subset \Omega$,*

$$\mu(A) \geqslant \limsup_{n\to\infty} \mu_n(A).$$

3. *Construction of the measure μ_s*

We take a function $f \in B$ with $J(f) \neq \mathbb{C}$ and a value R' satisfying

$$R' \geqslant R_3(f), \quad S(f) \subset B(0, R'/2), \quad \log(9R'/10) > 1600\pi^2 L^{26} C, \tag{3·1}$$

where $R_3(f)$ is as defined in Lemma 2·7 and $C > 4800L^3$. Now take a point $z_0 \in [J(f) \cap I(f) \cap D(4R')]$ with $|f^n(z_0)| \geqslant 4R'$ for each $n \in \mathbb{N}$ and $B(z_0, |z_0|/4) \cap E(f) = \varnothing$. This is possible by Lemma 2·9. We put

$$R = |z_0|, \quad A = B(z_0, R/C), \quad E = B(z_0, 2R/C).$$

These definitions remain in force for the whole of Sections 3, 4 and 5.

LEMMA 3·1. *There exist $w \in \mathbb{C}$, $r > 0$ such that*
(i) $U = B(w, r) \subset N(f) \cap A$,

(ii) *if, for* $i = 1, 2$, g_i *is a branch of* $f^{-n(i)}$ *satisfying*

$$f^k g_i(U) \subset D(R'),$$

for each $0 \leqslant k \leqslant n(i)$, *and* $g_1|_U \neq g_2|_U$ *then*

$$g_1(U) \cap g_2(U) = \varnothing.$$

Proof. As $J(f) \neq \mathbb{C}$, there is a component N_0 of $N(f)$. We claim that there exists an open set $V \subset N(f)$ with $f^n(V) \cap V = \varnothing$ for each $n \in \mathbb{N}$. If $f^n(N_0) \cap N_0 = \varnothing$ for each $n \in \mathbb{N}$ then we can take V to be any open set in N_0. If N_0 is periodic (i.e. $f^p(N_0) \subset N_0$ for some $p \in \mathbb{N}$) then there are two possibilities (see, for example, [**4**, theorem 2·2]).

Case I. If $f^{np}(z) \to c$, for some constant c, as $n \to \infty$, for all $z \in N_0$, then N_0 contains a set V of the required form.

Case II. If $f^{np}(z) \not\to c$ for all $z \in N_0$ then N_0 is a Siegel disc or a Herman ring and so there exists a component $N_1 \neq f^{p-1}(N_0)$ of $N(f)$ with $f(N_1) \subset N_0$. Thus $f^n(N_1) \cap N_1 = \varnothing$ for each $n \in \mathbb{N}$ and so we can take V to be any open set in N_1.

We now take a point $z \in V \backslash E(f)$. It follows from Lemma 2·11 that there exist $w \in A$, $n' \in \mathbb{N}$ such that $f^{n'}(w) = z$. It then follows that there exists $r > 0$ such that $U = B(w, r) \subset A$ and $f^{n'}(U) \subset V$.

If $f^{-(m+k)}(U) \cap f^{-k}(U) \neq \varnothing$ for some $m, k \in \mathbb{N}$, then $f^{n'}(U) \cap f^{n'+m}(U) \neq \varnothing$ and hence $V \cap f^m(V) \neq \varnothing$ which is a contradiction. Thus

$$f^{-(m+k)}(U) \cap f^{-k}(U) = \varnothing$$

for all $k, m \in \mathbb{N}$.

Finally, suppose that, for some $n \in \mathbb{N}$, there are two branches g_1 and g_2 of f^{-n} such that

$$f^k(g_i(U)) \subset D(R') \subset D(R_2(f)),$$

for $i = 1, 2$, $0 \leqslant k \leqslant n$. As $U = B(w, r) \subset A$ we must have $r < |w|/4$ and so it follows from Lemma 2·6 part (ii) that g_1 and g_2 are each univalent in U. Thus

$$g_1|_U = g_2|_U \quad \text{or} \quad g_1(U) \cap g_2(U) = \varnothing.$$

We put

$$M(r) = \max_{|z|=r} |f(z)|$$

and define a sequence R'_n by putting

$$R'_1 = 2R', \quad R'_{n+1} = M(R'_n).$$

As $S(f) \subset B(0, R'_1)$ and (see, for example, [**2**, lemma 2])

$$S(f^n) = \bigcup_{k=0}^{n-1} f^k(S(f)),$$

it follows that $S(f^n) \subset B(0, R'_n)$.

We now define two sequences R''_n, r_n by

$$R''_n = \max\{R'_n, R_1(f^n)\}, \quad r_n = |f^n(z_0)|,$$

where R_1 is as defined in Lemma 2·5. We also take an analytic curve Γ joining a point z_R to ∞ such that $|f(z)| = R = |z_0|$ for each $z \in \Gamma$. Such a curve exists by Lemma 2·7.

LEMMA 3·2. *There exist* $m, N \in \mathbb{N}$ *such that:*

(i) $\Gamma \cap C(r_N) \neq \varnothing$,

(ii) $G = \{z: 9r_N/10 \leqslant |z| \leqslant 10r_N/9\} \subset D(2R''_m)$,

(iii) *there exists a branch* g_0 *of* f^{-1} *such that*

$$w_0 = g_0(z_0) \in g_0(E) \subset G' = \{z: 17r_N/18 \leqslant |z| \leqslant 19r_N/18\},$$

(iv) *for each* $z \in G$ *there exist two distinct points* $w_1, w_2 \in G'$ *such that* $f^m(w_1) = f^m(w_2) = z$.

Proof. As $z_0 \in A \subset B(z_0, R/4) \subset \mathbb{C} \setminus E()f)$, it follows from Lemma 2·11 that there exist two distinct points $z_1, z_2 \in O^-(z_0) \cap A$. As $z_0 \in J(f)$, we must also have $z_1, z_2 \in J(f)$ and so it follows from Lemma 2·10 that there exist r', $m > 0$ such that

$$\bar{B}(z_1, r') \cap \bar{B}(z_2, r') = \varnothing, \quad \bar{B}(z_i, r') \subset A, \quad f^m(\bar{B}(z_i, r')) \supset A,$$

for $i = 1, 2$.

As $z_0 \in I(f)$, there exists $N \in \mathbb{N}$ such that

$$r_N = |f^N(z_0)| > \max\{20R''_m, |z_R|\},$$

and so these values of N, m satisfy (i) and (ii).

We now take a point $z' \in \Gamma \cap C(r_N)$ so that $w' = f(z') \in C(R)$. If we take $\tilde{C}(R)$ to be the shortest segment of $C(R)$ joining w' to z_0 and g_0 to be the branch of f^{-1} that maps w' to z' then we can continue g_0 univalently along $\tilde{C}(R)$ to E and so, from Lemma 2·3, for $w \in \tilde{C}(R) \cup E$ we have

$$|g'_0(w)| \leqslant L^{26} |g'_0(w')|.$$

As each $w \in E$ can be joined to w' by a curve of length less than $2\pi R$ which lies in $\tilde{C}(R) \cup E$, we have, for $w \in E$,

$$|g_0(w) - z'| < 2\pi R L^{26} |g'_0(w')|.$$

As $|w'| = R > R_1(f)$, it follows from Lemma 2·5 that

$$|g'_0(w')| < 8\pi |g_0(w')|/[|w'| \log |w'|] = 8\pi |z'|/(R \log R) = 8\pi r_N/(R \log R)$$

and so, as $R > R'$, it follows from (3·1) that, for each $w \in E$,

$$|g_0(w) - z'| < 16\pi^2 R L^{26} r_N/(R \log R) < r_N/36. \tag{3·2}$$

Thus $w_0 = g_0(z_0) \in g_0(E) \subset B(z', r_N/36) \subset G'$ which proves (iii).

We now take g_i to be the branch of f^{-m} that maps A univalently into $B(z_i, r') \subset A$ and put $h_i = g_0 g_i f$, for $i = 1, 2$. It follows from (3·2) that h_i is a branch of f^{-m} satisfying

$$h_i(w_0) \in g_0(A) \subset B(z', r_N/36) \tag{3·3}$$

for $i = 1, 2$.

If $z \in G$ then there exists a simple curve $C \subset G$ of length less than $2\pi r_N$ which joins $w_0 = g_0(z_0)$ to z. For each $w \in C$ we have

$$4|w_0|/5 < (9/10)^2 |w_0| \leqslant |w| \leqslant (10/9)^2 |w_0| < 5|w_0|/4.$$

As $|w_0| > 2R''_m > R_1(f^m)$ and $S(f^m) \subset B(0, R'_m) \subset B(0, R''_m)$, it follows from Lemma 2·3 and Lemma 2·5 that we can continue h_i univalently along C and, for each $w \in C$,

$$|(h_i)'(w)| \leqslant L^{26} |(h_i)'(w_0)| \leqslant 8\pi L^{26} |h_i(w_0)|/[|w_0| \log |w_0|].$$

As $h_i(w_0)$, $w_0 \in G$ and $r_N \geqslant R'$ it follows from (3·1) that

$$|h_i(z) - h_i(w_0)| \leqslant 16\pi^2 L^{26} r_N[10r_N/9]/[(9r_N/10)\log(9r_N/10)] < r_N/36.$$

Together with (3·3) this shows that

$$h_i(z) \in B(z', r_N/18) \subset G'$$

for $i = 1, 2$. As $f^m h_i(z) = z$, this proves (iv).

We now put

$$I_n = \{g(z_0) \in A \cup G : g \text{ is a branch of } f^{-n} \text{ with } f^k g(E) \subset D(R') \text{ for each } 0 \leqslant k \leqslant n\}$$

and
$$I = \bigcup_{n=1}^{\infty} I_n.$$

LEMMA 3·3. (i) *If* $|f^{mk}(z)| \geqslant R'_m$ *for each* $0 \leqslant k \leqslant n$ *then* $|f^k(z)| \geqslant 2R'$ *for each* $0 \leqslant k \leqslant nm$;

(ii) *If, in addition,* $z \in A \cup G$ *and* $f^{nm+1}(z) = z_0$ *then* $z \in I_{nm+1}$.

Proof. (i) If, for some $0 \leqslant k < n$, $0 < p < m$, we have $|f^{mk+p}(z)| < 2R'$ then $|f^{m(k+1)}(z)| < \max_{|w|=2R'} |f^{m-p}(w)| \leqslant R'_{m-p+1} \leqslant R'_m$, which is a contradiction. As $R'_m > 2R'$, we have

$$|f^k(z)| \geqslant 2R'$$

for $0 \leqslant k \leqslant nm$.

(ii) If $f^{nm+1}(z) = z_0$ then, taking g to be the branch of $f^{-(nm+1)}$ that maps z_0 to z and noting that

$$E \subset B(z_0, |z_0|/4), \quad |f^k(z)| \geqslant 2R' \geqslant R_2(f)$$

for $0 \leqslant k \leqslant nm+1$, it follows from Lemma 2·6 part (i) that

$$f^k g(E) \subset B(f^k g(z_0), |f^k g(z_0)|/16) = B(f^k(z), |f^k(z)|/16) \subset D(R')$$

and hence $z \in I_{nm+1}$ as claimed.

LEMMA 3·4. *For each* $n, k \in \mathbb{N}$, $I_n \cap I_{n+k} = \varnothing$.

Proof. If, for some $z \in \mathbb{C}$, $n, k \in \mathbb{N}$, we have $z \in I_n \cap I_{n+k}$ then

$$f^n(z) = f^{n+k}(z) = z_0$$

and hence

$$f^k(z_0) = f^{k+n}(z) = z_0$$

which implies that $f^{pk}(z_0) = z_0$ for each $p \in \mathbb{N}$. This contradicts the fact that $f^p(z) \to \infty$ as $p \to \infty$.

For each $z \in I_n$ we put

$$d(z) = \frac{1}{(f^n)^{\times}(z)}.$$

It follows from Lemma 3·4 that $d(z)$ is a single-valued function. We then take s to be the value defined by

$$s = \inf\{t : \sum_{z \in I} d(z)^t < \infty\}. \tag{3·4}$$

LEMMA 3·5. *The value s defined by* (3·4) *satisfies* $0 < s \leqslant 2$.

Proof. Take a set U which satisfies the conditions of Lemma 3·1. If $z \in I_n$, for some $n \in \mathbb{N}$, then $z = g(z_0)$ for some branch g of f^{-n}. We put

$$U(z) = g(U).$$

As $R' \geqslant R_2(f)$ and $U \subset E \subset B(z_0, |z_0|/8)$, it follows from part (iv) of Lemma 2·6 that

$$\operatorname{Diam}(U(z)) \leqslant 3Lg^{\times}(z_0)\operatorname{Diam}(U) = 3Ld(z)\operatorname{Diam}(U)$$

and so there exists $K > 0$ such that, for each $z \in I$, the spherical area of $U(z)$ is at most $K(d(z))^2$.

If $z_1, z_2 \in I$ and $z_1 \neq z_2$, then it follows from Lemma 3·1 that $U(z_1) \cap U(z_2) = \varnothing$. As the area of the sphere is finite it follows that

$$K \sum_{z \in I} d(z)^2 < \infty$$

and so $s \leqslant 2$.

From Lemma 3·2 part (iii) we know that there exists $w_0 \in G'$ with $f(w_0) = z_0$. We put

$$J_n = \{z : f^{mn}(z) = w_0, f^{mr}(z) \in G' \quad \text{for} \quad 0 \leqslant r < n\}.$$

As $G' \subset D(R''_m) \subset D(R'_m)$, it follows from Lemma 3·3 that $J_n \subset I_{mn+1}$. If we take

$$K > \max\{1, \sup_{z \in G'} (f^m)^{\times}(z)\}$$

and put

$$K' = f^{\times}(w_0)$$

then, for $z \in J_n$, we have

$$d(z) = \frac{1}{(f^{mn+1})^{\times}(z)} = \frac{1}{f^{\times}(w_0)} \prod_{k=0}^{n-1} \frac{1}{(f^m)^{\times}(f^{mk}(z))} \geqslant 1/(K'K^n).$$

From Lemma 3·2 part (iv) we know that J_n contains at least 2^n points and so

$$\sum_{z \in I} d(z)^t \geqslant \sum_{n=0}^{\infty} \sum_{z \in I_{mn+1}} d(z)^t \geqslant \sum_{n=0}^{\infty} \sum_{z \in J_n} d(z)^t \geqslant \sum_{n=0}^{\infty} 2^n/(K'K^n)^t.$$

If $t \leqslant \log 2/\log K$ then $2 \geqslant K^t$ and so $\sum_{z \in I} d(z)^t = \infty$. As $K > 1$, it follows that

$$s \geqslant \log 2/\log K > 0.$$

For $0 < t \leqslant 2$, $t \neq s$, we define functions μ_t such that

$$\mu_t(B) = c_t \sum_{z \in I \cap B} d(z)^t$$

for each set $B \subset \mathbb{C}$. If $t < s$ we put $c_t = 1$ and if $t > s$ we take c_t to be the value which gives $\mu_t(\mathbb{C}) = 1$. Clearly, for $t > s$, μ_t is a measure which is supported on $A \cup G$. We now take a weak limit of the measures μ_t as $t \searrow s$ to give a measure μ_s which will be supported on $\bar{A} \cup G$ with $\mu_s(\bar{A} \cup G) = 1$.

We conclude this section with some results which will be useful for the proofs of Theorems 3 and 4.

LEMMA 3·6. (i) *There exists* $\epsilon > 0$ *such that, for* $0 < t \leqslant 2$, $\mu_t(\bar{A}) \geqslant \epsilon\mu_t(G)$,
(ii) *for* $s \leqslant t \leqslant 2$, $\mu_t(\bar{A}) \geqslant \epsilon/(1+\epsilon)$.

Proof. We claim that, for each $z \in G$, there exists $w \in A$ such that $z = f^N(w)$ and $|f^p(w)| \geqslant 2R'$ for each $0 \leqslant p \leqslant N$. Let g be the branch of f^{-1} that maps $f^N(z_0)$ to $f^{N-1}(z_0)$ and continue g univalently to $z \in G$ along a simple curve C lying in G of length less than $2\pi r_N$. From Lemma 3·2 part (ii) we know that $G \subset D(2R''_m) \subset D(2R')$. As $S(f) \subset B(0, R'/2)$ and $R' \geqslant R_1(f)$, it follows from Lemma 2·3 and Lemma 2·5 that, for each $w \in C$,

$$|g'(w)| \leqslant L^{26} |g'(f^N(z_0))| \leqslant \frac{8\pi L^{26} |f^{N-1}(z_0)|}{|f^N(z_0)| \log |f^N(z_0)|} = \frac{8\pi L^{26} r_{N-1}}{r_N \log r_N}$$

and so, as $\log r_N > \log R' > 16\pi^2 L^{26} C$,

$$g(z) \in B\left(f^{N-1}(z_0), \frac{16\pi^2 L^{26} r_{N-1} r_N}{r_N \log r_N}\right) \subset B(f^{N-1}(z_0), r_{N-1}/C). \tag{3·5}$$

As $|f^p(z_0)| \geqslant 4R' \geqslant R_2(f)$ for $0 \leqslant p \leqslant N-1$, it follows from Lemma 2·6 that, taking h to be the branch of $f^{-(N-1)}$ that maps $f^{N-1}(z_0)$ to z_0,

$$f^p h(B(f^{N-1}(z_0), r_{N-1}/C) \subset B(f^p(z_0), r_p/C) \subset B(f^p(z_0), r_p/4) \tag{3·6}$$

for $0 \leqslant p \leqslant N-1$.

From (3·5) together with (3·6) in the case $p = 0$ we see that there exists $w \in A$ with $f^N(w) = z$. As $|f^p(z_0)| \geqslant 4R'$ we see from (3·6) that $|f^p(w)| > 2R'$ for $0 \leqslant p \leqslant N-1$. As $f^N(w) = z \in G \subset D(2R')$ our claim is proved.

Now suppose that $z \in I_n \cap G$ for some $n \in \mathbb{N}$. Take $w \in A$ such that $f^N(w) = z$ and $|f^p(w)| > 2R'$ for $0 \leqslant p \leqslant N$ and let g be the branch of $f^{-(n+N)}$ that maps z_0 to w. As $z \in I_n$ we know that

$$f^{N+k} g(z_0) \in f^{N+k} g(E) \subset D(R') \tag{3·7}$$

for $0 \leqslant k \leqslant n$ and so, as $E \subset B(z_0, |z_0|/4)$, it follows from Lemma 2·6 that

$$f^N g(E) \subset B(f^N g(z_0), |f^N g(z_0)|/4) = B(f^N(w), |f^N(w)|/4).$$

As $|f^p(w)| > 2R'$ for $0 \leqslant p \leqslant N$, it follows from Lemma 2·6 that

$$f^p g(E) \subset B(f^p(w), |f^p(w)|/4) \subset D(R')$$

for $0 \leqslant p \leqslant N$. Together with (3·7) this shows that $w \in I_{n+N}$.

Clearly

$$d(w) = d(z)/(f^N)^\times(w)$$

and so, taking

$$K = \max\{1, \sup_{w \in A} (f^N)^\times(w)\},$$

we have, for $0 < t \leqslant 2$, $t \neq s$,

$$\mu_t(\bar{A}) = \mu_t(A) = c_t \sum_{w \in I \cap A} d(w)^t \geqslant c_t \sum_{\substack{w \in I \cap A \\ f^N(w) \in I \cap G}} d(w)^t$$

$$\geqslant c_t \sum_{z \in I \cap G} d(z)^t / K^t = \mu_t(G)/K^t \geqslant \mu_t(G)/K^2.$$

This proves part (i) for $t \neq s$.

If $2 \geqslant t > s$ then

$$\mu_t(\mathbb{C}) = \mu_t(A) + \mu_t(G) = 1$$

and so, from (i),

$$1 \leqslant \mu_t(A) + \mu_t(A)/\epsilon = \mu_t(A)(\epsilon+1)/\epsilon.$$

This proves part (ii) for $t > s$. It now follows from Lemma 2·13 that the results must also be true for $t = s$.

LEMMA 3·7. *There exist $K_1 = K_1(f) > 0, K_2 = K_2(f) > 1$ such that, for each $n \in \mathbb{N}$ and each $z \in I_n$,*

$$d(z) < K_1/(K_2)^n.$$

Proof. If $z \in I_n$ then $|f^k(z)| \geqslant R' \geqslant R_1(f)$, for $0 \leqslant k \leqslant n$, and so, from Lemma 2·5,

$$1/d(z) = (f^n)^{\times}(z) = \frac{1+|z|^2}{1+|f^n(z)|^2}|(f^n)'(z)|$$

$$\geqslant \frac{(1+|z|^2)\,|f^n(z)|}{(1+|f^n(z)|^2)\,|z|}\prod_{k=1}^{n}\frac{\log|f^k(z)|}{8\pi} \geqslant \frac{(1+|z|^2)\,|z_0|}{(1+|z_0|^2)\,|z|}\left(\frac{\log R'}{8\pi}\right)^n.$$

As $I_n \subset G \cup A$ we must have $|z| > R/2$ and so, as $\log R' > 8\pi$, the result follows.

LEMMA 3·8. *Given $K > 0$, $0 < a < s$, there exist infinitely many values of $n \in \mathbb{N}$ for which*

$$\sum_{z \in I_n \cap A} d(z)^{s-a} \geqslant K.$$

Proof. We note that

$$\mu_{s-a/2}(A) + \mu_{s-a/2}(G) = \sum_{z \in I \cap A} d(z)^{s-a/2} + \sum_{z \in I \cap G} d(z)^{s-a/2} = \sum_{z \in I} d(z)^{s-a/2} = \infty.$$

From Lemma 3·6 we know that

$$\mu_{s-a/2}(G) \leqslant \mu_{s-a/2}(\bar{A})/\epsilon = \mu_{s-a/2}(A)/\epsilon$$

and so we must have

$$\mu_{s-a/2}(A) = \sum_{n=1}^{\infty}\sum_{z \in I_n \cap A} d(z)^{s-a/2} = \infty. \tag{3·8}$$

If there are only finitely many $n \in \mathbb{N}$ for which

$$\sum_{z \in I_n \cap A} d(z)^{s-a} \geqslant K$$

then there must exist $K' \geqslant K$ such that

$$\sum_{z \in I_n \cap A} d(z)^{s-a} < K'$$

for each $n \in \mathbb{N}$. It follows from Lemma 3·7 that, for each $n \in \mathbb{N}$,

$$\sum_{z \in I_n \cap A} d(z)^{s-a/2} \leqslant \sup_{z \in I_n \cap A} d(z)^{a/2} \sum_{z \in I_n \cap A} d(z)^{s-a} < K'[K_1/(K_2)^n]^{a/2},$$

where $K_1 > 0$, $K_2 > 1$, and so

$$\sum_{n=1}\sum_{z\in I_n\cap A} d(z)^{s-a/2} < K'(K_1)^{a/2}\sum_{n=1}^{\infty}(1/K_2)^{na/2} < \infty$$

which contradicts (3·8).

We put

$$\mathscr{A}_n = \{g(A): g \text{ is a branch of } f^{-n} \text{ with } g(z_0)\in I_n\},$$
$$\mathscr{E}_n = \{g(E): g \text{ is a branch of } f^{-n} \text{ with } g(z_0)\in I_n\}.$$

LEMMA 3·9. *There exists $n_0 \in \mathbb{N}$ such that*

$$\operatorname{diam}(A_n) < \operatorname{diam}(E_n) < 1$$

for each $A_n \in \mathscr{A}_n, E_n \in \mathscr{E}_n$, with $n \geqslant n_0$.

Proof. Let g denote the branch of f^{-n} which maps E to E_n and A to A_n. As $g(z_0) = z' \in I_n$ we have

$$|f^k g(z_0)| > R'$$

for $0 \leqslant k \leqslant n$ and so, from Lemma 2·6,

$$\operatorname{Diam}(A_n) < \operatorname{Diam}(E_n) \leqslant \sup_{z\in E} g^{\times}(z)\operatorname{Diam}(E)$$
$$\leqslant 3L\, g^{\times}(z_0)\operatorname{Diam}(E) = 3L\, d(z')\operatorname{Diam}(E).$$

From Lemma 3·7 we have

$$d(z') < K_1/(K_2)^n$$

and so
$$\operatorname{Diam}(A_n) < \operatorname{Diam}(E_n) < 3LK_1 \operatorname{Diam}(E)/(K_2)^n. \tag{3·9}$$

As $E_n \cap I_n \neq \emptyset$ and $I_n \subset A \cup G \subset \{z: |z| \leqslant 10r_N/9\}$ we see from (3·9) that, for large n,

$$E_n \subset \{z: |z| \leqslant 11r_N/9\}$$

and hence, for these values of n, there exists $K > 0$ such that

$$\operatorname{diam}(A_n) < \operatorname{diam}(E_n) < K\operatorname{Diam}(E_n) < 3KLK_1\operatorname{Diam}(E)/(K_2)^n$$

and, as $K_2 > 1$, the result follows.

LEMMA 3·10. *There exists $K_3 > 0$ such that, for each $A_n \in \mathscr{A}_n$ with $n \geqslant n_0$,*

$$K_3\,\mu_s(\bar{A}_n) \geqslant (\operatorname{diam}(A_n))^s.$$

Proof. If $A_n \in \mathscr{A}_n$ then, for each $z \in I_p \cap A$, there exists $w \in A_n$ such that $f^n(w) = z$. We claim that $w \in I_{n+p}$. If h denotes the branch of f^{-p} that maps z_0 to $z \in A$ and g denotes the branch of f^{-n} that maps z_0 to $z' \in A_n$, then gh is the branch of $f^{-(n+p)}$ that maps z_0 to w. As

$$f^{n+k}gh(z_0) \in f^{n+k}gh(E) = f^k h(E) \subset D(R') \tag{3·10}$$

for $0 \leqslant k \leqslant p$ and as $z \in A$, it follows from Lemma 2·6 that

$$f^n gh(E) = h(E) \subset B(h(z_0), 2|h(z_0)|/(4C)) = B(z, |z|/(2C)) \subset E.$$

As $g(z_0) \in I_n$ it follows that, for $0 \leqslant k \leqslant n$,

$$f^k gh(E) \subset f^k g(E) \subset D(R').$$

Together with (3·10) this shows that $w \in I_{n+p}$.

As $f^n(A_n) = A \subset B(z_0, |z_0|/8)$ and $g(z_0) = z' \in I_n$, it follows from Lemma 2·6 that, for each $w \in A_n$,

$$(f^n)^\times(z')/(3L) \leqslant (f^n)^\times(w) \leqslant 3L(f^n)^\times(z') \tag{3·11}$$

and so
$$\mathrm{Diam}(A) \geqslant (f^n)^\times(z')\,\mathrm{Diam}(A_n)/(3L). \tag{3·12}$$

If $w \in A_n$ and $f^n(w) = z \in I$ then, from (3·11) and (3·12),

$$d(w) = d(z)/(f^n)^\times(w) \geqslant d(z)/[3L(f^n)^\times(z')] \geqslant d(z)\,\mathrm{Diam}(A_n)/[9L^2\,\mathrm{Diam}(A)]$$

and so, for $t > s$,

$$\mu_t(A_n) = c_t \sum_{w \in I \cap A_n} d(w)^t \geqslant c_t \sum_{\substack{w \in I \cap A_n \\ f^n(w) \in I \cap A}} d(w)^t \geqslant \left(\frac{\mathrm{Diam}(A_n)}{9L^2\,\mathrm{Diam}(A)}\right)^t c_t \sum_{z \in I \cap A} d(z)^t$$
$$= (\mathrm{Diam}(A_n))^t \mu_t(A)/[9L^2\,\mathrm{Diam}(A)]^t.$$

From Lemma 3·6 we have

$$\mu_t(A_n) \geqslant (\mathrm{Diam}(A_n))^t \epsilon/[(9L^2\,\mathrm{Diam}(A))^t(1+\epsilon)]. \tag{3·13}$$

If $n \geqslant n_0$ then, from Lemma 3·9, $\mathrm{diam}(A_n) < 1$. As $A_n \cap I_n \neq \emptyset$ and $I_n \subset A \cup G$, it follows that, for some $K' > 0$, we have

$$\mathrm{Diam}(A_n) \geqslant \mathrm{diam}(A_n)/K'$$

and so, taking the limit of (3·13) as $t \searrow s$, we see from Lemma 2·13 that

$$\mu_s(\bar{A}_n) \geqslant (\mathrm{diam}(A_n))^s \epsilon/[(9L^2K'\,\mathrm{Diam}(A))^s(1+\epsilon)].$$

4. *Proof of Theorem* 3

Recall that Γ is an analytic curve joining a point z_R to ∞ and that $|f(z)| = R$ for each $z \in \Gamma$.

Lemma 4·1. *For each $r > 0$, the length of $\Gamma \cap B(0,r)$ is finite.*

Proof. It is clear that there are only a finite number of branches $g_1, g_2, \ldots, g_n$ of f^{-1} satisfying $g_i(z_0) \in B(0, 2r)$. We cut $C(R)$ at z_0 and, for $1 \leqslant i \leqslant n$, continue g_i univalently in an anticlockwise direction around the cut curve $C(R)$. The length of $\bigcup_{i=1}^n g_i(C(R))$ is clearly finite.

Now suppose that $z \in \Gamma \cap B(0,r)$. There exists $w' \in C(R)$ and a branch g of f^{-1} such that $z = g(w')$. We continue g analytically in a clockwise direction from w' along $C(R)$ to z_0. As $R \geqslant 4R' \geqslant R_1(f)$ and $S(f) \subset B(0, R'/2)$, it follows from Lemma 2·3 and Lemma 2·5 that, for each point w on this segment of $C(R)$,

$$|g'(w)| \leqslant L^{26}|g'(w')| \leqslant \frac{8\pi L^{26}|g(w')|}{|w'|\log|w'|} \leqslant \frac{8\pi L^{26} r}{R\log R}$$

and so, as $\log R > \log R' > 16\pi^2 L^{26}$,

$$|g(w') - g(z_0)| \leqslant \frac{16\pi^2 L^{26} R r}{R\log R} < r.$$

As $z = g(w') \in B(0,r)$, it follows that $g(z_0) \in B(0,2r)$ and hence $g = g_i$ for some $1 \leqslant i \leqslant n$. Thus

$$\Gamma \cap B(0,r) \subset \bigcup_{i=1}^{n} g_i(C(R))$$

and hence has finite length.

We take m, N to be values which satisfy Lemma 3·2.

LEMMA 4·2. *For each $n \in \mathbb{N} \cup \{0\}$ there exist 2^n curves $\gamma_{n,i}$, $1 \leqslant i \leqslant 2^n$, each of which joins $C(19r_N/18)$ to ∞ and lies in $D(17r_N/18)$ with*

(i) $\gamma_{0,1} \subset \Gamma$,

(ii) *for each $0 \leqslant r \leqslant n$,*

$$f^{rm}(\gamma_{n,i}) \subset \gamma_{n-r,j}$$

for some $1 \leqslant j \leqslant 2^{n-r}$,

(iii) $\gamma_{n,i} \cap \gamma_{n,j} = \varnothing$, *if $i \neq j$.*

Proof. We begin by considering the case $n = 0$. From Lemma 3·2 part (i) we have $\Gamma \cap C(r_N) \neq \varnothing$ and so Γ joins $C(19r_N/18)$ to ∞. If there does not exist a segment $\gamma_{0,1} \subset \Gamma \cap N(19r_N/18)$ joining $C(19r_N/18)$ to ∞ then the length of

$$\Gamma \cap \{z\colon 17r_N/18 \leqslant |z| \leqslant 19r_N/18\}$$

must be infinite which contradicts Lemma 4·1.

We now assume that the result is true for $n-1$ and, for some $1 \leqslant i \leqslant 2^{n-1}$, consider the curve $\gamma_{n-1,i}$. We take $z' \in C(19r_N/18)$ and note from Lemma 3·2 part (iv) that there exist two points w_1, $w_2 \in G'$ such that $f^m(w_1) = f^m(w_2) = z'$. Let h_k denote the branch of f^{-m} that maps z' to w_k. As we know from Lemma 3·2 part (ii) that

$$S(f^m) \subset B(0,R'_m) \subset B(0,R''_m) \subset B(0,9r_N/10)$$

we can continue h_k univalently along $\gamma_{n-1,i}$. As $w_1 \in G'$, the curve $\Gamma_{n,2i} = h_1(\gamma_{n-1,i})$ must join $C(19r_N/18)$ to ∞. If $\Gamma_{n,2i}$ does not contain a curve $\gamma_{n,2i} \subset D(17r_N/18)$ which joins $C(19r_N/18)$ to ∞ then the length of

$$\Gamma_{n,2i} \cap \{z\colon 17r_N/18 \leqslant |z| \leqslant 19r_N/18\}$$

must be infinite. As $f^{nm}(\Gamma_{n,2i}) \subset \gamma_{0,1} \subset \Gamma$, it follows that the length of $\Gamma \cap B(0,r)$, where

$$r = \sup_{|z| \leqslant 19r_N/18} |f^{nm}(z)|,$$

is infinite, which contradicts Lemma 4·1. In the same way we can show that $\Gamma_{n,2i-1} = h_2(\gamma_{n-1,i})$ contains a curve $\gamma_{n,2i-1} \subset N(17r_N/18)$ which joins $C(19r_N/18)$ to ∞.

Recall that, for each $n \in \mathbb{N}$,

$$\mathscr{A}_n = \{g(A)\colon g \text{ is a branch of } f^{-n} \text{ with } g(z_0) \in I_n\}.$$

For each $n \in \mathbb{N}$, $1 \leqslant i \leqslant 2^n$, we put

$$F_{n,i} = \{z\colon z \in \gamma_{n,i} \cap G,\ f^{nm+1}(z) = z_0\},$$

$$H_{n,i} = \{g(A)\colon g \text{ is a branch of } f^{-(nm+1)} \text{ with } g(z_0) \in F_{n,i}\}.$$

LEMMA 4·3. *For each $n \in \mathbb{N}$, $1 \leqslant i \leqslant 2^n$, $F_{n,i} \subset I_{mn+1}$ and hence $H_{n,i} \subset \mathscr{A}_{mn+1}$.*

Proof. If $z \in F_{n,i}$ then, from Lemma 4·2 and Lemma 3·2, for $0 \leqslant r \leqslant n$ there exists $1 \leqslant j \leqslant 2^{n-r}$ such that

$$f^{rm}(z) \in \gamma_{n-r,j} \subset D(17r_N/18) \subset D(R''_m) \subset D(R'_m)$$

and so, from Lemma 3·3, $z \in I_{mn+1}$.

LEMMA 4·4. *There exists $K_4 > 0$ such that, for each $n \in \mathbb{N}$, $1 \leqslant i \leqslant 2^n$,*

$$\sum_{A_{mn+1} \in H_{n,i}} \operatorname{diam}(A_{mn+1}) \geqslant K_4.$$

Proof. From Lemma 4·2 we know that, for each $n \in \mathbb{N}$, $1 \leqslant i \leqslant 2^n$, $\gamma_{n,i}$ contains a segment $\gamma'_{n,i}$ joining $C(19r_N/18)$ to $C(39r_N/36)$ such that

$$\gamma'_{n,i} \subset \{z \colon 17r_N/18 \leqslant |z| \leqslant 39r_N/36\}.$$

We take a point $z \in \gamma'_{n,i}$ and let h denote the branch of f^{-1} that maps $w' \in C(R)$ to $f^{mn}(z)$. We cut $C(R)$ at $-z_0$ and continue h univalently to the whole of the cut curve $C(R)$ and to A. If $-z_0 = R\exp(i\phi)$ then we define $h(-z_0)$ by

$$h(-z_0) = \lim_{\theta \nearrow \phi} h(R\exp(i\theta)).$$

It follows from Lemma 2·3 that, for each $w \in C(R) \cup A$,

$$|h'(w)| \leqslant L^{26}|h'(w')|$$

and hence

$$h(C(R) \cup A) \subset B(h(w'), 2\pi RL^{26}|h'(w')|). \tag{4·1}$$

From Lemma 2·5 we have

$$|h'(w')| < \frac{8\pi|h(w')|}{|w'|\log|w'|}$$

and so, as $\log R > \log R' > 1600\pi^2L^{26}$,

$$2\pi RL^{26}|h'(w')| < 16\pi^2L^{26}|h(w')|/\log R < |h(w')|/100 < |h(w')|/8. \tag{4·2}$$

As $z \in \gamma_{n,i}$ we know that, for $0 \leqslant p \leqslant n$,

$$f^{mp}(z) \in \gamma_{n-p,j} \subset D(R'_m),$$

for some $1 \leqslant j \leqslant 2^{n-p}$, and so it follows from Lemma 3·3 that

$$|f^p(z)| > R' \tag{4·3}$$

for $0 \leqslant p \leqslant mn+1$. Thus, from Lemma 2·6, the branch H of f^{-mn} that maps $h(w')$ to z is univalent in $B(h(w'), |h(w')|/4)$. It follows from Lemma 2·2, (4·1) and (4·2) that

$$g(C(R)) = Hh(C(R)) \subset B(z, 2\pi RL^{27}|g'(w')|) \tag{4·4}$$

and, from Lemma 2·6, (4·1), (4·2) and (4·3), that

$$g(C(R)) \subset B(z, |z|/100) \subset G. \tag{4·5}$$

Also, from (4·3) and Lemma 2·6, we know that g is univalent in A and hence, from Lemma 2·1, (4·1) and (4·2),

$$g(A) \supset B(g(z_0), |g'(z_0)|R/(4C)) \supset B(g(z_0), |g'(w')|R/(4CL^{27})). \tag{4·6}$$

We now take a collection $G_{n,i}$ of disjoint curves γ_k such that $\gamma_k \subset \gamma_{n,i}$, $\gamma_k \cap \gamma'_{n,i} \neq \emptyset$, f^{mn+1} maps γ_k univalently onto $C(R)$ and

$$\bigcup_{\gamma_k \in G_{n,i}} \gamma_k \supset \gamma'_{n,i}. \tag{4·7}$$

From (4·5) we know that $\gamma_k \subset G$ and so, if $\gamma_k = g_k(C(R))$, then $g_k(A) \in H_{n,i}$. From (4·4), (4·6) and (4·7) we see that

$$\sum_{A_{mn+1} \in H_{n,i}} \operatorname{diam}(A_{mn+1}) \geqslant \sum_{g_k(C(R)) \in G_{n,i}} \operatorname{diam}(g_k(A))$$

$$\geqslant [1/(8\pi L^{54} C)] \sum_{g_k(C(R)) \in G_{n,i}} \operatorname{diam}(g_k(C(R)))$$

$$\geqslant \operatorname{diam}(\gamma'_{n,i})/(8\pi L^{54} C) \geqslant r_N/(288\pi L^{54} C).$$

We are now in a position to prove Theorem 3. If $A_{mn+1} = g(A) \in \mathscr{A}_{nm+1}$, then it follows from Lemma 2·6 that g is univalent in E and so all the closures of the sets in $\bigcup_{i=1}^{2^n} H_{n,i}$ are disjoint. It therefore follows from Lemma 3·10 that, for each $n \in \mathbb{N}$ with $n \geqslant n_0$,

$$\sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,i}} \operatorname{diam}(A_{mn+1})^s \leqslant K_3 \sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,i}} \mu_s(\bar{A}_{mn+1}) \leqslant K_3. \tag{4·8}$$

If $n \geqslant n_0$ then it follows from Lemma 3·9 that, for $1 \leqslant i \leqslant 2^n$ and each $A_{mn+1} \in H_{n,i}$, $\operatorname{diam}(A_{mn+1}) < 1$ and so, if $s \leqslant 1$, it follows from (4·8) that

$$\sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,i}} \operatorname{diam}(A_{mn+1}) \leqslant \sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,i}} \operatorname{diam}(A_{mn+1})^s \leqslant K_3. \tag{4·9}$$

Finally, it follows from Lemma 4·4 that, for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{2^n} \sum_{A_{mn+1} \in H_{n,i}} \operatorname{diam}(A_{mn+1}) \geqslant 2^n K_4. \tag{4·10}$$

Combining (4·9) and (4·10) we see that, for $n \geqslant n_0$,

$$2^n K_4 \leqslant K_3.$$

As there are arbitrarily large values of $n \in \mathbb{N}$ satisfying $n \geqslant n_0$ this is clearly a contradiction and so we must have $s > 1$ as claimed.

5. *Proof of Theorem 4*

We take a value $a \in (0, s)$ and recall that

$$\mathscr{E}_n = \{g(E) : g \text{ is a branch of } f^{-n} \text{ with } g(z_0) \in I_n\}.$$

LEMMA 5·1. *There exists $M > 0$ such that*

$$\sum_{\substack{E_M \in \mathscr{E}_M \\ E_M \subset E}} (\operatorname{diam}(E_M))^{s-a} \geqslant L^{2(s-a)} (\operatorname{diam}(E))^{s-a}.$$

Proof. Take a value $n \geqslant n_0$ and a set $E_n = g(E) \in \mathscr{E}_n$ with $g(z_0) \in A$. We know from Lemma 3·9 that $\operatorname{diam}(E_n) < 1$ and hence $E_n \subset E$. As $g(z_0) \in I_n$, we know that $|f^r g(z_0)| \geqslant R' \geqslant R_2(f)$, for $0 \leqslant r \leqslant n$ and so, from Lemma 2·6,

$$\operatorname{Diam}(E_n) \geqslant \inf_{z \in E} g^{\times}(z) \operatorname{Diam}(E) \geqslant g^{\times}(z_0) \operatorname{Diam}(E)/(3L).$$

As $E_n \subset E$ it follows that there exists $K > 1$ such that

$$\operatorname{diam}(E_n) \geqslant g^{\times}(z_0) \operatorname{diam}(E)/(3LK) = d(z) \operatorname{diam}(E)/(3LK),$$

where $z = g(z_0) \in I_n$. Thus, for $n \geqslant n_0$,

$$\sum_{\substack{E_n \in \mathscr{E}_n \\ E_n \subset E}} (\operatorname{diam}(E_n))^{s-a} \geqslant (\operatorname{diam}(E)/(3LK))^{s-a} \sum_{z \in A \cap I_n} d(z)^{s-a}. \tag{5·1}$$

From Lemma 3·8 we know that there exists $M \geqslant n_0$ such that

$$\sum_{z \in A \cap I_M} d(z)^{s-a} \geqslant (3L^3 K)^{s-a}. \tag{5·2}$$

Combining (5·1) with (5·2) we see that

$$\sum_{\substack{E_M \in \mathscr{E}_M \\ E_M \subset E}} (\operatorname{diam}(E_M))^{s-a} \geqslant L^{2(s-a)} (\operatorname{diam}(E))^{s-a}.$$

We now put

$$\mathscr{F}_n = \{g(E_M) : E_M \in \mathscr{E}_M, E_M \subset E,\ g \text{ is a branch of } f^{-n} \text{ with } g(z_0) \in I_n\}.$$

LEMMA 5·2. *There exists a set $F \in \mathscr{F}_1$ with $F \subset G$.*

Proof. From Lemma 3·2 part (iii) we know that there exists a branch g of f^{-1} with $g(E) \subset G' \subset G$. Clearly $g(z_0) \in I_1$ and so, for each $E_M \in \mathscr{E}_M$ with $E_M \subset E$, we have $g(E_M) \in \mathscr{F}_1$ and $g(E_M) \subset g(E) \subset G$.

LEMMA 5·3. *If $F_n \in \mathscr{F}_n$ and $F_n \subset G$ for some $n \in \mathbb{N}$ then*

$$\sum_{\substack{F_{n+M} \in \mathscr{F}_{n+M} \\ F_{n+M} \subset F_n}} (\operatorname{diam}(F_{n+M}))^{s-a} > (\operatorname{diam}(F_n))^{s-a}.$$

Proof. Take a set $F_n = g(E'_M) \in \mathscr{F}_n$ and a set $E_M \in \mathscr{E}_M$ such that $E_M \subset E$. Taking h to be the branch of f^{-M} that maps E to E'_M, we note that

$$gh(E_M) \subset gh(E) = g(E'_M) = F_n \subset G. \tag{5·3}$$

As $h(z_0) \in I_M$ we see that, for $0 \leqslant k \leqslant M$,

$$f^{n+k} gh(E) = f^k h(E) \subset D(R').$$

As $g(z_0) \in I_n$ we see that, for $0 \leqslant k \leqslant n$,

$$f^k gh(E) = f^k g(E'_M) \subset f^k g(E) \subset D(R').$$

Thus, for $0 \leqslant k \leqslant n+M$,

$$f^k gh(E) \subset D(R'). \tag{5·4}$$

From (5·3) we know that $gh(z_0)\in gh(E)\subset G$ and so, together with (5·4), we see that $gh(z_0)\in I_{n+M}$ and hence $gh(E_M)\in\mathscr{F}_{n+M}$.

As $z_0\in E$, it follows from (5·4) together with Lemma 2·6 that

$$\operatorname{diam}(F_n)=\operatorname{diam}(g(E'_M))=\operatorname{diam}(gh(E))\leqslant L|(gh)'(z_0)|\operatorname{diam}(E)$$

and, as $E_M\subset E$,

$$\operatorname{diam}(gh(E_M))\geqslant|(gh)'(z_0)|\operatorname{diam}(E_M)/L.$$

Together with Lemma 5·1 this gives

$$\sum_{\substack{F_{n+M}\in\mathscr{F}_{n+M}\\F_{n+M}\subset F_n}}(\operatorname{diam}(F_{n+M}))^{s-a}\geqslant\sum_{\substack{E_M\in\mathscr{E}_M\\E_M\subset E}}(\operatorname{diam}(gh(E_M))^{s-a}$$

$$\geqslant(|(gh)'(z_0)|/L)^{s-a}\sum_{\substack{E_M\in\mathscr{E}_M\\E_M\subset E}}(\operatorname{diam}(E_M))^{s-a}$$

$$\geqslant(L|(gh)'(z_0)|\operatorname{diam}(E))^{s-a}\geqslant(\operatorname{diam}(F_n))^{s-a}.$$

We are now in a position to prove one of the two main results of this section. We put

$$J'=\{z\colon z\in J(f)\cap G,|f^n(z)|\geqslant R'\text{ for each }n\in\mathbb{N}\}.$$

LEMMA 5·4. *There exists* $r_0>0$, $K_5>0$ *such that, if* $r<r_0$ *and* $z\in J'$, *there exists* $P\in\mathbb{N}$ *for which*

$$\sum_{\substack{F_{PM+1}\in\mathscr{F}_{PM+1}\\F_{PM+1}\cap B(z,r)\neq\varnothing}}(\operatorname{diam}(F_{PM+1}))^{s-a}<K_5(\operatorname{diam}(B(z,r))^{s-a}.$$

Proof. We take $r>0$ and $z\in J'$. Let q be the smallest value of $n\in\mathbb{N}$ for which

$$|(f^n)'(z)|r>|f^n(z)|/50 \tag{5·5}$$

and take P to be the integer satisfying

$$q-M\leqslant PM+1<q. \tag{5·6}$$

Note that the existence of q follows from Lemma 2·12. As $|f^q(z)|\geqslant R'$, it follows from (5·5) that $|(f^q)'(z)|r>R'/50$. If

$$r<r_0=\inf_{z\in G}\{R'/(50|f'(z)|),R'/(50|(f^2)'(z)|),\dots,R'/(50|(f^{M+1})'(z)|)\},$$

then we must have $q>M+1$ and hence $P\geqslant 1$.

For each $E_M\in\mathscr{E}_M$ with $E_M\subset E$ we put

$$H(E_M)=\{F_{PM+1}\colon F_{PM+1}\in\mathscr{F}_{PM+1},f^{PM+1}(F_{PM+1})=E_M,F_{PM+1}\cap B(z,r)\neq\varnothing\}.$$

We note from (5·6) that $0\leqslant q-2-PM\leqslant M-1$ and consider two separate cases.

Case I.

$$f^{q-2-PM}(E_M)\cap f^{q-1}(B(z,r))=\varnothing$$

for some $E_M\in\mathscr{E}_M$ satisfying $E_M\subset E$.

Suppose that there exists a set $F_{PM+1} \in \mathscr{F}_{PM+1}$ with $f^{PM+1}(F_{PM+1}) = E_M$ such that $F_{PM+1} \cap B(z,r) \neq \varnothing$. Then $f^{q-1}(F_{PM+1}) \cap f^{q-1}(B(z,r)) \neq \varnothing$. But $f^{q-1}(F_{PM+1}) = f^{q-2-PM}(E_M)$ and so we have a contradiction. Thus

$$\sum_{F_{PM+1} \in H(E_M)} (\mathrm{diam}(F_{PM+1}))^{s-a} = 0.$$

Case II.

$$f^{q-2-PM}(E_M) \cap f^{q-1}(B(z,r)) \neq \varnothing$$

for some $E_M \in \mathscr{E}_M$ satisfying $E_M \subset E$.

We begin by showing that

$$\mathrm{diam}(f^{q-2-PM}(E_M)) < \mathrm{diam}(f^{q-1}(B(z,r))). \tag{5·7}$$

Let g denote the branch of f^{-M} that maps E to E_M. As $g(z_0) \in I_M$ we know that

$$|f^p g(z_0)| \geqslant R' \geqslant R_2(f), \tag{5·8}$$

for $0 \leqslant p \leqslant M$. As $0 < q-1-PM \leqslant M$, it follows from Lemma 2·6 that

$$\begin{aligned} &B(f^{q-1-PM}g(z_0), |f^{q-1-PM}g(z_0)|/(200L)) \\ &\quad \supset f^{q-1-PM}g(B(z_0, R/(200L))) \supset B(f^{q-1-PM}g(z_0), R\,|(f^{q-1-PM}g)'(z_0)|/(200L^2)). \end{aligned} \tag{5·9}$$

Let h denote the branch of f^{-1} that maps $f^{q-1-PM}(z_0)$ to $f^{q-2-PM}(z_0)$. From (5·8) we have $|f^{q-1-PM}g(z_0)| \geqslant R'$ and so h is univalent in $B(f^{q-1-PM}g(z_0), |f^{q-1-PM}g(z_0)|/(200L))$. It follows from (5·9) and Lemma 2·1 that

$$\begin{aligned} &h(B(f^{q-1-PM}g(z_0), |f^{q-1-PM}g(z_0)|/(200L))) \\ &\quad \supset h(B(f^{q-1-PM}g(z_0), R\,|(f^{q-1-PM}g)'(z_0)|/(200L^2))) \\ &\quad \supset B(f^{q-2-PM}g(z_0), R\,|(f^{q-2-PM}g)'(z_0)|/(800L^2)). \end{aligned} \tag{5·10}$$

As $0 \leqslant q-2-PM < M$ and $C > 4800L^3$, it follows from Lemma 2·6 and (5·8) that

$$\begin{aligned} f^{q-2-PM}(E_M) &= f^{q-2-PM}g(E) = f^{q-2-PM}g(B(z_0, 2R/C)) \\ &\subset B(f^{q-2-PM}g(z_0), 2RL\,|(f^{q-2-PM}g)'(z_0)|/C) \\ &\subset B(f^{q-2-PM}g(z_0), R\,|(f^{q-2-PM}g)'(z_0)|/(2400L^2)). \end{aligned}$$

Thus, if $\mathrm{diam}(f^{q-1}(B(z,r)) \leqslant \mathrm{diam}(f^{q-2-PM}(E_M))$, we have

$$f^{q-1}(B(z,r)) \subset B(f^{q-2-PM}g(z_0), R\,|(f^{q-2-PM}g)'(z_0)|/(800L^2))$$

and hence, from (5·10),

$$f^q(B(z,r)) \subset B(f^{q-1-PM}g(z_0), |f^{q-1-PM}g(z_0)|/(200L)).$$

Thus

$$f^q(B(z,r)) \subset B(f^q(z), |f^q(z)|/(50L)).$$

As $|f^p(z)| \geqslant R' \geqslant R_2(f)$ for $0 \leqslant p \leqslant q$, it follows from Lemma 2·6 that

$$B(z,r) \subset B(z, |f^q(z)|/(50\,|(f^q)'(z)|))$$

and so $|(f^q)'(z)|\,r \leqslant |f^q(z)|/50$ which contradicts (5·5). Thus (5·7) must indeed be true.

As $|f^p(z)| \geqslant R'$ for $0 \leqslant p \leqslant q-1$ and

$$|(f^{q-1})'(z)|\,r \leqslant |f^{q-1}(z)|/50, \tag{5·11}$$

it follows from Lemma 2·6 that the branch h_1 of $f^{-(q-1)}$ that maps $f^{q-1}(z)$ to z is univalent in $B(f^{q-1}(z), 4|(f^{q-1})'(z)|r)$ and so, from Lemma 2·1,

$$f^{q-1}(B(z,r)) \subset B(f^{q-1}(z), 4|(f^{q-1})'(z)|r). \tag{5·12}$$

It follows from (5·7) and (5·11) that

$$f^{q-2-PM}(E_M) \subset B(f^{q-1}(z), 12|(f^{q-1})'(z)|r) \subset B(f^{q-1}(z), |f^{q-1}(z)|/4)$$

and so, as $|f^{q-1}(z)| \geqslant R'$, it follows from Lemma 2·6, (5·7) and (5·12) that

$$\begin{aligned}\sum_{F_{PM+1}\in H(E_M)} (\mathrm{diam}(F_{PM+1}))^{s-a} &= (\mathrm{diam}(h_1 f^{q-2-PM}(E_M)))^{s-a}\\ &< (L\,\mathrm{diam}(f^{q-2-PM}(E_M))/|(f^{q-1})'(z)|)^{s-a}\\ &< (L\,\mathrm{diam}(f^{q-1}(B(z,r))/|(f^{q-1})'(z)|)^{s-a}\\ &\leqslant (4L)^{s-a}(\mathrm{diam}(B(z,r)))^{s-a} < 16L^2(\mathrm{diam}(B(z,r)))^{s-a}.\end{aligned}$$

As there are only a finite number of sets $E_M \in \mathscr{E}_M$ with $E_M \subset E$, combining the results of Case I and Case II gives the desired result.

LEMMA 5·5. *If, for some* $n \in \mathbb{N}$, $F_n \in \mathscr{F}_n$ *and* $F_n \subset G$ *then* $F_n \cap J' \neq \varnothing$.

Proof. Take a set $F_n \in \mathscr{F}_n$ with $F_n \subset G$ and the set $E_M = f^n(F_n) \in \mathscr{E}_M$ with $E_M \subset E$. Let w' denote the point in $I_M \cap E_M$ and z' denote the point in F_n which satisfies $f^n(z') = w'$. As $z_0 \in J(f)$ and $|f^p(z_0)| \geqslant R'$ for each $p \in \mathbb{N}$, it follows that $z' \in J(f)$ and

$$|f^{p+n+M}(z')| = |f^p(z_0)| \geqslant R', \tag{5·13}$$

for each $p \in \mathbb{N}$.

Let h denote the branch of f^{-M} that maps E to E_M and g denote the branch of f^{-n} that maps E_M to F_n. Then

$$f^{p+n}(z') \in f^{p+n}(F_n) = f^{p+n}gh(E) = f^p h(E) \subset D(R'),$$

for $0 \leqslant p \leqslant M$, and

$$f^p(z') \in f^p(F_n) = f^p g(E_M) \subset f^p g(E) \subset D(R'),$$

for $0 \leqslant p \leqslant n$. Together with (5·13) this shows that $|f^p(z')| \geqslant R'$ for each $p \in \mathbb{N}$ and so $z' \in F_n \cap J'$.

We are now in a position to prove the other main result of this section.

LEMMA 5·6. $\dim J' \geqslant s-a$.

Proof. Take a set $F \in \mathscr{F}_1$ such that $F \subset G$. The existence of such a set follows from Lemma 5·2. If $\dim J' < s-a$ then there exists a cover of J' with sets $B_i = B(z_i, r_i)$, $i \in I$, such that, for each $i \in I$, we have $z_i \in J'$, $r_i < r_0$ and

$$\sum_{i\in I} (\mathrm{diam}(B_i))^{s-a} < (\mathrm{diam}(F))^{s-a}/(2K_5). \tag{5·14}$$

From Lemma 5·4 we know that, for each $i \in I$, there exists $P(i) \in \mathbb{N}$ such that

$$\sum_{\substack{F_{P(i)M+1}\in\mathscr{F}_{P(i)M+1}\\ F_{P(i)M+1}\cap B_i \neq \varnothing}} (\mathrm{diam}(F_{P(i)M+1}))^{s-a} < K_5(\mathrm{diam}(B_i))^{s-a}. \tag{5·15}$$

As J' is compact we can cover J' with a finite number of the sets B_i. We label these sets $B_1, B_2, \ldots, B_n$ in such a way that $P(i) \leqslant P(i+1)$, for $1 \leqslant i \leqslant n$. It follows from Lemma 5·3 and (5·15) that

$$\begin{aligned}
(\operatorname{diam}(F))^{s-a} &\leqslant \sum_{\substack{F_1 \in \mathscr{F}_1 \\ F_1 \subset G}} \operatorname{diam}(F_1))^{s-a} \leqslant \sum_{F_{P(1)M+1} \subset G} (\operatorname{diam}(F_{P(1)M+1}))^{s-a} \\
&\leqslant \sum_{F_{P(1)M+1} \subset (G \backslash B_1)} (\operatorname{diam}(F_{P(1)M+1}))^{s-a} + \sum_{F_{P(1)M+1} \cap B_1 \neq \varnothing} (\operatorname{diam}(F_{P(1)M+1}))^{s-a} \\
&\leqslant \sum_{F_{P(2)M+1} \subset (G \backslash B_1)} (\operatorname{diam}(F_{P(2)M+1}))^{s-a} + K_5 (\operatorname{diam}(B_1))^{s-a} \\
&\leqslant \sum_{F_{P(2)M+1} \subset [G \backslash (B_1 \cup B_2)]} (\operatorname{diam}(F_{P(2)M+1}))^{s-a} \\
&\quad + \sum_{F_{P(2)M+1} \cap B_2 \neq \varnothing} (\operatorname{diam}(F_{P(1)M+1}))^{s-a} + K_5 (\operatorname{diam}(B_1))^{s-a} \\
&\leqslant \sum_{F_{P(3)M+1} \subset [G \backslash (B_1 \cup B_2)]} (\operatorname{diam}(F_{P(3)M+1}))^{s-a} + K_5 [(\operatorname{diam}(B_1))^{s-a} + (\operatorname{diam}(B_2))^{s-a}] \\
&\leqslant \\
&\vdots \\
&\leqslant \sum_{F_{P(n)M+1} \subset [G \backslash (B_1 \cup B_2 \cup \cdots \cup B_n)]} (\operatorname{diam}(F_{P(1)M+1}))^{s-a} + K_5 \sum_{i=1}^{n} (\operatorname{diam}(B_i))^{s-a}, \qquad (5·16)
\end{aligned}$$

where $F_{P(i)M+1}$ denotes a set in $\mathscr{F}_{P(i)M+1}$ in each of the above statements. As $J' \subset \bigcup_{i=1}^{n} B_i$, it follows from Lemma 5·5 that

$$\left\{ F_{P(n)M+1} : F_{P(n)M+1} \in \mathscr{F}_{P(n)M+1}, F_{P(n)M+1} \subset \left(G \setminus \bigcup_{i=1}^{n} B_i \right) \right\} = \varnothing$$

and so, from (5·14) and (5·16) we have

$$(\operatorname{diam}(F))^{s-a} \leqslant K_5 \sum_{i=1}^{n} (\operatorname{diam}(B_i))^{s-a} < (\operatorname{diam}(F))^{s-a}/2$$

which is a contradiction.

As a was an arbitrary value in $(0, s)$ it follows from Lemma 5·6 that $\dim J' \geqslant s$. As $J' \subset J(f)$, this proves Theorem 4.

6. *Examples*

As mentioned in the introduction, in [**10**] we studied a family of functions f_k satisfying Theorem 1. To obtain the functions f_k we used the function defined by

$$E(z) = \frac{1}{2\pi i} \int_L \frac{\exp(e^t)}{t-z} dt,$$

where L is the boundary of the region

$$G = \{z : Re(z) > 0, -\pi < \operatorname{Im}(z) < \pi\}$$

described in a clockwise direction, for $z \in \mathbb{C}\backslash\bar{G}$, and by analytic continuation for $z \in \bar{G}$. The functions f_k were then defined by

$$f_k(z) = E(z) - K.$$

For large K we showed that $J(f_k) \subset G$. From the properties of the functions f_k given in [**10**, section 3], it can easily be seen that, for large K, $f_k \in B$ and hence $\dim J(f_k) > 1$.

As $\dim J(f_k)$ is very close to one for large K, it has been suggested that if a function f could be constructed in such a way that its Julia set, $J(f)$, was contained in a domain of finite Lebesgue measure then $J(f)$ might be expected to have dimension equal to one. We construct a family of such functions, F_k, which belong to B and hence satisfy $\dim J(F_k) > 1$.

Consider the functions defined by

$$E_n(z) = \frac{1}{2\pi i}\int_{L_n} \frac{\exp(e^{e^t})}{t-z}\,dt,$$

where L_n is the boundary of the region

$$G_n = \{z = x+iy : x > n, |y| < \pi\, e^{-x}\}$$

described in a clockwise direction, for $z \in \mathbb{C}\backslash\bar{G}_n$. By the residue theorem we see that $E_n(z) = E_1(z)$ for each $z \in \mathbb{C}\backslash\bar{G}_1$ and each $n > 1$. Thus the functions E_n, $n > 1$, give an analytic continuation of E_1 to a transcendental entire function F.

LEMMA 6·1. *There exists $C_1 > 0$ such that $|F(z)| < C_1$ for each $z \in \mathbb{C}\backslash G_1$.*

Proof. It follows from the residue theorem that, for $z \in \mathbb{C}\backslash\bar{G}_1$,

$$F(z) = \frac{1}{2\pi i}\int_{L_2} \frac{\exp(e^{e^t})}{t-z}\,dt.$$

Let

$$\Gamma_1(x) = \{z = x+iy : 5\pi e^{-x}/6 \leqslant y \leqslant \pi e^{-x}\},$$

$$\Gamma_2(x) = \{z = x+iy : -\pi e^{-x} \leqslant y \leqslant -5\pi e^{-x}/6\}.$$

As

$$\left|\frac{1}{2\pi i}\int_{\Gamma_i(x)} \frac{\exp(e^{e^t})}{t-z}\,dt\right| \to 0$$

as $x \to \infty$, for $i = 1, 2$, it follows from the residue theorem that, for $z \in \mathbb{C}\backslash\bar{G}_1$,

$$F(z) = \frac{1}{2\pi i}\int_{L_2'} \frac{\exp(e^{e^t})}{t-z}\,dt,$$

where L_2' is the boundary of

$$G_2' = \{z = x+iy : x > 2, |y| < 5\pi\, e^{-x}/6\}.$$

It is not difficult to show that there exists $C_1 > 0$ such that, for $z \in \mathbb{C}\backslash\bar{G}_1$,

$$\left|\frac{1}{2\pi i}\int_{L_2'} \frac{\exp(e^{e^t})}{t-z}\,dt\right| < C_1/2$$

and so, for $z \in \mathbb{C}\backslash G_1$, $|F(z)| \leqslant C_1/2 < C_1$.

We define the functions F_k by

$$F_k(z) = F(z) - K.$$

If $K > C_1$ then it follows from Lemma 6·1 that, for $z \in \mathbb{C}\backslash\bar{G}_1$,

$$F_k(z) \in B(-K, C_1) \subset \{z: Re(z) < 0\} \subset \mathbb{C}\backslash\bar{G}_1 \tag{6·1}$$

and so $(\mathbb{C}\backslash\bar{G}_1) \subset N(F_k)$. Thus $J(F_k) \subset \bar{G}_1$. It is not difficult to see that the plane Lebesgue measure of $\bar{G}_1$ is equal to $2\pi/e < \infty$. It remains to show that, for large K, $F_k \in B$.

LEMMA 6·2. *There exists $g_1(z)$ such that $F(z) = \exp(e^{e^z}) + g_1(z)$ for each $z \in G_1$ and $|g_1(z)| \to 0$ as $z \to \infty$.*

Proof. If $z \in G_1$ and $Re(z) < n$ then

$$F(z) = E_n(z) = \frac{1}{2\pi i}\int_{L_n} \frac{\exp(e^{e^t})}{t-z}\,dt. \tag{6·2}$$

By the residue theorem,

$$\frac{1}{2\pi i}\int_{L_n} \frac{\exp(e^{e^t})}{t-z}\,dt - \frac{1}{2\pi i}\int_{L_0} \frac{\exp(e^{e^t})}{t-z}\,dt = \exp(e^{e^z}). \tag{6·3}$$

Let

$$\gamma_1(x) = \{z = x+iy: \pi e^{-x} \leqslant y \leqslant 5\pi e^{-x}/4\},$$
$$\gamma_2(x) = \{z = x+iy: -5\pi e^{-x}/4 \leqslant y \leqslant -\pi e^{-x}\}.$$

As
$$\left|\frac{1}{2\pi i}\int_{\gamma_i(x)} \frac{\exp(e^{e^t})}{t-z}\,dt\right| \to 0$$

as $x \to \infty$, for $i = 1, 2$ it follows from the residue theorem that, for $z \in G_1$,

$$\frac{1}{2\pi i}\int_{L_0} \frac{\exp(e^{e^t})}{t-z}\,dt = \frac{1}{2\pi i}\int_{L_0'} \frac{\exp(e^{e^t})}{t-z}\,dt, \tag{6·4}$$

where L_0' denotes the boundary of

$$G_0' = \{z = x+iy: x > 0, |y| < 5\pi\, e^{-x}/4\}.$$

Combining the results of (6·2), (6·3) and (6·4) we see that, for each $z \in G_1$,

$$F(z) = \exp(e^{e^z}) + g_1(z),$$

where
$$g_1(z) = \frac{1}{2\pi i}\int_{L_0'} \frac{\exp(e^{e^t})}{t-z}\,dt.$$

It is not difficult to show that $g_1(z) \to 0$ as $z \to \infty$ in G_1.

Thus, for $z \in G_1$,

$$F_k(z) = \exp(e^{e^z}) - K + g_1(z), \tag{6·5}$$

where $|g_1(z)| < C_2$, and so from Cauchy's inequalities

$$F_k'(z) = e^z e^{e^z} \exp(e^{e^z}) + g_2(z), \tag{6·6}$$

where $|g_2(z)| < C_2$.

LEMMA 6·3. *For $K > C_1$ we have $F_k \in B$.*

Proof. The transcendental singularities of F_k^{-1} are the asymptotic values of F_k. The only finite asymptotic value of $\exp(e^{e^z})$ in G_1 is 0 and so from Lemma 6·1 and Lemma 6·2 we see that, for $K > C_1$, the finite transcendental singularities of F_k^{-1} are contained in $B(-K, C_1+C_2)$.

The remaining singularities of F_k^{-1} are the images of points z such that $F_k'(z) = 0$. If $z \in G_1$ and $F_k'(z) = 0$ then, from (6·6),

$$|\exp(e^{e^z})| \leqslant |e^z e^{e^z} \exp(e^{e^z})| < C_2$$

and hence, from (6·5), $F_k(z) \in B(-K, 2C_2)$. Together with Lemma 6·1 and the results of the first paragraph this shows that, for $K > C_1$,

$$S(F_k) \subset B(-K, 2C_2 + C_1)$$

and so $F_k \in B$.

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