

- [11] Lasse Rempe: *Dynamics of exponential maps*. Ph.D. thesis, Christian-Albrechts-Universität Kiel (2003); http://e-diss.uni-kiel.de/diss_781.
- [12] Günter Rottenfusser: *Dynamic rays for a family of entire transcendental maps*. Diploma thesis, Technische Universität München (2002).
- [13] Günter Rottenfusser, *On the dynamical fine structure of entire transcendental functions*, doctoral thesis, Internationales University Bremen (2005).
- [14] Günter Rottenfusser, Johannes Rückert, Lasse Rempe, Dierk Schleicher, *Dynamic rays of bounded-type entire functions*, submitted.
- [15] Dierk Schleicher: *The dynamical fine structure of iterated cosine maps and a dimension paradox*. Duke Mathematics Journal **136** 2 (2007), 343–356.
- [16] Dierk Schleicher: *Hausdorff dimension, its properties, and its surprises*. American Mathematical Monthly **114** 6 (2007), 509–528. ArXiv math.DS/0505099.
- [17] Dierk Schleicher and Michael Thon, *Hausdorff dimension of dynamic rays for certain entire functions of bounded type*, manuscript.
- [18] Dierk Schleicher and Johannes Zimmer: *Escaping points of exponential maps*. Journal of the London Mathematical Society **67** 2 (2003), 380–400.
- [19] Hendrik Schubert: *Area of Fatou sets of trigonometric functions*. Proceedings of the American Mathematical Society, to appear.

GÖRRESSTR. 20, 80798 MÜNCHEN, GERMANY

E-mail address: guenter.rottenfusser@gmx.net

SCHOOL OF ENGINEERING AND SCIENCE, RESEARCH I, JACOBS UNIVERSITY (FORMERLY INTERNATIONAL UNIVERSITY), POSTFACH 750 561, D-28725 BREMEN, GERMANY

E-mail address: dierk@iu-bremen.de

DIMENSIONS OF JULIA SETS OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS

GWYNETH M. STALLARD

Dedicated to the memory of Professor Noel Baker

Abstract. We look at the relationships between the different dimensions that can be used to describe the size of the Julia set of a transcendental meromorphic function and also look at the different values that these dimensions can take. We summarise the main results in this area and indicate some of the techniques that are used.

1. INTRODUCTION

Let f be a meromorphic function which is not rational of degree one and denote by f^n , $n \in \mathbb{N}$, the n th iterate of f . The Fatou set, $F(f)$, is defined to be the set of points, $z \in \mathbb{C}$, such that $(f^n)_{n \in \mathbb{N}}$ is well-defined, meromorphic and forms a normal family in some neighbourhood of z . The complement, $J(f)$, of $F(f)$ is called the Julia set of f . An introduction to the properties of these sets can be found in, for example, [2] for rational functions and in [3] for transcendental meromorphic functions.

Most Julia sets are very intricate and their size cannot be described well in terms of classical geometry. There are a number of definitions of dimension that are well suited to describing the size of fractals such as Julia sets and in Section 2 we discuss the advantages and disadvantages of these different dimensions.

In Section 3 we look at the relationships between these different dimensions — when they give the same value for the size of a Julia set and when they give different values.

In Section 4 we look at the possible values for the dimension of the Julia set of a transcendental entire function and summarise some of the main results concerning exponential functions.

In Section 5 we look at the possible values for the dimension of the Julia set of a transcendental meromorphic function with poles.

We use the following notation:

$$S(f) = \{z : z \text{ is a finite singularity of } f^{-1}\},$$

$$P(f) = \{z : z \text{ is a finite singularity of } f^{-n} \text{ for some } n \in \mathbb{N}\},$$

$$B = \{\text{transcendental meromorphic } f : S(f) \text{ is bounded}\},$$

$$I(f) = \{z : f^n(z) \neq \infty, f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

2. DEFINITIONS OF DIMENSIONS

In this section we define the main definitions of dimension that are used to describe the size of Julia sets. We discuss the advantages and disadvantages of the different dimensions and look at the relationships between them. This material is covered in more detail in [10].

2.1. Box dimensions. First we define the box dimension of a bounded set $A \subseteq \mathbb{C}$ — this is probably the simplest ‘respectable’ definition of a dimension and is widely used. Suppose that we have a grid composed of lines parallel to the axes so that each box in the grid has sides of length d . We count the number $N_d(A)$ of boxes in this grid that meet the set A . The box dimension of A is defined to be

$$\dim_B A = \lim_{d \rightarrow 0} \frac{\log N_d(A)}{-\log d},$$

if this limit exists. There are several equivalent definitions of the box dimension of a set.

It is important to note that the box dimension of a set is only well defined if the limit in the definition exists — this is not always the case. Because of this, we often look at the *upper* and *lower* box dimensions of a set. The upper box dimension of A is defined to be

$$\overline{\dim}_B A = \limsup_{d \rightarrow 0} \frac{\log N_d(A)}{-\log d},$$

and the lower box dimension of A is defined to be

$$\underline{\dim}_B A = \liminf_{d \rightarrow 0} \frac{\log N_d(A)}{-\log d}.$$

The box dimension of a set exists precisely when the upper and lower box dimensions are equal.

2.2. Hausdorff dimension. Another definition of dimension that is widely used is that of Hausdorff dimension. Here, instead of covering A with sets that all have the same size, we look at coverings of A with sets of diameter at most equal to d . For each $t > 0$, we put

$$H_t(A) = \liminf_{d \rightarrow 0} \sum_i r_i^t,$$

where the infimum is taken over all possible covers of A with sets of diameter $r_i < d$. Thus

$$(2.1) \quad H_t(A) \leq \lim_{d \rightarrow 0} N_{d(A)}(\sqrt{2d})^t.$$

The Hausdorff dimension $s = \dim_H A$ of the set A is defined to be the unique value satisfying

$$(2.2) \quad H_t(A) = \begin{cases} \infty & \text{for } t < s \\ 0 & \text{for } t > s. \end{cases}$$

It can be shown that each of the functions H_t is a measure and this enables us to use results from measure theory to obtain estimates of the Hausdorff dimension of a set. The following result which is part of Frostman’s Lemma is often used to obtain a lower bound for the Hausdorff dimension of a set.

Lemma 2.1. *Let A be a compact subset of $\mathbb{R}^n$ and let $s > 0$. Suppose that there exist a measure μ supported on A and constants $C, d > 0$ such that every ball B of radius $r < d$ satisfies*

$$\mu(B) \leq Cr^s.$$

Then

$$\dim_H A \geq s.$$

Note that if the box dimension of the set A exists then

$$(2.3) \quad \lim_{d \rightarrow 0} N_{d(A)} d^t = \begin{cases} \infty & \text{for } t < \dim_B A \\ 0 & \text{for } t > \dim_B A. \end{cases}$$

It follows from (2.1), (2.2) and (2.3) that

$$(2.4) \quad \dim_H A \leq \dim_B A.$$

A simple example of a set which has different Hausdorff and box dimensions is the set $A = \{1/n : n \in \mathbb{N}\}$. This is just a countable collection of points, each of which has zero dimension. As we would expect, $\dim_H A = 0$. However, by covering A with intervals of length $1/n^2$, it is easy to check that $\dim_B A = 1/2$. More generally, Hausdorff dimension is stable under countable unions; that is,

$$\dim_H \bigcup_{i=1}^{\infty} A_i = \sup_{1 \leq i \leq \infty} \dim_H A_i,$$

whereas, as this example shows, box dimension is not stable under countable unions — a significant drawback.

2.3. Modified box dimensions. In order to overcome this drawback, a more recent definition of dimension has been introduced — the *modified* box dimension. The modified upper box dimension of a set A is defined as

$$\overline{\dim}_{MB} A = \inf_{i \in \mathbb{N}} \{ \sup_{i \in \mathbb{N}} \overline{\dim}_{BA_i} : A \subset \bigcup_{i=1}^{\infty} A_i \},$$

where the infimum is over all possible countable covers of A with bounded sets A_i , $i \in \mathbb{N}$. The modified lower box dimension is defined similarly. It follows from the definition that

$$\overline{\dim}_{MB} A \leq \overline{\dim}_B A.$$

By using the fact that Hausdorff dimension is stable under countable unions, together with (2.4), it can also be shown that

$$\dim_H A \leq \overline{\dim}_{MB} A.$$

The modified box dimensions are stable under countable unions and are defined for unbounded sets as well as bounded ones. Unlike Hausdorff dimension, they are not defined using measures. The modified upper box dimension can, however, be shown to be equal to the packing dimension which is defined by measures. (This is shown in [10, Proposition 3.8] for a bounded set A . This proof applies equally well when A is unbounded.)

2.4. Packing dimension. When defining the Hausdorff dimension of a set A , we considered coverings of A with sets of small diameter and tried to make these coverings as efficient as possible. When defining the packing dimension of a set A we consider packings of A with disjoint balls of small radius and try to make these packings as dense as possible.

For each $t > 0$, we put

$$P_t^0(A) = \limsup_{d \rightarrow 0} \sum_i r_i^t,$$

where the supremum is taken over all possible collections of disjoint sets of diameter $r_i < d$ with centres in A . Unfortunately, the functions P_t^0 are not measures and so we modify them as follows to produce measures:

$$P_t(A) = \inf \left\{ \sum_{i=1}^{\infty} P_t^0(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

The packing dimension $s = \dim_P A$ of the set A is defined to be the unique value satisfying

$$P_t(A) = \begin{cases} \infty & \text{for } t < s \\ 0 & \text{for } t > s. \end{cases}$$

3. RELATIONSHIPS BETWEEN DIMENSIONS OF JULIA SETS

We have now given the main definitions of dimension that are used to describe the size of Julia sets and seen that they are related as follows.

$$\dim_H A \leq \overline{\dim}_{MB} A = \dim_P A \leq \overline{\dim}_B A.$$

The lower box dimension can be less than or greater than the packing dimension.

The Hausdorff and packing dimensions have several nice properties — they are defined for every set, they are stable under countable unions and are defined using measures — and so these are the dimensions that we are particularly interested in. Although the box dimensions do not have these properties, they can sometimes be easy to estimate and so it is useful to have conditions which ensure that these different dimensions are equal.

3.1. Box and packing dimensions. For a transcendental meromorphic function, f , the Julia set is unbounded since, by Picard's Theorem, all but at most two points in the Julia set must have pre-images accumulating at the essential singularity at infinity. Thus, strictly speaking, the box dimensions of $J(f)$ are not well-defined. There are, however, well-defined 'local' upper and lower box dimensions of the Julia set. In [19, Theorem 2] we proved that, apart from at the exceptional set $E(f)$ which contains at most two points, these local box dimensions are *constant*. Moreover, this enabled us to show that the local upper box dimension of the Julia set (away from $E(f)$) is equal to the packing dimension of the Julia set — this result was proved for rational functions in [31, Theorem 7.1].

Theorem 3.1. *Let f be a meromorphic function. Then there exist $\bar{d} > 0$ and $\underline{d} > 0$ such that, for any bounded open connected set U satisfying $U \cap J(f) \neq \emptyset$ and $\overline{U} \cap E(f) = \emptyset$,*

$$\overline{\dim}_B(U \cap J(f)) = \bar{d},$$

and

$$\underline{\dim}_B(U \cap J(f)) = \underline{d}.$$

Further,

$$\dim_P J(f) = \bar{d}.$$

In Theorem 3.1,

$$E(f) = \{z : O^-(z) \text{ is finite}\},$$

where

$$O^-(z) = \{w : f^n(w) = z, \text{ for some } n \in \mathbb{N}\}.$$

Theorem 3.1 enables us to use simple box dimension calculations to estimate the size of the packing dimension of the Julia set.

3.2. Hyperbolic functions. There is a large class of *rational* functions for which all the definitions of dimension that we have described give the same value for the size of the Julia set. Indeed, the following result follows fairly easily from results of Sullivan [30, Theorems 3 and 4].

Theorem 3.2. *Let f be a hyperbolic rational function. Then*

$$\dim_H J(f) = \dim_P J(f) = \dim_B J(f).$$

An obvious question to ask is whether this result is true for transcendental meromorphic functions. Before answering this question, we must establish what is meant by the word hyperbolic. A rational function f is hyperbolic if f satisfies the following equivalent conditions (see [2, Section 9.7] and [4, Section 5.2]):

- $\bar{P}(f) \cap J(f) = \emptyset$;
- $\bar{P}(f)$ is a compact subset of $F(f)$;
- f is expanding, in the sense that there exist $K > 1$ and $c > 0$ such that $|(f^n)'(z)| > cK^n$ for each $z \in J(f)$, $n \in \mathbb{N}$.

These conditions are not equivalent for a transcendental meromorphic function f , since the Julia set of such a function is unbounded, but in [18] we showed that, if f is in the class

$$\hat{B} = \{f : f \in B \text{ and } J(f) \cap \bar{P}(f) = \emptyset\},$$

then

- $P(f)$ is bounded;
- f is expanding in the sense that there exist $K > 1$ and $c > 0$ such that $|(f^n)'(z)| > cK^n(|f^n(z)| + 1)/(|z| + 1)$ for each $n \in \mathbb{N}$ and each $z \in J(f)$ such that f^n is analytic at z .

Thus it makes sense to say that functions in the class $\hat{B}$ are hyperbolic. The result of Theorem 3.2, however, does not hold for functions in $\hat{B}$. In [28], [29] and [19] we showed that there are examples of transcendental meromorphic functions in the class $\hat{B}$ (both with and without poles) whose Julia sets have packing dimensions that are strictly greater than their Hausdorff dimensions.

We discuss these functions later in the paper.

The Hausdorff and packing dimensions of the Julia set must be equal when the Hausdorff dimension of the Julia set is equal to two. The only other example of a transcendental meromorphic function that we are aware of for which the Hausdorff and packing dimensions of the Julia set are known to be equal is $\tan z$ — in this case the Julia set is simply equal to the real line and so has dimension one.

Question 1 Does there exist a transcendental meromorphic function f for which $\dim_H J(f) = \dim_P J(f) = d$, where d is not an integer?

4. TRANSCENDENTAL ENTIRE FUNCTIONS

In this section we restrict our attention to transcendental *entire* functions. For such a function, the Julia set must contain continua since any multiply connected component of the Fatou set is a bounded wandering domain (of the type sometimes described as a Baker wandering domain) — this was proved by Baker [1]. Thus we have the following result.

Theorem 4.1. *Let f be a transcendental entire function. Then*

$$1 \leq \dim_H J(f) \leq \dim_P J(f) \leq 2.$$

Later in this section we will give examples which show that, for each $d \in (1, 2]$, there is a transcendental entire function f for which $\dim_H J(f) = d$.

Question 2 Does there exist a transcendental entire function f for which $\dim_H J(f) = 1$?

We also give examples of transcendental entire functions for which the packing dimension of the Julia set is equal to two even though the Hausdorff dimension of the Julia set is arbitrarily close to one.

4.1. Exponential functions. It was proved by Misiurewicz [16] that the Julia set of the exponential function e^z is equal to the whole plane and so, trivially, has Hausdorff dimension equal to two. More interesting examples, from the point of view of dimension, were given by McMullen who proved the following result [15, Theorems 1.2 and 1.3].

Theorem 4.2. *Let $f_\lambda(z) = \lambda e^z$, where $\lambda \neq 0$. Then*

$$\dim_H J(f_\lambda) = 2.$$

Further, if $0 < \lambda < 1/e$, then the area of $J(f_\lambda)$ is zero.

In order to prove this result, McMullen first proved the following general result which is useful for obtaining a lower bound for the Hausdorff dimension of a set. The proof of this result uses Lemma 2.1.

Lemma 4.3. *For each $n \in \mathbb{N}$, let A_n be a finite collection of disjoint compact subsets of $\mathbb{R}^d$, each of which has positive d -dimensional measure, and define*

$$\mathcal{U}_n = \bigcup_{A_n \in \mathcal{A}_n} A_n, \quad A = \bigcap_{n=1}^{\infty} \mathcal{U}_n.$$

Suppose also that, for each $A_n \in \mathcal{A}_n$, there exists $A_{n+1} \in \mathcal{A}_{n+1}$ and a unique $A_{n-1} \in \mathcal{A}_{n-1}$ such that

$$A_{n+1} \subseteq A_n \subseteq A_{n-1}.$$

If Δ_n, d_n are such that, for each $A_n \in \mathcal{A}_n$,

$$\text{vol}(\mathcal{U}_{n+1} \cap A_n) / \text{vol}(A_n) \geq \Delta_n$$

and

$$\text{diam } A_n \leq d_n, \quad d_n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then

$$\dim_H A \geq d - \limsup_{n \rightarrow \infty} \sum_{i=1}^n \frac{|\log \Delta_i|}{|\log d_n|}.$$

Functions of the form $f_\lambda(z) = \lambda e^z$ were studied by Devaney and Krych [6]. They showed that there is a unique attracting fixed point p_λ and a unique repelling fixed point q_λ with $0 < p_\lambda < 1 < q_\lambda$. The half plane $\{z : \Re(z) < q_\lambda\}$ is contained in the attracting basin of p_λ and

$$(4.1) \quad J(f_\lambda) = \{z : \Re(f_\lambda^n(z)) \geq q_\lambda, \text{ for all } n \in \mathbb{N}\}.$$

About half of the points in the right half-plane

$$H = \{z : \Re(z) \geq q_\lambda\}$$

remain there after applying f — this enabled McMullen to use Lemma 4.3. To do this, take the collection of strips

$$S_k = \{z \in H : |\Im(z) - 2k\pi| \leq \pi/4\}, \quad k \in \mathbb{Z}$$

and pack them with boxes

$$B_k^j = \{z \in S_k : |\Re(z) - q_\lambda - j\pi| \leq \pi/4\}, \quad j \in \mathbb{N}.$$

Lemma 4.3 can then be applied to the collection of sets given by

$$\mathcal{A}_n = \{A_n : f_\lambda^n(A_n) = B_k^j \text{ for some } k \in \mathbb{Z}, j \in \mathbb{N}, \text{ and } A_n \subset A_{n-1}\}$$

for some $A_{n-1} \in \mathcal{A}_{n-1}$,

for $n \geq 2$, with A_1 being one of the boxes B_k^j . The set A defined in Lemma 4.3 is then clearly a subset of $J(f_\lambda)$ by (4.1). It follows from Koebe's distortion

theorem (see, for example, [8]), that there exists $C > 0$ such that, for each $A_n \in \mathcal{A}_n$ and each $n \in \mathbb{N}$,

$$\text{vol}(\mathcal{U}_{n+1} \cap A_n) / \text{vol}(A_n) \geq C \sum_{B_k^j \cap f_\lambda^{n+1}(A_n) \neq \emptyset} \text{vol}(B_k^j) / \text{vol } f_\lambda^{n+1}(A_n).$$

This implies that the quantities Δ_n in Lemma 4.3 are all greater than some constant, for each $n \in \mathbb{N}$. (This is where we use the fact that roughly a half of all points in any right half-plane remain there after applying f_λ .) It is then straightforward to check that Lemma 4.3 implies that the set A (which is a subset of $J(f_\lambda)$) has Hausdorff dimension two.

In fact $J(f_\lambda)$ is actually a 'Cantor bouquet' of curves whose endpoints are the only accessible points from the basin of attraction (see [5], [6] and [7]). Karpínska ([11] and [12]) proved the following remarkable result.

Theorem 4.4. *Let $f_\lambda(z) = \lambda e^z$, where $0 < \lambda < 1/e$, and let $\mathcal{C}_\lambda$ denote the set of endpoints of the curves that make up $J(f_\lambda)$. Then*

$$\dim_H \mathcal{C}_\lambda = 2$$

while

$$\dim_H (J(f_\lambda) \setminus \mathcal{C}_\lambda) = 1.$$

To prove the first part of this result, Karpínska modified McMullen's proof of Theorem 4.2. The key change was to restrict the sets A_n such that, for a given $A_n \in \mathcal{A}_n$, the sets $A_{n+1} \subset A_n$ are roughly those for which $f_\lambda^{n+1}(A_{n+1})$ is a box in the upper and lower quarters of the set $f_\lambda^{n+1}(A_n)$. This still produces a set A which has Hausdorff dimension two. Karpínska showed that each point in this set A is accessible from the basin of attraction and is therefore contained in $\mathcal{C}_\lambda$. Note that this subset of $\mathcal{C}_\lambda$ consists of points that tend to infinity very quickly under iteration.

By using methods of thermodynamic formalism, Karpínska [11, Theorem 2] showed that the set

$$J_{bd}(f_\lambda) = \{z \in J(f_\lambda) : \{f_\lambda^n(z)\} \text{ is bounded}\}$$

has Hausdorff dimension which tends to 1 from above as λ tends to zero. This set is also a subset of the set of endpoints $\mathcal{C}_\lambda$.

Theorem 4.5. *Let $f_\lambda(z) = \lambda e^z$, where $0 < \lambda < 1/e$. Then*

$$\dim_H J_{bd}(f_\lambda) \geq 1$$

and, if λ is sufficiently small, then

$$1 + \frac{1}{\log(\log 1/\lambda)} < \dim_H J_{bd}(f_\lambda) < 1 + \frac{1}{\log(\log 1/\lambda)}.$$

Urbański and Zdunik [32] used conformal measures to show that the set

$$J_r(f_\lambda) = J(f_\lambda) \setminus I(f_\lambda),$$

consisting of those points in the Julia set that do not tend to infinity under iteration, has Hausdorff dimension less than two. This set contains $J_{bd}(f_\lambda)$ and is also a subset of the set of endpoints $\mathcal{C}_\lambda$. (They also give a direct proof of this result using Karpíńska's Theorem 4.5 together with other results.) By using the methods of thermodynamic formalism, Urbański and Zdunik [33] show that the Hausdorff dimension of $J_r(f_\lambda)$ is actually a real analytic function of λ for $\lambda \in (0, 1/e)$.

Theorem 4.6. *Let $f_\lambda(z) = \lambda e^z$, where $0 < \lambda < 1/e$. Then*

$$\dim_H J_r(f_\lambda) < 2$$

and $\dim_H J_r(f_\lambda)$ is a real analytic function of λ .

These results show that, for the exponential family, the main contribution to the size of the Julia set is from those points whose iterates tend to infinity under iteration. Kotus and Urbański [14] showed that this is not the case for elliptic functions.

Theorem 4.7. *Let f be an elliptic function and let q be the maximal multiplicity of the poles of f . Then*

$$\dim_H J(f) > \frac{2q}{1+q} \geq \dim_H I(f).$$

4.2. Julia sets with small Hausdorff dimension. Recall that, for a transcendental entire function f , we have $1 \leq \dim_H J(f) \leq 2$. We now give examples which show that 1 is the best possible lower bound here. Recall that, in McMullen's proof of Theorem 4.2, the key observation is that for an exponential function f , the Julia set consists of those points that remain in a right half plane under iteration and approximately half of the points in a right half plane remain there after applying f . This suggests that in order to obtain a function whose Julia set has Hausdorff dimension less than two it would be a good idea to construct a function whose Julia set lies in a relatively small part of the plane. This was the motivation in [21] for looking at the following function which is described in [17]. The transcendental entire function E is defined by

$$E(z) = \frac{1}{2\pi i} \int_L \frac{\exp(e^t)}{t - z} dt,$$

where L is the boundary of the region

$$G = \{z : \Re(z) > 0, |\Im(z)| < \pi\}$$

described in a clockwise direction, for $z \in \mathbb{C} \setminus \overline{G}$, and by analytic continuation elsewhere. It is shown in [17] that $E(z)$ is close to $\exp(e^z)$ for $z \in G$ and is close to $-1/z$ elsewhere.

In [21], we studied the family of functions defined by

$$f_K(z) = E(z) - K, \text{ where } K \in \mathbb{R}.$$

For large K , $F(f_K)$ is the attracting basin of a fixed point in the left half plane and $J(f_K)$ is a Cantor bouquet of curves in the half-strip

$$G_K = G \cap \{z : \Re(z) > K'\},$$

where $K' > \log \log K/2$. Clearly $J(f_K)$ is contained in a relatively small part of the plane which becomes progressively smaller as K gets larger. We have

$$J(f_K) = \{z : f_K^n(z) \in G_K \text{ for all } n \in \mathbb{N}\}$$

and the proportion of points in a vertical cross-section of G_K that remain in G_K after applying f_K tends to zero as we move across to the right of G_K . This is the key difference between these functions and the exponential functions that allowed us to prove the following result in [21].

Theorem 4.8. *The functions f_K defined above satisfy*

$$1 \leq \dim_H J(f_K) \leq 1 + \delta(K),$$

where $\delta(K) \rightarrow 0$ as $K \rightarrow \infty$.

In order to obtain an upper bound for the Hausdorff dimension of a set it is sufficient to consider just one sequence of covers with sets of progressively smaller diameter. A 'natural' way to do this for a Julia set is to begin with a cover $\mathcal{C}$ and then take successive pre-images of the sets in $\mathcal{C}$ to give a sequence of covers. This method was used in the proof of Theorem 4.8 and also by Karpíńska to prove the second part of Theorem 4.4.

In Theorem 4.8, the 'natural' sequence of covers is defined as follows. We begin by covering the strip G_K with the collection of boxes B_n , $n \in \mathbb{N}$, defined by

$$B_n = \{z : |\Re(z - z_n)| < 3C/2, |\Im(z - z_n)| < 3C/2\},$$

where $z_n = K' + nC$ and C is some constant with $3C/2 > \pi$. Now take an arbitrary set B_N and define the following sequence of covers of $J(f_K)$.

$$\mathcal{A}_k = \{A_k : A_k \text{ is a component of } f_K^{-k}(B_n) \text{ for some } n \in \mathbb{N}, A_k \cap B_N \neq \emptyset\}.$$

It is not difficult to show that $\max\{\text{diam } A_k : A_k \in \mathcal{A}_k\} \rightarrow 0$ as $k \rightarrow \infty$. Thus, to prove Theorem 4.8, it is sufficient to show that, for each $A_k \in \mathcal{A}_k$ and each $k \in \mathbb{N}$,

$$\sum_{A_{k+1} \in \mathcal{A}_{k+1}, A_{k+1} \cap A_k \neq \emptyset} (\text{diam } A_{k+1})^{1+\delta(K)} \leq (\text{diam } A_k)^{1+\delta(K)}.$$

The set $P(f_K)$ lies in the left half-plane and so we can use Koebe's distortion theorem to show that there exists a constant $c > 0$ such that, for each $A_k \in \mathcal{A}_k$ and each $k \in \mathbb{N}$,

$$\sum_{A_{k+1} \in \mathcal{A}_{k+1}, A_{k+1} \cap A_k \neq \emptyset} \frac{(\text{diam } A_{k+1})^{1+\delta(K)}}{(\text{diam } A_k)^{1+\delta(K)}} = c \sum_{U \in H_s} \frac{(\text{diam } U)^{1+\delta(K)}}{(\text{diam } B_s)^{1+\delta(K)}},$$

where $B_s = f_K^k(A_k)$ and

$$H_s = \{U : U \text{ is a component of } f_K^{-1}(B_n) \text{ for some } n \in \mathbb{N}, U \cap B_s \neq \emptyset\}.$$

The right hand quantity can be shown to be less than one thus proving Theorem 4.8.

4.3. Lower bounds for the Hausdorff dimension of a Julia set. We have seen how a 'natural' sequence of covers of a Julia set, $J(f)$, can be used to obtain an upper bound for $\dim_H J(f)$. We now show how a lower bound for $\dim_H J(f)$ can be obtained by showing that a natural choice of covers is the best possible choice of covers. We prove the following general result.

Theorem 4.9. *Let E be a compact set and, for each $n \in \mathbb{N}$, let $\mathcal{A}_n = \{A_n\}$ be a collection of disjoint sets such that, for some $t > 0$,*

- (1) $A_n \cap E \neq \emptyset$ for each $A_n \in \mathcal{A}_n$, $n \in \mathbb{N}$;
- (2) there exists $M \in \mathbb{N}$ such that

$$\sum_{A_n + M \subset A_n, A_n + M \in \mathcal{A}_{n+M}} (\text{diam } A_{n+M})^t > (\text{diam } A_n)^t,$$

for each $A_n \in \mathcal{A}_n$, $n \in \mathbb{N}$;

- (3) there exist $r_0 > 0$, $C > 0$, $m \in \mathbb{N}$ such that for all $r < r_0$, $z \in E$ there exists $P \in \mathbb{N} \cup \{0\}$ such that

$$\sum_{A_{PM+m} \cap B(z,r) \neq \emptyset, A_{PM+m} \in \mathcal{A}_{PM+m}} (\text{diam } A_{PM+m})^t < C(\text{diam } B(z,r))^t.$$

Then $\dim_H E \geq t$.

Proof. Suppose that $\dim_H E < t$. Then there exists a cover of E with sets $B_i = B(z_i, r_i)$, $i \in I$, such that, for each $i \in I$, we have $z_i \in E$, $r_i < r_0$ and

$$(4.2) \quad \sum_{i \in I} (\text{diam } B_i)^t < \sum_{A_m \in \mathcal{A}_m} (\text{diam } A_m)^t / C.$$

It follows from condition (3) that, for each $i \in I$, there exists $P(i) \in \mathbb{N} \cup \{0\}$ such that

$$(4.3) \quad \sum_{A_{P(i)M+m} \cap B_i \neq \emptyset, A_{P(i)M+m} \in \mathcal{A}_{P(i)M+m}} (\text{diam } A_{P(i)M+m})^t < C(\text{diam } B_i)^t.$$

Since E is compact, we can cover E with a finite number of the sets B_i . We label these sets $B_1, B_2, \dots, B_n$ in such a way that $P(i) \leq P(i+1)$ for $1 \leq i < n$. It follows from (4.3) and condition (2) that

$$\begin{aligned} & \sum_{A_m \in \mathcal{A}_m} (\text{diam } A_m)^t \\ & \leq \sum_{A_{P(1)M+m}} (\text{diam } A_{P(1)M+m})^t \\ & = \sum_{A_{P(1)M+m} \cap B_1 = \emptyset} (\text{diam } A_{P(1)M+m})^t + \sum_{A_{P(1)M+m} \cap B_1 \neq \emptyset} (\text{diam } A_{P(1)M+m})^t \\ & \leq \sum_{A_{P(2)M+m} \cap B_1 = \emptyset} (\text{diam } A_{P(2)M+m})^t + C(\text{diam } B_1)^t \\ (4.4) \quad & \leq \sum_{A_{P(2)M+m} \cap (B_1 \cup B_2) = \emptyset} (\text{diam } A_{P(2)M+m})^t \\ & \quad + \sum_{A_{P(2)M+m} \cap B_2 \neq \emptyset} (\text{diam } A_{P(2)M+m})^t + C(\text{diam } B_1)^t \\ & \leq \sum_{A_{P(3)M+m} \cap (B_1 \cup B_2) = \emptyset} (\text{diam } A_{P(3)M+m})^t + C[(\text{diam } B_1)^t + (\text{diam } B_2)^t] \\ & \leq \vdots \\ & \leq \sum_{A_{P(n)M+m} \cap \bigcup_{i=1}^n B_i = \emptyset} (\text{diam } A_{P(n)M+m})^t + C \sum_{i=1}^n (\text{diam } B_i)^t, \end{aligned}$$

where $A_{P(i)M+m}$ denotes a set in $\mathcal{A}_{P(i)M+m}$ wherever it appears in the above equation. Since the sets B_i , $1 \leq i \leq n$, form a cover of E , it follows from condition (1) that

$$\{A_{P(n)M+m} \in \mathcal{A}_{P(n)M+m} : A_{P(n)M+m} \cap \bigcup_{i=1}^n B_i = \emptyset\} = \emptyset.$$

Thus, from (4.2) and (4.4) we have

$$\sum_{i \in I} (\text{diam } B_i)^t < \sum_{A_m \in \mathcal{A}_m} (\text{diam } A_m)^t / C \leq \sum_{i=1}^n (\text{diam } B_i)^t$$

which is a contradiction. This completes the proof.

We used this method in [27] to obtain exact values for the Hausdorff dimensions of the Julia sets of a family of transcendental entire functions, thus showing that every value in $(1, 2]$ is attained by the Hausdorff dimension of such a Julia set. The examples considered in [27] are similar to those described in Section 4.2 but have their Julia sets in a larger area of the plane — larger than a strip but smaller than an angle.

For each $p \in (0, \infty)$, the transcendental entire function E_p is defined by

$$E_p(z) = \frac{1}{2\pi i} \int_{L_p} \frac{\exp(e^{(\log t)^{1+p}})}{t - z} dt,$$

where L_p is the boundary of the region

$$G_p = \{z = x + iy : |y| < \frac{\pi x}{(1+p)(\log x)^p}, x \geq 3\}$$

described in a clockwise direction, for $z \in \mathbb{C} \setminus \overline{G_p}$, and by analytic continuation elsewhere. The function E_p is close to $\exp(e^{(\log z)^{1+p}})$ for $z \in \overline{G_p}$ and is bounded elsewhere. (Note that many other functions can be defined in this way by taking the boundary to be a curve on which the integrand is small.)

In [27], we studied the family of functions defined by

$$f_{p,K}(z) = E_p(z) - K, \text{ where } K \in \mathbb{R}.$$

Here $(\log t)^{1+p}$ and $(\log x)^{1+p}$ are taken to be the values which are real for $t > e$. For large K , $F(f_{p,K})$ is the attracting basin of a fixed point in the left half plane and $J(f_{p,K})$ is a Cantor bouquet of curves in $G_{p,K} = G \cap \{z : \Re(z) \geq K'\}$, where $K'_p \rightarrow \infty$ as $K \rightarrow \infty$. We have

$$J(f_{p,K}) = \{z : f_{p,K}^n(z) \in G_{p,K} \text{ for all } n \in \mathbb{N}\}.$$

We proved the following result.

Theorem 4.10. *For each $p \in (0, \infty)$, there exists $C(p) > 0$ such that, for each $K > C(p)$, the functions $f_{p,K}$ defined above satisfy*

$$\dim_H J(f_{p,K}) = 1 + \frac{1}{1+p}.$$

For a function $f_{p,K}$, a ‘natural’ sequence of covers of $J(f_{p,K})$ is defined as follows. We begin by covering $G_{p,K}$ with the collection of boxes B_n , $n \in \mathbb{N}$, defined by

$$B_n = \{z = x + iy : x_n \leq x \leq x_n + R_n, |y| \leq R_n\},$$

where

$$x_1 = K'_p, \quad x_{n+1} = x_n + R_n, \quad R_n = \frac{\pi x_n}{(1+p)(\log x_n)^p}.$$

Now define the following collection of sets inductively:

$$\mathcal{A}_1 = \{A_1 : A_1 \text{ is a component of } f^{-1}(B_n) \text{ for some } n \in \mathbb{N}, A_1 \subset \bigcup_{n \in \mathbb{N}} B_n\},$$

$$\begin{aligned} \mathcal{A}_{k+1} &= \{A_{k+1} : A_{k+1} \text{ is a component of } f^{-(k+1)}(B_n) \text{ for some } n \in \mathbb{N} \\ &\text{and } A_{k+1} \subset \bigcup_{A_k \in \mathcal{A}_k} A_k\}. \end{aligned}$$

In [27], we show that, if K is sufficiently large, then the following is true.

$$\lim_{k \rightarrow \infty} \sum_{A_k \in \mathcal{A}_k} (\text{diam } A_k)^t < \infty \quad \Leftrightarrow \quad t > 1 + \frac{1}{1+p}.$$

This shows that $1 + \frac{1}{1+p}$ is an upper bound for $\dim_H J(f_{p,K})$. To show that it is also a lower bound, we apply Theorem 4.9 to the collections $\mathcal{A}_n$ defined above but with

$$B_n = \{z = x + iy : x_n \leq x \leq x_n + R_n/\pi, |y| \leq R_n/\pi\},$$

to ensure that sets in $\mathcal{A}_n$ are disjoint. This shows that, in some sense, the natural sequence of covers is also the most efficient sequence of covers.

It is very unusual to be able to obtain an exact value for the Hausdorff dimension of a Julia set. For these functions, the sum $\sum_{A_k \in \mathcal{A}_k} (\text{diam } A_k)^t$ is approximately equal to

$$\sum_{n=1}^{\infty} (\text{diam } B_n)^t \sum_{f(b)=x_n} \frac{1}{|f'(b)|^t} \sum_{f^{k-1}(z)=b} \frac{1}{|(f^{k-1})'(z)|^t},$$

by Koebe’s distortion theorem. The expression $\sum_{n=1}^{\infty} (\text{diam } B_n)^t \sum_{f(b)=x_n} \frac{1}{|f'(b)|^t}$ can be approximated by the integral

$$\int_{x_1}^{\infty} \frac{1}{x(\log x)^{(p+1)(t-1)}} dx$$

which converges precisely when $t > 1 + \frac{1}{1+p}$.

We also used the method of Theorem 4.9 in [23] to obtain the following result.

Theorem 4.11. *If f is an entire function in the class B then*

$$\dim_H J(f) > 1.$$

Let f be an entire function in the class B . To prove Theorem 4.11 we took $R > 0$ such that $S(f) \subset B(0, R)$ and a point $z_0 \in J(f) \cap I(f)$ such that $|f^n(z_0)| \geq 8R$. We then took a set

$$I_n \subset \{z : f^n(z) = z_0\}$$

chosen in such a way that the points in I_n lie in a compact part of the plane and

$$z \in I_n \Rightarrow |f^k(z)| \geq 2R, \text{ for } 0 \leq k \leq n.$$

Thus

$$I = \bigcup_{n=1}^{\infty} I_n \subset J(f) \cap I(f).$$

We defined

$$s = \inf \left\{ t : \sum_{n \in \mathbb{N}} \sum_{z \in I_n} \left(\frac{1 + |z_0|^2}{|(f^n)'(z)|(1 + |z|^2)} \right)^t < \infty \right\}.$$

The proof of Theorem 4.11 in [23] is then split into two parts. Firstly, we showed that $s > 1$ and secondly we showed that $\dim_H J(f) > s$. The proof of this second part was based on the method described in Theorem 4.9 applied to the following sets:

$$A_n = \{g(A) : g \text{ is a branch of } f^{-n} \text{ with } g(z_0) \in I_n\},$$

where A is a disc centered at z_0 .

To prove that $s > 1$, we defined measures μ_t by

$$\mu_t(B) = c_t \sum_{n \in \mathbb{N}} \sum_{z \in I \cap B} \left(\frac{1 + |z_0|^2}{|(f^n)'(z)|(1 + |z|^2)} \right)^t, \quad t > s,$$

such that $\mu_t(\mathbb{C}) = 1$ and then let μ_s be a weak limit of the measures μ_t as $t \rightarrow s$. The measure μ_s is supported on a bounded part of the plane E . We use bounded distortion to show that there exists $C_1 > 0$ such that

$$(4.5) \quad \mu_s(A_n) \geq C_1(\text{diam } A_n)^s, \text{ for each } A_n \in \mathcal{A}_n, n \in \mathbb{N}.$$

We then look at pre-images of the curve $\gamma = \{z : |z| = |z_0|\}$ and show that there exists $m > 0$ such that, for each $n \in \mathbb{N}$, there are 2^n disjoint unbounded curves $\gamma_{n,i}$, $1 \leq i \leq n$, crossing E with $f^{mn+1}(\gamma_{n,i}) \subset \gamma$. The sets A_{mn+1} cover a certain proportion of the curves $\gamma_{n,i} \cap E$ and so there exists $C_2 > 0$ such that

$$\sum_{A_{mn+1} \in \mathcal{A}_{mn+1}, A_{mn+1} \cap E \neq \emptyset} \text{diam } A_n \geq 2^n C_2, \text{ for each } n \in \mathbb{N}.$$

Together with (4.5), this shows that $s > 1$ thus proving Theorem 4.11.

4.4. Packing dimensions of Julia sets of transcendental entire functions. In Section 4.2, we introduced a family of transcendental entire functions f_K for which $\dim_H J(f_K) \rightarrow 1$ as $K \rightarrow \infty$. In [29] we showed that $\dim_P J(f_K) = \dim_B J(f_K) = 2$, for large K , and so the Hausdorff and packing dimensions of the Julia sets are as different as they could be for a transcendental entire function. The functions f_K are certainly hyperbolic since $\overline{P}(f_K)$ is bounded and disjoint from the Julia set — this shows that the result of Theorem 3.2 does not hold for transcendental entire functions.

Recall that $J(f_K)$ is contained in a half-strip

$$G_K = \{z : \Re(z) > K', |\Im(z)| < \pi\}.$$

In order to obtain an estimate for $\dim_B J(f_K)$, we use the fact that $f_K^{-1}(\mathbb{R} \cap G_K) \subset J(f_K)$. The set $f_K^{-1}(\mathbb{R} \cap G_K)$ comprises of unbounded curves which, near a point $x \in \mathbb{R}$, are separated by a vertical distance of approximately $1/e^x$. So, there exists some $c_1 > 0$ such that every box of side c_1/e^x within a distance 1 of x must contain a point in $J(f_K)$.

If we fix $X \in \mathbb{R} \cap J(f_K)$, then this result is true for $f^n(X)$, for each $n \in \mathbb{N}$. Pulling back by f_K^n and using bounded distortion, we show that, close to X , there are at least $c_2 e^{2f_K^n(X)}$ disjoint boxes of side $c_3/(e^{f_K^n(X)}(f_K^n)'(X))$ which meet $J(f_K)$, for some $c_2, c_3 > 0$. The result then follows from the fact that

$$\lim_{n \rightarrow \infty} \frac{\log c_2 e^{2f_K^n(X)}}{\log e^{f_K^n(X)}(f_K^n)'(X)/c_3} = 2.$$

In [19] we developed these ideas further to prove the following result.

Theorem 4.12. *If f is an entire function in the class B then*

$$\dim_P J(f) = 2.$$

The proof of this result uses the fact that, if f is an entire function in the class B , then $J(f)$ contains an unbounded continuum Γ . We use a logarithmic change of variables, as introduced by Eremenko and Lyubich in [9], and note that the set $\exp^{-1}(\Gamma)$ comprises of unbounded continua which are separated by a vertical distance of 2π . We pull these back under $f^n \circ \exp$ and, using bounded distortion, we show that there exist $c_4, c_5 > 0$ such that, close to a point $z_0 \in J(f)$, there are at least $c_4 |\log f^n(z_0)|^2$ boxes of side $c_5 |f^n(z_0)|/|(f^n)'(z_0)|$ which meet $f^{-n} \circ \exp(\exp^{-1}(\Gamma)) = f^{-n}(\Gamma) \subset J(f)$, for each $n \in \mathbb{N}$. Thus

$$\overline{\dim}_B J(f) \geq 2 \lim_{n \rightarrow \infty} \frac{\log \log |f^n(z_0)|}{\log |(f^n)'(z_0)/f^n(z_0)|}.$$

By using Wiman-Valiron theory, we are able to show that, for each $\epsilon > 0$, there exists a point $z_0 \in J(f)$ for which this limit is at least $1 - \epsilon$. This is sufficient to prove Theorem 4.12.

(Note that in [19] we actually prove the stronger result that Theorem 4.12 holds for the set $A(f)$ which consists of those points that escape to infinity as quickly as possible — this is a subset of $J(f)$ for functions in the class B .)

Question 3 Does there exist a transcendental entire function f for which $\dim_P J(f) < 2$?

5. MEROMORPHIC FUNCTIONS WITH POLES

In this section we consider those transcendental meromorphic functions that have at least one pole. If such a function has only *finitely* many poles, then it follows from [20, Theorem 4] that the Julia set contains a continuum and so we have the following result.

Theorem 5.1. *Let f be a transcendental meromorphic function with finitely many poles. Then*

$$1 \leq \dim_H J(f) \leq \dim_P J(f) \leq 2.$$

We also showed in [20] that if f is a transcendental meromorphic function with finitely many poles then $J(f)$ contains no free Jordan arc.

If a function has *infinitely* many poles, however, then the situation is rather different. For example, if $f(z) = \tan z$, then the Julia set is the real axis, the upper and lower half planes are invariant components of the Fatou set and 0 is a parabolic fixed point. If $f_\lambda(z) = \lambda \tan z$, with $0 < \lambda < 1$, then 0 is an attracting fixed point and the Julia set is a Cantor set in the real line.

In [22] we proved the following result.

Theorem 5.2. *Let f be a transcendental meromorphic function. Then*

$$0 < \dim_H J(f) \leq 2.$$

This result can be proved by using Ahlfors' Five Islands Theorem to construct a disc D containing two disjoint simply connected domains G_1 and G_2 such that f^2 is a univalent map of G_i onto D for $i = 1, 2$. This sets up an iterated function scheme for which the invariant set (or attractor) is a subset of $J(f)$ with positive Hausdorff dimension.

It is in fact possible to obtain transcendental meromorphic functions whose Julia sets have Hausdorff dimension arbitrarily close to 0. Consider again the functions $f_\lambda(z) = \lambda \tan z$, with $0 < \lambda < 1$. For such a function, the upper and

lower half planes are invariant and are so contained in the Fatou set. Thus the Julia set is contained in the real line. Since 0 is attracting, an interval containing 0, and all the pre-images of this interval must also be in the Fatou set. Thus the Julia set is a Cantor set in the real line.

As $\lambda \rightarrow 0$, the size of the interval around 0 increases. The size of the pre-images also increase and it might be expected that the Hausdorff dimension of $J(f_\lambda)$ would tend to zero. This is not, however, the case as is shown by the following result of Kotus [13, Example 1].

Theorem 5.3. *Let $f_\lambda(z) = \lambda \tan z$ with $\lambda \in \mathbb{C}$. Then*

$$\dim_H J(f_\lambda) \geq 1/2.$$

Kotus actually proves a more general result than this in her paper. She considers transcendental meromorphic functions with infinitely many poles with some regularity conditions on the location of the poles and the behaviour of the function near the poles. She obtains a lower bound for the Hausdorff dimension of the Julia set — this lower bound depends on the positions of the poles. To do this she uses a natural sequence of covers of the Julia set (as described in Section 4) and Lemma 2.1.

In [22] we proved the following result.

Theorem 5.4. *For each $m \in \mathbb{N}$ with $m \geq 4$, the function f_m defined by*

$$f_m(z) = \lambda_m \tan z / \prod_{r=1}^m (z - r\pi),$$

where $\lambda_m = (2m)^{-m}$, satisfies

$$\dim_H J(f_m) \leq 4/m.$$

This shows that 0 is indeed the best possible lower bound for the Hausdorff dimension of the Julia set of a transcendental meromorphic function. The reason for dividing $\lambda \tan z$ by a polynomial was to ensure that the first pre-images of the interval around the attracting fixed point at 0 become larger the further away from 0 they are — this makes the Julia set more sparse in the real line. The polynomial was chosen so as not to introduce any further poles. The proof of Theorem 5.4 involves looking at a natural sequence of covers of the Julia set as before. The initial cover here is a collection of intervals containing the poles of f .

In [28] we proved the following result.

Theorem 5.5. For each $m \in \mathbb{N}$ with $m \geq 4$, the function f_m defined by

$$f_m(z) = \lambda_m \tan z / \prod_{r=1}^m (z - r\pi),$$

where $\lambda_m = (2m)^{-m}$, is hyperbolic and satisfies

$$\dim_P J(f_m) \geq 1/2.$$

This gives another class of functions whose Julia sets have different Hausdorff and packing dimensions and shows that the result of Theorem 3.2 does not hold for transcendental meromorphic functions with poles.

The proof of Theorem 5.5 involves identifying a subset A of $J(f_m)$ with box dimension equal to half and then applying Theorem 3.1. The set A consists of those first pre-images of the poles of f_m that are just to the left of the pole of f_m at $(2m + 1/2)\pi$. The points in A are very close to the points $1/(c_m(n + 1/2)\pi)$ for each $n \in \mathbb{N}$ and some constant c_m . Thus

$$\dim_B A = \dim_B \{1/n : n \in \mathbb{N}\} = 1/2.$$

In [25] we considered the functions defined by

$$f_{p,\lambda}(z) = \lambda \sum_{n=p^2}^{\infty} \frac{2z}{n^{2p} - z^2}, \quad p \in \mathbb{N}, \lambda > 0.$$

These functions are similar to the functions defined by $f_\lambda(z) = \lambda \tan z$ except that the poles are at $\pm n^p$ instead of at $\pm(n + 1/2)\pi$. As before, we have $J(f_{p,\lambda}) \subset \mathbb{R}$. Spreading the poles out by increasing the values of p gives an alternative way of obtaining Julia sets of small Hausdorff dimension and in [25] we proved the following result.

Theorem 5.6. Let $f_{p,\lambda}$ be as defined above. If $p \geq 6$ and $0 < \lambda < 1/6^p$, then

$$\dim_H J(f_{p,\lambda}) \leq 1/p.$$

For a given value of p , we were also able to show that taking λ to be large causes the Hausdorff dimension of the Julia set to be close to 1. More precisely, we proved the following.

Theorem 5.7. Let $f_{p,\lambda}$ be as defined above. If $p \geq 6$ and $p^{4p-1}/10^5 \log p < \lambda < p^{4p-1}/10^4 \log p$, then

$$\dim_H J(f_{p,\lambda}) \geq 1 - (30 \log \log p / \log p).$$

In order to prove Theorems 5.6 and 5.7, we used the following result which we proved in [24].

Theorem 5.8. Let f be a hyperbolic function in the class B . Then there exists $s(f)$ such that

$$\lim_{n \rightarrow \infty} \sum_{f^n(z)=a} \left(\frac{1 + |f^n(z)|^2}{|(f^n)'(z)|(1 + |z|^2)} \right)^t = \begin{cases} \infty & \text{for } t < s(f) \\ 0 & \text{for } t > s(f), \end{cases}$$

for each $a \in J(f)$.

For the functions $f_{p,\lambda}$ defined above, we showed in [24] that $\dim_H J(f_{p,\lambda}) = s(f_{p,\lambda})$ and then used this to prove Theorems 5.6 and 5.7. Further, in [26], we showed that, for a fixed $p \geq 6$, $s(f_{p,\lambda})$ is a continuous function of λ for $0 < \lambda < \frac{p^{4p-1}}{10^4 \log p}$. Together with Theorems 5.6 and 5.7, this shows that each value in $(0, 1)$ is attained by $\dim_H J(f_{p,\lambda})$ for some p, λ . (Note that the results in [26] are proved for a class of families of functions depending on a parameter — this class includes the functions $f_{p,\lambda}$ that we have just discussed.)

Finally, we note that it was shown by Sullivan [30] that for a hyperbolic rational function f , the Hausdorff dimension of the Julia set is always equal to $s(f)$. In [24] we showed that this is not the case for transcendental meromorphic functions — for the hyperbolic functions $f_{p,K}$ described in Section 4.3, $s(f_{p,K})$ is strictly less than $\dim_H J(f_{p,K})$.

REFERENCES

- [1] I.N. Baker. The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 1, (1975), 277-283.
- [2] A.F. Beardon. *Iteration of Rational Functions* (Springer, 1991).
- [3] W. Bergweiler. Iteration of meromorphic functions. *Bull. Amer. Math. Soc.*, 29 (1993), 151-188.
- [4] L. Carleson and T.W. Gamelin. *Complex Dynamics* (Springer, 1993).
- [5] R.L. Devaney and L. Goldberg. Uniformization of attracting basins for exponential maps. *Duke Math. J.* 2 (1987), 253-266.
- [6] R. Devaney and M. Krych. Dynamics of $\exp(z)$. *Ergod. Th. Dynam. Sys.*, 4 (1984), 35-52.
- [7] R.L. Devaney and F. Tangerman. Dynamics of entire functions near the essential singularity. *Ergod. Th. and Dynam. Sys.*, 6 (1986), 489-503.
- [8] P.L. Duren. *Univalent Functions* (Springer, 1953).
- [9] A.E. Eremenko and M.Yu Lyubich. Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier (Grenoble)*, 42 (1992), 989-1020.
- [10] K.J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications* (Wiley, 1990).
- [11] B. Karpińska. Area and Hausdorff dimension of the set of accessible points of the Julia sets of λe^z and $\lambda \sin z$. *Fund. Math.*, 159 (1999), 269-287.
- [12] B. Karpińska. Hausdorff dimension of the hairs without endpoints for $\lambda \exp z$. *C. R. Acad. Sci. Paris*, 328 (1) (1999), 1039-1044.
- [13] J. Kotus. On the Hausdorff dimension of Julia sets of meromorphic functions II. *Bull. Soc. Math. France*, 123 (1995), 33-46.

- [14] J. Kotus and M. Urbański. Hausdorff dimension and Hausdorff measures of Julia sets of elliptic functions. *Bull. London Math. Soc.*, 35 (2003), 269-275.
- [15] C. McMullen. Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.*, 300 (1987), 329-342.
- [16] M. Misiurewicz. On iterates of e^z . *Ergod. Th. Dynam. Sys.*, 1 (1981), 103-106.
- [17] G. Pólya and G. Szegő. *Problems and Theorems in Analysis I* (Part III, problems 158-160). Springer, New York, 1972.
- [18] P.J. Rippon and G.M. Stallard. Iteration of a class of hyperbolic meromorphic functions. *Proc. Amer. Math. Soc.*, 127 (1999), 3251-3258.
- [19] P.J. Rippon and G.M. Stallard. Dimensions of Julia sets of meromorphic functions. *J. London Math. Soc.*, (2) 71 (2005), 669-683.
- [20] P.J. Rippon and G.M. Stallard. Escaping points of meromorphic functions with a finite number of poles. *J. d'Analyse Math.*, 96 (2005), 225-245.
- [21] G.M. Stallard. The Hausdorff dimension of Julia sets of entire functions. *Ergod. Th. and Dynam. Sys.*, 11 (1991), 769-777.
- [22] G.M. Stallard. The Hausdorff dimension of Julia sets of meromorphic functions. *J. London Math. Soc.*, (2) 49 (1994), 281-295.
- [23] G.M. Stallard. The Hausdorff dimension of Julia sets of entire functions II. *Math. Proc. Camb. Phil. Soc.*, 119 (1996), 513-536.
- [24] G.M. Stallard. The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions. *Math. Proc. Camb. Phil. Soc.*, 127 (1999), 271-288.
- [25] G.M. Stallard. The Hausdorff dimension of Julia sets of meromorphic functions II. *J. London Math. Soc.*, (2) 60 (1999), 847-859.
- [26] G.M. Stallard. The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions II. *Ergod. Th. and Dynam. Sys.*, 20 (2000), 895-910.
- [27] G.M. Stallard. The Hausdorff dimension of Julia sets of entire functions IV. *J. London Math. Soc.*, (2) 61 (2000), 471-488.
- [28] G.M. Stallard. Dimensions of Julia sets of hyperbolic meromorphic functions. *Bull. London Math. Soc.*, 33 (2001), 689-694.
- [29] G.M. Stallard. Dimensions of Julia sets of hyperbolic entire functions. *Bull. London Math. Soc.*, 36 (2004), 263-270.
- [30] D. Sullivan. 'Conformal dynamical systems', *Geometric dynamics*, Lecture Notes in Math., 1007 (Springer, New York, 1983) 725-752.
- [31] M. Urbański. Geometry and ergodic theory of conformal non-recurrent dynamics. *Ergod. Th. and Dynam. Sys.*, 17 (1997), 1449-1476.
- [32] M. Urbański and A. Zdunik. The finer geometry and dynamics of the hyperbolic exponential family. *Michigan Math. Journal*, 51 (2003), 227-250.
- [33] M. Urbański and A. Zdunik. Real analyticity of Hausdorff dimension of finer Julia sets of exponential family. *Ergod. Th. and Dynam. Sys.*, 24 (2004), 279-315.

THE OPEN UNIVERSITY, DEPARTMENT OF MATHEMATICS, WALTON HALL, MILTON KEYNES MK7 6AA, UK

E-mail address: g.m.stallard@open.ac.uk

ABEL'S FUNCTIONAL EQUATION AND ITS ROLE IN THE PROBLEM OF CROISSANCE RÉGULIÈRE.

GEORGE SZEKERES

Professor Szekeres completed the manuscript of this paper in April 2005, sadly just a few months before he died, aged 94.

In memory of I.N. Baker

From the earliest years of his mathematical life Noel Baker shared with me a strong interest in Abel's functional equation

$$(1) \quad A(f(x)) = A(x) + 1$$

where f is a continuous strictly increasing function of a real variable x . Yet the nearest we ever came to a joint publication were two papers published side by side in the Journal of the Australian Mathematical Society; see [5] and [1]. Inspired by an earlier work of Noel, I showed that if f is an entire or rational function with $f'(0) = 1$ and having a Taylor expansion

$$f(z) = z + \sum_{n \geq 2} a_n z^n$$

then with the exception of $z/(a+az)$ the formal fractional iterates

$$(2) \quad f_\sigma(z) = z + \sigma a_2 z^2 + \sum_{n \geq 2} a_n^{(\sigma)} z^n, \quad \sigma \in \mathbb{R}$$

of f (the coefficients $a_n^{(\sigma)}$ being formally determined from $f \circ f_\sigma = f_\sigma \circ f$) cannot be holomorphic at $z = 0$ for all real σ . In fact, by results of Baker, Écalle and Liverpool (see, for example, [4]), the set of σ for which the series (2) converges forms a discrete one-dimensional lattice in $\mathbb{R}$. Noel immediately generalized my result to all meromorphic f . At that time he was strongly influenced by the works of the French mathematicians G. Julia and P. Fatou [2], long before their work was resurrected as part of the subject of dynamical systems.

The basic tool of fractional iterations is Abel's functional equation (1) or some of its variants. This simple but intriguing functional equation has an interesting history. Abel himself never published it in his short lifetime (he was barely 27 when he died) but his brief note, in which he clearly states the role of the equation in fractional iterations, was fortunately rescued for posterity by his teacher Holmboe and was included in the second volume of Abel's collected works under the cryptic title *Détermination d'une fonction*