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BAKER DOMAINS

P.J. RIPPON

Abstract. This paper surveys the current state of knowledge about Baker domains, including: the rate of growth of iterates in a Baker domain, the connection with singular values, and the various different types of Baker domain. Also, many examples of functions with Baker domains are given and it is proved that transcendental meromorphic functions of the form $f(z) = z + R(z)e^{Q(z)}$, where R is rational and Q is a polynomial, have infinitely many invariant Baker domains.

Dedicated to the memory of Professor Noel Baker.

1. Introduction

The extension of the Fatou-Julia theory of complex dynamics from rational functions to transcendental meromorphic functions leads to the occurrence of two new types of dynamical phenomena, known as Baker domains and wandering domains. Noel Baker made extensive contributions to our understanding of both these phenomena, and Baker domains are named after him. The aim of this paper is to survey the current state of knowledge about Baker domains.

Throughout this paper $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a transcendental meromorphic function and we denote by f^n, $n = 0, 1, 2, \ldots$, the nth iterate of f. The *Fatou set* $F(f)$ is defined to be the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ is well-defined, meromorphic and forms a normal family in some neighborhood of z. The complement of $F(f)$ is called the *Julia set* $J(f)$ of f. An introduction to the properties of these sets can be found in [19], and also [48] for the case of entire functions.

The set $F(f)$ is completely invariant, so for any component U of $F(f)$ there exists, for each $n = 0, 1, 2, \ldots$, a component of $F(f)$, which we call U_n, such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a *periodic* component of *period* p (or p-*periodic* component), assuming p to be minimal, and $U_0, \ldots, U_{p-1}$ is a p-*cycle*. There are then five possible types of periodic components, as follows:

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- U contains an attracting periodic point z_0 of period p. In this case, $f^{np}(z) \to z_0$ as $n \to \infty$ for $z \in U$, and U is called an *attracting component.*
- ∂U contains a periodic point z_0 of period p and $f^{np}(z) \to z_0$ as $n \to \infty$ for $z \in U$. In this case, U is called a *parabolic component* or *Leau domain.*
- $f^p : U \to U$ is analytically conjugate to a Euclidean rotation (through an angle that is an irrational multiple of π) of the unit disc onto itself. In this case, U is called a *Siegel disc.*
- $f^p : U \to U$ is analytically conjugate to a Euclidean rotation (through an angle that is an irrational multiple of π) of an annulus onto itself. In this case, U is called a *Herman ring.*
- There exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$ but $f^p(z_0)$ is not defined. In this case, U is called a *Baker domain.*

See [19, Theorem 6] for this result, including detailed historical references. This classification can be extended to various other cases such as analytic self-maps of the punctured plane $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ (see [8]) and functions meromorphic outside a small exceptional set (see [10] and [28]).

If U_n is not eventually periodic, then we say that U is a *wandering component* of $F(f)$, or a *wandering domain.*

The name Baker domain was introduced by Eremenko and Lyubich; see their fundamental paper [35]. Before that, names such as *domain of attraction of* ∞ [37] and *essentially parabolic domain* [13] were used. Both [19] and [48] include an introduction to Baker domains and this survey aims to build on these accounts.

If f is a transcendental *entire* function with a Baker domain U, then $f^n(z) \to \infty$ as $n \to \infty$, for $z \in U$. If f is a meromorphic function with a p-cycle of Baker domains $U_0, \ldots, U_{p-1}$, then there is at least one component U_k, $0 \leq k \leq p-1$, with the property that $f^{np}(z) \to \infty$ as $n \to \infty$, for $z \in U_k$.

In [4, Theorem 3.1], Baker showed that if f is a transcendental entire function, then any multiply connected component of $F(f)$ must be a wandering domain. Thus, if U is a Baker domain of an *entire* function f, then U is simply connected. This is not true in general for meromorphic functions, even those with only finitely many poles, as we see in the next section. Also, we note that any multiply connected Baker domain must be infinitely connected, by a result of Baker, Kotus and Lü [13].

The study of Baker domains has revealed that they are in a sense more complicated than the other types of invariant Fatou components. In particular, there are different types of Baker domains and they are related to singular values in various different ways.

The plan of the paper is as follows. In Section 2 we give many examples of Baker domains. In Section 3 we describe the rate of growth of iterates in a Baker domain and in Section 4 we discuss the relationship between Baker domains and singular values (this section has considerable overlap with the survey [24, Section 2.2]). In Section 5 we describe results which classify Baker domains into different types and discuss the boundaries of Baker domains. Finally, in Section 6 we show that certain types of meromorphic functions have infinitely many Baker domains.

2. Examples of Baker domains

The first example of a Baker domain was given by Fatou [37, Example I].

Example 2.1. Fatou observed that for the function

$$f(z) = z + 1 + e^{-z}$$

we have $\Re(f(z)) > \Re(z)$ and $f^n(z) \to \infty$ whenever $\Re(z) > 0$. Hence the right half-plane $\{z : \Re(z) > 0\}$ is contained in an invariant Baker domain U. Fatou showed that for this function U is the only component of $F(f)$.

This example illustrates a common way to prove the existence of a Baker domain for a meromorphic function f, namely, to use analytic estimates for f in order to find a domain V which is invariant under f^p, for some $p \in \mathbb{N}$, and which contains at least one point z such that $f^{np}(z) \to z_0$ but $f^p(z_0)$ is not defined. Then, by the classification of periodic components, V is contained in a periodic component U of $F(f)$ (with period some divisor of p) and U must be a Baker domain. The following examples were all constructed this way.

Example 2.2. Baker [3] showed that, for sufficiently large positive a, the function

$$f(z) = z + \frac{\sin\sqrt{z}}{\sqrt{z}} + a$$

has an invariant Baker domain which contains an invariant parabola-shaped domain $V = \{x + iy : y^2 < 4(x+1), x > a^2\}$. Figure 1 shows an approximate picture of this Baker domain in the case $a = 6$ (the white points are slowly escaping), kindly provided by Dominique Fleischmann, who has recently shown that this function has an invariant Baker domain for all $a > 0$ [38].

Example 2.3. Baker, Kotus and Lü [13, page 606] showed that

$$f(z) = \frac{1}{z} - e^z$$

has a 2-cycle U_0, U_1 of Baker domains such that

$$f^{2n}(z) \to \infty \text{ for } z \in U_0, \quad f^{2n}(z) \to 0 \text{ for } z \in U_1,$$

and U_0 contains a set of the form $V = \{z : 3\pi/4 < \arg z < 5\pi/4, |z| > R\}$, where $R > 0$, which is invariant under f^2.

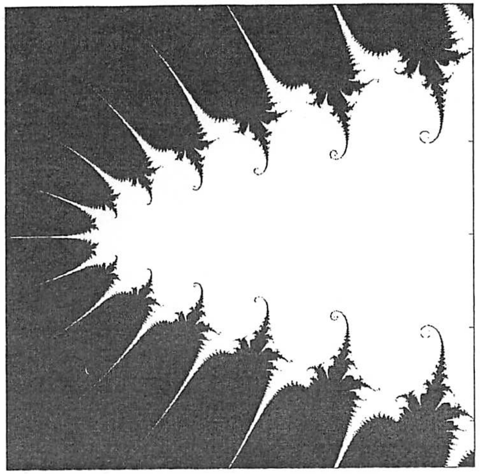

FIGURE 1

Example 2.4. König [44, Example 3] showed that, for certain values of the real constants a and b,

$$f(z) = \frac{1}{z} + ae^{-z} + b$$

has a 3-cycle U_0, U_1, U_2 of Baker domains such that

$$f^{3n}(z) \to \infty \text{ for } z \in U_0, \quad f^{3n}(z) \to b \text{ for } z \in U_1, \quad f^{3n}(z) \to 0 \text{ for } z \in U_2;$$

and U_0 contains a right half-plane which is invariant under f^3.

Example 2.5. Rippon and Stallard [53, Theorem 3] showed that

$$f(z) = az(1 + e^{-z^p}), \quad \text{where } a > 1,\ p \in \mathbb{N},$$

has p invariant Baker domains, one in each sector $\{z : |\arg z - 2k\pi/p| < \pi/p\}$, $k = 0, 1, \dots, p-1$, and that $g(z) = e^{2\pi i/p} f(z)$ has a p-cycle of Baker domains. A similar example was given by Morosawa [47].

Example 2.6. Domínguez [33, Example 1] showed that

$$f(z) = z + 2 + e^{-z} + \frac{1/100}{z - (1 + \pi i)},$$

has an invariant infinitely connected Baker domain, which contains an invariant right half-plane.

Another way to prove the existence of certain Baker domains is to start with an analytic self-map g of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, with known behaviour near 0 or ∞, and lift the dynamics to $\mathbb{C}$ as follows.

Let $\pi(z) = e^{az}$, where $a \neq 0$, be a projection map from $\mathbb{C}$ to $\mathbb{C}^*$. Since $g(\pi(z)) \neq 0$, for $z \in \mathbb{C}$, there is an entire function f such that $\pi(f(z)) = g(\pi(z))$, for $z \in \mathbb{C}$. It was proved by Bergweiler [22] that in this situation we have

$$J(f) = \pi^{-1}(J(g)).$$

In particular, the set $J(f)$ is periodic with period $2\pi i/a$.

This process is often called a *logarithmic change of variable* or a *logarithmic lift*. We now show how this technique can be used to give several examples of Baker domains.

Example 2.7. On applying the above process to

$$g(w) = cwe^{-w}, \quad \text{where } c \neq 0,$$

with $\pi(z) = e^{-z}$, we obtain

$$f(z) = z - \log c + e^{-z}.$$

If $0 < |c| < 1$, then g has an attracting fixed point at 0 with attracting basin V, so $U = \pi^{-1}(V)$ is a single component of $F(f)$ which contains a right half-plane in which $f^n \to \infty$. Such a Baker domain is a generalisation of the type given by Fatou (Example 2.1).

Example 2.8. In [40], Herman applied this process with $g(w) = e^{2\pi i\theta} we^w$. For suitable irrational $\theta \in (0, 1)$, the function g has an invariant Siegel disc V with centre 0, on which g is conjugate to an irrational rotation. Using $\pi(z) = e^z$, we obtain

$$f(z) = z + 2\pi i\theta + e^z,$$

and $U = \pi^{-1}(V)$ is a single component of $F(f)$ which contains a left half-plane. In this case, the dynamics of g around 0 lifts to give $\Im(f^n) \to \infty$ in U, so this component of $F(f)$ must be a Baker domain and f is univalent in U. Moreover, for some choices of θ the Siegel disc V is bounded by a quasicircle, so the boundary of the corresponding Baker domain U is a Jordan curve in $\hat{\mathbb{C}}$; see [15].

Example 2.9. Starting with

$$g(w) = cw^2 e^{-w}, \quad \text{where } c \neq 0,$$

we obtain, with $\pi(z) = e^{-z}$ again, the entire function

$$f(z) = 2z - \log c + e^{-z}.$$

In this case, g has a super-attracting fixed point at 0, for all $c \neq 0$. If V is the corresponding basin of attraction, then $U = \pi^{-1}(V)$ is a Baker domain which contains a right half-plane. In [21, Theorem 1] Bergweiler considered the function $f(z) = 2z + 2 - \log 2 - e^z$, which is of this type (apart from a trivial change of variable), and showed (using the theory of polynomial-like maps) that ∂U is a Jordan curve in $\hat{\mathbb{C}}$ and f is univalent on U.

Example 2.10. Baker and Domínguez [8, Theorem 5.1] took

$$g(w) = we^{-w},$$

which, with $\pi(z) = e^{-z}$, gives

$$f(z) = z + e^{-z}.$$

The function g has a parabolic fixed point at $0 \in J(g)$, with parabolic component $V \subset F(g)$ containing $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$; see [48, page 17], for example. Also, $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\} \subset J(g)$ since $g(\mathbb{R}^-) \subset \mathbb{R}^-$, $g^n \to \infty$ on $\mathbb{R}^-$ and g belongs to the class $\mathcal{B}$; see Theorem 4.1. Lifting these properties of g by π, we deduce that $\pi^{-1}(V)$ has infinitely many components $U_k, k \in \mathbb{Z}$, all congruent under translation by integer multiples of $2\pi i$, such that $U_k \subset \{z : (2k-1)\pi < \Im(z) < (2k+1)\pi\}$ and ∂U_k is asymptotic to the lines $y = (2k \pm 1)\pi$. For each $k \in \mathbb{Z}$, we have $f^n \to \infty$ and $\Re(f^n) \to \infty$ in U_k, and the lines $y = (2k \pm 1)\pi$ lie in $J(f)$. Therefore the U_k are distinct invariant Baker domains.

In Section 6 we describe various types of meromorphic functions with infinitely many invariant Baker domains; these can be considered as generalisations of Example 2.10.

Example 2.11. Barański and Fagella [16, page 422] started with

$$g(w) = e^{2\pi i\theta} w e^{\frac{1}{2}\beta(w-1/w)}, \quad \text{where } 0 < \theta < 1,\ 0 < \beta < 1,$$

which is a self-map of $\mathbb{C}^*$ that does not extend analytically to $\mathbb{C}$. With $\pi(z) = e^{iz}$, this gives

$$f(z) = z + 2\pi\theta + \beta \sin z.$$

It is known that for suitable values of θ and β, the map g has an invariant Herman ring V, symmetric with respect to the unit circle, on which g is conjugate to an irrational rotation; see [40] or [6]. Thus $U = \pi^{-1}(V)$ is an invariant Baker domain, symmetric with respect to the real axis.

Finally, we mention a method for obtaining Baker domains due to Eremenko and Lyubich [34], which uses results from approximation theory. The idea of the method is to construct:

- a function g that is analytic on one or more unbounded closed sets V_j, with the property that, for each j, $f(V_j) \subset V_j$ and $f^n \to \infty$ in V_j, where f is any function that is uniformly close to g on $\bigcup_j V_j$;
- a function h that is analytic on one or more unbounded closed sets W_j, with the property that, for each j, $f^n(z) = O(1)$ for $z \in W_j$, where f is any function that is uniformly close to h on $\bigcup_j W_j$.

Then, we use Arakelyan's theorem [39] to construct an entire function f which is uniformly close to g on each V_j and to h on each W_j. We deduce that each V_j is contained in a Baker domain of f which does not meet any of the sets W_j. Moreover, if g is chosen to be univalent in a large enough neighbourhood of V_j, then we can arrange that f is univalent in V_j. Using other approximation results, such as Nersejan's theorem [39], we can also apply this method to obtain meromorphic examples. The following examples of Baker domains were all constructed using this approximation method.

Example 2.12. Eremenko and Lyubich [34, Example 3] constructed an entire function f with a Baker domain containing a right half-plane in which f is univalent.

Example 2.13. König [44, Example 1] constructed a meromorphic function f with a finite number of poles and an invariant infinitely connected Baker domain containing a half-plane.

Example 2.14. Rippon and Stallard [53, Theorem 4] constructed an entire function f with a p-cycle of Baker domains, each shaped like a sector, in each of which f is univalent.

Example 2.15. Barański and Fagella [16, page 424] constructed an entire function f with a spiral-shaped Baker domain in which f is univalent.

3. Growth in Baker domains

For many of the examples described in Section 2 it is clear that, at most points z of the Baker domain, $|f(z)|$ is no greater than a multiple of $|z|$ and the iterates $f^n(z)$ do not tend to ∞ rapidly. The following result [51, Theorem 1] shows that this property always holds in a Baker domain.

Theorem 3.1. *Let U be a p-periodic Baker domain of a meromorphic function f in which $f^{np} \to \infty$. Then, for any compact subset K of U, there exist constants $C > 0$ and $n_0 \in \mathbb{N}$ such that*

$$|f^{np}(z')| \le C|f^{np}(z)|, \quad \text{for } z, z' \in K, n \ge n_0; \tag{3.1}$$

for all $z \in U$,

$$\ln|f^{np}(z)| = O(n) \quad \text{as } n \to \infty; \tag{3.2}$$

for any $z_0 \in U$ and any path $\Gamma = \bigcup_{n=0}^{\infty} f^{np}(\Gamma_0)$, where Γ_0 joins z_0 to $f^p(z_0)$ in U and $0 \notin \Gamma$, there is a constant $C > 0$ such that

$$\frac{1}{C}|z| \le |f(z)| \le C|z|, \quad \text{for } z \in \Gamma. \tag{3.3}$$

Note that the path Γ in this result lies in U and tends to ∞.

The rate at which iterates tend to infinity in a Baker domain was first studied by Baker [3], in connection with the question of whether a transcendental entire function of small growth has no unbounded components of $F(f)$. In [7], he proved Theorem 3.1 for an entire function f in the case $p = 1$ by using classical estimates for the density of the hyperbolic metric in a simply connected domain; recall that a Baker domain of an entire function is simply connected.

In any hyperbolic domain D (that is, a domain which omits at least 2 points of $\mathbb{C}$), we can define the hyperbolic metric $[z,z']_D$, for $z,z' \in D$; see [31], for example. If D is also simply connected, then the density ρ_D of this metric satisfies:

$$\frac{1}{2\text{dist}(z,\partial D)} \le \rho_D(z) \le \frac{2}{\text{dist}(z,\partial D)}. \tag{3.4}$$

In the case of a simply connected Baker domain, the result (3.1) is proved using the contracting property of the hyperbolic metric and the lower bound in (3.4), and then (3.2) and (3.3) follow by applying (3.1) to suitable compact subsets of U; see [19, Lemma 7].

In a multiply connected hyperbolic domain, however, we have a weaker lower bound for the density of the hyperbolic metric. Using this, correspondingly weaker growth estimates than those in Theorem 3.1 can be obtained in a multiply connected Baker domain; see [13, Lemma 4.1] and [19, Lemma 7].

The proof of Theorem 3.1 for a general meromorphic function was based on work of Bonfert [29]. Briefly, Bonfert used results from the theory of Fuchsian groups to show that if g is an analytic self-map of an unbounded hyperbolic domain D in which $g^n \to \infty$, and g does not extend analytically to ∞, then

- a lower bound for the hyperbolic density of the form (3.4) holds at points on and near any orbit $g^n(z_0)$, $n = 0,1,2,\ldots$, where $z_0 \in D$;
- whether $[g^{n+1}(z_0), g^n(z_0)]_D \to 0$ does not depend on the choice of $z_0 \in D$.

Note that for $z_0 \in D$, the quantity $[g^{n+1}(z_0), g^n(z_0)]_D$ is non-increasing by the contraction property of the hyperbolic metric.

Bonfert's results can also be used to relate the geometric properties of a p-periodic Baker domain U of a meromorphic function f to the proximity of f^p to the identity function in U, as stated below; see [51, Theorem 2].

Theorem 3.2. *Let U be a p-periodic Baker domain of a meromorphic function f in which $f^{np} \to \infty$ and for $z_0 \in U$ let $z_n = f^n(z_0)$ and $\Gamma = \bigcup_{n=0}^{\infty} f^{np}(\Gamma_0)$, where Γ_0 is a path in U joining z_0 to z_p.*

(a) *There exists $C_1 > 0$ such that*

$$\text{dist}(z,\partial U) \le C_1 \frac{|f^p(z) - z|}{[f^p(z), z]_U}, \quad \textit{for } z \in \Gamma. \tag{3.5}$$

If we also have $[z_{(n+1)p}, z_{np}]_U \not\to 0$ as $n \to \infty$, then there exists $C_2 > 0$ such that

$$\text{dist}(z,\partial U) \le C_2 |f^p(z) - z|, \quad \textit{for } z \in \Gamma. \tag{3.6}$$

(b) *There exist $c_1 > 0$ and $c_2 > 0$ such that*

$$\text{dist}(z,\partial U) \ge c_1 \frac{|f^p(z) - z|}{[f^p(z), z]_U} \ge c_2 |f^p(z) - z|, \quad \textit{for } z \in \Gamma. \tag{3.7}$$

It can be seen from this result that the closer f^p is to the identity in a p-periodic Baker domain the thinner is the Baker domain, and vice versa.

A less general result of this type was proved in [53, Theorem 3] and used there to show the *non-existence* of Baker domains under certain circumstances; see [53, Theorem 3]. Also, a version of this result was used in [16, Lemma 3.3] to classify univalent Baker domains of entire functions; see Theorem 5.3.

4. Baker domains and singular values

Let f be a transcendental meromorphic function. For $p \in \mathbb{N}$, we denote by $\text{sing}(f^{-p})$ the set of finite *inverse function singularities* of f^p; that is, the set of points $w \in \mathbb{C}$ such that some branch of $(f^p)^{-1}$ can be analytically continued (along some path) up to but not through w. We also call the points of $\text{sing}(f^{-p})$ the *singular values* of f^p. The set $\text{sing}(f^{-1})$ of singular values of f consists of the *critical values* of f (those values $f(z_0)$, where $f'(z_0) = 0$) and the finite *asymptotic values* of f (those values α such that $f(z) \to \alpha$ as $z \to \infty$ along a path Γ).

For $p \in \mathbb{N}$, we can relate $\text{sing}(f^{-p})$ to $\text{sing}(f^{-1})$ as follows:

$$f^{p-1}\left(\text{sing}(f^{-1}) \setminus A_{p-1}\right) \subset \text{sing}(f^{-p}) \subset \bigcup_{j=0}^{p-1} f^j\left(\text{sing}(f^{-1}) \setminus A_j\right), \tag{4.1}$$

where $A_j = A_j(f) = \{z : f^j \text{ is not analytic at } z\}$; see [41, Theorem 7.1.2] and also [1, Lemma 2] for the case of a transcendental entire function.

The *postsingular set* $P(f)$ is defined as

$$P(f) = \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1}) \setminus A_n),$$

or, in some texts, the closure of this set. By (4.1), the set $P(f)$ is the union of all the sets $\text{sing}(f^{-p})$, $p \in \mathbb{N}$.

Attracting components, parabolic components, Siegel discs and Herman rings of a meromorphic function f all have close connections with the singular values of f; see [19, Theorem 7]. For Baker domains there are also connections with singular values, but they are less precise. In [35] Eremenko and Lyubich introduced the class

$$\mathcal{B} = \{f : \text{sing}(f^{-1}) \text{ is bounded}\}$$

and proved the following fundamental result.

Theorem 4.1. *If f is a transcendental entire function and $f \in \mathcal{B}$, then there is no component of $F(f)$ in which $f^n \to \infty$. In particular, the function f has no Baker domains.*

The idea of the proof of Theorem 4.1 is as follows. Let $f \in \mathcal{B}$ and choose R so large that $\text{sing}(f^{-1}) \subset \{z : |z| < R/2\}$ and $|f(0)| < R$. Now put $D_R = \{z : |z| > R\}$. If Δ_0 is an open disc in $F(f)$ in which $f^n \to \infty$, then we can assume that $|f^n| > R$ in Δ_0, for $n \geq 0$, so each $\Delta_n = f^n(\Delta_0)$ lies in a component V_n of $f^{-1}(D_R)$. Now each V_n is simply connected and does not contain 0, so we can consider a component W_n of $\log V_n$ and take T_n to be the component of $\log \Delta_n$ in W_n. Then for some branch of the logarithm $F_n(s) = \log f(e^s)$ maps W_n univalently onto the half-plane $\{t : \Re(t) > \log R\}$ and T_n univalently onto T_{n+1}. By Koebe's $\frac{1}{4}$-theorem, F_n is expanding on T_n for all large values of n (since W_n contains no disc of radius greater than π and $\inf_{s \in T_n} \Re(s) \to \infty$ as $n \to \infty$), but $\operatorname{diam} T_n \leq 2\pi$, for $n \geq 0$, which gives a contradiction.

The following result extends Theorem 4.1 to p-cycles of Baker domains.

Theorem 4.2. *If f is a transcendental meromorphic function and sing(f^{-p}) is bounded, then there is no component of $F(f)$ in which $f^{np} \to \infty$. In particular, the function f has no p-periodic Baker domains.*

This was proved in [54, Theorem A], under the slightly stronger assumption that

$$\bigcup_{j=0}^{p-1} f^j\left(\text{sing}(f^{-1}) \setminus A_j\right) \text{ is bounded,}$$

but the proof in [54] also gives Theorem 4.2. This proof builds on Eremenko and Lyubich's method of proof of Theorem 4.1, and on the partial proof given in [19, Theorem 16]. The extra ingredient used in [54] is that, for R large enough, each component of $f^{-p}(\{z : |z| > R\} \cup \{\infty\})$ is simply connected; for a meromorphic function such components can be bounded or unbounded.

As a corollary of Theorem 4.2, we note that if f belongs to the Speiser class S of transcendental meromorphic functions with only finitely many singular values, then $\text{sing}(f^{-p})$ is bounded for all $p \in \mathbb{N}$, by (4.1), so f has no Baker domains. Such functions are also known to have no wandering domains; see [14].

Bargmann [17, Theorem 4] used a fundamental result of König (see Theorem 5.1) and results on normal families of covering maps to give a stronger version of Theorem 4.1 for Baker domains. This shows that if f has a Baker domain, then the (unbounded) set of singular values of f cannot be too sparse.

Theorem 4.3. *If f is an entire function with an invariant Baker domain, then there exist constants $C > 1$ and $r_0 > 0$ such that every annulus of the form $\{z : r/C < |z| < Cr\}$ with $r \geq r_0$ meets sing(f^{-1}).*

As part of the proof of Theorem 4.3, Bargmann showed that an invariant Baker domain of a transcendental entire function contains a path Γ tending to ∞ on which f' is bounded and also (3.3) holds.

Theorem 4.3 has been generalised to meromorphic functions with a finite number of poles [57], by adapting the argument used by Eremenko and Lyubich to prove Theorem 4.1.

Theorem 4.4. *If f is a meromorphic function with a finite number of poles and with a p-periodic Baker domain, then there exist constants $C > 1$ and $r_0 > 0$ such that every annulus $\{z : r/C < |z| < Cr\}$ with $r \geq r_0$ meets sing(f^{-p}).*

The paper [57] also gives examples to show that this generalisation does not extend to meromorphic functions with infinitely many poles.

However, it should be noted that Baker domains need not contain singular values. For example it is straightfoward to check that this is the case in each of Examples 2.8 and 2.11. Moreover, the example considered by Bergweiler $f(z) = 2z + 2 - \log 2 - e^z$ has an invariant Baker domain U in which f is univalent and $\text{dist}(P(f), U) > 0$, where $\text{dist}(\,.\,,\,.\,)$ denotes the Euclidean distance; see [21, Theorem 1]. In the same paper, Bergweiler [21, Theorem 3] showed that a Baker domain cannot lie too far from the set $P(f)$ in the following sense.

Theorem 4.5. *Let f be a transcendental entire function with an invariant Baker domain U. If $U \cap$ sing$(f^{-1}) = \emptyset$, then there exists a sequence (p_n) such that $p_n \in P(f)$, $|p_n| \to \infty$, $|p_{n+1}/p_n| \to 1$ and dist$(p_n, U) = o(|p_n|)$ as $n \to \infty$.*

For some classes of meromorphic functions, we can assert that any cycle of Baker domains must contain a singular value. For example, Bergweiler used an observation of Herman [40, page 609] to show that this is the case for certain functions arising from Newton's method (see [20]) and pointed out that similar reasoning applies to functions of the form $f(z) = z + p(z)e^{q(z)}$, where p and q are polynomials; see also [26] for other examples.

We end this section by listing several other particular types of meromorphic function for which we can assert the existence of Baker domains containing singular values. The first is due to Hinkkanen [42].

Theorem 4.6. *Let f be a transcendental meromorphic function such that*

$$f(z) = z + az^{-m} + O(z^{-m-\delta}) \quad \text{as } z \to \infty,\ z \in V,$$

where $a \in \mathbb{C}^*$, $\delta > 0$ and $V = \{z : |\arg z - t_k| < \pi/(m+1)\}$ for some $k \in \mathbb{Z}$ with $t_k = (2\pi k + \arg a)/(m+1)$. Then f has an invariant Baker domain U, which contains an unbounded Jordan domain in V whose boundary is tangent to ∂V at ∞, and $U \cap$ sing(f^{-1}) is non-empty.

Sometimes it is possible to prove that there is a Baker domain containing infinitely many singular values, as in the following result of Rippon and Stallard [53, Theorem 2].

Theorem 4.7. *Let f be a transcendental meromorphic function such that*

$$f(z) = az + bz^k e^{-z}(1 + o(1)) \quad \text{as } \Re(z) \to \infty, \tag{4.2}$$

where $k \in \mathbb{N}$, $a > 1$ and $b > 0$. Then f has an invariant Baker domain U, which contains $\{z : |z^k e^{-z}| < \rho, |z| > R\}$ for each $\rho > 0$ and large enough values of $R > 0$, and $U \cap sing(f^{-1})$ is unbounded.

Recently the author showed that if (4.2) holds, then the sets of critical points of f in U and critical values of f in U are unbounded; see [51, Theorem 3].

Note that if f is of the form (4.2) with $k \in \mathbb{Z} \setminus \mathbb{N}$, then f has an invariant Baker domain U, which contains an invariant set of the form $\{z : \Re(z) > R\}$ for large enough values of R. However, for such k we cannot deduce that $U \cap \text{sing}(f^{-1})$ is non-empty. For example, the function $f(z) = 2z + 2e^{-2}e^{-z}$ has such an invariant Baker domain U and f is univalent in U. This follows from the corresponding properties of the function $f(z) = 2z + 2 - \log 2 - e^z$ mentioned in Example 2.9, after a suitable linear change of variable.

Finally we give another result of this type, due to Bergweiler [23, Theorem 3].

Theorem 4.8. *Let f be a transcendental entire function of finite order such that*

$$f(z) = z + a + o(1) \quad \text{as } z \to \infty, \ |\arg z| \le \eta. \tag{4.3}$$

where $a, \eta > 0$. Then f has an invariant Baker domain U, which contains $\{z : |\arg z| \le \eta, \Re(z) > R\}$ for some $R > 0$, and $U \cap sing(f^{-1})$ is unbounded.

A detailed discussion of the background to this result can also be found in the survey article [24].

Recently, Lauber [45] has studied the Baker domains of various families of entire functions that satisfy the hypotheses of Theorem 4.8 (after a trivial change of variable), in particular the functions $g_b(z) = z - 1 + bze^z$, where $b \in \mathbb{C}$. For the functions g_b, there is a single invariant Baker domain U_b which contains all the critical points of g_b with at most one exception; also, the set of parameters b for which there is such an exceptional critical point contains a copy of the Mandelbrot set.

5. Types of Baker domains

Let U be a simply connected Baker domain of f and let ψ be a Riemann map from the open unit disc $\mathbb{D}$ onto U. Then $g = \psi^{-1} \circ f \circ \psi$ is an analytic self-map of $\mathbb{D}$. Hence by the Denjoy-Wolff theorem (see [31], for example), there is a point $\zeta \in \overline{\mathbb{D}}$ such that $g^n \to \zeta$ locally uniformly in $\mathbb{D}$. The fact that $f^n \to \infty$ in U implies that $\zeta \in \partial\mathbb{D}$. In [32], Cowen introduced a remarkable classification of such self-maps g of $\mathbb{D}$ whose Denjoy-Wolff point lies in $\partial\mathbb{D}$, proving that g must be semi-conjugate in $\mathbb{D}$ to a Möbius transformation in one of three different ways. Cowen's result leads to a corresponding classification of the original simply connected Baker domain. This classification is expressed in terms of the following notation, introduced by König in [44].

Let G be a domain and let $f : G \to G$ be analytic. Then a domain $V \subset G$ is *absorbing* (or *fundamental*) for f if: V is simply connected, $f(V) \subset V$ and for each compact set $K \subset G$ there exists $N = N_K$ such that $f^N(K) \subset V$.

Now let $\mathbb{H} = \{z : \Re(z) > 0\}$. The triple (V, ϕ, T) is called a *conformal conjugacy* (or *eventual conjugacy*) of f in G if:

(a) V is *absorbing* for f;
(b) $\phi : G \to \Omega \in \{\mathbb{H}, \mathbb{C}\}$ is analytic and univalent in V;
(c) $T : \Omega \to \Omega$ is a bijection, and $\phi(V)$ is absorbing for T;
(d) $\phi(f(z)) = T(\phi(z))$, for $z \in G$.

In this situation we write $f \sim T$. We note that properties (b) and (d) imply that f is univalent in V.

Cowen proved that any self-map g of $\mathbb{D}$ whose Denjoy-Wolff point lies in $\partial\mathbb{D}$ has a conformal conjugacy in $\mathbb{D}$, and the Riemann map enables us to transfer this to any meromorphic function with a simply connected Baker domain. In [44, Theorem 1], König succeeded in generalising this result to some multiply connected Baker domains.

Theorem 5.1. *Let f be a meromorphic function with a finite number of poles, and with a p-periodic Baker domain U. Then there exists a conformal conjugacy of f^p in U, which is one of the following (mutually exclusive) types:*

(a) *$T(z) = z + 1$ and $\Omega = \mathbb{C}$;*
(b) *$T(z) = z \pm i$ and $\Omega = \mathbb{H}$;*
(c) *$T(z) = az$, for $a > 1$, and $\Omega = \mathbb{H}$.*

Case (a) is called *parabolic of type I*, case (b) is called *parabolic of type II*, and case (c) is called *hyperbolic*.

Moreover, König gave geometric criteria for these three cases, as follows [44, Theorem 3].

Theorem 5.2. *Let U be a p-periodic Baker domain of a meromorphic function f in which $f^{np} \to \infty$ and on which f^p has a conformal conjugacy. For $z_0 \in U$, put*

$$c_n = c_n(z_0) = \frac{|f^{(n+1)p}(z_0) - f^{np}(z_0)|}{\text{dist}(f^{np}(z_0), \partial U)}.$$

Then exactly one of the following cases holds:

(a) *$f^p \sim z + 1$, which is equivalent to*

$$\lim_{n\to\infty} c_n = 0, \quad \text{for } z_0 \in U;$$

(b) $f^p \sim z \pm i$, *which is equivalent to*

$$\liminf_{n\to\infty} c_n > 0 \quad \text{for } z_0 \in U, \text{ but } \inf_{z_0 \in U} \limsup_{n\to\infty} c_n = 0;$$

(c) $f^p \sim az$ *for some* $a > 1$, *which is equivalent to*

$$c_n > c \quad \text{for } z_0 \in U,\ n \in \mathbb{N}, \text{ where } c = c(f) > 0.$$

In case (a), there is at least one singular value of f in U.

These geometric criteria enable us to classify some of the Baker domains in Section 2. For example, it is easy to check that the function $f(z) = z+1+e^{-z}$, studied by Fatou, satisfies

$$\frac{f^{n+1}(z_0) - f^n(z_0)}{\operatorname{dist}(f^n(z_0), \partial U)} = O\left(\frac{1}{n}\right).$$

So this example is parabolic of type I, as are Examples 2.2, 2.6 and 2.10. On the other hand, Example 2.8 is parabolic of type II, and Examples 2.4 and 2.9 are hyperbolic; see [46] for a discussion of the stability of these three types.

In [16], Barański and Fagella considered invariant *univalent* Baker domains U; that is, invariant Baker domains of entire functions in which f is univalent. In this situation, the restriction of f to U is conjugate to a conformal mapping from $\mathbb{D}$ onto $\mathbb{D}$, and this implies that parabolic Baker domains of type I cannot occur. The paper [16] studies the remaining two cases and divides the hyperbolic univalent Baker domains into two types, depending on whether backward iterates of $f|_U$ approach a finite point of ∂U, called *hyperbolic of type I*, or approach ∞, called *hyperbolic of type II*. The Bergweiler function $f(z) = 2z + 2 - \log 2 - e^z$ is hyperbolic of type I, whereas Example 2.11 is hyperbolic of type II and Example 2.8 is parabolic (of type II).

The following necessary and sufficient condition from [16] is closely related to Theorems 3.2 and 5.2.

Theorem 5.3. *Let U be an invariant univalent Baker domain of an entire function f. Then U is hyperbolic if and only if*

$$\sup_{z\in U} \frac{\operatorname{dist}(z, \partial U)}{|f(z) - z|} < \infty.$$

Bergweiler [23] and independently Fagella and Henriksen [36] have studied the Baker domains of functions for which the conjugate self-map of $\mathbb{D}$ extends analytically to a neighbourhood of the Denjoy-Wolff point on $\partial\mathbb{D}$. Such Baker domains include those for which the restriction of f to U is a proper map (see [59] for results about proper maps), in which case f is conjugate to a finite Blaschke product on $\mathbb{D}$. A classification of such Baker domains expressed in terms of the hyperbolic metric is given in [23], and one in terms of the nature of the 'invariant petals' which exist in U is given in [36].

Finally in this section, we discuss some of the work that has been done to describe the geometric properties of the boundaries of Baker domains. This work began with the paper of Baker and Weinreich [15], which studied the nature of the boundary of *any* unbounded invariant Fatou component of a transcendental entire function f. They showed that in most cases such boundaries are highly irregular, in the sense that ∞ belongs to the impression of every prime end of U. In particular, if ∂U is a Jordan curve in $\hat{\mathbb{C}}$, then U must be a Baker domain and f must be univalent in U. Examples 2.8 and 2.9 show that the latter possibility can occur.

The key tool in this study of boundaries is the fact that if ψ is a conformal map from the unit disc $\mathbb{D}$ onto U, then $g = \psi^{-1} \circ f \circ \psi$ is an *inner function*, that is, an analytic self-map of $\mathbb{D}$ whose angular limits have modulus 1 almost everywhere on $\partial\mathbb{D}$. The paper [15] initiates a version of the Fatou-Julia theory for inner functions, a topic taken further in [18].

More precise information about such unbounded invariant components U can be obtained by studying the set

$$\Theta = \{e^{i\theta} : \psi(re^{i\theta}) \to \infty \text{ as } r \to 1\},$$

introduced by Kisaka [43]. The idea here is that each point in Θ corresponds to a different *access* to ∞ from within U. For a Baker domain U the set $\Theta \neq \emptyset$, by Theorem 3.1. In [9], Baker and Domínguez obtained the following result.

Theorem 5.4. *Let U be an invariant Baker domain of an entire function f such that f is not univalent in U. Then $\overline{\Theta}$ contains a perfect set.*

Thus, in this situation U has infinitely many accesses to ∞, so ∂U has infinitely many components. In [18] Bargmann further develops the Fatou-Julia theory for inner functions and proves that if U is a Baker domain of an entire function f, which is parabolic of type I, then $\overline{\Theta} = \partial\mathbb{D}$.

In [16] Barański and Fagella applied their classification of univalent Baker domains to show that univalent Baker domains which are hyperbolic of type II must have disconnected boundaries. However, it is an open question whether such Baker domains can have infinitely many accesses to ∞.

6. Functions with infinitely many Baker domains

In [52] various families of meromorphic functions were given, each function having infinitely many invariant Baker domains in which the dynamical behaviour near ∞ is analogous to that in Leau domains near a parabolic fixed point. Here we describe the results of [52] in a slightly different way which makes it easier to deduce similar results for further families of functions. First we describe the underlying dynamics near ∞ of functions analytic in a right half-plane with a certain asymptotic behaviour there. Roughly speaking, for these functions there are infinitely many strip-like invariant domains, which play the role of attracting petals with respect to ∞, and they are 'separated' by strip-like sets which play the role of repelling petals.

First we define

$$R(t,s) = \{z : \Re(z) \geq t, |\Im(z)| \leq s\}, \quad t > 0,\ s > 0,$$

$$T_k(\theta) = \{x + iy : x > t_k(\theta), |y - 2k\pi| < \min\{\theta, \pi/4 + (x - t_k(\theta))\cot\theta/2\}\},$$

where $0 < \theta < \pi$ and $t_k(\theta) > 0$ is to be chosen, and

$$S_k = \{x + iy : x \geq s_k, |y - (2k+1)\pi| \leq \pi/4\},$$

where $s_k > 0$ is to be chosen.

Theorem 6.1. *Let $f(z) = z + \phi(z)e^{-z}$ be analytic in $H = \{z : \Re(z) > x_0\}$, where $x_0 > 0$, and suppose that, for each $s > 0$,*

$$\text{(6.1)} \qquad \sup\{|\arg\phi(z)| : z \in R(t,s)\} \to 0 \text{ as } t \to \infty.$$

Then, for each $k \in \mathbb{Z}$ and $3\pi/4 < \theta < \pi$, and for $t_k(\theta) > 0$, $s_k > 0$ sufficiently large, we have

(a) *$f(T_k(\theta)) \subset T_k(\theta)$;*
(b) *if $z \in T_k(7\pi/8)$, then*

$$\Re(f^n(z)) \to \infty, \quad \Im(f^n(z)) - 2k\pi \to 0, \quad f^{n+1}(z) - f^n(z) \to 0$$

as $n \to \infty$;

(c) *there exist paths γ_k and Γ_k in S_k such that γ_k is bounded, Γ_k joins γ_k to ∞ and*

$$\text{if } z \in \Gamma_k, \text{ then } f^n(z) \in \gamma_k, \text{ for some } n \geq 0;$$

(d) *if $z \in S_{k-1} \cup S_k$, then $f^n(z) \notin S_{k-1} \cup S_k$, for some $n > 0$.*

We refer to the sets $T_k(\theta)$ with properties (a) and (b) as *attracting petals* and the sets S_k with properties (c) and (d) as *repelling petals*.

Remark If $f(z) = z + \phi(z)e^{bz}$, $b \neq 0$, is analytic in the half-plane $\rho(H)$, where $\rho(z) = -z/b$, and ϕ satisfies

$$\text{(6.2)} \qquad \sup\{|\arg\phi(z) - \theta| : z \in \rho(R(t,s))\} \to 0 \text{ as } t \to \infty,$$

then the change of variables $w = -bz - i(\theta + \arg(-b))$, gives a function of the form in Theorem 6.1.

The proofs of the various parts of Theorem 6.1 are all contained in the proof of [52, Theorem 1], which is similar to Theorem 6.1 but with a slightly less general formulation. In [52] the function f is assumed to be meromorphic in $\mathbb{C}$ but in Theorem 6.1 it only needs to be analytic in a half-plane, so the notion of a Baker domain of f may not be defined. Also, parts (c) and (d) were not included in the statement of [52, Theorem 1] but were part of the proof; see [52, Lemma 2.4]. These parts of Theorem 6.1 are expressed in a way that

is preserved under a univalent change of variable. This allows us to apply Theorem 6.1 to various types of meromorphic functions, after a change of variable if necessary, and conclude that they have infinitely many attracting and repelling petals. We can then deduce that each attracting petal lies in an invariant Baker domain by using the following lemma.

Lemma 6.1. *Let V_n, $n = 0, 1, \ldots$, be unbounded simply connected domains, each bounded by a simple curve tending to ∞ in both directions, such that*

$$\text{(6.3)} \qquad V_n \subset V_0, \quad \text{for } n = 1, 2, \ldots, \quad \text{and} \quad \bigcap_{n=0}^{\infty} \overline{V_n} = \emptyset,$$

and let f be analytic in V_0 such that

$$\text{(6.4)} \qquad f(V_n) \subset V_n, \quad \text{for } n = 0, 1, \ldots.$$

Then $f^n(z) \to \infty$ as $n \to \infty$, for $z \in V_0$. Thus, if f is also a transcendental meromorphic function, then V_0 is a subset of an invariant Baker domain of f.

Proof. Let ϕ be a Riemann map from the open unit disc $\mathbb{D}$ onto V_0. Since V_0 is a Jordan domain in $\hat{\mathbb{C}}$, the map ϕ extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{V_0} \cup \{\infty\}$; see [50, Theorem 2.6]. Now put $g = \phi^{-1} \circ f \circ \phi$. Then g is a self-map of $\mathbb{D}$, and so has a Denjoy-Wolff point $\zeta \in \overline{\mathbb{D}}$, such that $g^n(t) \to \zeta$ for $t \in \mathbb{D}$; see [31], for example. Hence $f^n(z) \to \phi(\zeta) \in \partial V_0 \cup \{\infty\}$ for $z \in V_0$. If $\phi(\zeta) \neq \infty$, then $\phi(\zeta) \in \overline{V_n}$ for all n, by (6.4), so we obtain a contradiction to (6.3). □

The question then arises as to whether each different attracting petal lies in a different Baker domain. To illustrate how this can be deduced from Theorem 6.1, we prove the following result from [52].

Theorem 6.2. *Let f be a meromorphic function such that*

$$\text{(6.5)} \qquad f(z) = z + az^k e^{-z}(1 + o(1)) \quad \text{as } \Re(z) \to \infty,$$

where $a > 0$. Then f has infinitely many distinct invariant Baker domains U_k, $k \in \mathbb{Z}$, such that, for each $k \in \mathbb{Z}$ and $3\pi/4 < \theta < \pi$, and for $t_k(\theta)$ sufficiently large:

(a) *$U_k \supset T_k(\theta)$;*
(b) *if $z \in U_k$, then $f^n(z)$ eventually lies in $T_k(7\pi/8)$, so*

$$\Re(f^n(z)) \to \infty, \quad \Im(f^n(z)) - 2k\pi \to 0, \quad f^{n+1}(z) - f^n(z) \to 0$$

as $n \to \infty$;

(c) *U_k contains at least one singularity of f^{-1}.*

Proof. It is clear that f satisfies (6.1), so we can apply Theorem 6.1 immediately, without any change of variable. By Theorem 6.1(a) and Lemma 6.1, each set $T_k(\theta)$, $3\pi/4 < \theta < \pi$, with $t_k(\theta)$ large enough, lies in an invariant Baker domain U_k.

We now prove part (b), which also shows that the Baker domains U_k, $k \in \mathbb{Z}$, are distinct. Suppose that $z \in U_k$. Choose $z' \in T_k = T_k(7\pi/8)$ and join z to z' by a path σ in U_k. Then $f^n \to \infty$ uniformly on σ. Thus if R is so large that

$$\gamma_{k-1} \cup L_k \cup \gamma_k \subset \{z : |z| \le R\}, \tag{6.6}$$

where γ_k are the paths defined in Theorem 6.1(c) and

$$L_k = \partial T_k \cap \{x + iy : |y - 2k\pi| < 7\pi/8\},$$

then there exists $N \in \mathbb{N}$ such that

$$f^n(\sigma) \subset \{z : |z| > R\}, \quad \text{for } n \ge N. \tag{6.7}$$

Now suppose that for some $n \ge N$, we have

$$f^n(z) \notin T_k \cup S_{k-1} \cup S_k.$$

Since $f^n(z') \in T_k(7\pi/8)$, we deduce by (6.6) and (6.7) that there is a point $w \in f^n(\sigma) \cap (\Gamma_{k-1} \cup \Gamma_k)$, where Γ_k are the paths defined in Theorem 6.1. Thus $f^m(w) \in \gamma_{k-1} \cup \gamma_k$ for some $m \ge 0$, by Theorem 6.1(c), which contradicts (6.7).

Hence $f^n(z) \in T_k \cup S_{k-1} \cup S_k$, for $n \ge N$, and it follows from Theorem 6.1(d) that $f^n(z)$ lies eventually in T_k, as required.

Finally, to prove part (c) we note that if U_k contains no singularity of f^{-1}, then $f : U_k \longrightarrow U_k$ is a univalent map. Thus, for $z \in U_k$ the hyperbolic distance between $f^{n+1}(z)$ and $f^n(z)$ remains constant, and this is contradicted by the fact that $f^{n+1}(z) - f^n(z) \to 0$ as $n \to \infty$ and $\operatorname{dist}(f^n(z), \partial U_k) > \pi/2$ for large n.

Alternatively, we can apply König's criterion, Theorem 5.2(a), to deduce that U_k is parabolic of type I and so U_k contains a singular value of f. □

Remark If

$$f(z) = z + az^k e^{bz}(1 + o(1)) \quad \text{as } \Re(-bz) \to \infty,$$

where $a, b \in \mathbb{C} \setminus \{0\}$, then the change of variable $w = -bz - i \arg(a/(-b)^{k-1})$, gives a function satisfying (6.5) with $a > 0$.

Next, we consider meromorphic functions of the form

$$f(z) = z + R(z)e^{Q(z)},$$

where R is rational and Q is a non-constant polynomial. Functions of this form were studied by Stallard [58], who showed that they do not have wandering domains. A key step in the proof of this result is to establish that all but a finite number of critical values of f lie in invariant domains of f. These invariant domains are constructed, after the change of variable $w = Q(z)$, in a similar way to the sets $T_k(\theta)$ in Theorem 6.1, and it is natural to expect therefore that f has infinitely many invariant Baker domains. Indeed, using Theorem 6.1 we can obtain the following result. The proof is closely related to that of [58, Theorem 2.1], but here we also need to prove that the invariant domains found there correspond to infinitely many *distinct* invariant Baker domains.

Theorem 6.3. *Let*

$$f(z) = z + R(z)e^{Q(z)},$$

where R is rational and Q is a polynomial with $\deg Q = m \ge 1$. Then f has m infinite families of invariant Baker domains, each family lying for large values of z in a sector of angle $2\pi/m$. Each such Baker domain is parabolic of type I and contains at least one critical value of f.

Proof. On substituting $z = cz'$, where c is a complex constant, we can assume that

$$f(z) = z + \lambda \frac{S(z)}{T(z)} e^{Q(z)},$$

where S, T and Q are monic polynomials and $\lambda \ne 0$. Write $w = Q(z)$ and introduce m inverse branches of Q, defined in $A = \{w : |w| > M, w \notin \mathbb{R}^+\}$ for M large enough, of the form

$$z = Q_j^{-1}(w) = w^{1/m} + c_0 + c_1 w^{-1/m} + \cdots, \quad 1 \le j \le m.$$

In the definition of $Q_j^{-1}(w)$ we take $\arg w^{1/m} \in (2\pi(j-1)/m, 2\pi j/m)$.

For $j = 1, \ldots, m$, we put $A_j = Q_j^{-1}(A)$, $B_j = Q_j^{-1}(A \cap \{w : \Re(w) < 0\})$. Note that ∂A_j is a simple curve which tends to ∞ in the two directions $\arg z = (2j-1)\pi/m \pm \pi/m$ and ∂B_j is a simple curve which tends to ∞ in the two directions $\arg z = (2j-1)\pi/m \pm \frac{1}{2}\pi/m$.

Now put $F_j = Q \circ f \circ Q_j^{-1}|_A$, $n = \deg S - \deg T$ and $k = m + n - 1$. Then

$$\begin{aligned} f(Q_j^{-1}(w)) &= Q_j^{-1}(w) + \lambda w^{n/m} e^w (1 + O(w^{-1/m})) \quad \text{as } w \to \infty \\ &= Q_j^{-1}(w) + o(1) \quad \text{as } \Re(w) \to -\infty,\ \Im(w) \text{ bounded.} \end{aligned} \tag{6.8}$$

Hence

$$\begin{aligned} F_j(w) &= Q\left(Q_j^{-1}(w) + \lambda w^{n/m} e^w (1 + O(w^{-1/m}))\right) \\ &= w + Q'(Q_j^{-1}(w)) \lambda w^{n/m} e^w (1 + O(w^{-1/m}))(1 + o(1)) \\ &= w + m w^{(m-1)/m} \lambda w^{n/m} e^w (1 + o(1)) \\ &= w + \lambda m w^{k/m} e^w (1 + o(1)) \quad \text{as } \Re(w) \to -\infty,\ \Im(w) \text{ bounded.} \end{aligned}$$

To justify the second line above we can either use the Taylor series for Q about $Q_j^{-1}(w)$ and apply Cauchy's estimate to the coefficients or use the fact that Q is univalent on the set A_j so, by the Koebe distortion theorem,

$$\frac{Q(\zeta) - Q(z)}{Q'(z)(\zeta - z)} = 1 + o(1),$$

uniformly as $\zeta \to z$, for $z \in B_j$; see [30, proof of Lemma 6.3] for a similar application.

Now the function $\phi_j(w) = \lambda m w^{k/m}$ is analytic in the left half-plane and satisfies (6.2) there, with $\theta = \arg\lambda + (2j-1)k\pi/m$. Thus we can apply Theorem 6.1 and the remark following it (with $b = 1$) to F_j. We deduce that F_j has infinite families of attracting and repelling petals in the left half-plane, of the form $T'_k(\theta) = \tau(T_k(\theta))$ and $S'_k = \tau(S_k)$, where $T_k(\theta)$ and S_k are attracting and repelling petals with the properties given in Theorem 6.1, and

$$\tau(w) = -w - i(\arg\lambda + (2j-1)k\pi/m + \pi).$$

For $j = 1, \dots, m$ and $k \in \mathbb{Z}$, we put $V_{j,k}(\theta) = Q_j^{-1}(T'_k(\theta))$ and $W_{j,k} = Q_j^{-1}(S'_k)$. Then

$$\text{(6.9)} \quad V_{j,k}(\theta) \cup W_{j,k} \subset B_j \subset A_j, \quad \text{for } j = 1, \dots, m,\ k \in \mathbb{Z},\ 3\pi/4 < \theta < \pi.$$

By (6.8) and (6.9), we can assume that $t_k(\theta)$ and s_k are chosen so large when applying Theorem 6.1 that

$$\text{(6.10)} \qquad \text{if } w \in \bigcup_{k\in\mathbb{Z}} (T'_k(\theta) \cup S'_k), \quad \text{then } f(Q_j^{-1}(w)) \in A_j.$$

We now claim that

$$f(V_{j,k}(\theta)) \subset V_{j,k}(\theta), \quad \text{for } j = 1, \dots, m,\ k \in \mathbb{Z},\ 3\pi/4 < \theta < \pi.$$

For if $z \in V_{j,k}(\theta)$, then $z = Q_j^{-1}(w)$, where $w \in T'_k(\theta)$, so $Q(f(z)) = F_j(w)$ is in $T'_k(\theta)$. Also, $f(z) \in A_j$ by (6.10), so $f(z) = Q_j^{-1}(Q(f(z)))$ is in $V_{j,k}(\theta)$, as required. Each such invariant domain $V_{j,k}(\theta)$ lies in a component $U_{j,k}$ of $F(f)$ and, by Theorem 6.1(a) and Lemma 6.1, each $U_{j,k}$ is an invariant Baker domain.

We can now prove that these Baker domains are distinct and that all orbits $f^n(z)$, $z \in U_{j,k}$, eventually land in $V_{j,k}(7\pi/8)$. For the function F_j, there are paths γ_k and Γ_k in the sets S_k with the properties stated in Theorem 6.1(c) and (d). The images under the univalent map $Q_j^{-1} \circ \tau$ of these paths and sets have similar properties, since $f(z) = Q_j^{-1}(Q(f(z))$ for $z \in W_{j,k} = Q_j^{-1}(S'_k)$, by (6.10), and these properties enable us to use a similar argument to that in the proof of Theorem 6.2(b); we omit the details.

Finally, we note that any orbit of F_j in one of the invariant domains $T'_j(7\pi/8)$ satisfies

$$\frac{F_j^{n+1}(w) - F_j^n(w)}{\operatorname{dist}(F_j^n(w), \partial T'_k(7\pi/8))} \to 0 \quad \text{as } n \to \infty.$$

Since the mapping $z = Q_j^{-1}(w)$ is univalent on A, it has uniformly bounded distortion on any square of side 4π in $\{w : \Re(w) < 0, |w| > 2M\}$. Thus we deduce that, for $z \in V_{j,k}(7\pi/8)$ and hence for $z \in U_{j,k}$, we have

$$\frac{f^{n+1}(z) - f^n(z)}{\operatorname{dist}(f^n(z), \partial U_{j,k})} \to 0 \quad \text{as } n \to \infty,$$

so $U_{j,k}$ is parabolic of type I, by König's criterion. Hence $U_{j,k}$ must contain a critical value of f, because f has no finite asymptotic values; see [58, page 218]. □

Remark It follows from the above proof that if Q is a monic polynomial, then for each j, $1 \le j \le m$, the Baker domains $U_{j,k}$, $k \in \mathbb{Z}$, are all asymptotic at ∞ to the ray $\{z : \arg z = (2j-1)\pi/m\}$, $j = 1, \dots, m$.

The following special case of Theorem 6.3 was also given in [52] (apart from a trivial change of variable).

Corollary 6.1. *For $p \in \mathbb{N}$, let $f(z) = z(1 + e^{z^p})$. Then*

(a) *the function f has infinitely many invariant Baker domains in each sector $\{z : 2\pi(j-1)/p < \arg z < 2\pi j/p\}$, $j = 1, \dots, p$;*
(b) *the function $g(z) = e^{2\pi i/p} f(z)$ has infinitely many p-cycles of Baker domains.*

Proof. Part (a) holds by Theorem 6.3 and the above remark, together with the fact that no Baker domain can meet $\{z : \arg z = 2\pi(j-1)/p\}$, $j = 1, \dots, p$, since these rays are invariant and $f^n(z)$ tends to infinity too rapidly on them, by Theorem 3.1. (In fact, these rays lie in $J(f)$, as can be seen by considering the fast rate at which the orbits of nearby pairs of points on these rays separate; see [55, Theorem 4].)

To obtain part (b), we use the fact that $g^p = f^p$, so $J(g) = J(f)$. □

Theorem 6.3 is related to an interesting question of Xavier Buff, namely, is it possible to find a transcendental entire function with only finitely many fixed points and no Baker domains? This question is related to results in the paper [30], where it is shown that if a transcendental entire function f has a logarithmic singularity over 0, then the Newton map $N_f(z) = z - f(z)/f'(z)$ has a Baker domain associated with that singularity; see [27] for related results.

Since any transcendental entire function of finite order with only finitely many fixed points must be of the form $f(z) = z + P(z)e^{Q(z)}$, where P and Q are polynomials with $\deg Q \ge 1$, Theorem 6.3 gives the answer 'yes' at least in this case. The general question, however, concerns a function of the form $f(z) = z + P(z)e^{g(z)}$, where g is any entire function. This question remains open. However, there are many examples of functions of this form with infinite order which have infinitely many Baker domains. The following example was mentioned in [52].

Corollary 6.2. *Let $f(z) = z + \exp(-e^z)$. Then the function f has infinitely many invariant Baker domains in each strip $\{z : |\Im(z) - 2\pi j| < \pi\}$, $j \in \mathbb{Z}$.*

Proof. We follow the approach in the proof of Theorem 6.3, but the details are simpler. Write $w = e^z$ and introduce inverse branches

$$z = L_j(w) = \log|w| + i\arg_j(w), \quad \text{where } (2j-1)\pi < \arg_j(w) < (2j+1)\pi,$$

defined on $A = \{w : |\arg w| < \pi\}$. Then consider $F_j(w) = \exp(f(L_j(w))) = w\exp(e^{-w})$. By Theorem 6.1, the function F_j has infinitely many attracting and repelling petals, $T_k(\theta)$ and S_k, $k \in \mathbb{Z}$, in the right half-plane (moreover, by Theorem 6.2, the function F_j has infinitely many invariant Baker domains).

Now put $V_{j,k}(\theta) = L_j(T_k(\theta))$. Then the sets $V_{j,k}(\theta)$, $k \in \mathbb{Z}$, are asymptotic in the right half-plane to the line $\Im(z) = 2\pi j$. Since $T_k(\theta)$ is invariant under F_j and

$$f(L_j(w)) = L_j(w) + e^{-w} = L_j(w) + o(1) \quad \text{as } \Re(w) \to \infty,$$

the set $V_{j,k}(\theta)$ is invariant under f. Hence, for each $k, j \in \mathbb{Z}$, the function f has an invariant Baker domain $U_{j,k}$ containing $V_{j,k}(\theta)$. We can then use a similar argument to that in the proof of Theorem 6.2(b), but here involving the sets $L_j(S_k)$ and their associated paths, to show that these Baker domains are distinct. Also, these Baker domains do not meet the lines $\Im(z) = (2j\pm1)\pi$, $j \in \mathbb{Z}$, since these lines are invariant and $f^n(z)$ tends to infinity too rapidly on them. This completes the proof. □

Similar reasoning (but using more techniques from the proof of Theorem 6.3) can be applied to other functions of the form $f(z) = z + R(z)e^{g(z)}$, whenever g has infinitely many inverse branches g_j^{-1}, $j \in \mathbb{Z}$, defined on a left half-plane $B = \{w : \Re(w) < -M\}$, where $M > 0$, and the inverse image sets $g_j^{-1}(B)$ are sufficiently large that the following property holds:

$$\text{if } z \in g_j^{-1}(B), \text{ then } f(z) \notin g_i^{-1}(B), \text{ where } i \neq j.$$

For example, the following result can be obtained by using this approach; we omit the details.

Corollary 6.3. *Let $f(z) = z + e^{\sin z}$. Then, for every $j \in \mathbb{Z}$, the function f has infinitely many invariant Baker domains asymptotic in the upper half-plane to the line $\Re(z) = 2\pi j - \pi/2$.*

Remark Bergweiler [25] has recently given an example of a transcendental meromorphic function which has no fixed points and no invariant Baker domains, thus providing a partial answer to Buff's question.

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