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DISTRIBUTION OF VALUES AND SINGULARITIES OF ANALYTIC FUNCTIONS

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1. Introduction

1. From the classical theorems of Weierstrass and Picard, the theory of the distribution of values and of the singularities of analytic functions has been enormously developed and generalized in various directions. In this survey lecture, I must, therefore, make very strong restrictions in order to include more detailed considerations. I shall treat exclusively meromorphic functions $f(z)$ which are *single-valued* in a domain of the complex *plane*. Even with this limitation, it is, of course, impossible to achieve completeness. In the following, I shall attempt to describe certain main features of the theory, emphasizing points which are either wholly open or only partially solved. Moreover, I shall deal with certain recent results and open questions which I personally have found interesting.

2. Nevanlinna Theory

2. Let us start with the classical assumption that $f(z)$ is non-constant and meromorphic for $|z|<R\leqq\infty$.

Value distribution theory can conveniently be built up by starting from the classical argument principle

$$\int\limits_{|z|=r} d \arg \frac{f(z)-a}{f(z)-\zeta}=2\pi(n(r, a)-n(r, \zeta)),$$

where $n(r, a)$ denotes the number of a-points, counted according to their multiplicity, in the disc of radius r.

Multiplication of both sides by $d\mu(\zeta)$, where μ is a set function completely additive with respect to the Borel subsets of an arbitrary closed set E, and subsequent integrations with respect to $\mu(\zeta)$ and r, yield (This technique is due to Frostman [8], cf. also [19].)

$$(1) \qquad \frac{1}{2\pi}\int_0^{2\pi} u(f(re^{i\varphi}))d\varphi-u(f(0))+N(r, a)\mu(E)=\int_E N(r, \zeta)d\mu(\zeta).$$

Here

$$u(w) = \int_E \log |w-\zeta| d\mu(\zeta) - \mu(E) \log |w-a| ,$$

and

$$N(r, a) = \int_0^r \frac{n(r, a)}{r} dr$$

denotes, as usual, the counting function.

The relation (1), which is simply an integrated form of the argument principle, provides an easy and unified approach to several important questions in the value distribution theory.

3. If the above set function μ is non-negative, i.e., if μ is a mass distribution, and if this distribution is not too irregular, the relation (1) is practically Nevanlinna's first Main Theorem. In particular, if E is the whole plane and $d\mu = dS$ the spherical element of area, with the normalization $S(E)=1$, (1) is the first Main Theorem in the spherically invariant form of Shimizu and Ahlfors ([27]),

$$m(r, a) + N(r, a) = T(r) . \tag{2}$$

Here the »Schmiegungsfunktion» $m(r, a)$ measures the approximation to the value a, and $T(r)$, the mean value of $N(r, a)$ with respect to the spherical metric, is the characteristic function of $f(z)$.

If $T(r)$ is unbounded, (2) expresses the invariance of the total »affinity» of $f(z)$ with respect to an arbitrary value a.

4. The fundamental relation (2) opens up an extensive complex of new problems. The immediate question is that of the relative magnitude of the components $m(r,a)$ and $N(r, a)$. This question is answered by Nevanlinna's second Main Theorem, which, in almost amazingly exact terms, states that in general $m(r, a)$ is asymptotically small compared to $N(r, a)$.

There exist to-day numerous proofs to this Theorem, but none offers an easy approach to the final result. I regard it appropriate to give here a brief account of the nature of the difficulties involved, as it is by no means impossible that certain improvements or modifications of the proof could still throw new light upon these highly investigated problems.

5. Making use of the representation

$$T(r) = \int N(r, \zeta) dS(\zeta) ,$$

we wish to show that, in general, $N(r, a)$ and the mean value $T(r)$ are of the same order of magnitude.

Having Picard's Theorem in mind, one starts by taking $q(\geqq 3)$ different complex numbers $a_1, a_2, \ldots, a_q$ and tries to compare the asymptotic behaviour of $\sum N(r, a_\nu)$ with that of $T(r)$. In order to get a common link between this sum and $T(r)$, it is natural to integrate $N(r, a)$ with respect to a suitably chosen auxiliary mass distribution. This mass must, on the one hand, be singular at the q points a_ν, while, on the other hand, it has to be sufficiently uniform so as to yield a mean value not much different from $T(r)$. F. Nevanlinna [26] was the first to point out that the hyperbolic metric of the complement of the points $w = a_\nu$ provides a mass distribution appropriate for this purpose.

Since $N(r, a)$ is subharmonic in a ([18]), its mean values with respect to fairly regular unit masses do not differ very much from each other. Especially, if dH denotes the element of area in the hyperbolic metric of the complement of the points a_ν, and if we write

$$T_H(r) = \int N(r, \zeta) dH(\zeta),$$

then, considering that the total mass is now equal to $\pi(q-2)/2$, it is not difficult to show (Virtanen [34]) that

$$T_H(r) \sim \frac{\pi}{2}(q-2)T(r). \tag{3}$$

6. The difficult part of the problem is to establish a relation between $T_H(r)$ and $\sum N(r, a_\nu)$. In [34], Virtanen has indicated that this can be made in three steps.

First, one gets an upper estimate of $T_H(r)$ by returning from the function plane to the z-plane and by applying at this stage the principle of hyperbolic measure. This yields

$$T_H(r) \leqq \int_{|z|<r} \log \frac{r}{|z|}\, dH^*(z), \tag{4}$$

where H^* refers to the hyperbolic metric of $|z|<R$, punctured at all points at which $f(z) = a_\nu$, $\nu = 1, 2, \ldots, q$.

After this, the majorant in (4) is transformed with the help of Green's formula. With respect to the singularities of the H^*-metric, this step gives

$$\int_{|z|<r} \log \frac{r}{|z|}\, dH^*(z) = \frac{\pi}{2} \sum_{\nu=1}^{q} \bar{N}(r, a_\nu) + A(r). \tag{5}$$

Here $\bar{N}$ is the counting function counting all a-points only once, irrespective of possible multiplicity, and $A(r)$ admits a simple representation in terms of the density of the H^*-metric.

The third and remaining step, in order to obtain the second Main Theorem, is the purely technical problem (See [27]) of showing that in (5), $A(r)$ in general has the character of a remainder term, i.e., that $A(r)$ is small compared to the sum $\sum \bar{N}(r, a_\nu)$. Considering (3), (4), and (5), we then obtain the desired result.

7. Provided that $f(z)$ takes at least one value a_ν in $|z|<R$, an inaccuracy in the above estimates is committed in step (4): on applying the principle of hyperbolic measure, equality holds if and only if $w=f(z)$ gives a one-one mapping of the universal covering surface of the punctured disc $|z|<R$ onto the universal covering surface of $w \neq a_1, a_2, \ldots, a_q$. Hence, the extremal function possesses logarithmic singularities in $|z|<R$, while the considered $f(z)$ is meromorphic there.

If the characteristic function of $f(z)$ is of sufficiently rapid growth, the inaccuracy in (4) is of no essential significance, i.e., $A(r)$ in (5) is of a lower order of magnitude than $T(r)$. As is well known, this is always the case for $R=\infty$, while for $R<\infty$, this is no longer true if

$$T(r)=O(-\log (R-r)). \tag{6}$$

8. It is certainly an interesting problem to try to establish relations, corresponding to the second Main Theorem, for meromorphic functions satisfying the condition (6). The starting point, of course, must be different, and the hyperbolic metric will no longer do as an auxiliary mass distribution. With the help of various extremal problems one can, however, readily find mass distributions which, like the hyperbolic measure, are conformally invariant, monotonic with respect to the domain, and satisfy a similar majorant principle (Cf. [2]). One can ask whether, after suitable modifications, appropriate mass distributions could provide a solution along the above lines.

Let it be remarked that a modification of the second Main Theorem which yields a more beautiful final form is not entirely improbable either.

9. One of the most striking consequences of the second Main Theorem is the so-called *deficiency relation* of Nevanlinna. In the customary notation

$$\delta(a) = 1 - \overline{\lim_{r \to R}} \frac{N(r, a)}{T(r)} \tag{7}$$

for the deficiency, it follows, excluding the exceptional case (6), that

$$\sum \delta(a) \leqq 2;$$

a far-reaching refinement of Picard's Theorem.

The deficiency relation gives immediately rise to the following *Inversion Problem:* Let there be given a sequence $a_1, a_2, \ldots$ of complex numbers and a sequence $\delta_1, \delta_2, \ldots$ of positive numbers $\leqq 1$ with $\sum \delta_\nu \leqq 2$. Does there exist a function $f(z)$, meromorphic for $z \neq \infty$[1]), such that $\delta_\nu(a) = \delta_\nu$ and $\delta(a) = 0$ for $a \neq a_\nu$? An affirmative answer would complement the value distribution theory in quite a beautiful manner.

The solution of the problem, which has been attacked by numerous mathematicians, has been sought by geometric methods, i.e., one has attempted to find the solution $f(z)$ by preassigning suitable ramification properties to the Riemann surface of its inverse function. Progress has gradually been made towards the affirmative solution of the problem, but to my knowledge, the general case is still open.

As for bibliography, early development is described in [27], and Wittich's monography [35] gives an account of the results attained up to 1955. In addition to the references in [35], attention must also be called to some remarkable recent results of Goldberg [9].

10. One sees almost immediately that there is a certain connection between the deficient values (i.e., values a for which $\delta(a) > 0$), the asymptotic behaviour with respect to these values near the boundary, and the structure of the Riemann surface of the inverse function. The problems concerning these mutual relations have given rise to most extensive studies.

If $f(z)$ omits a value, in the parabolic case $R = \infty$ this value is always asymptotic (Iversen [15]). One can ask about the situation if, instead of assuming that $f(z)$ omits the value, the value is only supposed to be deficient. Conversely, is an asymptotic value always deficient?

Both these questions can be answered in the negative. The latter case can easily be solved, and in the former case, counterexamples have been given by Teichmüller and Mme Schwartz for meromorphic functions and by Hayman for entire functions ([35]). Hence, some of the most striking results in this field are constituted by interesting counterexamples[2]).

As regards the results concerning the relations between $\delta(a)$ and the structure of the Riemann surface onto which $w = f(z)$ maps $|z| < R$, a somewhat similar situation is prevailing: In spite of much important work and interesting results obtained (For bibliography, see [27], [35], and Collingwood [4]), really general results in the positive direction scarcely exist.

This somewhat unfortunate situation derives from the fact that, in spite of its great naturalness, the very concept of deficiency also contains serious drawbacks, which are due to the special exhaustion process in the definition

[1]) In the hyperbolic case, the Inversion Problem has hitherto not been treated.
[2]) In the hyperbolic case, a function of unbounded characteristic may even omit a value, without this being asymptotic (Bagemihl and Seidel [3]).

(7). In addition to the above negative results, Dugué [6] has pointed out that $\delta(a)$ is not necessarily invariant with respect to so simple transformations as translations of the z-plane, i.e., that $f(z)$ and $f(z+c)$ do not necessarily possess the same deficiencies.

There is reason to ask whether a modification of the definition of deficiency could be introduced so as to avoid at least some of the above handicaps. In particular, is it possible to define it without any exhaustion process and yet essentially preserve its original meaning?

3. Functions of Bounded Characteristic

11. The basis of all the above considerations, Nevanlinna's first Main Theorem (2), loses its meaning if $T(r)=O(1)$. In order to develop a theory of value distribution in this case also, we return to the generalized argument principle (1) and apply it in the following manner: Let G be a domain in the function plane with a boundary of positive capacity (and hence possessing Green's functions g), and take μ in (1) as the harmonic measure at $w=a$ $(a \in G)$ of the boundary of G. The relation (1) then becomes

$$\Phi(r, a, G)+N(r, a)=P(r, a, G)\,, \tag{8}$$

where

$$\Phi(r, a, G)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{g}(f(re^{i\varphi}), a, G)d\varphi \qquad \left(\overset{+}{g}=\begin{cases}g, f\in G\\ 0, f\notin G\end{cases}\right)$$

and $P(r,a,G)$ is the least harmonic majorant of $N(r,a)$ in G. The relation (8) is an identity, holding for all meromorphic $f(z)$, whether of bounded characteristic or not.

12. We shall now specialize the situation by assuming that for $f(z)$, $T(r)=O(1)$. Requiring $f(z)$ to be non-constant implies that only the case $R<\infty$ comes into question.

On studying the value distribution of functions of bounded characteristic, it seems essential to take account of the boundary values. It is well known that the radial (or even angular) limits exist almost everywhere on $|z|=R$ and that in the function plane they constitute a set of positive capacity. In the following, Γ denotes the closure of the radial limits of $f(z)$ which correspond to a set on $|z|=R$ of measure $2\pi R$.

Provided that a Γ with a non-void complement exists, we now choose G in (8) to be an arbitrary domain in the exterior of Γ. Moreover, we proceed to the limit, $r\to R$; the condition $T(r)=O(1)$ guarantees that Φ, N, and P then tend separately to finite limits.

Under these hypotheses, we can prove, again with the help of the general relation (1) by choosing μ as the difference of two suitably chosen harmonic measures, that the »Schmiegungsfunktion» $\Phi(R, a, G)$ vanishes, except perhaps for a set of values a of capacity zero ([19]). Hence, up to such a set, $N(R, a)$ coincides with its least harmonic majorant in G. In particular, $f(z)$ either takes no value in G, or it takes all values there with the possible exception of the above null-set[1]).

The result $N(R, a) = P(R, a)$ »almost everywhere», which generalizes previous results on bounded functions with radial limits equal to 1 a.e., obtained by Seidel [32] and Frostman [8] (See also Noshiro [28], Ohtsuka [29], Tsuji [33], Lohwater [23]), can be regarded as a counterpart of the second Main Theorem. A further analogy is obtained if we introduce the natural definition of deficiency $\delta(a) = 1 - N(R, a)/P(R, a, G)$. From $\Phi(r, a, G) = m(r, a) + O(1)$ it follows, by (8), that $T(r) = P(r, a, G) + O(1)$. Hence, this definition is formally equivalent with the classical one.

In the particular case that all values of $f(z)$ lie in G (and hence, the radial limits a.e. lie on the boundary of G), $P(R,a,G) = g(a, f(0), G)$. Hence, in this case $P(R,a,G)$ is a domain function independent of $f(z)$, and the situation indicates a certain analogy with the first Main Theorem. It follows especially that for any two $f(z)$, satisfying these conditions, the counting functions coincide, except perhaps for a set of values of capacity zero.

The above results can be extended to far more general cases, whereby also new interesting phenomena are encountered. (Heins [12], Parreau [31]).

13. Exactly as in the Nevanlinna theory, the above results lead to problems concerning the relations between the value distribution, the boundary behaviour, and the structure of the Riemann surface of the inverse function.

An example of Frostman [8] shows that an asymptotic value is not always deficient. In contrast to the classical theory, the converse, however, is true: a deficient value is always asymptotic ([19]).

Following the classical example of Iversen, direct and indirect critical singularities can be defined on the Riemann surface of the inverse function. Direct critical singularities always correspond to deficient values, whereas this need not be the case for indirect critical singularities ([19]). Huckemann [14], who recently studied these questions, has shown, however, that also indirect critical singularities may give rise to deficient values.

1) If Γ is the closure of *all* radial limits, $f(z)$ either omits all values or at most one value in G ([20]). In a special case, Lohwater [24] has recently proved that this result holds also if Γ contains all radial limits except for those corresponding to a set of capacity zero.

4. Normal Functions

14. Except for the Nevanlinna theory, the behaviour of a meromorphic $f(z)$ near singular points admits other important characterizations. I mention in passing Ahlfors' elegant theory of covering surfaces with its applications and relations with Nevanlinna's theory. (See e.g. Ahlfors [1], Nevanlinna [27], Dinghas [5]). Instead, I shall deal rather with certain ideas based on Montel's concept of *normal families*.

As is well known, a family of analytic functions, uniformly bounded in a domain, is normal. By means of the modular function, this result can immediately be generalized: a family of meromorphic functions, omitting three fixed values, is also normal[1]). Roughly speaking, this is the fundamental result which is used on applying the theory of normal families to problems related to Picard's Theorem.

It is well known that many interesting results have been obtained in this direction, particularly by Julia [16], Ostrowski [30], and Montel [25]. These applications, however, seem not to have exhausted the power of the theory of normal families. I shall now illustrate briefly a new approach to the study of the singularities with the help of normal families.

15. Let us consider, for a moment, a function $f(z)$, meromorphic in an arbitrary simply-connected domain G. Denoting by $z'=S(z)$ a one-one conformal mapping of G onto itself, we consider, for fixed $f(z)$, all functions of the form $f(S(z))$.[2]) If this family is normal, $f(z)$ itself is said to be a *normal function* ([21]).

The definition is extended to multiply connected domains by calling $f(z)$ normal if $f(z)$ is normal on the corresponding universal covering surface. In multiply connected domains G, another form of normality can be defined as follows: $f(z)$ is said to be *weakly normal* in G, if it is normal in every simply connected subdomain of G. Normality always implies weak normality (cf. condition (9) below).

16. Because convergence is defined in the spherical metric, it is not very surprising that normal functions admit a simple characterization in terms of the spherical derivative

$$\varrho(f(z)) = \frac{|f'(z)|}{1+|f(z)|^2}.$$

As a matter of fact, a non-constant $f(z)$ is normal in G if and only if

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1) Convergence is defined with respect to the spherical metric.

2) For $|z|<1$, Hayman [11] has also considered normal families of the form $f(S(z))$, calling them uniformly normal.

(9) $$\frac{\varrho(f(z))}{\lambda(z)} = O(1),$$

$\lambda(z)|dz|$ denoting the element of length in the hyperbolic metric of G, which is necessarily of hyperbolic type ([21]).

17. Normal functions are involved in a natural manner in several considerations concerning the boundary behaviour of meromorphic functions ([21]). Here I shall restrict myself to two applications.

Let us first assume that $f(z)$ is meromorphic in a neighbourhood of the essential singularity $z=0$. By Picard's Theorem, $f(z)$ takes all values in this neighbourhood, except at most two. But can $f(z)$ be normal there?

In [21], it was proved that this is impossible. Hence, it follows from the condition (9) by a simple computation that

$$\varrho(f(z)) \neq O\left(\frac{1}{|z|\,\log\dfrac{1}{|z|}}\right).$$

One is thus led to the following question ([22]): How rapidly at least does $\varrho(f(z))$ grow in the neighbourhood of the singularity $z=0$?[1])

This question admits a precise answer. In fact, for all meromorphic $f(z)$ with $z=0$ as an isolated essential singularity,

(10) $$\overline{\lim_{z\to 0}}\; |z|\varrho(f(z)) \geqq \tfrac{1}{2}.$$

The bound $\frac{1}{2}$ is sharp, i.e., there exist meromorphic functions for which (10) holds as an equality.

The inequality (10) can be quite briefly established by using the following device. Remembering that two points w_1 and w_2 are diametrically opposite on the Riemann sphere if $w_1\overline{w}_2=-1$, we construct the meromorphic function $F(z)=f(z)\overline{f}(\overline{z})$ (or $f(ze^{i\vartheta})\overline{f}(\overline{z})$, $\vartheta>0$, if $z=0$ does not happen to be a singular point for $F(z)$), and consider the behaviour of $F(z)$ with respect to the value $w=-1$. If $F(z)$ takes the value -1 infinitely often in the neighbourhood of $z=0$, it follows from the above that there exist in every neighbourhood of $z=0$ circles $|z|=$const. with points z such that $w=f(\overline{z})$ and $w=f(z)$ are diametrically opposite on the Riemann sphere. Hence, the length L of the images of such circles on the Riemann sphere satisfies the inequality $L\geqq\pi$. On the other hand,

$$L \leqq \int\limits_{|z|=\text{const.}} \varrho(f(z))\,|dz| \leqq 2\pi|z|\max\varrho(f(z)).$$

Combining these two inequalities, we obtain (10).

[1]) If $f(z)$ is meromorphic at $z=0$, then of course $\varrho(f(z))=O(1)$.

If $F(z)$ does not take the value -1 infinitely often around $z=0$, it approximates it, by Weierstrass' Theorem, and (10) follows as above.

To prove that (10) is sharp, one has to construct an example for which the equality holds. For instance,

$$f(z)=\prod_{\nu=0}^{\infty}\frac{z-e^{-\nu^{\lambda}}}{z+e^{-\nu^{\lambda}}}\,,\ \lambda>1\,,$$

provides such an example. Because $f(-z)=1/f(z)$ and $\varrho(f)=\varrho(1/f)$, this function must only be considered in the right half-plane, where $|f(z)|<1$. Points near the real axis are handled with the help of Schwarz's Lemma, while some computations cannot be avoided in dealing with the remaining part of the half-plane.

18. An interesting class of meromorphic functions with slow growth is distinguished by the property

$$\overline{\lim_{z\to 0}}\ |z|\,\varrho(f(z))<\infty\,. \tag{11}$$

In [22], we proved that this is exactly the class of functions weakly normal in a neighbourhood of $z=0$. For the study of these functions, this is often quite a useful characterization. It follows almost immediately, e.g., that these functions cannot possess any asymptotic values.

This function class has already been thoroughly studied. Applying a criterion of Ostrowski [30], we see that (11) is nothing but a necessary and sufficient condition for a function $f(z)$ to be *exceptional* in the sense of Julia.

Several questions treated by Julia and Ostrowski can thus be viewed from a new angle. For example, if $f(z)$ is not exceptional, $f(z)$ cannot be normal in all »angles» with vertex at $z=0$, and the result concerning Julia directions follows. If more than $\overline{\lim}_{z\to 0}\ |z|\,\varrho(f(z))=\infty$ is known about the growth of $\varrho(f(z))$, this result can be sharpened in an obvious manner.

19. As another example of the applicability of the theory of normal functions, I present here a theorem for meromorphic functions which can be said to correspond the Phragmén-Lindelöf theorem for regular functions.

Let us consider a function $f(z)$, meromorphic in a simply-connected domain G bounded by a Jordan curve. Furthermore, let us suppose that on the boundary, $|f(z)|\leq A$, with the possible exception of one point P.

The classical Phragmén-Lindelöf Theorem states that if $f(z)$ is *regular*, then either $|f(z)|\leq A$ holds throughout in G, or the maximum modulus of $f(z)$ in the neighbourhood of P tends to infinity with a certain minimal rapidity determined by the geometric configuration.

If $f(z)$ is *meromorphic* in G, the following somewhat analogous situation is prevailing[1]): Either $|f(z)| \leqq A$ holds in the whole domain, in which case

$$\frac{\varrho(f(z))}{\lambda(z)} \leqq A,$$

or

$$\sup \frac{\varrho(f(z))}{\lambda(z)} \geqq B_A.$$

The constant B_A is obtained as the unique solution of the equation

$$(1+\sqrt{1+x^2})e^{-\sqrt{1+x^2}} = Ax,$$

and is thus independent of the configuration. The bounds are sharp.

Let us remark that for $A \to 0$, $B_A \to \infty$. In particular, if A is smaller than the universal constant $A_0 = 0.796\ldots$, determined as the root of the equation

$$(1+\sqrt{1+x^2})e^{-\sqrt{1+x^2}} = x^2,$$

then $B_A > A$.

5. General Domains of Existence

20. Hitherto, we have considered $f(z)$ either in a simply-connected domain or in the neighbourhood of an isolated essential singularity. In the case that $f(z)$ is meromorphic in the whole plane, except for n points $z_1, z_2, \ldots, z_n$, Dugué [7] has established an interesting extension of Picard's Theorem. Clearly, $f(z)$ can omit two values in the vicinity of each singularity. These values, however, are not independent: at most $n+1$ of them can be different. Dugué also poses the question whether *deficient* values belonging to different singularities are dependent on each other in some manner.

21. In the general case that the essential singularities of $f(z)$ constitute an arbitrary closed set E, relatively little is known about the value distribution of $f(z)$. In spite of important work by af Hällström [10], many Japanese mathematicians (for bibliography, see Kuroda [17]), Hervé [13], and others, several fundamental questions are still quite open.

One does not know, for instance, whether closed sets E without isolated points exist in whose complement Picard's Theorem is valid, i.e., every $f(z)$ omits at most two values. A problem which is presumably closely related is to characterize sets E whose complements tolerate normal functions.

1) In a slightly different formulation, this result was proved in [21] for G being the half-plane. The general case follows by conformal mapping.

If E is of capacity zero, it is known that $f(z)$ cannot omit a set of values of positive capacity. It is not known, however, whether this result is sharp. Conjectures have been expressed that the set omitted is at most countable or even that Picard's Theorem is valid.

It is not until E is positive in the sense of Painlevé's problem that the existence of functions omitting large sets of values is known.

I have expressly chosen these examples as closing remarks to this lecture, in order to point out that there are still several unsolved, interesting, concrete problems in the theory of value distribution and singularities in the complex plane.

References

[1] Ahlfors, L.: Zur Theorie der Überlagerungsflächen. - Acta Math. 65 (1935).

[2] Ahlfors, L. and Beurling, A.: Conformal invariants and function-theoretic null-sets. - Acta Math. 83 (1950).

[3] Bagemihl, F. et Seidel, W.: Valeurs évitées, mais non asymptotiques, pour les fonctions holomorphes dans le cercle-unité. - C. R. Acad. Sci. Paris 241 (1955).

[4] Collingwood, E. F.: Exceptional values of meromorphic functions. - Trans. Amer. Math. Soc. 66 (1949).

[5] Dinghas, A.: Zu Nevanlinna's zweitem Fundamentalsatz in der Theorie der meromorphen Funktionen. - Ann. Acad. Sci. Fenn. 151 (1953).

[6] Dugué, D.: Le défaut au sens de M. Nevanlinna dépend de l'origine choisie. - C. R. Acad. Sci. Paris 225 (1947).

[7] —»— Vers un théorème de Picard global. - Ann. Sci. Ecole Norm. Sup. 69 (1952).

[8] Frostman, O.: Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. - Meddel. Lunds Univ. Mat. Sem. 3 (1935).

[9] Goldberg, A. A.: On defects of meromorphic functions. - Dokl. Akad. Nauk S.S.S.R. 98 (1954). (Russian.)

[10] af Hällström, G.: Über meromorphe Funktionen mit mehrfach zusammenhängenden Existenzgebieten. - Acta Acad. Aboensis 12 (1940).

[11] Hayman, W. K.: Uniformly normal families. - Lectures on Functions of a complex variable, The University of Michigan Press, Ann Arbor (1955).

[12] Heins, M.: Lindelöfian maps. - Annals of Math. 62 (1955).

[13] Hervé, M.: Contribution à l'étude d'une fonction méromorphe au voisinage d'un ensemble singulier de capacité nulle. - J. Math. Pures Appl. 35 (1956).

[14] Huckemann, F.: Über den Defekt von mittelbaren Randstellen auf beschränktartigen Riemannschen Flächen. - Proc. International Colloquium on the Theory of Functions, Ann. Acad. Sci. Fenn. 250 (1958).

[15] Iversen, F.: Recherches sur les fonctions inverses des fonctions méromorphes. - Thèse, Helsingfors (1914).

[16] Julia, G.: Leçons sur les fonctions uniformes à point singulier essentiel isolé. - Collection Borel, Paris (1924).

[17] Kuroda, T.: On the uniform meromorphic functions with the set of capacity zero of essential singularities. - Tohoku Math. J. 3 (1951).

[18] Lehto, O.: A majorant principle in the theory of functions. - Math. Scand. 1 (1953)

[19] —»— Value distribution and boundary behaviour of a function of bounded characteristic and the Riemann surface of its inverse function. - Ann. Acad. Sci. Fenn. 177 (1954).

[20] —»— Boundary theorems for analytic functions. - Ann. Acad. Sci. Fenn. 196 (1954).

[21] Lehto, O. and Virtanen, K. I.: Boundary behaviour and normal meromorphic functions. - Acta Math. 97 (1957).

[22] —»— —»— On the behaviour of meromorphic functions in the neighbourhood of an isolated singularity. - Ann. Acad. Sci. Fenn. 240 (1957).

[23] Lohwater, A. J.: The boundary values of a class of meromorphic functions. - Duke Math. J. 19 (1952).

[24] —»— The boundary behavior of meromorphic functions. - Proc. International Colloquium on the Theory of Functions, Ann. Acad. Sci. Fenn. 250 (1958).

[25] Montel, P.: Leçons sur les familles normales de fonctions analytiques et leurs applications. - Gauthier-Villars, Paris (1927).

[26] Nevanlinna, F.: Über die Anwendung einer Klasse von uniformisierenden Transzendenten zur Untersuchung der Wertverteilung analytischer Funktionen. - Acta Math. 50 (1927).

[27] Nevanlinna, R.: Eindeutige analytische Funktionen. - Springer-Verlag, Berlin (1936 u. 1953).

[28] Noshiro, K.: Contributions to the theory of singularities of analytic functions. - Jap. J. Math. 19 (1948).

[29] Ohtsuka, M.: On the cluster sets of analytic functions in a Jordan domain. - J. Math. Soc. Japan 2 (1950).

[30] Ostrowski, A.: Über Folgen analytischer Funktionen und einige Verschärfungen des Picardschen Satzes. - Math. Zeitschr. 24 (1924).

[31] Parreau, M.: Fonction caractéristique d'une application conforme. - Ann. Fac. Sci. Univ. Toulouse 19 (1956).

[32] Seidel, W.: On the distribution of values of bounded analytic functions. - Trans. Amer. Math. Soc. 36 (1934).

[33] Tsuji, M.: Fundamental theorems in potential theory. - J. Math. Soc. Japan 4 (1952).

[34] Virtanen, K. I.: Eine Bemerkung über die Anwendung hyperbolischer Massbestimmungen in der Wertverteilungslehre der meromorphen Funktionen. Math. Scand. 1 (1953).

[35] Wittich, H.: Neuere Untersuchungen über eindeutige analytische Funktionen. - Springer-Verlag, Berlin (1955).

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