CHAPTER 1

Guiding Principles

1.0. Chapter Overview

Good teaching is a product of preparation, effort, knowledge, insight, and a good attitude on the part of the instructor. If you treat your teaching as analogous to having root canal treatments, then it's a certainty that you will not enjoy it. And your students probably will not either.

This chapter discusses broad issues connected with the teaching process: respect, preparation, time management, voice control, and so forth. The ideas presented here are ones that you will want to internalize. You want them to be a natural part of your teaching habit.

1.1. Respect

You cannot be a good teacher if you do not respect yourself. If you are going to stand up in front of thirty people or three hundred people and try to teach them something, then you had better

- Believe that you are well qualified to do so.
- Want to do so.
- Be *prepared* to do so.
- Make sure that these characteristics are evident to your audience.

It is a privilege to stand before a group of people—whether they be young adults or your own peers—and to share your thoughts with them. You should acknowledge this privilege by **(a)** dressing appropriately for the occasion, **(b)** making an effort to communicate with your audience, **(c)** respecting the audience's point of view.

One completely obvious fact is that you should have your material *absolutely mastered* before you enter the classroom.[1] If you *do* possess this mastery, then you can expend the majority of your effort and attention on conveying the material to the audience. If, instead, you have a proof or an example that is not quite right, and if you stand in front of the group trying to fix it, then you will lose all but the diehards quickly. (Note that this last statement is true both when you are giving a calculus lecture and when you are giving a colloquium talk at Harvard.)

When I was a novice teacher I often spent only a few minutes preparing my lectures. I would go to class and wing it. And, as you might predict, I made mistakes. Now that I am sliding into my dotage, I in fact prepare every class *very carefully*. I write out complete notes. And now I hardly ever make mistakes in class.

[1] I am going to emphasize this point throughout the book—almost *ad nauseum*. After more than forty years of teaching, I can tell you confidently that this above all else is the key to success in the classroom. The students need to know that you are in control of the situation. That you are on top of the material. That you can answer any question. They want to believe in you and trust you. So you *must* know what you are talking about.

On the rare occasions when I do, I can make the class feel like it's "our" mistake, fix it quickly, and move on. This gives the student the very strong impression that I am in complete control of the situation. The fact that I can *always* answer *all of* their questions (I only rarely have to say, "let me get back to you on that one") only serves to reinforce that impression.[2] And that is what I want.

It is easy to rationalize that, if the students were more able, then they could roll with the ups and downs of your lecture. This is strictly illogical. How do you behave when you are listening to a colloquium or seminar and the lecturer goes off into orbit—either to fix an incorrect argument or into a private conversation with his buddy in the front row or, worse, into a private conversation with himself? All right then, now that you have admitted honestly how *you* behave, then how can you expect unseasoned freshmen to be tolerant when you do not seem to be able to do the examples that *they* are expected to do? How about more seasoned sophomores?

One of the best arguments for even elementary college mathematics courses to be taught by people with advanced degrees is this: Because the material is all trivial and obvious to the professor, he/she can maintain a broad sense of perspective, he/she will not be thrown by questions, and he/she can concentrate on the act of *teaching.*

If you respect yourself then, it follows logically, you will respect your audience. You should prepare for your class. That way you will not be surprised by gaps in your thinking, you will not have to cast around for a necessary idea, and you will not lose your train of thought in class. You will be receptive to questions. You will sense immediately when the students are not understanding, and you will do something about it.

Throughout this book, I will repeatedly exhort you to prepare your classes. I do not necessarily have in mind that you should spend an inordinate amount of *time* preparing. Consider by analogy the psychology of sport. Weightlifters, for example, are taught to meditate in a certain fashion before a big lift. Likewise, preparing is a way to collect your thoughts and put yourself in the proper frame of mind to give a class. You might prepare by walking to the student union and buying a cup of coffee (thinking on the way about your lecture). Or you might prepare by browsing through some calculus books (or even—horror of horrors—browsing through the *actual text* for that particular class). I have sometimes prepared for a class by staring bleakly out the window pondering the notion that I have nothing to say. No matter—this is a yoga. It helps you to pull yourself together. It makes good sense. See also Section 1.2.

To me, preparation is the core of effective teaching. While this may sound like a tautology, and not worth developing, there is in fact more here than meets the eye. Just as being a bit organized relieves you of the stress and nuisance of spending hours looking for a postage stamp or a pair of scissors when you need them, so being the master of your subject gives you the ability to cope with the unexpected, to handle questions creatively, and to give proper stimulus to your students. An experienced and knowledgeable teacher who is comfortable with his/her craft is constantly adjusting the lecture, in real time, to suit the expressions on the students' faces, to suit their responses to queries and prods, to suit the rate and thoroughness

[2] It will happen very occasionally that a student will ask (for instance) for a counterexample which, even if you know it off the top of your head, is too recondite to present in class. Then the thing to do is to tell the student to see you after class.

with which they are absorbing the material. Just as a good driver (of a car) is constantly (and unconsciously) making little adjustments in steering in response to road conditions, to other drivers, to weather, and so forth, it is also the case that a good teacher (just as unconsciously) is engaging in a delicate give and take with the audience. Complete mastery is the unique tool that gives you the freedom to develop this skill.

You should treat questions with respect. I go into every class that I teach knowing full well that I am probably much smarter, and certainly much better informed, than most of the people in the room. But I do not need to use a room full of eighteen-year-olds as a vehicle for bolstering my ego. If a student asks a question, even a stupid one, then I treat it as an event. A wrong question can be turned into a good one with a simple turn of phrase from the instructor (see Section 1.5). If the question requires a lengthy answer, then give a short one and encourage the student to see you after class. If you insult—even gently—the questioner, then you not only offend that person but you perhaps offend everyone else in the room. Once the students have turned hostile it is difficult to win them back, both on that day and on subsequent days.[3]

In fact one of the most dynamic and inspiring things that you can do for your students is to help them think that their questions are steering the class. Even though students know in their heart of hearts that you are the boss, and you are the person with all the knowledge, they deeply resent the notion that you are the high priest dispensing wisdom. It is so much more stimulating to respond to a student question by saying, "Yes, that was my next point ..." or "You are anticipating the ideas. That's very good ..." or "You have raised a crucial issue. That's great. Let me pursue it for a moment ..."

Sometimes you can actually develop an idea in dialogue with a student—right in front of the class. For example, you say, "This point is not a boundary point. Why not?" The student replies, "Some neighborhoods have no points from the complement." "Good," you say. "So, in a certain sense, this point is *away* from the complement of the set. Instead we might say it's ..." "Interior?" interjects the student. How could a student *not* pay attention if he/she thinks that his/her thoughts, his/her questions, have shaped the discussion?

1.2. Prepare

Some people rationalize not teaching well by saying (either to themselves or to others), "My time is too valuable. I am not going to spend it preparing my calculus lecture. I am so smart that I can just walk into the classroom and wing it. And the students will benefit from watching a mathematician think on his feet." (As a student, I actually had professors who announced this nonsense to the class on a regular basis. And, as you can imagine, these were professors who royally botched up their lectures on a regular basis.)

It is true that most of us can walk into the room most of the time and mostly wing it. But most of us will not be very successful if we do so. Thirty minutes can be

[3] One of the legendary teachers of twentieth century mathematics was Walter Rudin. He began every class by telling the students that questions are important. Questions are wonderful. He liked all questions. There was no such thing as a dumb question. You should never hesitate to ask a question in my class. On one particular first day, after the lecture had been going on for a while, a student timidly raised his hand and posed a question. Walter looked him in the eye and said, "That's the stupidest question that I've ever heard in my life."

sufficient time for an experienced instructor to prepare a calculus lecture. A novice instructor, especially one teaching an unfamiliar subject for the first time, may need considerably more preparation time.[4] Make sure that you have the definitions and theorems straight. Read through the examples to make sure that there are no unpleasant surprises. It is a good idea to have a single page of notes containing the key points and also briefly listing your motivating ideas. To write out every word that you will say, write out a separate page of anticipated questions, have auxiliary pages of extra examples, have inspirational quotes drawn from the works of Thomas Carlyle, make up a new notational system, make up your own exotic examples, and so forth, is primarily an exercise in self-abuse. Over-preparation can actually stultify a lecture or a class. But you've got to know your stuff.

I cannot emphasize too strongly the fact that preparation is of utmost importance if you are going to deliver a stimulating class. However it is also true that the more you prepare the more you lose your spontaneity. You must strike a balance between **(i)** knowing the material cold and **(ii)** being able to "talk things through" with your audience.

My own experience is that there is a "right amount" of preparation that is suitable for each type of course. I want to be confident that I'm not going to screw up in the middle of lecture. But I also want to be actually thinking the ideas through as I present them. I want to feel that my lecture or class has an edge. It *is* possible to over-prepare. To continue to prepare after you have already prepared sufficiently is a bit like hitting yourself in the head with a hammer because it feels so good when you stop.

You must be sufficiently confident that you can field questions on the fly, can modify your lecture (again on the fly) to suit circumstances, can tolerate a diversion to address a point that has been raised. I always try to give the class the impression that I am more than willing to take time to address any question or issue. That I am *not* in a rush. That I really care about addressing their concerns and thoughts.

The ability to do these things well is largely a product of experience. But you can *cultivate* this ability too. You cannot learn to play the piano by accident. And you will not learn to teach well by accident. You must be aware—in detail—of what it is that you are trying to master and then consciously hone that skill.

If you do not prepare—I mean *really* do not prepare—and louse up two or three classes in a row, then you will experience one or more of the following consequences: **(i)** Students will take up your time after class and during your office hour (in order to complain and ask questions), **(ii)** Students will stop coming to class, **(iii)** Students will complain to the undergraduate director and to the chair, **(iv)** Students will (if you are *really bad*) complain to the Dean and write letters to the student newspaper, **(v)** Students will write bad teaching evaluations for your course. Now student teaching evaluations are not gospel (see Section 2.8). They contain some remarks that are of value and some that are not. Getting bad teaching evaluations does not necessarily mean that you did a bad job. And I know that the Dean will only slap me on the wrist if he gets a complaint about my teaching (however, if there are ten complaints, then I had better look out). Finally, I know that the chair will give me the benefit of the doubt and allow me every opportunity to put any

[4]A couple of years ago I taught an upper division course in graph theory. And I am *not* a graph theorist by any stretch of the imagination—even though I wrote a paper with Erdős about graph theory. I often had to spend a couple or more hours preparing any given lecture.

difficult situation in perspective. But if I spend thirty minutes preparing each of my classes then I will avoid all this grief and, in general, find the teaching experience pleasurable rather than painful. What could be simpler?

As well as preparing for a class, you would be wise to debrief yourself after class. Ask yourself how it went. Were you sufficiently well prepared? Did you handle questions well? Did you present that difficult concept as clearly as you had hoped? Did you draw that hyperboloid of one sheet clearly and efficiently? Was there room for improvement? Be as tough on yourself as you would be after any exercise that you genuinely care about—from playing the piano to engaging in a tennis match. It will result in real improvement in your teaching.

Read your teaching evaluations (Section 2.8). Many are insipid. Others are puerile. Most, however, are thoughtful and well-meant. If ten of your students say that your writing is unclear, or that you do too many calculations in your head, or that you talk too quickly, or that you are impatient with questions, then maybe there is a problem that you should address.[5] Teaching is a yoga. Your mantra is "Am I getting through to them?" Or it could be "Am I giving them something to take away?"

It is a good idea to try to anticipate questions that students will ask. But you cannot do this artificially, as a platonic exercise late at night over a cup of coffee. It comes with experience. Assuming that you have adopted the attitude that you actually care whether your students learn something, then after some experience of teaching you will know by instinct what points are confusing and why. This instinct enables you to prepare a cogent lecture—to know what to emphasize, where to slow down, where to provide extra examples. It helps you to be receptive to student questions. It helps you to have a good attitude in the classroom.

An easy way to cut down on your preparation time for a class is to present examples straight out of the book. The weak students will appreciate this repetition. Most students will not, and you will probably be criticized for this policy. On the other hand, it is rather tricky to make up good examples of maximum-minimum problems or graphing problems or applications of Stokes's theorem. It can be time-consuming as well. If you need more examples for your calculus class, then pick up another calculus book and borrow some. Develop a file of examples that you can dip into each time you teach calculus. You will learn quickly that making up your own examples is hard work. Do you ever wonder why most calculus books are so disappointing? All right, *you* try to make up eight good examples (with algebra that works out nicely) to illustrate the divergence theorem.

1.3. Speak Up

If you are going to be a successful teacher, then you have to find a way to fill the room with yourself. If you stand in front of the class (be it a class of ten or a class of a hundred) and just mumble and look at your shoes,[6] then you will not successfully convey the information. Even the most dedicated students will have

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[5] A fact of life these days is the Web site *Rate My Professors* at `www.ratemyprofessors.com`. Going there, you can look up yourself or most any other professor in the country. You may be amused and surprised to see what sort of teachers many of your friends and colleagues are. You also may be amused or surprised or humiliated to see what sort of teacher you are. And this information is out there for everyone to see!!

[6] An introverted mathematician is one who looks at his/her shoes when he/she talks to you. An extroverted mathematician is one who looks at *your* shoes when he/she talks to you.

trouble paying attention. You will not have stimulated anyone to think critically. In fact you will have lost your audience.

I must stress that the way you comport yourself in *your* classroom should be consistent with your personality and your way of doing things. But you still need to get through to those kids. You will want to spend some time figuring out how to make this work.

You do not need to be a showoff or a ham or a joke teller to fill a room with your presence. You can be dignified and reserved and old-fashioned and still be a successful instructor with today's students. But you must let the students know that you are there. You must establish eye contact. You must let them know that you are *talking to them.*

Before I start a class session, especially with a large group of students, I engage some of them in informal conversation. I get them to talk about themselves. I ask them how they are doing on the homework assignment. I comment about the weather, or about something that is going on around campus. Then I make a smooth transition into the more formal class activities. That way I already have half a dozen people on my side. The others soon follow.

Some new instructors—especially those who are naturally soft-spoken or shy—may need some practice with voice modulation and projection. If you are such a person, then get together a group of friends and give a practice lecture for them. Ask for their criticism. Make a tape recording of your practice lecture and listen to it critically.

If there is any doubt in your mind as to whether you are reaching your audience during a particular class, then *ask* about it. Say "Can you hear me? Am I talking loud enough? Are there any questions?" This is one of many simple devices for changing the pace of a class, giving note-takers a break, allowing students to wake up.

Think of a good movie that you've seen recently. Now remove the changes of scenery; remove the voice modulation and changes of emotion; remove the changes in focal length; remove the skillful use of silence as a counterpoint to sound; remove the musical soundtrack. What would remain? Could you stay awake during a showing of what is left of this movie? Now think about your class in these terms.

1.4. Lectures

In an empty room sits a violin.

One person walks in, picks it up, draws the bow across the strings, and a horrible screeching results. He leaves in bewilderment.

A second person walks in, attempts to play, and the notes are all off key.

A third player picks up the instrument and produces heavenly sounds that bring tears to the eyes. He is Isaac Stern and the instrument is a Stradivarius.

Wouldn't it have been foolish to say, after hearing the first two players, that this instrument is outmoded, that it doesn't work? That it should be abandoned to the scrap heap? Yet this is what many are saying today about the method of teaching mathematics with lectures. Citing statistics that students are not learning calculus sufficiently well, or in sufficiently large numbers, government-sponsored projects nationwide assert that the lecture doesn't work, that we need new teaching techniques.

Whether you like lectures or not, we have to face some facts here. Most of us don't lecture very well. After all, when were we taught to lecture, and in what forum? How many excellent lecturers do you know? Where did they learn to lecture?

OK, so we admit it: The lecture doesn't work very well because most of us aren't very good at it. The trouble is that most of us aren't very good at any other method of teaching either. We are not skilled at any of these methods because we have received no training in them, and because we have not given them careful thought. While one possible solution to the problem is "The lecture is dead so let's move on to something else," another possible solution is "Let's learn how to lecture." See also Section 4.2 on classroom learning vs. Online learning.

Those who say that "the use of the lecture as an educational device is outmoded" rationalize their stance, at least in part, by noting that we are dealing with a generation raised on television and computers. Our students have spent a good deal of their lives with `Twitter` and `Google` and `Hulu`. They argue that today's students are too ready to fall into the passive mode when confronted with a television-like environment. The kids simply don't have the patience or the attention span to sit through a lecture. It follows that we must teach them interactively, or in groups, or using cooperative learning. Perhaps we should use computers and software to bring students to life.

Lectures have been used to good effect for more than 3000 years. I am hesitant to abandon them in favor of a technology (personal computers, videos) that has existed for just thirty years. In spite of popular rumors to the contrary, a lecture does not need to be a bone-dry desultory disquisition. It can have wit, erudition, and sparkle. It can arouse curiosity, inform, and amuse. It is a powerful teaching device that has stood the test of time.[7] The ability to give a good lecture is a valuable art, and one that you should cultivate.

However, you really have to work at making your lectures reach your students. It is true that mathematics teaching in this country is not, overall, a great success. The reason is not that the lecture method is "broken." Rather, we tend not to put a lot of effort into our teaching because the value-reward system is often not set up to encourage putting a lot of effort into it. Many of us spend semester after semester facing down rooms full of calculus students who are there only because the course is required for their major. And it is not required for their major because their department wants them to know calculus. Rather, it is required for their major because their department hopes that half of them will flunk. It is a sorry situation. And it is easy for us, the underpaid and overworked faculty, to become demoralized. Certainly "The lecture is dead." is one way to rationalize an already dreary reality.

Of course many of us content ourselves with internal rewards, with the sense of satisfaction in doing a good job. No matter what rewards you seek, you must identify and learn to use the tools that will make you an effective teacher. You must learn to develop eye contact with your audience, to fill the room with your voice and your presence, and to present your ideas with enthusiasm and clarity. Other sections of this book discuss these techniques in detail.

Turn on your television and watch a self-help program, or a television evangelist, or a get-rich-quick real estate huckster. These people are not using overhead

[7] And it goes without saying that an Online course will have none of these attributes.

projectors, or computer simulations, or `Mathematica`. (Incidentally, they are also not using group learning or self-discovery!) In their own way they are lecturing, and *very powerfully*. They can persuade people to donate money, to change religions, or to join their cause. Of course your calculus lecture should not literally emulate the methods of any of these television personalities. But these people and their methods are living proof that the lecture is not dead, and that the traditional techniques of Aristotelian rhetoric are as relevant and effective as ever. By watching even a charlatan do his/her stuff, you can learn something about how to engage an audience, how to answer questions, how to interact with people.[8]

I like to startle people by telling them that I have refined my teaching technique by watching David Letterman (of late night television fame). This is not an exaggeration. Letterman is a master at communication, at dealing with many different types of people, at taking a situation that is going sour and turning it around. *I am not talking here about telling jokes.* I am instead talking about skill at handling people. At taking a conversation which is going dead and bringing it to life. At taking a shy, reticent person and making him/her wake up and contribute.

My friend Glenn Schober was once teaching a class to help train graduate students to teach. For the first day he carefully crafted a lecture on elementary mathematics in which he purposely made 25 cardinal teaching errors. He walked into class on the first day, told the students he was going to give a sample lecture, and did so. At the end he said, "There were a number of important teaching errors that I intentionally committed in this lecture. See how many you can identify." The students found 32 errors.

This story is amusing, but it is also an important object lesson. There are two kinds of "important" in this world—your kind and my kind. Any classroom situation has a promulgator and an audience. If the lecturer and the audience work together, and share the same goals, then they will reach the same place at the same time, with mutual satisfaction as the result. If not, then they will be working at cross purposes, with mutual frustration as an outcome. These statements apply no matter *what* teaching method you are using. The key to success is that you must *communicate* with your audience.

There are other useful teaching environments besides lectures. Although less common in mathematics than in some humanities courses, group discussions can be useful. If you want to get students interested in what the boundary of a set in a metric space ought to be, then you can begin with a discussion in which students offer various suggestions. Before you define what a finite set is, ask the students to suggest a definition.

It is not difficult to see that putting a student with a group of four or five peers, so that they can conduct an intimate, one-on-one discussion of a mathematical topic, is bound to generate student interest. It also may tend to help timid students to open up, and to engage in communication. But we must acknowledge that educational activities that, in effect, make the classroom the venue where all learning takes place use time differently than learning activities that make the

[8]Some of the most popular and successful lecturers today are sports coaches or military officers who give standing-room-only motivational presentations about self-esteem and motivation and success. They often have the audience—usually consisting of mature adults who have solid, middle-class lives—weeping openly because they are so moved. Can you imagine having such an effect with your calculus lecture? If you have ever seen one of these people in action, as I have, you will be quite impressed. They really know something about how to connect with an audience.

student's dorm room (or the library) the place where learning takes place. To be specific, the old-fashioned paradigm for student learning was that the student would sit in class for an hour listening to a lecture and taking notes. Then he/she would go home and spend three to five hours deciphering the notes, filling in the gaps, and doing the homework. The new paradigm has the student learning *right now*, either before a computer screen or interacting with a group of other students. The reform teacher makes sure that students are engaged in the learning process because he/she actually *manages the engagement*. The traditional teacher leaves more of the responsibility for engagement with the student himself/herself.[9]

To repeat, many of the reform methods use time in a different, and less familiar, manner than do the traditional (lecture) methods (see also Section 3.6). When you are giving a lecture class, you know just what is going on during the allotted class time, and you also know what is *supposed* to be going on during out-of-class study time. If you instead teach with group work, self-discovery, computer labs, `Mathematica` notebooks, and other new devices, then you must be retrained as an instructor. You must learn anew how to monitor what is going on in your class and how to evaluate student progress.

Reformers, among them Ed Dubinsky, assert that they work their students much harder than do the traditionalists—and that the students are so fulfilled by the learning process that they are glad to do the work. Since reform teachers have many more student contact hours than do traditional teachers, they are probably well qualified to make this assertion (see [**ASI**] for more on this idea). The jury is still out on the question of whether students taught with reform methods or students taught with traditional (lecture) methods derive the most from their education. Which students learn more? Which retain more? Which have greater self-esteem? Which feel more empowered? Which have greater interest in the learning process? Which teaching method encourages more students to become math majors? How many students taught by the reform methods go on to graduate school and become professional mathematicians? Frankly, we don't know.

Of course it is as unfair to group "reformers" together as it is to group "liberals" or "anti-vivisectionists" together. I have heard reformers say in a public venue that, in the reform environment, only half as much material can be presented to the students (as is/was presented in the traditional lecture environment). At the same time, they assert that that's about as much as the students ever learned anyway. So nothing is really lost. Traditionalists, remembering the lectures they attended and the way that they worked when they were students, might bridle at this reasoning. A traditionalist will watch a reform class, perceive unmanaged use of time, and conclude that less learning is taking place. The reformer will argue that his/her students are working harder and internalizing more.

Clearly we must weigh any teaching method according to both its merits and its demerits. Traditional techniques (lectures) can have the effect on today's students of making learning and erudition seem to be dry, dusty, uninteresting, and

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[9]It has been pointed out (by Steven Zucker among others) that the main difference between high school learning and college learning is this: In high school, learning takes place at the moment of impact, in the classroom, and the primary responsibility for learning lies with the teacher. In college, learning takes place (primarily) outside the classroom, and the primary responsibility for learning lies with the student. It is because students do not understand this distinction, and how to deal with it, that they have such a struggle with the first year of college. They simply do not know how to study.

irrelevant. However, they are efficient at promulgating a great deal of information. Some of the new techniques are terrific at getting students involved in the material, at making the ideas come alive, and perhaps at aiding student retention. But these methods use time in new ways, and it seems possible that less material will be taught when they are used. Do check out Section 4.5 on the flipped classroom, which suggests a new reform-style teaching method that does use time effectively and well.

It has been natural, in this section on the lecture, to digress about reform. For one of the battle cries of the reform movement has been that "The lecture is dead." Let us now conclude this section by mentioning some teaching devices that should be of interest both to reformers and to traditionalists.

A casual reading of the present discussion might mislead the reader into thinking that lectures cover more material but are boring while reform methods cover less material but are more engaging. There is more to it than that. Lectures do not need to be boring. They can be engaging, fascinating, and even exciting. Reform methods do not necessarily *have to* cover less material, but they can easily do so if they are not carefully managed. A smart math teacher will learn from both points of view, and craft his/her teaching methodology accordingly.

It can be instructive to have students volunteer to do problems at the blackboard. Once in a great while—when I am lecturing—if a student offers an alternative proof of a proposition or another point of view, I hand him/her the chalk. Everyone is usually quite surprised, but the results are generally pleasing and it provides a nice change of pace.

Computer labs (Section 4.6) can also be a useful instructional device. The subject of sophomore-level differential equations lends itself well to helping students explore the interface between what we can do by hand and what the machine can teach us. Let's be frank. We do not know how to solve most differential equations explicitly, or in closed form. Thus it is important for students to see how much analysis one can do with traditional methodologies and then to see how the machine can use phase plane analysis, numerical methods, and graphing to provide further concrete data.

We should continue to seek new and better methods and technologies for teaching. This author, and this book, has a built-in bias toward traditional methods, such as lectures. That is because he has watched them work and used them successfully for more than forty years. I hope that other writings will describe and explore some of the new teaching techniques. The new Chapter 4 of this edition of *How to Teach Mathematics* explores various uses of the Internet inside and outside the classroom.

1.5. Questions

In a programmed learning environment, whether the interface is with a PC or with `Mathematica` notebooks or with a MAC, the student cannot ask questions. The give and take of questions and answers is a critical aspect of the human part of the teaching process. Teachers are *supposed* to answer questions.

There is more to this than meets the eye. When I say that a teacher answers questions I do not envision the student saying, "What is the area of a circle?" and the teacher saying "πr^2." I instead envision the student struggling to articulate some confusion and the experienced teacher turning this angst into a cogent question and then answering it. To do this well requires experience and practice. I frequently find myself responding to a student by saying, "Let's set your question aside for a minute and consider the following." I then put the student at ease by quickly running through something that I know the student knows cold, and that serves as a setup for answering the original question. With the student on my side, I can answer the primary problem successfully. The point is that some questions are so ill-posed that they literally cannot be answered. It is the teacher's job to make the question an answerable one and then to answer it.[10]

A similar, but alternative, scenario is one in which the student asks a rather garbled question and I respond by saying, "Let me play the question back for you in my own words and then try to answer it . . . ." The point is that the responses "Your question makes no sense" or "I don't know what you mean" are both insulting and a cop-out. To be sure, it is the easy answer. But you will pay for it later. It takes some courage for the student to ask a question in class, in front of his/her peers. By treating questions with respect, you are both acknowledging this fact and helping someone to learn. If instead you stare at the student as though he has weevils in his eyebrows, then you will gain no allies and will most likely lose several friends and make a few enemies.

Yet another encouraging response to a student question is to say, "Thank you. That question leads naturally to our next topic . . . " Of course you must be quick on your feet in order to be able to pull this off. It is worth the trouble. Students respond well when they are treated as equals—see the discussion of this technique at the end of Section 1.1.

There are complex issues involved here. A teacher does not just lecture and answer questions. A good teacher helps students to discover the ideas. There are few things more stimulating and rewarding than a class in which the students are anticipating the ideas because of seeds that you have planted. The way that you construct your lecture and your course is one device for planting those seeds. The way that you answer questions is another.

When I discuss teaching with a colleague who has become thoroughly disenchanted with the process, I frequently hear complaints of the following sort: "Students these days are impossible. The questions that they pose are unanswerable. Suppose, for example, that I am doing a problem with three components. I end up writing certain fractions with the number 3 in the denominator. Some student will ask 'Do we always put a 3 in the denominator when doing a problem from this section?' How am I supposed to answer a question like that?"

Agreed, it is not obvious how to answer such a question, since the person asking it either **(i)** has not understood the discussion, **(ii)** has not been listening, or **(iii)** has no aptitude for the subject matter. It is tempting to vent your spleen against the student asking such a question. Do not do so. The student asking this question probably needs some real help with analytical thinking, and you cannot give the required private tutorial in the middle of a class hour. But you can provide guidance. Say something like "When a problem has three components it is logical

[10]It is almost always a good idea to repeat the student question and/or to write it on the board. Because a lot of people in the room will not have heard the question, or will not have understood the question. So you just answering it cold will omit much of the class from the discussion. You must take control of the situation.

that factors of 1/3 will come up. This can happen with certain problems in this section, or in any section. But it would be wrong to make generalizations and to say that this is what we do in all problems. If you would like to discuss this further, please see me after class." In a way, you are making the best of a bad situation. But at least you are doing something constructive, and providing an avenue for further help if the student needs it.

The Dalai Lama once visited the headquarters of *Time* magazine in Chicago. He was given the chef's tour, and then there was a grand formal lunch at which the various executives of the enterprise pontificated *ad nauseum*. The Dalai Lama—an elfin man—sat swathed in his saffron robe, an inscrutable smile on his face, saying nothing. After about an hour, the CEO of *Time* turned to the Dalai Lama and said, "Do you have any questions about *Time*, the nation's premiere news magazine? Go ahead, ask us anything at all." The Dalai Lama bowed his head for a moment, apparently deep in thought. Then he looked up and said, "Why do you publish it?"

We mathematicians are very much like the executives of *Time* in that story. We are wrapped up in our own world, we all speak the same language, and we suffer intruders with pained resignation. Sadly, our students are like intruders. They come to us with questions we would never have dreamed of, often expressed in language that is obscure at best, and they expect answers. They do not speak our language, and they do not necessarily respect our mores. Yet it is our job to talk to our students, to engage them in discourse, to answer their questions. We must exercise patience in order to gain their trust. And we must try to speak to them in their language, rather than in our own.

Let us consider some further illustrations of the principle of making a silk purse from a sow's ear—that is, answering unexpected or awkward questions in a constructive manner. The first example is a simple one.

> **Q:** Why isn't the product rule $(f \cdot g)' = f' \cdot g'$?

The answer is *not* "Here is the correct statement of the product rule and here is the proof." Consider instead how much more receptive students will be to this answer:

> **A:** Leibniz, one of the fathers of calculus, thought that this is what the product rule should be. He recorded this thought in his diary. Ten days later, he gave the correct form—with a proof and the cryptic statement that he had known this to be the correct form "for some time." Because we have the language of functions, we can see quickly that Leibniz's first idea for the product rule could not be correct. If we set $f(x) = x^2$ and $g(x) = x$ then we can see rather quickly that $(f \cdot g)'$ and $f' \cdot g'$ are unequal. So the simple answer to your question is that the product rule that you suggest gives the wrong answer. Instead, the rule $(f \cdot g)' = f' \cdot g + g' \cdot f$ gives the *right* answer and can be verified mathematically.

The second example is more subtle.

> **Q:** Why don't we divide vectors in three-space?

The *wrong* answer is to tell about Stiefel-Whitney classes and that the only Euclidean spaces with a division ring structure are $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^4$, and $\mathbb{R}^8$. A better answer is as follows.

> **A:** J. Willard Gibbs invented vectors to model physical forces. There is no sensible physical interpretation of "division" of physical forces. The nearest thing would be the operations of projection and cross product, which we will learn about later.

Notice that in both illustrations an attempt is made to turn the question into more than what it is—to make the questioner feel that he has made a contribution to the discussion.

> **Q:** Why isn't the concept of velocity in two and three dimensions a number, just like it is in one dimension?

If you are in a bad mood, you will be tempted to think that this person has been dreaming for the past hour (or the past week!) and has understood absolutely nothing that you have been saying. Bear up. Resist the temptation to voice your frustrations. Instead try this:

> **A:** Let me rephrase your question. Instead let's ask, "Why don't we use vectors in one dimension to represent velocity just as we do in two and three dimensions?" One of the most important features of vector language is that a vector has *direction* as well as *magnitude*. In one dimension there are only two directions—right and left. We can represent those two directions rather easily with either a plus sign or a minus sign. Thus positive velocity represents motion from left to right and negative velocity represents motion from right to left. The vector language is *implicit* in the way that we do calculus in one dimension, but we need not articulate it because positivity and negativity are adequate to express the directions of motion.
>
> In dimensions two and higher there are infinitely many different directions and we therefore require the explicit use of vectors to express velocity.

As the author of this book, I have the luxury of being able to sit back and drink coffee and think carefully about how to formulate these "ideal" answers to poor questions. When you are actually teaching you must be able to do this on your feet, either during your office hour or in front of a class. At first you will not be so articulate. This is an acquired skill. But it is one *worth acquiring*. It is a device for showing respect for your audience, and in turn winning *its* respect.

Large lectures pose special problems with the issue of student questions. Obviously you cannot let each student ask his/her little question. You cannot let your lecture get bogged down with questions like "How do you do problem 6?" or "Will this be on the test?" See Section 2.13 for a discursive consideration of questions in the large lecture context.

A final note about questions. Even though you are an authority in your field, there are certainly things that you do not know. Occasionally these lacunae in your knowledge will be showcased by a question asked in class or during your office hour (it does not happen often, so don't get chills). The sure and important attribute of an intelligent, educated individual is an ability to say, "I don't know the answer to that question. Let me think about it and tell you next time." On the (rare) occasions when you have to say this, be sure to follow through. If the item that you don't know is an integral part of the class—and this had better not be the case very often—get it down cold because the question is liable to come up again in a different guise later in the course. If it is not an integral part of the course, then you have no reason to feel bad. Just get it straight and report back.

The main point is that you should never, under any circumstances, try to fake it. If you do, then you will look bad, your interlocutor will be frustrated and annoyed, and you will have served no good purpose. If there is any circumstance in which honesty is the best policy, this is it. Professor of Economic History Jonathan R. T. Hughes was wise to observe that "There is no substitute for knowing what you are talking about."

1.6. Time

There are several aspects of teaching that require time management skills. When you are giving a lecture, you must cover a certain amount of material in the allotted time—and at a reasonable rate. When you give a course, you must cover a certain amount of material in one semester or term. When you give an exam, it must be doable by an average student in the given time slot. When you answer a question, the length of the answer should suit the occasion.

All of these topics will be touched upon in other parts of this book. They require some thought, and some practice and experience, so that they become second nature to you.

Nobody can design a class so that (as in the movies) the last 'QED' is being written on the blackboard just as the bell rings. There are certain precepts to follow in this regard:

- Have some extra material prepared to fill up extra time.
- If you finish your lesson with five minutes to spare, don't rocket into a new topic. You will have to repeat it all next time anyway, and students find this practice confusing. You can use the spare five minutes to summarize what you have covered that day, or to give the students a "thought" problem to take home, or to just tell them to have a nice day. That is a much more gracious way to end the class.
- If the clock shows that just five minutes remain, and you have ten or fifteen minutes of material left to present, then you will have to find a comfortable place to quit. Don't race to fit all the material into the remaining time. At the same time—if possible—don't just stop abruptly and in mid-thought, thinking that you can pick up a calculation cold in the following class.

  An experienced teacher will know which will be the last example or topic in the hour, and that he/she might get caught for time. Therefore the instructor will plan in advance for this eventuality and think of several junctures at which he might bring the hour to a graceful close. With enough experience, you will know intuitively how to identify the comfortable places to stop; thus end-of-the-hour time management problems can be handled on the fly. In particular, if five minutes remain then *do not begin* a ten-minute example!
- If you prepare (the last part of) your lesson in units of five minutes' duration, and if you are on the ball, then you should never have to run over by more than two minutes nor finish more than two minutes ahead of time. (The idea here is if there are three minutes remaining, then you

can include another five-minute chunk without running over by more than two minutes. If there are just two minutes remaining, then you should stop.)

- *If you run out of time*, do not keep lecturing past the end of the period—at least not by more than a minute or two. Students have other classes to attend, and they will not be listening. If the time is gone, then just quit. Make up for your lapse in the next class (this will require some careful planning on your part). Best is to plan your class so that you do not get caught short of time.

A special note about buzzers: Some math buildings have a loud buzzer or bell that sounds at the end of the class period.[11] Once that buzzer sounds, all is lost. Most students will instantly start packing up their books and heading for the exit. If there is a clock on the wall, then you will know when the buzzer is going to sound, and you can be adjusting your lecture and zeroing in on a suitable conclusion as you go along. At a school without a buzzer (especially one that also has no clocks in its classrooms!), you have a bit of slack since no two wristwatches are in agreement. You may want to interpret the advice in this section according to the physical environment in which you are teaching.

Some students have the annoying habit of setting the alarms on their electronic watches to chime precisely on each hour. Worse, some students set their alarms for five minutes before the hour or three minutes before the hour or two minutes after the hour. Since no two watches agree, what you experience at the end of class is a cacophony of electronic beeping. This distracts the students, it disrupts your class, and it is a damned nuisance. Tell the students in advance—on the first day of class—that you want all electronic alarms turned off during your class. Be stern about it, and nail those who fail to comply.

- If a student asks a question that requires a long answer, don't let your answer eat up valuable class time. Tell the student that the question is a good one, but ancillary to the main subject matter of the course (it had better be, or else you evidently forgot to cover an important topic) and that the question can best be treated after class. However, do not let the student get the impression that the question is being given the brush-off.
- On the other hand, if a student asks a question for which a brief answer is appropriate (such as "Shouldn't that 2 be a 3?" or "When is the next homework assignment due?") then do give a suitably brief answer. Anecdotes about your childhood in Shropshire are probably out of place.

By the way, this last is more than a frivolous remark. As we slide into our golden years, we seem to be irretrievably moved to share with our students various remembrances of things past—"It seems to me that twenty years ago students worked much harder than you people are willing to work." or "When I was a student, we put in five hours of study for each hour of class time." or "I used to

[11] I once taught in a classroom in which there was a buzzer that went off three minutes before the end of the class period, and another that went off at the end of the class period. This was just the pits. Because, as soon as the first buzzer went off, students started packing up their books and turning off their brains. It was really counterproductive.

walk six miles barefoot through the snow to attend calculus class, and it was uphill both ways." Trust me: Students hate this sort of emotional slobbering. You will defeat all the other good things that you do by giving in to this temptation to prattle.

If you have the time problem under control at the level of individual class meetings, then you will have the ability to pace your course in the large as well. You should have a good idea how much material you want to cover. And when you plan the course you should allot a certain number of class periods for each topic. If you are teaching undergraduates, they depend on your course for learning a certain body of material (that may be prerequisite for a later course). Don't shortchange them.

A test should be designed for the allotted time slot. You can rationalize giving a two-hour exam in a one-hour time slot by saying to yourself that there is so much material in the course that you simply *had* to make the test this long. This is nonsense. The point of the exam is not to *actually test* the students on every single point in the course, but to make the students *think* that they are being tested on every point in the course. Ideally, the students will study everything—but your test amounts to a spot check. Even if you had a four-hour time slot in which to give the exam, you couldn't really test them on everything, now could you? See Section 2.9 on exams.

If you give a two-hour exam in a one-hour time slot, then you run several risks: that students will become angry, demoralized, alienated, or all three. Telling a student not to worry about his/her grade of 37/100 because the average was 32/100 does not work. Students are unable to put such information into perspective.

1.7. Applications

One of the most chilling things that can happen to an unprepared, unseasoned faculty member is to have a belligerent (engineering) student raise his/her hand and say, "What is all this stuff good for?" And one of the most irresponsible things that a faculty member can say in response is, "I don't know. That is not my problem." If you do not have an answer for this student question, then you are not doing your job.

I have found it useful in all of my undergraduate classes to tell the students about applications of the techniques being presented *before* the aforesaid chilling question ever comes up. This requires a little imagination. If I am lecturing about matrix theory, then I tell the students about Markov processes, or I say a little bit about image processing and data compression and the fast Fourier transform. Or I tell them about eigenvalue asymptotics for clamped beams and applications to the building of a space station (not coincidentally, this is a problem that I have worked on in my research). If I am lecturing about surfaces then I tell them about the many applications of surface design problems. If I am lecturing about uniform continuity or uniform convergence, then I tell them about some of the applications of Fourier analysis. The pedagogical technique that I am describing in effect defuses any potential belligerence from engineering or other students who have no patience for mathematical abstraction.

Carrying out this teaching technique requires a little forethought and a little practice. After a while it becomes second nature, and you will find yourself thinking of potential answers while taking a shower or walking to class. If it suits your style,