Making Waves

"You know, Kara, we're coming to the part of the Calculus course I hate most," remarked Louis as they hovered around the tea and cookies at the department Friday tea. "I've always dreaded having to cover Fourier series in this introductory class."

"There are plenty of parts of this course I find kind of tedious," replied Kara, "but I actually look forward to Fourier series. It's such a beautiful part of math, and it's something the students really haven't ever seen before – unlike, say, derivatives, which they all had in some form or other in high school."

"Sure, Fourier series are beautiful, but I don't see how they are appropriate at this level. Oh, we can explain the formulas to them, and they can calculate the first few Fourier coefficients of some simple functions, but we can't even begin to put the theory in the proper mathematical framework."

"What do you mean, proper mathematical framework?" Kara said, grinning. "It's not like we put derivatives in their proper mathematical framework. We try to illustrate what they're good for, and give some feel for how to calculate with them. Surely we can do the same with Fourier series."

"I don't know about that" said Louis. "To me, Fourier expansions are a nice example of orthogonal decomposition in Hilbert space. If you can't apply linear algebra to the picture, then I hardly see why you should bring them up at all. It's not like they have the practical applications that more basic Calculus does."

"You're kidding, right?" said Kara.

"What do you mean?"

"Fourier series have billions of practical applications. I mean, how can you understand anything about waves of any sort without Fourier series? Fourier wanted to understand heat conduction – that's why Fourier series were invented! Not to mention quantum mechanics. Surely you point this out to your students."

Louis felt somewhat affronted. He had the distinct impression that Kara was laughing at him.

"Look, I learned about Fourier series in the first year grad course on Analysis, in the context of Hilbert spaces. Fourier expansions made perfect sense to me. I believe you that the theory has applications, but I don't have the time or the training to get involved in learning them. It's different for you, maybe – you're interested in applied mathematics – but I'm working in low-dimensional topology, and quantum mechanics just isn't relevant." Even as Louis said this, he could feel that he'd come out sounding snooty, which he really hadn't intended.

"Whatever. Look, Louis, I'm not telling you to become a physicist. But I really do think you're doing your class a disservice by not giving some physical motivation for Fourier series. Plus, never mind your Calc class, you're missing a lot of interesting stuff yourself."

"OK, Kara, maybe you're right. But how exactly am I supposed to come up with this motivating material by Monday?"

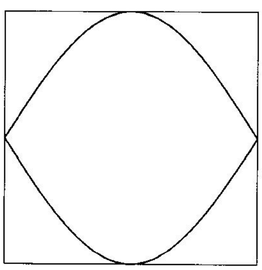
"Why not stop by my class at 9 a.m.? I'm going to give my little motivating speech right at the beginning of class. Maybe that will give you some idea of what you can do at your class."

"That sounds reasonable...I don't teach until 2 p.m. OK, I'll be there Monday morning at 9."

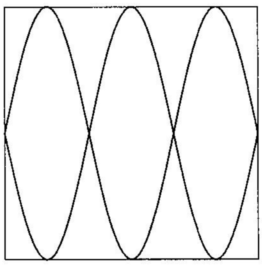
Monday Morning

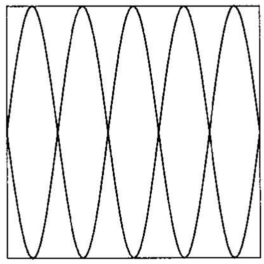
Louis slipped into the back of Kara's 9 a.m. Calculus class and tried to look inconspicuous. Sitting in the back put him among the sleepers and lost souls who congregate in the back of Calculus lectures. He was relieved when Kara ignored him and started her introduction.

"Good morning. Today we're starting a new topic, Fourier series." she said. "Fourier series are a crucial mathematical tool for the study of vibrations and waves. To give you some idea of what I mean, let's look for a moment at a guitar string stretched between two fixed points. When the guitar string vibrates, it produces a tone which we hear as a musical note. However, the guitar string can vibrate in many different ways. For example, it can vibrate up and down with just one hump:



This makes a sound which we call the "fundamental tone"; it is the lowest note this length of string can produce. However, it can also vibrate with 3 or with 5 humps:





These vibrations make tones which are multiples of the fundamental tone. They sound higher than the fundamental tone; they are called "overtones" of the fundamental.

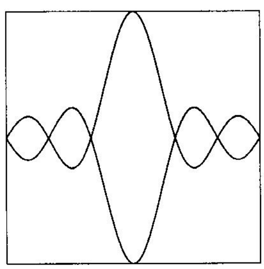
In each of these examples, the guitar string is shaped like a piece of the graph of a cosine function. In fact:

one hump	$y = \cos(x)$
three humps	$y = \cos(3x)$
five humps	$y = \cos(5x)$

These are called "pure tones."

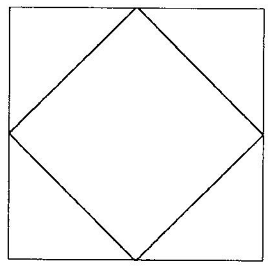
Now typically, when you pluck a guitar string, you don't produce a pure tone. Instead, you cause the string to vibrate in a complicated way built up out of some combination of the fundamental tone and the overtones. The "main note" that you hear is the fundamental, but the mixture of the overtones adds richness to the sound. The particular combination of fundamental and overtones which you hear produces the characteristic sound of that instrument.

For example, if plucked appropriately, the guitar string might vibrate like this:

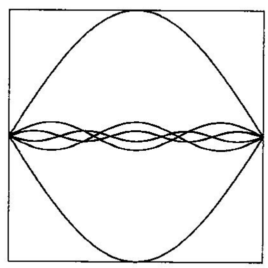


This is a combination of all three of the previous tones. This process of combining different tones to produce a complicated sound, or more generally of combining simple different periodic functions like cosine to produce a complicated periodic function is called *superposition.*

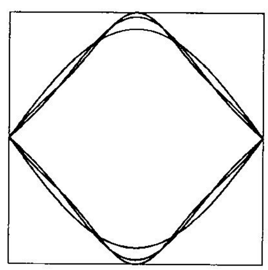
Fourier series are a technique for reversing this process. In other words, Fourier series are a method for taking a complicated waveform or "sound" and splitting it up into a sum of simpler, "pure" tones. For example, we might want to take a "sawtooth" vibration like this:



We could then ask what pure tones need to be combined to produce this kind of vibration. The different tones we need to add together are these:



When we add these together, we get a limiting situation like this:



Here you see a series of better and better approximations to the sawtooth made by combining more and more pure tones. It turns out that you need to combine infinitely many different pure tones to do this, and the theory of Fourier series tells you how. Now let's look at some formulas."

Louis slipped from his chair and left the classroom. He was really impressed – and really interested. He was also incredibly embarrassed that he'd never heard even that much explanation of the physical significance of Fourier series. He thought back to what he'd learned in his analysis class – $L^2(S^1)$, Hilbert spaces, all that stuff, which he'd thoroughly mastered – and realized that he had never heard the word "vibration" mentioned!

Right then, he made two resolutions: first, that he'd try to give his class the same motivation Kara had just given him; and second, that he'd go to the library right away and do a little reading on waves.

Friday Tea, One Week Later

"Kara, I want you to know how much I appreciate your having encouraged me to emphasize some of the applications of Fourier series. That little fifteen minute lesson I got from your class opened my eyes to all kinds of interesting stuff. I followed your lead and talked about vibrating strings," Louis said.

"Funny, Louis, I was just thinking that maybe you had the right idea after all. I sort of regret bringing up any of that stuff."

"How come?"

"Several reasons. First, I gave a quiz on Fourier series and I was pretty disappointed in the results. I asked the students to compute a few Fourier coefficients, and got pretty weak responses."

"Yeah, but that's typical, isn't it?" answered Louis.

"Maybe. But I also asked the students on the quiz to describe in their own words something about the significance of Fourier series for understanding vibrating strings. Only a few even tried that question, and of those only two were able to say something reasonable," said Kara.

"I'm not really surprised. It's always hard to get coherent answers to essay questions, especially on quizzes."

"OK. But here's the thing that really pushed me over the edge – when we were talking about the final exam, one of the students asked me if they needed to know any of that "physics stuff" I had covered – not just the

Fourier series, but the extra stuff on work and some other little things I had included as motivation earlier. It kind of put me in a difficult position. I know we're not going to ask any questions about applications on the final, partly because not everyone teaching the course covered that stuff, partly because of tradition, and partly because those kinds of questions are just too hard to grade. But something about the tone of this guy's question made it clear that if I told him that, he would just ignore anything I had said on those subjects and focus on what really matters: the formulas. So why do I bother?"

Louis paused to think. He had found Kara's little lecture on vibration fascinating, and he had naturally assumed that his students would too. But from what Kara was saying, this very material he had found so intriguing was seen by the students as "extra stuff" which got in the way of the "real material." He wondered what his own students would say about his new and exciting (to him) information about vibrating strings. Maybe he should just forget about trying to do anything fancy, and go back to concentrating on the mathematical formalism.

"I don't know what to do, Kara," he finally said, "you may not have done much for your real students, but you helped educate me. I hope that's worth something. Maybe some of it will sink into them, too."