Seeking Points

Daniel sighed as he dumped his books on his office desk. He'd just handed back the first midterm exam from his Calculus I class, and he could tell as he left the classroom that there were a lot of unhappy students. Still, the exam had been just like the practice exam he'd given out, and he was sure it was pretty straightforward. As he sat down to take a look at the paper on duality for fppf sheaves he was supposed to read, he heard a knock on his office door.

"Come in," he called, and he saw Sam, one of his Calculus students, push open the door hesitantly.

"Can I talk to you about my exam?" Sam said.

"I guess this was inevitable," thought Daniel to himself. To Sam, he said "What's up?"

"It's this question number 2," said Sam. "I don't think my answer was graded properly."

"Let me take a look," Daniel replied, "pull up a chair."

Sam sat down and passed his exam booklet over to Daniel. Daniel noticed that Sam had gotten 82 points out of 100 on the exam, which was a high B; but he had missed most of his points on problem 2. Then Daniel looked at the question, which said:

Problem 2 (20 points). Let $f(x) = x^3 - 5$. Use the definition of the derivative to compute the slope of the tangent line to the graph of $f(x)$ at the point where $x = 2$.

Then Daniel turned to Sam's exam paper. Sam had written the following:

Sam's Answer: $f(x) = x^3 - 5$. $f'(x) = 3x^2$. Slope=$f'(2) = 12$.

The grader of the problem had given Sam 5 out of the 20 points.

"Well, Sam," said Daniel, "you see you didn't do what the question asked. You are supposed to use the definition of the derivative to solve this problem, but you didn't give any method for deriving your answer. How did you do this problem?"

"I used the rule that the derivative of x^n is nx^{n-1}, which makes it really easy," replied Sam.

Daniel felt a little uncomfortable about this. He, like the rest of the Calculus teachers, was emphasizing understanding rather than algorithms for solving problems. He and his fellow instructors had specifically scheduled the first exam after a qualitative discussion of the derivative, and an introduction to the definition, but before discussing the various techniques of differentiation. He hadn't gone over the nx^{n-1} rule in class yet.

"Where did you get that from?" he asked Sam.

"I took Calc in High School, and we learned it there. We learned lots of other methods too. The answer is right, isn't it?"

"Yes, it's correct as far as it goes, but as I said it isn't what we asked for. We wanted you to show that you can use the definition of the derivative."

"You mean that thing with the limit?" said Sam.

"Yes," said Daniel, "Exactly, that thing with the limit – the difference quotient. In the review for the test I emphasized that if we asked you to use the definition of the derivative then we wanted you to use the difference quotient."

"Well," said Sam, "I didn't come to the review session. But it doesn't really seem fair to me that I got so many points off because I did the problem an easy way instead of a hard way."

"It isn't just a question of easy and hard," said Daniel. "We are trying to teach you to understand what the derivative means and where it comes from. We don't want you to just learn a bunch of formulas and how to make them go."

"Look, Professor, I know what the derivative means. It's the slope of the tangent line to the curve at the point, just like you asked. I knew that, because I knew what to calculate once I used my rule. Look at the rest of my exam – I got all the other problems basically right. I think I deserve more points on this problem."

"Sam, before we get into a discussion of points, let me ask you this. Do you know what the difference quotient is? Do you know WHY the formula you used gives you the slope of the tangent line?"

"Yeah, well, you did that in class a while ago, and I understood it then. It has something to do with secant lines and stuff, but I forget right now. I figured it doesn't really matter, 'cause I know these other, easier ways to do the problems. I just feel sorry for the other students who have to do it the

hard way. I taught my roommate in another section about the methods I learned and he really appreciated it."

"But Sam, that's just the point we are trying to get across. It IS just as important to know WHY the formula works as how to use it. The formulas you learned all had to be figured out by someone using the difference quotient. Let's take the problem from the test. What we wanted to see was the following:

$$
\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{((2+h)^3 - 5) - (2^3 - 5)}{h} \\
&= \lim_{h \to 0} \frac{2^3 + 3 \cdot 2^2 \cdot h + 3 \cdot 2 \cdot h^2 + h^3 - 5 - 2^3 + 5}{h} \\
&= \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h} \\
&= \lim_{h \to 0} 12 + 6h + h^2 \\
&= 12
\end{aligned}
$$

So the derivative is 12, and so is the slope. This calculation shows that the slope, 12, is the limiting value of the slopes of the secant lines."

"Well, maybe it shows that to you, but it looks like a bunch of formulas to me. Just different formulas. You really think all those people who wrote that instead of what I wrote know something I don't? They just went to the review session, which I admit I shoulda done. Look, Prof, I'm not here to argue about all of Mathematics. I promise from now on I'll come to your review sessions and do the problems just the way you want them. I just want 5 more points so I can get an A on this exam."

Daniel's heart sank. It was pretty clear this kid Sam didn't get Daniel's argument about "underlying ideas." And him promising to do whatever Daniel wanted on the next exam just made Daniel fell worse – that made it seem like the kid was just humoring him. As for more points – well, lots of people had made this mistake on the exam, and they'd all gotten five points. So Daniel couldn't really change this kid's point score without changing the others, too, though he did think Sam seemed pretty sharp.

"Sam, I'm afraid I can't give you any more points on this problem. We graded the exam consistently, and we gave everyone who made your mistake 5 points. I appreciate what you're telling me, and I get the impression you are following the course pretty well, so if you continue to do well you can get your A on the next midterm and the final and you'll get your A in the course."

"So you mean you graded lots of people unfairly, and you don't want to fix it? OK, you're the prof, I guess. And I'll be sure to come to the review session next time so I find out how you want us to do the problems."

Sam picked up his exam and left the room. Daniel stared after him for a minute or two, visibly upset, then took a deep breath and turned back to his desk. He had promised to read this paper before his next meeting with his advisor. Where was he? Oh, yes, he could see that the argument he was reading worked if he used the theorem on flat descent. He remembered sitting in on a lecture during his second year where his professor had described flat descent in detail; he couldn't exactly remember the proof of the theorem, but he did remember that you could apply it in this situation....