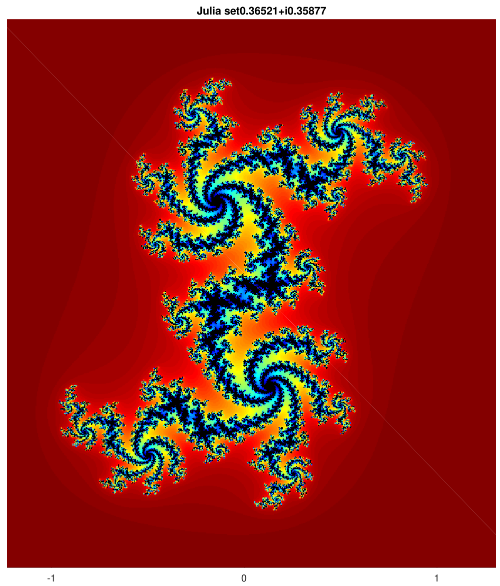
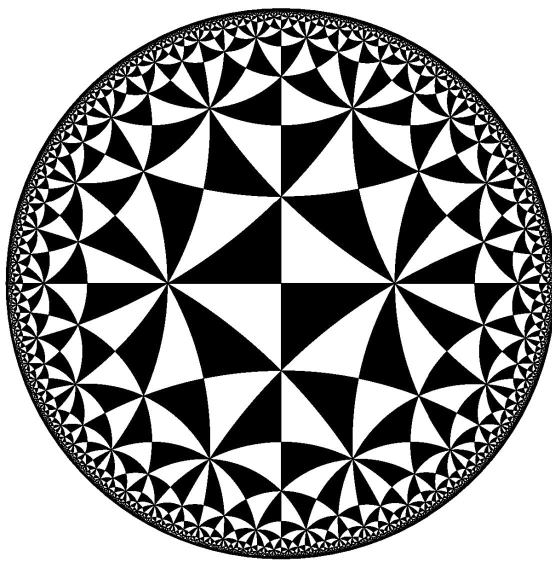
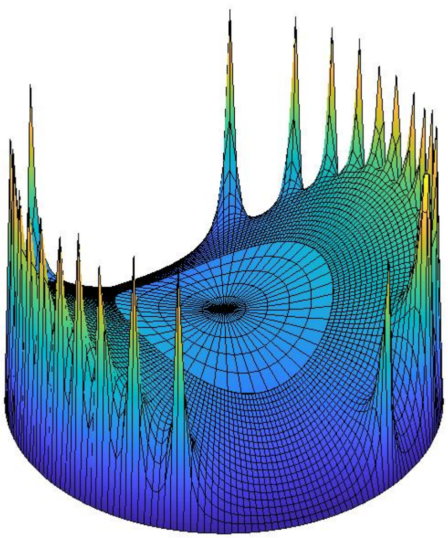
MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop
2024







Chapter 9: Calculus of Residues

Section 9.1: Contour Integration and Residues

Cauchy's theorem says that if f is analytic in a region Ω and if γ is a closed curve in Ω which is homologous to 0, then $\int_\gamma f(z)dz = 0$.

What happens if f has an isolated singularity at $a \in \Omega$?

Expanding f in its Laurent series about a, we have

$$f(z) = \sum_{n=-\infty}^{\infty} b_n(z-a)^n = \frac{b_{-1}}{z-a} + \frac{d}{dz}\left[\sum_{n=-\infty, n\neq -1}^{\infty} \frac{b_n}{(n+1)}(z-a)^{n+1}\right].$$

Thus

$$\int_{\partial\Delta} f(z)dz = b_{-1}\int_{\partial\Delta}\frac{dz}{z-a} = 2\pi i b_{-1}, \tag{9.1}$$

provided $\partial\Delta$ is oriented in the positive or counterclockwise direction.

Definition: If f is analytic in $\{0 < |z-a| < \delta\}$ for some $\delta > 0$, then the **residue of f at a**, written $\mathrm{Res}_a f$, is the coefficient of $(z-a)^{-1}$ in the Laurent expansion of f about $z = a$.

Theorem 9.2, Residue theorem: Suppose f is analytic in Ω *except for isolated singularities at* $a_1, \ldots, a_n$. *If* γ *is a cycle in* Ω *with* $\gamma \sim 0$ *and* $a_j \notin \gamma$, $j = 1, \ldots, n$, *then*

$$\int_\gamma f(z)dz = 2\pi i \sum_k \mathrm{n}(\gamma, a_k)\mathrm{Res}_{a_k} f.$$

Usually the residue theorem is applied to curves γ such that $\mathrm{n}(\gamma, a_k) = 0$ or 1, so that the sum on the right is $2\pi i$ times the sum of the residues of f at points enclosed by γ.

If f has infinitely many singularities clustering only on $\partial\Omega$ then we can shrink Ω slightly so that it contains only finitely many a_j and still have $\gamma \sim 0$.

Proof. Let Δ_k be a disk centered at a_k, $k = 1, 2, \ldots, n$, such that $\overline{\Delta}_m \cap \overline{\Delta}_k = \emptyset$ if $m \neq k$.

Orient $\partial\Delta_k$ in the counterclockwise direction. Then in the region $\Omega \backslash \{a_1, \ldots, a_n\}$,

$$\gamma - \sum_k \mathrm{n}(\gamma, a_k)\partial\Delta_k \sim 0$$

By Cauchy's theorem

$$\int_\gamma f(z)dz - \sum_{k=1}^{n} \mathrm{n}(\gamma, a_k) \int_{\partial\Delta_k} f(z)dz = 0.$$

Then Theorem 9.2 follows from (9.1). □

Section 9.2: Some examples

Example 9.3: $f(z) = \dfrac{e^{3z}}{(z-2)(z-4)}$

This has a simple pole at $z = 2$ and hence

$$\mathrm{Res}_2 f = \lim_{z \to 2} (z-2) f(z) = \frac{e^6}{-2}.$$

The residue at $z = 4$ can be calculated similarly.

Example 9.4: $g(z) = \dfrac{e^{3z}}{(z-2)^2}$.

Expand e^{3z} in a series expansion about $z = 2$:

$$g(z) = \frac{e^6 e^{3(z-2)}}{(z-2)^2} = \frac{e^6}{(z-2)^2} \sum_{n=0}^{\infty} \frac{3^n}{n!}(z-2)^n = \frac{e^6}{(z-2)^2} + \frac{3e^6}{z-2} + \ldots,$$

so that

$$\mathrm{Res}_2 g = 3e^6.$$

In this case $\lim_{z\to 2}(z-2)^2 g(z)$ is not the coefficient of $(z-2)^{-1}$ and $\lim_{z\to 2}(z-2)g(z)$ is infinite.

More generally, if $G(z)$ is analytic at $z=a$ then iiiii

$$\mathrm{Res}_a \frac{G(z)}{(z-a)^n} = \frac{G^{(n-1)}(a)}{(n-1)!}.$$

Example 9.5: Suppose we have a simple pole, and the pole is not already written as a factor of the denominator.

$$h(z) = e^{az}/(z^4 + 1).$$

Then h has simple poles at the fourth roots of -1. If $\omega^4 = -1$, then

$$\mathrm{Res}_\omega h = \lim_{z\to\omega} \frac{(z-\omega)e^{az}}{z^4+1} = \frac{e^{a\omega}}{\lim_{z\to\omega} \frac{z^4+1}{z-\omega}}.$$

Note that the denominator is the limit of difference quotients for the derivative of $z^4 + 1$ at $z = \omega$ and hence

$$\mathrm{Res}_\omega \frac{e^{az}}{z^4+1} = \frac{e^{a\omega}}{4\omega^3} = -\frac{\omega e^{a\omega}}{4}.$$

Example 9.6: Another method using series is illustrated by the example

$$k(z) = \frac{\pi \cot \pi z}{z^2}$$

To compute the residue of k at $z = 0$, note that $\cot \pi z$ has a simple pole at $z = 0$ and hence k has a pole of order 3, so that

$$\frac{\pi \cot \pi z}{z^2} = \frac{b_{-3}}{z^3} + \frac{b_{-2}}{z^2} + \frac{b_{-1}}{z} + b_0 + \ldots.$$

Then

$$\pi \cos \pi z = \left(\frac{\sin \pi z}{z}\right)(b_{-3} + b_{-2}z + b_{-1}z^2 + b_0 z^3 + \ldots.$$

Inserting the series expansions for cos and sin we obtain

$$\pi(1 - \frac{\pi^2}{2}z^2 + \ldots) = (\pi - \frac{\pi^3}{6}z^2 + \ldots)(b_{-3} + b_{-2}z + b_{-1}z^2 + \ldots,$$

Equating coefficients

$$\pi = \pi b_{-3} \quad - \frac{\pi^3}{2} = -\frac{\pi^3}{6}b_{-3} + \pi b_{-1},$$

and $\mathrm{Res}_0 k = b_{-1} = -\frac{\pi^2}{3}$.

Example 9.7: If γ is the circle centered at 0 with radius 3, then

$$\int_\gamma \frac{e^{3z}}{(z-2)(z-4)}dz = -2\pi i\frac{e^6}{2},$$

by the residue theorem and Example 9.3.

Example 9.8: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$.

Construct a contour γ consisting of the interval $[-R, R]$ followed by the semi-circle C_R in $\mathbb{H}$ of radius R, with $R > 1$.

By the residue theorem with $f(z) = 1/(z^4 + 1)$,

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i(\mathrm{Res}_{z_1} f + \mathrm{Res}_{z_2} f), \tag{9.2}$$

where z_1 and z_2 are the roots of $z^4 + 1 = 0$ in the upper half-plane $\mathbb{H}$.

Note that

$$\left|\int_{C_R} f(z)dz\right| \leq \int_0^{2\pi} \frac{Rd\theta}{R^4 - 1} \to 0$$

as $R \to \infty$. Since the integral $\int_{\mathbb{R}} (x^4 + 1)^{-1} dx$ is convergent, it equals

$$\lim_{R\to\infty} \int_{-R}^{R} (x^4 + 1)^{-1} dx,$$

so that by (9.2), and Example 9.5

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx = -\frac{2\pi i}{4}(z_1 + z_2) = \frac{\pi}{\sqrt{2}}.$$

The technique above can be used to compute the integral of any rational function with no poles on $\mathbb{R}$ if the degree of the denominator is at least 2 plus the degree of the numerator.

This latter condition is needed for the absolute convergence of the integral.

Example 9.9: $\displaystyle\int_0^{2\pi} \frac{1}{3+\sin\theta}d\theta$. Set $z = e^{i\theta}$. Then $dz = ie^{i\theta}d\theta = izd\theta$.

$$\int_0^{2\pi} \frac{1}{3+\sin\theta}d\theta = \int_{|z|=1} \frac{1}{(3+\frac{1}{2i}(z-1/z))}\frac{dz}{iz} = \int_{|z|=1} \frac{2dz}{z^2+6iz-1}.$$

The roots of $z^2+6iz-1$ occur at $z_1, z_2 = i(-3\pm\sqrt{8})$.

Only $i(-3+\sqrt{8})$ lies inside $|z|=1$.

By the residue theorem and computing residues as in Examples 9.3 or 9.5,

$$\int_0^{2\pi} \frac{1}{3+\sin\theta}d\theta = 2\pi i \mathrm{Res}_{i(-3+\sqrt{8})}\frac{2}{z^2+6iz-1} = \frac{2\pi}{\sqrt{8}}.$$

The technique in Example 9.9 can be used to compute

$$\int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta,$$

where $R(\cos\theta, \sin\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$, with no poles on the unit circle.

An integral on the circle as in Example 9.9, can be converted to an integral on the line using the Cayley transform $z = (i - w)/(i + w)$ of the upper half plane onto the disk.

It is interesting to note that you obtain the substitution $x = \tan\frac{\theta}{2}$ which you might have learned in calculus.

Example 9.10: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx.$

A first guess might be to write $\cos z = (e^{iz} + e^{-iz})/2$, but if $y = \mathrm{Im} z$ then $|\cos z| \sim e^{|y|}/2$ for large $|z|$.

This won't allow us to find a closed contour where the part off the real line makes only a small contribution to the integral.

Instead, we use $e^{iz}/(z^2+1)$ then take real parts of the resulting integral.

Using the same half-disk contour as in Example 9.8, we have the estimate

$$\left|\int_{C_R} \frac{e^{iz}}{z^2+1}dz\right| \le \int_{C_R} \frac{e^{-y}}{R^2-1}|dz| \le \frac{\pi R}{R^2-1} \to 0$$

as $R \to \infty$, where $y = \mathrm{Im} z > 0$. By the method in Example 9.5,

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx = 2\pi i \sum_{\mathrm{Im} a>0} \mathrm{Res}_a \frac{e^{iz}}{z^2+1} = 2\pi i \frac{e^{i\cdot i}}{2i} = \frac{\pi}{e}.$$

In this particular case, we did not have to take real parts. The integral itself is real because $\sin x/(x^2+1)$ is odd.

Section 9.3: Fourier and Mellin Transforms

The technique in Example 9.10 can be used to compute

$$\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx \text{ for } \lambda > 0 \tag{3.1}$$

provided f is meromorphic in the closed upper half-plane $\mathbb{H} \cup \mathbb{R}$ with no poles on $\mathbb{R}$ and $|f(z)| \leq K/|z|^{1+\epsilon}$ for some $\epsilon > 0$ and all large $|z|$ with $\text{Im}z > 0$.

If the integral (9.3) is desired for all real λ, then for negative λ use a contour in the lower half-plane, provided f is meromorphic and $|f(z)| \leq K/|z|^{1+\epsilon}$ in $\text{Im}z < 0$, e.g., f is rational and the degree of the denominator is at least 2 plus the degree of the numerator.

The integral in (9.3), usually with λ replaced by $-2\pi\lambda$, is called the **Fourier transform of** f, as a function of λ.

Example 9.11: $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \lambda x}{x^2+1} dx.$

This can not be done as in Example 9.10, because the integrand does not decay fast enough to prove $\int_{C_R} |f(z)||dz| \to 0$, where $f(z) = ze^{i\lambda z}/(z^2+1)$.

Indeed it is not even clear apriori that the integral in example (e) exists.

Example 9.11: $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \lambda x}{x^2+1} dx.$

We may suppose $\lambda > 0$, because sine is odd.

Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where

$$\begin{aligned}
\gamma_1 &= [-A, B], \ A, B > 0, \\
\gamma_2 &= \{B + iy : 0 \le y \le A + B\}, \\
\gamma_3 &= \{x + i(A+B) : B \ge x \ge -A\}, \\
\gamma_4 &= \{-A + iy : A + B \ge y \ge 0\},
\end{aligned}$$

orienting γ counter-clockwise.

Example 9.11: $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \lambda x}{x^2+1} dx.$

To prove convergence of this integral, we will let A and B tend to ∞ independently, and use the estimate

$$|z/(z^2+1)| \leq |z|/(|z|^2-1) \leq 2/|z|$$

when $|z|^2 > 2$. For A and B large,

$$\left|\int_{\gamma_3} \frac{ze^{i\lambda z}}{z^2+1} dz\right| \leq \int_{-A}^{B} \frac{2}{A+B} e^{-\lambda(A+B)} dx = \frac{2e^{-\lambda(A+B)}}{A+B}(A+B) \to 0,$$

as $A+B \to \infty$.

Also

$$\left| \int_{\gamma_2} \frac{ze^{i\lambda z}}{z^2+1} dz \right| \leq \int_0^{A+B} \frac{2}{B} e^{-\lambda y} dy \leq \frac{2}{B} \frac{\left(1 - e^{-\lambda(A+B)}\right)}{\lambda} \to 0,$$

as $B \to \infty$.

Example 9.11: $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \lambda x}{x^2+1} dx.$

A similar estimate holds on γ_4 as $A \to \infty$. By the residue theorem

$$\lim_{A,B\to\infty} \int_{-A}^{B} \frac{xe^{i\lambda x}}{x^2+1} dx = 2\pi i \ \mathrm{Res}_i \frac{ze^{i\lambda z}}{z^2+1} = \frac{2\pi i \cdot ie^{-\lambda}}{2i} = i\pi e^{-\lambda}. \tag{9.4}$$

By our estimates, the integrals over γ_2, γ_3, and γ_4 tend to 0 as A and B tend to ∞ so that the limit on the left side of (9.4) exists and (9.4) holds.

Example 9.11 follows from (9.4) by taking the imaginary parts.

Example 9.12: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$

The main difference between Example 9.12 and Example 9.11 is that the function $f(z) = e^{iz}/z$ has a simple pole on $\mathbb{R}$.

The function $\sin x/x$ is integrable near 0 since $\sin x/x \to 1$ as $x \to 0$, but $f(x)$ is not integrable.

However the real part of $f(x)$ is odd so that

$$\lim_{\delta\to 0} \int_{-1}^{-\delta} + \int_{\delta}^{1} \frac{e^{ix}}{x} dx$$

exists.

Definition 9.13: Suppose f is continuous on $\gamma \setminus \{a\}$, and suppose γ' is continuous at $\gamma^{-1}(a)$. Then the **Cauchy Principal Value** of the integral of f along γ is defined by

$$PV \int_{\gamma} f(z)dz = \lim_{\delta \to 0} \int_{\gamma \cap \{|z-a| \geq \delta\}} f(z)dz,$$

provided the limit exists.

Note that we have deleted points in a ball *centered* at a.

For example

$$PV \int_{-1}^{1} \frac{\cos x}{x} dx = 0,$$

because the integrand is odd, but the integral itself does not exist.

Proposition 9.14: Suppose f is meromorphic in $\{\mathrm{Im} z \geq 0\}$, such that

$$|f(z)| \leq \frac{K}{|z|} \quad \text{for} \mathrm{Im} z \geq 0 \text{ and} |z| > R.$$

Suppose also that all poles of f on $\mathbb{R}$ are simple. If $\lambda > 0$ then

$$PV \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx = 2\pi i \sum_{\mathrm{Im} a > 0} \mathrm{Res}_a e^{i\lambda z} f(z) + 2\pi i \sum_{\mathrm{Im} a = 0} \frac{1}{2} \mathrm{Res}_a e^{i\lambda z} f(z). \quad (9.5)$$

Proposition 9.14: Suppose f is meromorphic in $\{\text{Im}z \geq 0\}$, such that

$$|f(z)| \leq \frac{K}{|z|} \quad \text{for} \text{Im}z \geq 0 \text{ and} |z| > R.$$

Suppose also that all poles of f on $\mathbb{R}$ are simple. If $\lambda > 0$ then

$$PV \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx = 2\pi i \sum_{\text{Im}a>0} \text{Res}_a e^{i\lambda z} f(z) + 2\pi i \sum_{\text{Im}a=0} \frac{1}{2}\text{Res}_a e^{i\lambda z} f(z). \quad (9.5)$$

Part of the conclusion of Proposition 9.14 is that the integral exists even though the rate of decay at ∞ is possibly slower than our assumptions in (9.3).

If $\lambda < 0$, then a similar result holds using the lower half plane.

The integral may not exist if $\lambda = 0$ as the example $f(z) = 1/z$ shows.

One way to remember the conclusion of Proposition 3.2 is to think of the real line as cutting the pole at each a_j in half. The integra2 contributes half of the residue of f at a_j.

Proof. Note that f has at most finitely many poles in $\{\text{Im}z \geq 0\}$, so that both sums in Proposition 91.4 are finite.

Construct a contour similar to the rectangle Example 9.11, but avoiding the poles on $\mathbb{R}$ using small semicircles C_j of radius $\delta > 0$ centered at each pole $a_j \in \mathbb{R}$.

The integral of $f(z)e^{i\lambda z}$ along the top and sides of the contour tend to 0 as $A, B \to \infty$ as in Example 9.11.

The semi-circle C_j centered at a_j can be parameterized by $z = a_j + \delta e^{i\theta}$, $\pi > \theta > 0$ so that if

$$f(z)e^{i\lambda z} = \frac{b_j}{z - a_j} + g_j(z)$$

where g_j is analytic in an neighborhood of a_j, then

$$\int_{C_j} f(z)e^{i\lambda z}dz = \int_{\pi}^{0} \frac{b_j}{\delta e^{i\theta}}\delta i e^{i\theta}d\theta + \int_{C_j} g(z)dz.$$

Because g_j is continuous at a_j and the length of C_j is $\pi\delta$, we have

$$\lim_{\delta\to 0}\int_{C_j} f(z)e^{i\lambda z}dz = -i\pi b_j.$$

By the residue theorem

$$PV\int_{\mathbb{R}} f(z)e^{i\lambda z}dz - i\pi\sum_j b_j = 2\pi i\sum_{\mathrm{Im}\,a>0}\mathrm{Res}_a f(z)e^{i\lambda z}$$

and (9.5) holds. $\square$

For the present example $\int (\sin x)/x dx$,

$$PV \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i, \tag{3.4}$$

so that by taking imaginary parts

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

Note that $\sin x/x \to 1$ as $x \to 0$ so that $\int \sin x/x dx$ exists as an ordinary (improper) Riemann integral, if we set $\sin x/x = 1$ at $x = 0$.

For this reason we can drop the PV in front of the integral.

Example 9.15: $\displaystyle\int_0^\infty \frac{x^\alpha}{x^2+1}dx$, where $0 < \alpha < 1$.

An integral of the form

$$\int_0^\infty f(x)x^{\beta-1}dx$$

is called a **Mellin transform**.

By the change of variables $x = e^y$, the Mellin transform of f is the Fourier transform of $f(e^y)$ when β is purely imaginary.

Example 9.15: $\displaystyle\int_0^\infty \frac{x^\alpha}{x^2+1}dx$, where $0 < \alpha < 1$.

Define $z^\alpha = e^{\alpha \log z}$ in $\mathbb{C} \setminus [0, +\infty)$ where $0 < \arg z < 2\pi$ and set $f(z) = 1/(z^2+1)$.

Consider the "keyhole" contour γ consisting of a portion of a large circle C_R of radius R and a portion of a small circle C_δ of radius δ, both circles centered at 0, along with two line segments between C_δ and C_R at heights $\pm\epsilon$, oriented counterclockwise.

By the residue theorem, for R large and δ small,

$$\begin{aligned}\int_\gamma z^\alpha f(z)dz &= 2\pi i(\mathrm{Res}_i z^\alpha f(z) + \mathrm{Res}_{-i} z^\alpha f(z)) \\ &= 2\pi i\left(\frac{e^{\alpha \log i}}{2i} + \frac{e^{\alpha \log(-i)}}{-2i}\right) = \pi(e^{i\frac{\pi\alpha}{2}} - e^{i\frac{3\pi\alpha}{2}})\end{aligned}$$

We will first let $\epsilon \to 0$, then $R \to \infty$ and $\delta \to 0$.

Even though the integrals along the horizontal lines are in opposite directions, they do not cancel as $\epsilon \to 0$.

For $\epsilon > 0$

$$\lim_{\epsilon \to 0} (x + i\epsilon)^\alpha f(x + i\epsilon) = e^{\alpha \log |x|} f(x),$$

and

$$\lim_{\epsilon \to 0} (x - i\epsilon)^\alpha f(x - i\epsilon) = e^{\alpha (\log |x| + 2\pi i)} f(x),$$

because of our definition of $\log z$.

Thus the integral over the horizontal line segments tends to

$$\int_{\delta}^{R} (1 - e^{2\pi i\alpha}) x^{\alpha} f(x) dx. \tag{9.8}$$

For R large

$$\left| \int_{C_R} z^{\alpha} f(z) dz \right| \leq \int_{0}^{2\pi} \frac{R^{\alpha}}{R^2 - 1} R d\theta \to 0, \tag{9.9}$$

as $R \to \infty$ (since $\alpha < 1$).

Similarly

$$\left|\int_{C_\delta} z^\alpha f(z)dz\right| \leq \int_0^{2\pi} \frac{\delta^\alpha}{1-\delta^2}\delta d\theta \to 0, \tag{3.8}$$

as $\delta \to 0$ (since $\alpha > -1$).

By (9.7) to (9.10)

$$\int_0^\infty \frac{x^\alpha}{x^2+1}dx = \pi\frac{e^{i\frac{\pi\alpha}{2}} - e^{i\frac{3\pi\alpha}{2}}}{1-e^{2\pi i\alpha}} = \frac{\pi}{2\cos\alpha\pi/2}.$$

This line of reasoning works for meromorphic f satisfying $|f(z)| \leq C|z|^{-2}$ for large $|z|$ and with at worst a simple pole at 0.

The function z^α can be replaced by other functions which are not continuous across $\mathbb{R}$, such as $\log z$.

In this case real parts of the integrals along $[0, \infty)$ will cancel, but the imaginary parts will not.

Mellin transforms are used in applications to signal processing, image filtering, stress analysis and other areas.

Section 9.4: Series via Residues

Example 9:16: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

Set $f(z) = \frac{1}{z^2+1}$ and consider the meromorphic function $f(z)\pi \cot \pi z$. Write

$$\pi \cot \pi z = \pi i \left(\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right) = \pi i \left(\frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} \right). \tag{9.11}$$

Multiplying (9.11) by $z - n$ and letting $z \to n$ shows that $\pi \cot \pi z$ has a simple pole with residue 1 at each integer n.

Because the poles are simple, $f(z)\pi \cot \pi z$ has a simple pole with residue $f(n)$ at $z = n$.

Example 9:16: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

Consider the contour integral of $f(z)\pi \cot \pi z$ around the square S_N with vertices $(N + \frac{1}{2})(\pm 1 \pm i)$, where N is a large positive integer.

The function $\pi \cot \pi z$ is uniformly bounded on S_N, independent of N. Recall

$$\pi \cot \pi z = \pi i \left(\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right) = \pi i \left(\frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} \right). \tag{9.11}$$

The LFT $(\zeta + 1)/(\zeta - 1)$ maps the region $|\zeta - 1| < \delta$ onto a neighborhood of ∞ and is one-to-one, so it is bounded on $|\zeta - 1| > \delta$.

The estimate $|e^{2\pi i z} - 1| > 1 - e^{-\pi}$ holds on S_N (consider that $e^{2\pi i x z} = 1$ on on $\mathbb{Z}$ and the horizontal and vertical segments miss this set). Hence $\pi \cot \pi z$ is bounded on S_N.

Therefore, because $|f(z)| \leq C|z|^{-2}$, we have

$$\int_{S_N} f(z)\pi \cot \pi z dz \to 0.$$

By the residue theorem

$$0 = \text{Res}_i f(z)\pi \cot \pi z + \text{Res}_{-i} f(z)\pi \cot \pi z + \sum_{-\infty}^{\infty} f(n),$$

and hence

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{2}\left[\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right] + \frac{1}{2}.$$

This technique can be used to compute

$$\sum_{n=-\infty}^{\infty} f(n),$$

provided f is meromorphic with $|f(z)| \le C|z|^{-2}$ for $|z|$ large.

If some of the poles of f occur at integers, then the residue calculation at those poles is slightly more complicated because the poles of $f(z)\pi \cot \pi z$ will not have order 1 at these integers. See Example 9.6.

If only the weaker estimate $|f(z)| \le C|z|^{-1}$ holds, then f has a removable singularity at ∞ and so $g(z) = f(z) + f(-z)$ satisfies $|g(z)| \le C|z|^{-2}$ for large $|z|$. Applying the techique to g, we can find the symmetric limit

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n).$$