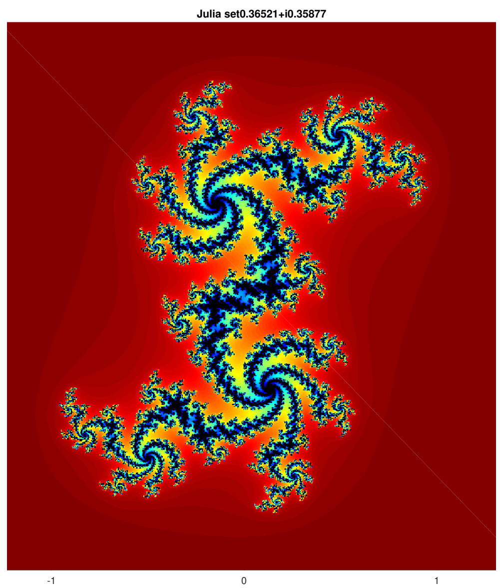
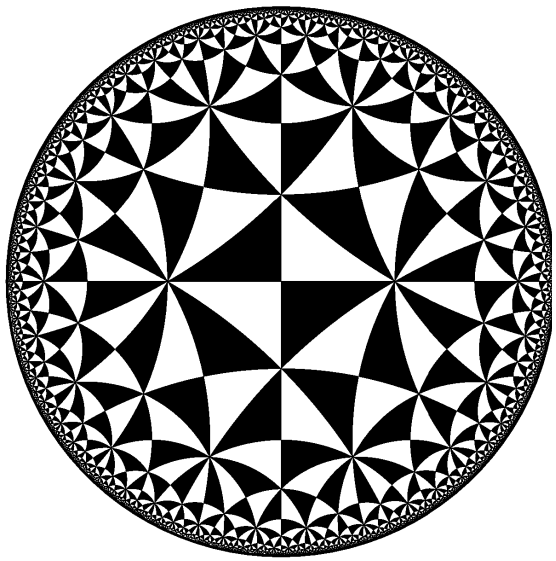
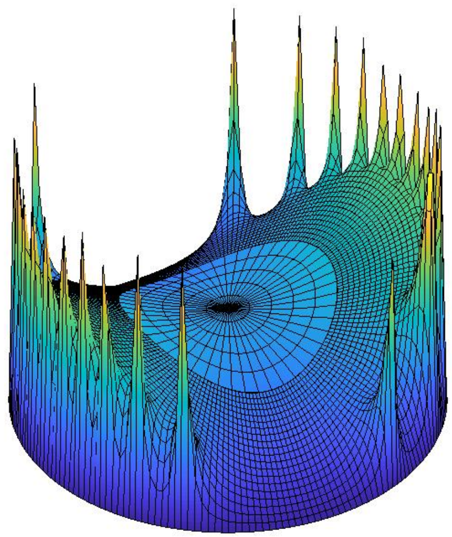
MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop







Chapter 7: Harmonic Functions

Section 7.1: The Mean-Value Property and the Maximum Principle

Definition 7.1: A continuous real-valued function u is **harmonic** on a region $\Omega \subset \mathbb{C}$ if for each $z \in \Omega$ there is an $r_z > 0$ (depending possibly on z) such that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})dt \tag{7.1}$$

for all $r < r_z$.

Equation (7.1) is called the **mean-value property**.

Definition 7.2: A continuous function u with values in $[-\infty, +\infty)$ is **subharmonic** on a region Ω if for each $z \in \Omega$ there is an $r_z > 0$ (depending possibly on z) such that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it}) dt \tag{1.2}$$

for all $r < r_z$.

In some texts, the continuity assumption is replaced by upper semi-continuity.

Definition: a real-valued function f is **lower semi-continuous** if $f^{-1}((y, \infty)) = \{x : f(x) > y\}$ is open for every $a \in \mathbb{R}$. Equivalently, if

$$\liminf_{x \to y} f(x) \geq y.$$

We allow a subharmonic function to take the value $-\infty$ but not $+\infty$.

- If u and $-u$ are subharmonic then u is harmonic.

- If u_1 and u_2 are harmonic then $A_1u_1 + A_2u_2$ is harmonic, for $A_1, A_2 \in \mathbb{R}$.

- If u is subharmonic then Au is subharmonic provided $A > 0$.

- If u_1 and u_2 are subharmonic then $u(z) = \max(u_1(z), u_2(z))$ is subharmonic.

- Uniform limits of harmonic functions are harmonic.

- Real and imaginary parts of an analytic function are harmonic.

- If f is analytic on Ω, then $\log|f|$ is harmonic on $\Omega \cap \{|f| > 0\}$ and $\log|f|$ is subharmonic on all of Ω. This is a VERY useful fact!

Theorem 7.3, Maximum Principle *Suppose u is subharmonic on a region Ω. If there exists $z_0 \in \Omega$ such that*

$$u(z_0) = \sup_{z \in \Omega} u(z) \qquad (7.3),$$

then u is constant.

The proof is almost identical to the proof for analytic functions.

Proof. Suppose (7.3) holds and set $E = \{z \in \Omega : u(z) = u(z_0)\}$. Since u is continuous, E is closed in Ω. By (7.3), the set E is non-empty. We need only show E is open, since Ω is connected.

If $z_1 \in E$, then by the mean value property

$$\frac{1}{2\pi}\int_0^{2\pi}[u(z_1) - u(z_1 + re^{it})]dt \leq 0, \tag{7.4}$$

for $r < r_{z_1}$. But the integrand is continuous and ≥ 0 and hence identically 0 for all t and all $r < r_{z_1}$. This proves E is open and hence equal to Ω. ☐

Corollary 7.4: *If u is a non-constant subharmonic function in a bounded region Ω and if u is continuous on $\overline{\Omega}$ then*

$$\max_{z\in\overline{\Omega}} u(z)$$

occurs on $\partial\Omega$ but not in Ω.

Equivalently,

$$\limsup_{z\to\partial\Omega} u(z) = \sup_{\Omega} u(z).$$

If Ω is unbounded, then ∞ must also be viewed as part of $\partial\Omega$. The function $u(z) = \mathrm{Re} z$ is harmonic on $\Omega = \{z : \mathrm{Re} z > 0\}$ and satisfies $u = 0$ on $\partial\Omega \cap \mathbb{C}$ but u is not bounded by 0.

Theorem 7.5: *If g is real-valued and continuous on $\partial\mathbb{D}$, set*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} g(e^{it})dt,$$

for $z \in \mathbb{D}$. Then u is harmonic in $\mathbb{D}$ and

$$\lim_{z\to\zeta} u(z) = g(\zeta), \tag{7.5}$$

for all $\zeta \in \partial\mathbb{D}$.

This is usually called the Poisson extension of g from $\mathbb{T}$ to $\mathbb{D}$.

We only need $g \in L^1(\mathbb{T})$ for extension to exist; can then show radial limit equals g almost everywhere. (Consequence of Hardy-Littlewood Maximal Theorem.)

Extension still works if g is replaced by a finite measure μ. Then radial limits equal absolutely continuous part of $d\mu = gd\theta + d\nu$.

Proof. The function

$$G(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} g(e^{it})dt$$

is analytic on $\mathbb{D}$ as can be seen by expanding the kernel

$$\frac{e^{it}+z}{e^{it}-z} = \frac{1+e^{-it}z}{1-e^{-it}z} = 1 + 2\sum_1^\infty e^{-int}z^n \tag{7.6}$$

and interchanging the order of summation and integration. The identity

$$\frac{1-|z|^2}{|e^{it}-z|^2} = \mathrm{Re}\left(\frac{\mathrm{e^{it}+z}}{\mathrm{e^{it}-z}}\right) \tag{7.7}$$

shows that $u = \mathrm{ReG}$ and hence u is harmonic. This proves harmonic extension exists; harder part is to show boundary values equal g.

If $g \equiv 1$, then $G \equiv 1$, since $\int e^{-int}dt = 0$ if $n \neq 0$. Thus for all $z \in \mathbb{D}$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}dt = 1. \tag{7.8}$$

To prove (7.5) fix t_0 and $\epsilon > 0$ then choose $\delta > 0$ so that $|g(e^{it}) - g(e^{it_0})| < \epsilon$ if $t \in I_\delta = \{t : |t - t_0| < \delta\}$. Then using (7.8)

$$|u(z) - g(e^{it_0})| = \Big|\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}(g(e^{it}) - g(e^{it_0}))dt\Big|$$

$$\leq \frac{\epsilon}{2\pi}\int_{I_\delta} \frac{1-|z|^2}{|e^{it}-z|^2}dt + M(z)\int_{\partial\mathbb{D}\setminus I_\delta} |g(e^{it}) - g(e^{it_0})|\frac{dt}{2\pi}.$$

where $M(z) = \sup_{\{t:|t-t_0|\geq\delta\}} \frac{1-|z|^2}{|e^{it}-z|^2}$.

The first term is at most ϵ by (7.8). Moreover $M(z) \to 0$ as $z \to e^{it_0}$ because $|e^{it} - z|$ is bounded below for $|t - t_0| \geq \delta$. Thus $u(z) \to g(e^{it_0})$ as $z \to e^{it_0}$. □

• The proof of Theorem 7.5 shows that we need only assume that g is integrable on $\partial\mathbb{D}$ and continuous at ζ for (1.5) to hold.

• The kernel

$$P_z(t) = \frac{1}{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}$$

is called the

Poisson kernel and u is the **Poisson integral of g**.

The Poisson kernel for the upper half plane $\mathbb{H}$ is given by

$$P_w^{\mathbb{H}}(s) = \frac{1}{\pi}\frac{v}{(u-s)^2+v^2} = \frac{1}{\pi}\mathrm{Im}\left(\frac{1}{\mathrm{s}-\mathrm{w}}\right),$$

where $w = u + iv$, $v > 0$, and $s \in \mathbb{R}$.

If Ω is a domain and $f \in C(\partial\Omega)$ suppose u is the harmonic function on Ω with boundary values f. If we fix $z \in \Omega$ then the map

$$f \to u(z)$$

is a linear map from $C(\partial\Omega)$ to $\mathbb{R}$ and

$$|u(z)| \leq \sup_{\partial\Omega} |f(x)|.$$

In the language of functional analysis, this is a bounded linear functional on the Banach space $C(\partial\Omega)$. By the Riesz representation theorem (MAT 533) there is a measure μ_z on $\partial\Omega$ so that

$$u(z) = \int_{\partial\Omega} f d\mu_z.$$

This is the Possion kernel in the case when $\Omega = \mathbb{D}$.

Also know as the "harmonic measure" on $\partial\Omega$ w.r.t. z.

Corollary 7.6, Harmonic extensions are unique: *If u is harmonic on $\mathbb{D}$ and continuous on $|z| \leq 1$, then for $z \in \mathbb{D}$*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} u(e^{it}) dt. \tag{7.9}$$

Proof. Let $U(z)$ denote the right side of (7.9). Then by Schwarz's theorem $u - U$ is harmonic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and equal to 0 on $\partial\mathbb{D}$. By the maximum principle applied to $u - U$ and $U - u$, we conclude $u = U$. $\square$

Corollary 7.7: *If u is harmonic on $|z| < 1$ and continuous on $|z| \le 1$, then*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} u(e^{it})dt$$

is the unique analytic function on $\mathbb{D}$ with $\mathrm{Re(f)} = \mathrm{u}$ and $\mathrm{Im(f(0))} = 0$.

The function

$$(e^{it}+z)/(e^{it}-z)$$

is called the **Herglotz kernel** and the integral is the **Herglotz integral**.

Proof. By the first part of the proof of Theorem 7.5, f is analytic. The real part of f is then equal to u by (7.7) and Corollary 7.6.

If g is another analytic function with Re(g) = u, then $f - g$ is purely imaginary and hence not an open mapping. Thus $f - g$ is constant. Finally note that $f(0) = \int u(e^{it})dt$ is real, so that if $g(0)$ is real then $g = f$. $\square$

Corollary: *If u is harmonic on a disk and if f is analytic then $u \circ f$ is harmonic.*

Proof. harmonicity is local and on a disk $u = \mathrm{Re(g)}$ for some analytic function g, by Corollary 7.7. Hence $u \circ f = \mathrm{Re(g \circ f)}$ harmonic. ☐

• Reverse is not true: if $u = \mathrm{Re(z)} = \mathrm{x}$ and $f = z^2$ then $f \circ u = x^2$ is not harmonic.

• The function $u = \log|z|$ is harmonic on $\Omega = \{z : 0 < |z| < \infty\}$ and is the real part of an analytic function on each disk that does not contain 0, but it is not the real part of an analytic function on all of Ω because $\arg(z(z))$ is not continuous on Ω.

Corollary 7.8, Jump Theorem: *Suppose f is an integrable function such that $f : \mathbb{T} \to \mathbb{C}$. Let*

$$F(z) = \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}\frac{d\zeta}{2\pi i}.$$

Then F is analytic on $\mathbb{C} \setminus \mathbb{T}$ and for $|\zeta| = 1$,

$$\lim_{z\to\zeta} \left[F(z) - F\left(\frac{1}{\bar{z}}\right) \right] = f(\zeta)$$

at all points of continuity ζ of f.

The function F is called the **Cauchy integral** or **Cauchy transform** of f.

The jump theorem says that the analytic function F jumps by $f(\zeta)$ as z crosses the unit circle at ζ. Notice that if f is analytic on $\overline{\mathbb{D}}$ then $F = f$ in $\mathbb{D}$ and $F = 0$ in $|z| > 1$ by Cauchy's integral formula.

Proof. We already proved that F is analytic off $\partial\mathbb{D}$. To prove the Corollary, just manipulate the integrals:

$$\begin{aligned} F(z) - F\left(\frac{1}{\overline{z}}\right) &= \int_{|\zeta|=1} f(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - 1/\overline{z}}\right)\frac{d\zeta}{2\pi i} \\ &= \int_{|\zeta|=1} f(\zeta)\frac{z - 1/\overline{z}}{(\zeta - z)(\zeta - 1/\overline{z})}\frac{d\zeta}{2\pi i} \\ &= \int_0^{2\pi} f(e^{it})\frac{1 - |z|^2}{|e^{it} - z|^2}\frac{dt}{2\pi}. \end{aligned}$$

Applying Schwarz's Theorem 1.5 to the real and imaginary parts of f completes the proof. $\square$

If g is integrable on $\mathbb{T}$, set $a_n = \frac{1}{2\pi}\int_0^{2\pi} g(e^{it})e^{-int}dt$. Then

$$\sum_{n=-\infty}^{\infty} a_n e^{int}$$

is called the **Fourier Series of g**.

Note that $|a_n| \le \frac{1}{2\pi}\int_0^{2\pi} |g(e^{it})|dt$. By (7.6) and (7.7)

$$\frac{1-|z|^2}{|e^{it}-z|^2} = 1 + \sum_{n=1}^{\infty} e^{-int} z^n + \sum_{n=1}^{\infty} e^{int}\, \overline{z}^n.$$

Interchanging the order of summation and integration, the harmonic "extension" of g to $\mathbb{D}$ is given by

$$G(z) \equiv PI(g)(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \overline{z}^n.$$

In other words, G is found from the Fourier series of g by replacing e^{it} with z and e^{-it} with $\overline{z}$.

The theory of Fourier series is an immense field that gave birth to harmonic analysis, geometric measure theory, set theory,...

One of the major theorems of 20th century analysis was Lennart Carleson's proof that the Fourier series of a continuous function (or more generally, an L^2 function) converges almost everywhere to that function. This can fail for L^1 functions.

See Lennart Carleson He was awarded the Abel prize in 2006.

There are many articles describing how problems in Fourier series led Cantor to develop modern set theory. Two such are:

How did Cantor Discover Set Theory and Topology? by S.M. Srivastava.

Trigonometric Series and Set Theory by A. Kechris.

Section 7.2: Cauchy-Riemann and Laplace Equations

If $f(z) = u(z) + iv(z)$ we will sometimes use the notation $f(x, y) = u(x, y) + iv(x, y)$ where $z = x + iy$ with x, y real and $u(x, y)$ and $v(x, y)$ are real-valued.

If f is analytic then by the definition of the complex derivative

$$\begin{aligned} f'(z) &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x(x, y) = u_x(x, y) + iv_x(x, y) \\ &= \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{ik} = \frac{1}{i} f_y(x, y) = v_y(x, y) - iu_y(x, y) \end{aligned}$$

Thus $u_x = v_y$ and $u_y = -v_x$.

These are called the **Cauchy-Riemann equations**.

Write $z = x + iy$ and define

$$f_z \equiv \frac{\partial f}{\partial z} \equiv \frac{1}{2}\left(f_x - i f_y\right)$$

and

$$f_{\overline{z}} \equiv \frac{\partial f}{\partial \overline{z}} \equiv \frac{1}{2}\left(f_x + i f_y\right).$$

Then it is an easy exercise to verify the chain rule:

$$(f \circ g)_z = f_z \circ g \; g_z + f_{\overline{z}} \circ g \; \overline{g}_z$$

and

$$(f \circ g)_{\overline{z}} = f_z \circ g \; g_{\overline{z}} + f_{\overline{z}} \circ g \; \overline{g}_{\overline{z}}.$$

The Cauchy-Riemann equations can be restated in this terminology as

$$f_{\overline{z}} = 0.$$

The derivative of f is just $f' = f_z$.

Theorem 7.9: *Suppose u and v are real-valued and continuously differentiable on a region Ω. Then u and v satisfy the Cauchy-Riemann equations if and only if $f = u + iv$ is analytic on Ω.*

Proof. Earlier we proved if $f = u + iv$ is analytic then u and v satisfy the Cauchy-Riemann equations.

Conversely, if u and v satisfy the Cauchy-Riemann equations, then by Taylor's theorem applied to the real-valued functions u and v,

$$\begin{aligned} f(x+h,y+k)-f(x,y) &= h\cdot u_x(x,y)+k\cdot u_y(x,y) \\ &\quad +i(hv_x(x,y)+kv_y(x,y) \\ &\quad +\epsilon(h,k) \end{aligned}$$

where $\epsilon(h,k)/\sqrt{|h|^2+|k|^2}\to 0$ as $|h|,|k|\to 0$.

Dividing by $h+ik$ and applying the Cauchy-Riemann equations we obtain

$$\lim_{h,k\to 0}\frac{f(x+h,y+k)-f(x,y)}{h+ik}=u_x(x,y)-iu_y(x,y).$$

So $f'(z)$ exists and is continuous, and therefore f is analytic. $\square$

Definition: If $f = u + iv$ is analytic in a region Ω then v is called a **harmonic conjugate** of u in Ω.

Because $-if$ is analytic, $-u$ is a harmonic conjugate of v.

The difference of two non-constant analytic functions is an open map, and. and hence cannot be purely imaginary on an open set. Thus if v_1 and v_2 are harmonic conjugates of u on a region Ω then $v_1 - v_2$ is constant.

Theorem 7.10:

(a) A function u is harmonic on a region Ω *if and only if* $2u_{\overline{z}} = u_x - iu_y$ *exists and is analytic on* Ω*.*

(b) If Ω *is simply-connected then u is harmonic on* Ω *if and only if* $u = \mathrm{Re(f)}$ *for some* f *analytic on* Ω*.*

Proof of (a):

Proof. If u is harmonic on Ω and if D is a disk contained in Ω then by Cor 7.7, $f = u + iv$ for some analytic function f on D. Moreover f' is analytic and $f' = u_x + iv_x = u_x - iu_y$ by the Cauchy-Riemann equations. This proves that $u_x - iu_y$ exists and is analytic on each B and hence on Ω.

Conversely, if $g = u_x - iu_y$ exists and is analytic on Ω, then g has a power series expansion on any disk $D \subset \overline{D} \subset \Omega$. Integrating the series term by term gives an analytic function f with $f' = g$.

If $w =$ Re(f) then by the Cauchy-Riemann equations, $w_x = u_x$ and $w_y = u_y$, so that $u = w + c$ on D where c is a constant. Since $w =$ Re(f) is harmonic, u must also be harmonic on B and hence on all of Ω. ◻

Proof of (b):

Proof. If u is harmonic on a simply-connected region Ω, then by previous argument, there is an analytic function f on all of Ω such that $f' = u_x - iu_y$.

By the Cauchy-Riemann equations $w = \mathrm{Re(f)}$ and u have the same partial derivatives on Ω and so $u = \mathrm{Re(f+c)}$ for some constant c. Hence u is harmonic.

□

If f is analytic then f has continuous partial derivatives of all orders. By the Cauchy-Riemann equations applied to $u = \mathrm{Re(f)}$ and $v = \mathrm{Im(f)}$,

$$u_{xx} = (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy}, \tag{2.2}$$

and hence $u_{xx} + u_{yy} = 0$. Similarly $v_{xx} + v_{yy} = 0$.

Definition 7.11: The Laplacian of u is the second-order derivative given by $\Delta u = u_{xx} + u_{yy}$.

We say that u **satisfies Laplace's equation** on a region Ω if u has continuous second-order partial derivatives (including the mixed partials) and $\Delta u = 0$ on Ω.

Theorem 7.12: *Suppose u is real-valued and continuous on a region Ω. Then the following are equivalent:*

(1) *u is harmonic on Ω*

(2) *u satisfies Laplace's equation on Ω*

(3) *If D is an open disk with $D \subset \overline{D} \subset \Omega$ and if v is harmonic on D, then $u - v$ and $v - u$ satisfy the maximum principle on D.*

We will prove

(1) $\Rightarrow$ (2) (already done above)

(1) $\Rightarrow$ (3) (already done by maximum principle)

(3) $\Rightarrow$ (1)

(2) $\Rightarrow$ (1)

Proof of (3) $\Rightarrow$ (1):

Proof. If (3) holds and if $D \subset \overline{D} \subset \Omega$, then let v be the Poisson integral of $u|_{\partial D}$ on D. Then v is harmonic on D and $u - v$ and $v - u$ are equal to 0 on ∂D by Theorem 75. By (3), $u = v$ on D. Thus u is harmonic, so (1) holds.

Proof of (2) $\Rightarrow$ (1):

Set $g = u_x - iu_y$. Now if R is a rectangle with sides parallel to the axes contained in Ω, we claim that $\int_{\partial R} g(\zeta)d\zeta = 0$. To see this, note that

$$\int_{\partial R} (u_x - iu_y)(dx + idy) = \int_{\partial R} u_x dx + u_y dy + i \int_{\partial R} u_x dy - u_y dx.$$

By the fundamental theorem of calculus applied to each segment in ∂R, the first integral is zero.

Also by the fundamental theorem of calculus, integrating along horizontal lines in R and vertical lines in R, the second integral can be rewritten as

$$\int_R (u_{xx} + u_{yy}) dx dy,$$

which is also equal to 0 by (2).

By Morera's theorem, $g = u_x - iu_y$ is analytic on Ω, so $u = \mathrm{Re(g)}$ is harmonic.

$\square$

Corollary 7.13: *If f is continuously differentiable (with respect to x and y) on a region Ω and if f preserves angles between curves at each point of Ω then f is analytic in Ω and $f' \neq 0$ on Ω.*

Proof. Suppose $z_0 \in \Omega$ and $\theta \in [0, 2\pi]$. Set $\gamma(t) = z_0 + te^{i\theta}$ and $w(t) = f(\gamma(t))$. Because f preserves angles between curves at z_0, the angle between $w(t)$ and $\gamma(t)$ at $t = 0$, $\arg(w'(0)/\gamma'(0))$, does not depend on θ.

By the chain rule

$$w'(t) \;=\; f_z\gamma'(t) + f_{\overline{z}}\,\overline{\gamma'}(t) = f_z e^{i\theta} + f_{\overline{z}}\, e^{-i\theta},$$

so that

$$\frac{w'(0)}{\gamma'(0)} = f_z + f_{\overline{z}}\, e^{-2i\theta},$$

But left side is constant in θ, so right side is too. Thus $f_{\overline{z}}(z_0) = 0$. By Theorem 7.9, f is analytic.

If f preserve angles at z, then $f'(z) \neq 0$ (otherwise angle are multiplied). $\square$

Section 7.3: Hadamard, Lindelöf and Harnack

The maximum principle has refinements that are very important in applications.

If u is harmonic in $\mathbb{D}$ and not constant then for interior points z, $u(z)$ is **strictly** smaller than $\sup_{\mathbb{T}} u$. We want to quantity how much smaller.

Theorem 7.14, Hadamard's Three-Circles Theorem *Suppose f is analytic in the annulus $A = \{z : r < |z| < R\}$. Let $m = \limsup_{|z|\to r} |f(z)|$ and $M = \limsup_{|z|\to R} |f(z)|$, and suppose $m, M < \infty$. If $z \in A$, then*

$$|f(z)| \leq M^{\omega(z)} m^{1-\omega(z)},$$

where $\omega(z) = \log(|z|/r)/\log(R/r)$.

Proof. The function $\omega \log M + (1-\omega)\log m$ is harmonic on A and equal to $\log M$ on $|z| = R$ and equal to $\log m$ on $|z| = r$. Thus

$$u = \log|f| - (\omega \log M + (1-\omega)\log m)$$

is subharmonic on A and $\limsup_{z\to\partial A} u(z) \le 0$. By the maximum principle, $u \le 0$ in A. □

The function ω is called "harmonic measure" of the annulus.

Given a domain Ω and a set $E \subset \partial\Omega$, $\omega(z, E, \Omega)$ is the harmonic function on Ω with boundary values 1 on E and 0 off E (boundary values in what sense, needs to be explained).

Intuitively, $\omega(z)$ is the probability that a Brownian motion started at z first hits $\partial\Omega$ in the set E. It is not hard to see (heuristically, at least) that this probability satisfies the mean value property and has correct boundary values.

Theorem, Lindelöf: *Suppose* Ω *is a region and suppose* $\{\zeta_1, \ldots, \zeta_n\}$ *is a finite subset of* $\partial\Omega$*, not equal to all of* $\partial\Omega$*. If* u *is subharmonic on* Ω *with* $u \leq M < \infty$ *on* Ω *and if*

$$\limsup_{z \in \Omega \to \zeta} u(z) \leq m,$$

for all $\zeta \in \partial\Omega \setminus \{\zeta_1, \ldots, \zeta_n\}$ *then* $u \leq m$ *on* Ω*.*

In the statement of Lindelöf's theorem, if Ω is unbounded, then we view ∞ as a boundary point, which may or may not be one of the exceptional points $\{\zeta_j\}$.

Proof. First suppose that Ω is bounded and let $d = \operatorname{diam}(\Omega)$. For $\epsilon > 0$ set

$$u_\epsilon(z) = u(z) + \epsilon \sum_{j=1}^{n} \log \left| \frac{z - \zeta_j}{d} \right|. \tag{7.13}$$

Then u_ϵ is subharmonic in Ω, $u_\epsilon \leq u$ and $u_\epsilon \to -\infty$ as $z \to \zeta_j$, for $j = 1, \ldots, n$. Thus $\limsup_{z \to \partial\Omega} u_\epsilon(z) \leq m$, and so by the maximum principle $u_\epsilon \leq m$ on Ω. Fix z and let $\epsilon \to 0$ in (7.13) to obtain $u(z) \leq m$.

If Ω is not bounded, we may suppose that $\zeta_j \neq \infty$ for $j = 1, \ldots, n$ by composing with an LFT if necessary. Given $\epsilon > 0$, we can choose R so that $R > \max_j |\zeta_j|$ and $u(z) \leq m + \epsilon$ for $z \in \Omega \cap \{|z| > R\}$. Now apply the bounded case to $u - \epsilon$ on $\Omega \cap \{|z| < R\}$ to conclude that $u - \epsilon \leq m$ on $\Omega \cap \{|z| < R\}$ and hence on Ω. Let $\epsilon \to 0$ to conclude $u \leq m$ on Ω. $\square$

The finite exceptional set can be replaced by any countable set, and by some uncountable sets, i.e., the sets of zero logarithmic capacity.

E has zero logarithmic capacity if there is a harmonic function u off E so that $u(z) \to \infty$ as $z \to E$. This is only property of finite sets that proof will use.

A consequence of Lindelöf maximum principle is the **Three-Lines** version of Hadamard's theorem, which plays an important role in complex interpolation theory of operators.

It follows from the proof of Hadamard's three-circles theorem and Lindelöf's maximum principle.

Corollary 7.16: *If u is subharmonic on the strip $S_0 = \{z = x + iy : 0 < x < 1\}$ set $m_0 = \limsup_{\mathrm{Re} z \to 0} u(z)$ and $m_1 = \limsup_{\mathrm{Re} z \to 1} u(z)$. If $u \leq M < \infty$ on S_0 then $u(z) \leq m_1 x + m_0(1 - x)$.*

Suppose g and h are continuous functions on a compact set X and suppose $d\mu$ is a positive measure on X. Then the function

$$F(z) = \int_X |g|^{pz}|h|^{q(1-z)}d\mu$$

is analytic on S_0 for $p, q > 0$. To prove this apply Morera's theorem after interchanging the order of integration.

Thus

$$\log\left|\int_X |g|^{pz}|h|^{q(1-z)}d\mu\right|$$

is subharmonic on S_0, and bounded above.

By Corollary 7.16, for $0 < x < 1$,

$$\log \left| \int_X |g|^{pz} |h|^{q(1-z)} d\mu \right| \leq x \log \int_X |g|^p d\mu + (1-x) \log \int_X |h|^q d\mu.$$

If $1/p + 1/q = 1$ with $p > 1$ then set $z = x = 1/p$ and exponentiate to obtain

$$\int_X |gh| d\mu \leq \left(\int_X |g|^p d\mu \right)^{\frac{1}{p}} \left(\int_X |h|^q d\mu \right)^{\frac{1}{q}},$$

which is called **Hölder's inequality**.

Riesz–Thorin interpolation theorem: *Let* $(\Omega_1.\Sigma_1, \mu_1)$ *and* $(\Omega_2.\Sigma_2, \mu_2)$ *be* σ*-finite measure spaces. Suppose* $1 \leq p_0, q_0, p_1, q_1 \leq \infty$*, and let*

$$T : L^{p_0}(\mu_1) + L^{p_1}(\mu_1) \to L^{q_0}(\mu_2) + L^{q_1}(\mu_2)$$

be a linear operator that boundedly maps $L^{p_0}(\mu_1)$ *to* $L^{q_0}(\mu_1)$ *and* $L^{p_1}(\mu_2)$ *to* $L^{q_1}(\mu_2)$*. For* $0 < \theta < 1$*, let* p_θ *and* q_θ *be define as*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T *boundedly maps* $L^{p_\theta}(\mu_1)$ *into* $l^{q_\theta}(\mu_2)$*, and satisfies the operator norm estimate*

$$\|T\|_{L^{p_\theta} \to L^{q_\theta}} \leq \|T\|^{1-\theta}_{L^{p_0} \to L^{q_0}} \cdot \|T\|^{\theta}_{L^{p_1} \to L^{q_1}}.$$

Proof can be found in several harmonic analysis texts.

Theorem 7.17, Harnack's Inequality: *Suppose u is a positive harmonic function on $\mathbb{D}$. Then for $|z| = r <$,*

$$\left(\frac{1-r}{1+r}\right) u(0) \leq u(z) \leq \left(\frac{1+r}{1-r}\right) u(0),$$

Proof. We may assume u is harmonic on $\overline{\mathbb{D}}$ by replacing u with $u(sz)$, $s < 1$ and then letting $s \to 1$.

$$\frac{1-r}{1+r} = \frac{1-r^2}{(1+r)^2} \leq \frac{1-|z|^2}{|e^{it}-z|^2} \leq \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r}.$$

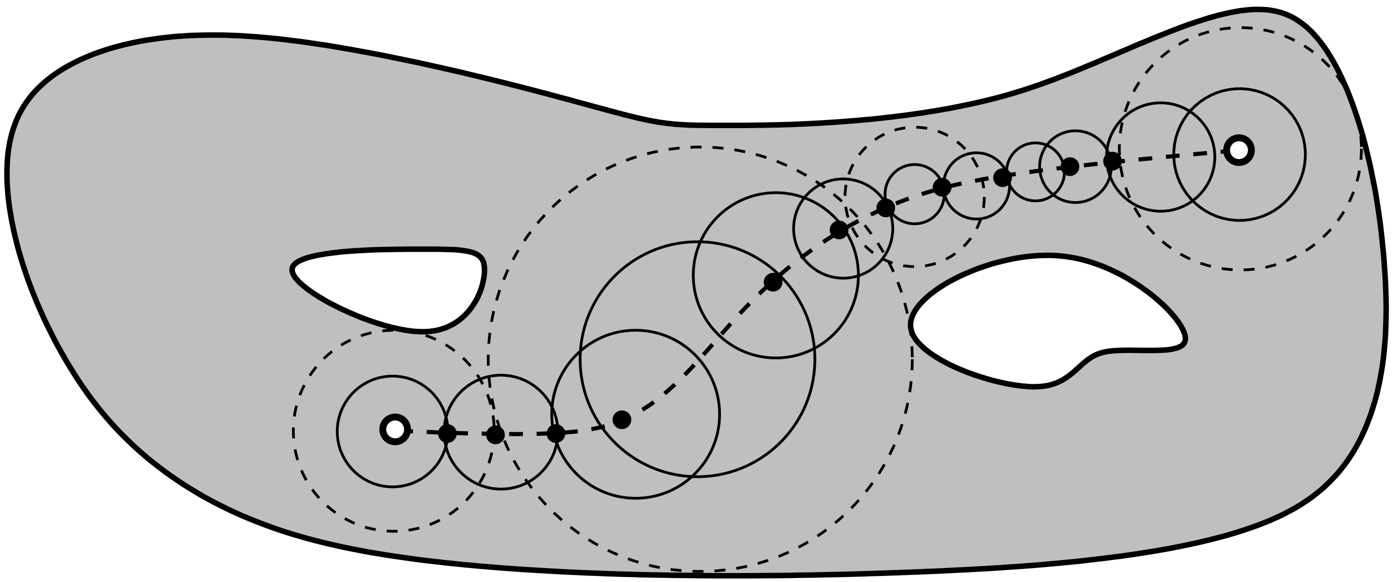
Then because u is positive and the mean-value property holds:

$$\left(\frac{1-r}{1+r}\right) u(0) \leq \int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} u(e^{it}) \frac{dt}{2\pi} \leq \left(\frac{1+r}{1-r}\right) u(0). \quad \square$$

Corollary 7.19 *Let K be a compact subset of a region Ω. Then there exists a constant C depending only on Ω and K such that if u is positive and harmonic on Ω then for all $z, w \in K$*

$$\frac{1}{C}u(w) \leq u(z) \leq Cu(w). \tag{3.2}$$

Proof. If D is a disk, let $2D$ be the disk with the same center as D and twice the radius. Suppose D is a disk such that $2D \subset \Omega$. Let φ be a linear map of $\mathbb{D}$ onto $2D$, then by Harnack's inequality applied to $u \circ \varphi$ we have that (7.14) holds for $z \in B$ and w equal to the center of B, with $C = 3$. Thus (7.14) holds for all $z, w \in B$ with $C = 9$.



Cover K by a finite collection of disks $\mathcal{D} = \{D_j\}$ with $2D_j \subset \Omega$. Add more disks if necessary so that $\cup D_j$ is connected. If $D_j, D_k \in \mathcal{D}$ with $D_j \cap D_k \neq \emptyset$ then (7.14) holds on $D_j \cup D_k$ with $C = 81$.

Because there are only finitely many disks and because their union is connected, (7.14) holds on $\cup\{D_j : D_j \in \mathcal{D}\}$, and therefore on K, with a constant C depending only on the number of disks in $\mathcal{D}$, and not on u. □

Theorem 7.20, Harnack's Principle: *Suppose* $\{u_n\}$ *are harmonic on a region* Ω *such that* $u_n(z) \leq u_{n+1}(z)$ *for all* $z \in \Omega$*. Then either*

(1) $\lim_{n\to\infty} u_n(z) \equiv u(z)$ *exists and is harmonic on* Ω*, or*

(2) $\lim_{n\to\infty} u_n(z) = +\infty$

where convergence is uniform on compact subsets of Ω*. this means that given* $K \subset \Omega$ *compact and* $M < \infty$ *there is an* $n_0 < \infty$ *so that* $u_n(z) \geq M$ *for all* $n \geq n_0$ *and* $z \in K$*.*

In the second case, convergence on compact sets means that given $K \subset \Omega$ compact and $M < \infty$, there is an $n_0 < \infty$ so that $u_n(z) \geq M$ for all $n \geq n_0$ and $z \in K$.

Proof. By assumption, if $n > m$ then $u_n - u_m \geq 0$, and by the maximum principle $u_n - u_m$ is strictly positive or identically 0.

Fix $z_0 \in K$ compact. By Cor 7.19 there is a C so that for all $z \in K$,

$$\frac{1}{C}(u_n(z_0) - u_m(z_0)) \leq u_n(z) - u_m(z) \leq C(u_n(z_0) - u_m(z_0)).$$

Thus $\{u_n(z_0)\}$ is Cauchy if and only if $\{u_n\}$ is uniformly Cauchy on K. Thus if the increasing sequence $\{u_n(z_0)\}$ converges, then $\{u_n\}$ converges uniformly on compact subsets of Ω.

Similarly if $u_n(z_0) \to \infty$ then $u_n(z) \to \infty$ uniformly on compact subsets of Ω. The limit function u is harmonic by the mean-value property. $\square$

One important consequence of the Harnack inequality is that all harmonic measures of a domain Ω are mutually absolutely continuous, for every choice of base point. In other words,

$$\omega(z_1, E, \Omega) = 0 \quad \Leftrightarrow \quad \omega(z_2, E, \Omega) = 0.$$

This may or may not be true base points are on different sides of a curve

Recall, two measures are **mutually absolutely continuous** if they have same sets of measure zero.

Two measures are **singular** if they give full mass to disjoint sets.

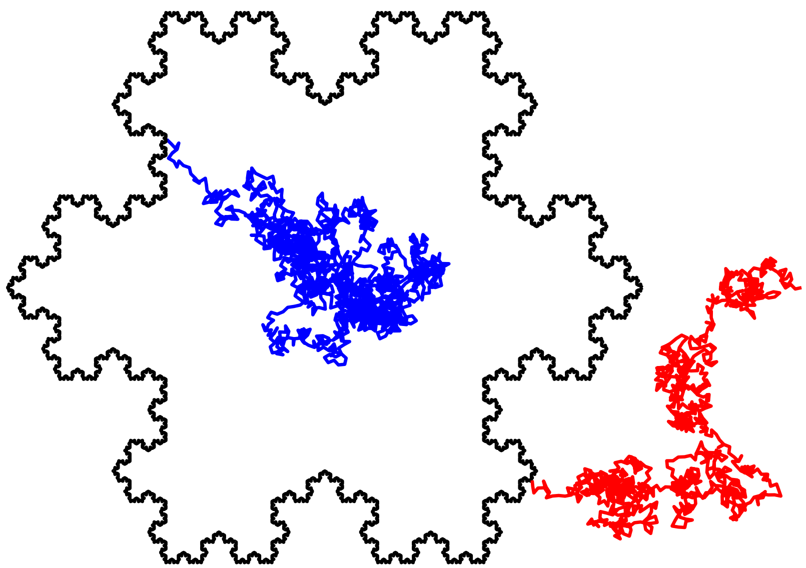
If γ is rectifiable closed curve, the F. and M. Riesz Theorem (1916) says that harmonic measure is mutually absolutely continuous to arclength on γ.

This is true for points on both sides of γ, so harmonic measures for both sides are mutually absolutely continuous.

Higher dimensional versions only proven quite recently (5-10 years ago). Uses singular integral theory and geometric measure theory.

It has been know from the 1930's (Lavrentiev) that for some curves, harmonic measures can be singular.

Theorem: Harmonic measures on opposite sides of γ are singular iff the set of tangent points has zero linear measure.



My PhD thesis

Theorem 7.18, Boundary Harnack inequality *Suppose u and v are positive harmonic functions on $\mathbb{D}$ which extend to be continuous and equal to 0 on a closed arc $I \subset \mathbb{T}$. Let $U_\delta = \{z \in \mathbb{D} : \mathrm{dist}(z, \mathbb{T} \setminus I) > \delta > 0\}$. Then for $z \in U_\delta$*

$$\frac{\delta^2}{4}\left(\frac{u(0)}{v(0)}\right) \le \frac{u(z)}{v(z)} \le \frac{4}{\delta^2}\left(\frac{u(0)}{v(0)}\right).$$

Proof. Fix $z \in U_\delta$ with $|z| < r < 1$ and set $\delta_r = \operatorname{dist}(z/r, \mathbb{T} \setminus I)$. By the Poisson integral formula

$$\begin{aligned}
\frac{u(z)}{1-|z/r|^2} &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{u(re^{it})}{|e^{it}-z/r|^2}dt \\
&\leq \frac{1}{2\pi}\int_{\mathbb{T}\setminus I} \frac{u(re^{it})}{\delta_r^2}dt + \frac{1}{2\pi}\int_{I} \frac{u(re^{it})}{|e^{it}-z/r|^2}dt \\
&\leq \frac{u(0)}{\delta_r^2} + \frac{1}{2\pi}\int_{I} \frac{u(re^{it})}{|e^{it}-z/r|^2}dt.
\end{aligned}$$

Similarly

$$\begin{aligned}\frac{v(z)}{1-|z/r|^2} &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{v(re^{it})}{|e^{it}-z/r|^2}dt \\ &\geq \frac{1}{2\pi}\int_{\mathbb{T}\setminus I} \frac{v(re^{it})}{4}dt + \frac{1}{2\pi}\int_I \frac{v(re^{it})}{|e^{it}-z/r|^2}dt \\ &= \frac{v(0)}{4} - \frac{1}{2\pi}\int_I \frac{v(re^{it})}{4}dt + \frac{1}{2\pi}\int_I \frac{v(re^{it})}{|e^{it}-z/r|^2}dt.\end{aligned}$$

Therefore

$$\frac{u(z)}{v(z)} = \frac{\frac{u(0)}{\delta_r^2} + \frac{1}{2\pi}\int_I \frac{u(re^{it})}{|e^{it}-z/r|^2}dt}{\frac{v(0)}{4} - \frac{1}{2\pi}\int_I \frac{v(re^{it})}{4}dt + \frac{1}{2\pi}\int_I \frac{v(re^{it})}{|e^{it}-z/r|^2}dt}$$

Letting $r \to 1$ we obtain the right-hand inequality, since $\delta_r \to \operatorname{dist}(z, \mathbb{T} \setminus I) > \delta > 0$, and $u(re^{it})$ and $v(re^{it})$ converge uniformly to 0 on I.

The left-hand inequality is proved by reversing the roles of u and v. $\square$