MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop
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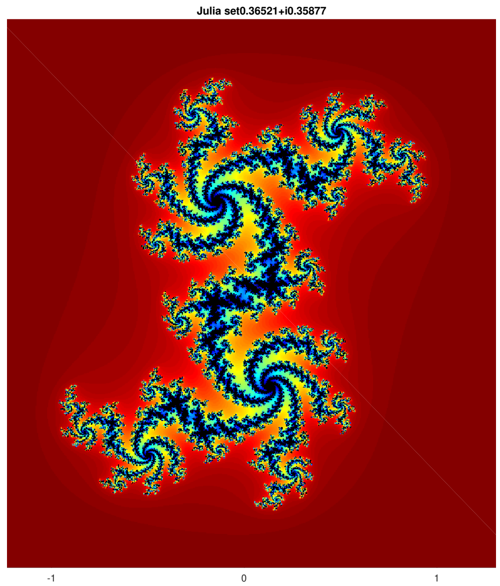
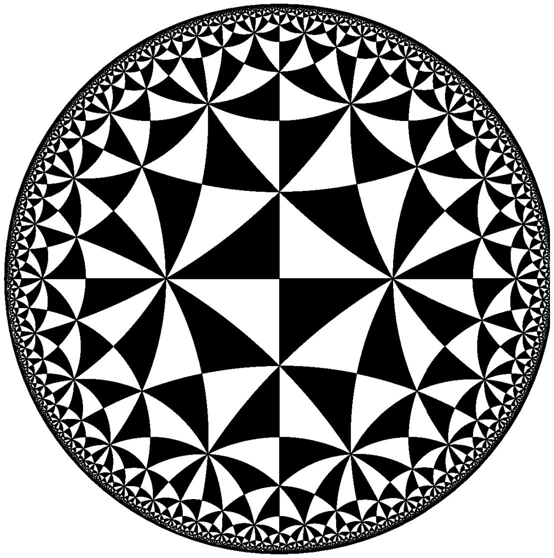
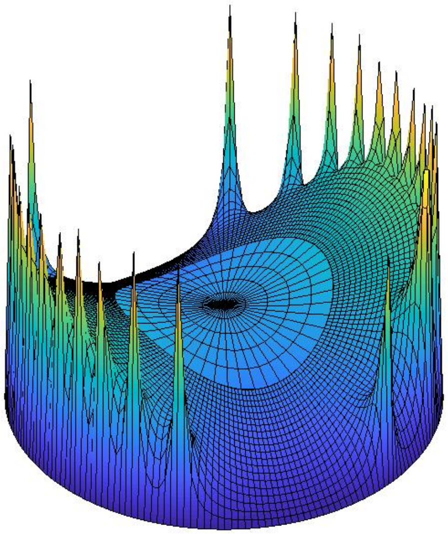






Chapter 5: Cauchy's Theorem

Section 5.1: Cauchy's Theorem

Recall that a **cycle** $\gamma = \sum_{j=1}^{n} \gamma_j$ is a finite union of closed curves $\gamma_1, \ldots, \gamma_n$.

Theorem 5.1, Cauchy's Theorem: *Suppose γ is a cycle contained in a region Ω and suppose*

$$\int_\gamma \frac{d\zeta}{\zeta - a} = 0 \tag{5.2}$$

for all $a \notin \Omega$. If f is analytic on Ω then

$$\int_\gamma f(\zeta) d\zeta = 0.$$

Proof. By Runge's theorem, we can find a sequence of rational function r_n with poles in $\mathbb{C} \setminus \Omega$ so that r_n converges to f uniformly on the compact set $\gamma \subset \Omega$.

Each rational functions has a partial fraction expansion

$$r(z) = \sum_{k=1}^{N} \sum_{j=1}^{n_k} \frac{c_{k,j}}{(z - p_k)^j} + q(z)$$

and integrating around a closed curve γ gives zero, except for the simple poles. But by our assumption (5.2), these integrals also vanish.

By uniform convergence,

$$|\int_\gamma f(z)dz| = |\int_\gamma (f(z) - r_n(z))dz| \leq \sup_\gamma |f - r_n| \ell(\gamma) \to 0. \quad \square$$

Theorem 5.2, Cauchy's Integral Formula *Suppose γ is a cycle contained in a region Ω and suppose*

$$\int_\gamma \frac{d\zeta}{\zeta - a} = 0$$

for all $a \notin \Omega$. If f is analytic on Ω and $z \in \mathbb{C} \setminus \gamma$ then

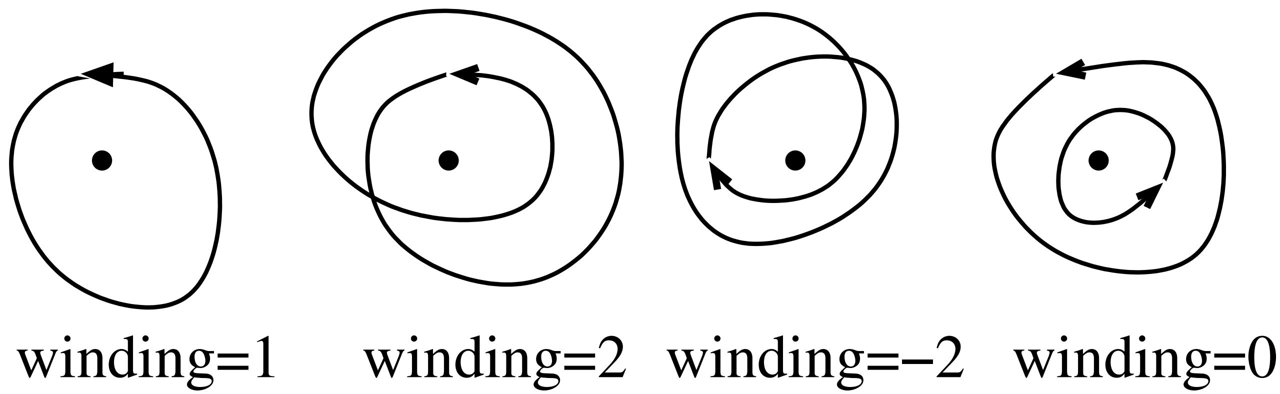
$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \cdot \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z} d\zeta.$$

Proof. For each $z \in \Omega$ the function $g(\zeta) = (f(\zeta) - f(z))/(\zeta - z)$ extends to be analytic on Ω, by Exercise 2.5.

By Cauchy's theorem it has integral over γ equal to 0. Theorem 5.2 follows by splitting the integral of g along γ into two pieces. □

Section 5.2: Winding number

Idea: intuitively, the winding number is the number of times a curve winds around a point.



It is the total change in $\arg(z-a)$ as z travels around γ.

Lemma 5.3 *If γ is a cycle and $a \notin \gamma$, then*

$$\frac{1}{2\pi i}\int_{\gamma}\frac{1}{\zeta - a}d\zeta$$

is an integer.

Proof. WLOG we may suppose γ is a closed curve parameterized by a continuous, piecewise differentiable function $\gamma : [0,1] \to \mathbb{C}$. Define

$$h(x) = \int_0^x \frac{\gamma'(t)}{\gamma(t) - a} dt.$$

Then $h'(x)$ exists and equals $\gamma'(x)/(\gamma(x) - a)$, except at finitely many points x. Then

$$\begin{aligned} \frac{d}{dx} e^{-h(x)}(\gamma(x) - a) &= -h'(x)e^{-h(x)}(\gamma(x) - a) + e^{-h(x)}\gamma'(x) \\ &= -\gamma'(x)e^{-h(x)} + \gamma'(x)e^{-h(x)} = 0, \end{aligned}$$

except at finitely many points.

Since $e^{-h(x)}(\gamma(x) - a)$ is continuous, it must be constant. Thus

$$e^{-h(1)}(\gamma(1) - a) = e^{-h(0)}(\gamma(0) - a) = 1 \cdot (\gamma(1) - a).$$

Since $\gamma(1) - a \neq 0$, $e^{-h(1)} = 1$ and $h(1) = 2\pi k i$, where k is an integer. Thus

$$\frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - a} d\zeta = \frac{h(1)}{2\pi i} = k,$$

an integer. □

Definition 5.4 If γ is a cycle, then the **index** or **winding number** of γ about a (or with respect to a) is

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - a},$$

for $a \notin \gamma$.

(1) $n(\gamma, a)$ is an analytic function of a, for $a \notin \gamma$, by Lemma 4.30 and Theorem 4.32. In particular it is continuous and integer-valued, and thus $n(\gamma, a)$ is constant in each component of $\mathbb{C} \setminus \gamma$.

(2) $n(\gamma, a) \to 0$ as $a \to \infty$. Thus $n(\gamma, a) = 0$ in the unbounded component of $\mathbb{C} \setminus \gamma$.

(3) $n(-\gamma, a) = -n(\gamma, a)$.

(4) $n(\gamma_1 + \gamma_2, a) = n(\gamma_1, a) + n(\gamma_2, a)$.

(5) If $\gamma(t) = e^{ikt}$, for $0 \le t \le 2\pi$, where k is an integer, then

$$n(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ike^{ikt}}{e^{ikt}} dt = k.$$

Definition 5.5 Closed curves γ_1 and γ_2 are **homologous** in a region Ω if $n(\gamma_1 - \gamma_2, a) = 0$ for all $a \notin \Omega$ and in this case we write

$$\gamma_1 \sim \gamma_2.$$

• Homology is an equivalence relation on the curves in Ω (Exercise 2). A closed curve $\gamma \subset \Omega$ is said to be **homologous to** 0 **in** Ω if $n(\gamma, a) = 0$ for all $a \notin \Omega$. In this case we write $\gamma \sim 0$.

• If γ is homotopic to zero, then it is homologous to zero, but not conversely.

• Cauchy's theorem says that if $\gamma \sim 0$ in Ω and if f is analytic in Ω then

$$\int_\gamma f(z)dz = 0,$$

Thus if γ_1 is homologous to γ_2 in Ω, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

• The most common application of Cauchy's integral formula is when $\gamma \subset \Omega$ with $\gamma \sim 0$ and $n(\gamma, z) = 1$. Then for f analytic on Ω,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta.$$

If Ω is a bounded region in $\mathbb{C}$ bounded by finitely many piecewise differentiable curves, then we can parameterize $\partial\Omega$ so that as you trace each boundary component, the region Ω lies on the left.

In other words $i\gamma'(t)$, where it exists, is an "inner normal", rotated counterclockwise by $\pi/2$ from the tangential direction $\gamma'(t)$.

In Exercise 5.4 you are asked to show that $n(\partial\Omega, a) = 1$ for all $a \in \Omega$ and $n(\partial\Omega, a) = 0$ for all $a \notin \overline{\Omega}$. We call this the **positive orientation** of $\partial\Omega$. Thus for all such regions, $\partial\Omega \sim 0$ in any region containing $\overline{\Omega}$.

So by Cauchy's theorem, if Ω is a bounded region, bounded by finitely many piecewise differentiable curves and if f is analytic on $\overline{\Omega}$, then

$$f(z) = \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}, \tag{5.4}$$

Definition 5.6: A region $\Omega \subset \mathbb{C}^*$ is called **simply-connected** if $\mathbb{C}^* \setminus \Omega$ is connected in $\mathbb{C}^*$.

Equivalently a region Ω is simply-connected if $\mathbb{S}^2 \setminus \pi(\Omega)$ is connected, where π is stereographic projection and $\pi(\infty)$ is defined to be the "North pole" $(0, 0, 1)$ in $\mathbb{S}^2$.

Simply-connected essentially means “no holes”. For example, the unit disk $\mathbb{D}$ is simply-connected. The vertical strip $\{z : 0 < \mathrm{Re} z < 1\}$ is simply-connected. The punctured plane $\mathbb{C} \setminus \{0\}$ is not simply-connected.

The set $\mathbb{C} \setminus \overline{\mathbb{D}}$ together with ∞ is simply-connected, but $\mathbb{C} \setminus \overline{\mathbb{D}}$ is not simply-connected.

Theorem 5.7: *A region* $\Omega \subset \mathbb{C}$ *is simply-connected if and only if every cycle in* Ω *is homologous to* 0 *in* Ω. *If* Ω *is not simply-connected then we can find a simple closed polygonal curve contained in* Ω *which is not homologous to* 0.

Proof. Suppose Ω is simply-connected and suppose γ is a cycle contained in Ω and suppose $a \notin \Omega$. Because $\Omega^c = \mathbb{C}^* \setminus \Omega$ is connected, it must be contained in one component of the complement of γ in $\mathbb{C}^*$. Because $\infty \in \Omega^c$, a must be in the unbounded component of $\mathbb{C} \setminus \gamma$, and $n(\gamma, a) = 0$.

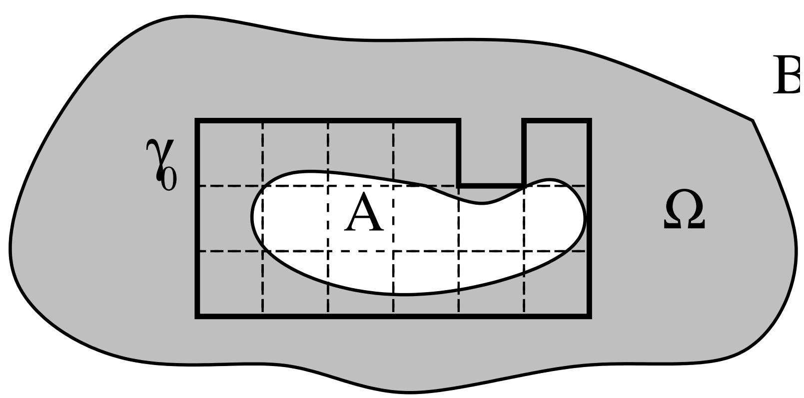
Conversely, suppose that $\mathbb{C}^* \setminus \Omega = A \cup B$ where A and B are non-empty closed sets in $\mathbb{C}^*$ with $A \cap B = \emptyset$. Without loss of generality $\infty \in B$. Since A is closed, a neighborhood of ∞ does not intersect A and hence A is bounded. Pick $a_0 \in A$. We'll construct a curve $\gamma_0 \subset \Omega$ such that $n(\gamma_0, a_0) \neq 0$, proving Theorem 2.5.

The construction is the same construction used to prove Runge's theorem.

Let $d = \text{dist}(A, B) = \inf\{|a - b| : a \in A, b \in B\} > 0$.

Pave the plane with squares of side $d/2$ such that a_0 is the center of one of the squares. Orient the boundary of each square in the positive, or counter-clockwise direction Shade each square S_j with $\overline{S_j} \cap A \neq \emptyset$.

Let γ_0 denote the cycle obtained from $\cup \partial S_j$ after performing all possible cancellations. Then $\gamma_0 \subset \Omega$ because γ_0 does not intersect either A or B, and $n(\gamma_0, a_0) = 1$. Because γ is a finite union of closed simple polygonal curves, at least one of these curves is not homologous to 0. □



Corollary 5.8 *Suppose f is analytic on a simply-connected region Ω. Then*

(1) $\int_\gamma f(z)dz = 0$ *for all closed curves* $\gamma \subset \Omega$.

(2) *There exists a function F analytic on Ω such that $F' = f$.*

(3) *If also $f(z) \neq 0$ for all $z \in \Omega$ then there exists a function g analytic on Ω such that $f = e^g$.*

Proof. The first statement follows from Cauchy's theorem and Theorem 5.7.

For the 2nd part, follow the proof of Morera's (Theorem 4.19).

Fix $z_0 \in \Omega$ and define $F(z) = \int_{\sigma_z} f(\zeta)d\zeta$ where σ_z is any curve contained in Ω connecting z_0 to z. This definition does not depend on the choice of σ_z because the integral along any closed curve is zero.

If $D \subset \Omega$ is a disk, then for $z \in D$ we can write $F(z)$ as an integral from z_0 to the center of D plus an integral along a horizontal then vertical line segment from the center of D to z. As in the proof of Morera's Theorem, F is analytic and $F' = f$ on D and hence on all of Ω.

to prove the third statement, note f'/f is analytic on Ω, so there is a function g analytic on Ω such that $g' = f'/f$ (previous argument).

To compare f and e^g, set

$$h = \frac{f}{e^g} = fe^{-g}.$$

Then

$$h' = f'e^{-g} - fg'e^{-g} = f'e^{-g} - f'e^{-g} = 0.$$

This implies h is a constant. Adding a constant to g, we may suppose $e^{g(z_0)} = f(z_0)$, so that $h \equiv 1$ and $f = e^g$. $\square$

Definition 5.9: If g is analytic in a region Ω and if $f = e^g$ then g is called a **logarithm** of f in Ω and written $g(z) = \log f(z)$. The function g is uniquely determined by its value at one point $z_0 \in \Omega$.

If g is a logarithm of f, so is $g + 2\pi i$.

Some sources treat log as a "multi-valued" function, taking countable many different values at once.

Section 5.3: Removable Singularities

Corollary 5.10, Riemann's Removable Singularity Theorem: *Suppose f is analytic in $\Omega = \{z : 0 < |z-a| < \delta\}$ and suppose*

$$\lim_{z \to a} (z-a) f(z) = 0.$$

Then f extends to be analytic in $\{z : |z-a| < \delta\}$.

In particular, this happens if f is bounded in a neighborhood of a.

This is the way the theorem is often stated.

Proof. Fix $z \in \Omega$ and choose ϵ and r so that $0 < \epsilon < |z - a| < r < \delta$. Let C_ϵ and C_r denote the circles of radius ϵ and r centered at a, oriented in the counter-clockwise direction.

The cycle $C_r - C_\epsilon$ is homologous to 0 in Ω, so that by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Note that

$$\left| \int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \leq \max_{\zeta \in C_\epsilon} |f(\zeta)| \frac{1}{|z - a| - \epsilon} 2\pi\epsilon.$$

But if $\zeta \in C_\epsilon$ then $|f(\zeta)|\epsilon = |f(\zeta)||\zeta - a| \to 0$ as $\epsilon \to 0$ and hence

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{5.6}$$

By Lemma 4.30, the right side of (5.6) is analytic for $z \in D(a, r)$. Thus if we define $f(a)$ as the value of the right side of (5.6) when $z = a$, then this extension is analytic at a and we have extended f to be analytic in $D(a, r)$. $\square$

Definition: we say that a compact set E has **one-dimensional Hausdorff measure equal to** 0 if for every $\epsilon > 0$ there are finitely many disks D_j with radius r_j so that

$$E \subset \cup_j D_j$$

and

$$\sum_j r_j < \epsilon.$$

Corollary 5.11, Painlevé: *Suppose $E \subset \mathbb{C}$ is a compact set with one-dimensional Hausdorff measure 0. If f is bounded and analytic on $U \setminus E$, where U is open and $E \subset U$, then f extends to be analytic on U.*

Proof. As in the proof of Runge's theorem and Theorem 15.7, we can find a cycle $\gamma \subset U \setminus E$ which is the boundary of a finite union of closed squares $\{S_j\}$ so that $n(\gamma, a) = 0$ or 1 for all $a \notin \gamma$ and $n(\gamma, b) = 1$ for all $b \in \cup S_j \setminus \gamma \supset E$, and $n(\gamma, b) = 0$ for $b \notin \cup S_j$ and hence for all $b \in \mathbb{C} \setminus U$.

Cover E by finitely many disks D_j of radius r_j so that $\sum r_j < \epsilon$.

We may assume each D_j intersects E so that for small ϵ, each D_j is contained in $\cup S_j \setminus \gamma$.

Let $V = \{z : n(\gamma, z) = 1\}$, let $\sigma = \partial\big(\cup D_j\big)$, and let $\Omega = V \setminus \cup\overline{D_j}$. Then $\gamma + \sigma = \partial\Omega$, which we parametrize so that $\partial\Omega$ has positive orientation.

Then as in (5.4), $\gamma + \sigma \sim 0$ in $U \setminus \cup\overline{D_j}$, so that by Cauchy's theorem

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_\sigma \frac{f(\zeta)}{\zeta - z}d\zeta, \text{ for } z \in V \setminus \cup\overline{D_j}.$$

Fix $z \in V \setminus \overline{\cup D_j}$. Then the second integral tends to 0 as $\epsilon \to 0$ because $\ell(\sigma) \leq \ell(\cup D_j) < 2\pi\epsilon$ and because f is bounded, exactly as in the proof of Riemann's theorem. Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

provides an analytic extension of f to E, by Lemma 4.30. □

For almost a century, it was an open problem to characterize removable sets for bounded analytic functions, i.e., if f is bounded and analytic on $\mathbb{C} \setminus E$ then f is bounded and analytic on $\mathbb{C}$ (hence constant by Liouville's theorem).

This was finally accomplished by the Xavier Tolsa. He showed that is E non-removable for bounded holomorphic functions if and only if it supports a positive measure μ so that (1)

$$\mu(D(x,r)) \leq Mr$$

(for some $M < \infty$ and all $x \in \mathbb{R}^2$ and $r > 0$) and (2) E has finite Menger curvature in the sense that

$$c^2(\mu) = \int\int\int c^2(x,y,z)d\mu(x)d\mu(y)d\mu(z) < \infty,$$

where $c(x,y,z)$ is the reciprocal of the radius of the unique circle passing thorough (x,y,z).

See Analytic capacity, rectifiability, and the Cauchy integral by Xavier Tolsa.

This problem motivated a great deal of analysis and geometric measure theory over the last 50 years.

Section 5.4: Laurent Series

Definition: An **annulus** is the region between two concentric circles

If f is analytic on the annulus $A = \{z : r < |z - a| < R\}$ then by Runge's theorem, we can approximate f by a rational function with poles only at a.

Theorem 5.12: Laurent series *Suppose* f *is analytic on* $A = \{z : r < |z-a| < R\}$. *Then there is a unique sequence* $\{a_n\} \subset \mathbb{C}$ *so that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where the series converges uniformly and absolutely on compact subsets of A. *Moreover*

$$a_n = \frac{1}{2\pi i} \int_{C_s} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta, \tag{5.8}$$

where C_s *is the circle centered at* a *with radius* s, $r < s < R$, *oriented counter-clockwise.*

Proof. WLOG, $a = 0$. Set

$$f_s(z) = \frac{1}{2\pi i} \int_{C_s} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where $C_s = \{se^{it} : 0 \leq t \leq 2\pi\}$.

By Lemma 4.30 f_s is analytic off C_s. If $r < |z| < s_1 < s_2 < R$ then $C_{s_2} - C_{s_1} \sim 0$ with respect to A and $n(C_{s_2} - C_{s_1}, z) = 0$. By Cauchy's integral formula, $f_{s_2}(z) - f_{s_1}(z) = 0$. This says that $f_s(z)$ does not depend on s, so long as $r < |z| < s < R$.

Expanding $\frac{1}{\zeta - z}$ in a power series expansion about 0, and interchanging the order of summation and integration, as we have done before, we conclude that f_s has a power series expansion

$$f_s(z) = \sum_{n=0}^{\infty} a_n z^n, |z| < s,$$

where a_n satisfies (5.8).

Likewise $f_s(z)$ does not depend on s so long as $r < s < |z| < R$.

Expanding $\frac{1}{\zeta - z}$ in a power series expansion about ∞, i.e. in powers of $1/z$, and interchanging the order of summation and integration we conclude that f_s has a power series expansion

$$f_s(z) = -\sum_{n=1}^{\infty} a_{-n} z^{-n},$$

valid in $|z| > s$, where a_{-n} satisfies (5.8).

If $r < s_1 < |z| < s_2 < R$, then $C_{s_2} - C_{s_1} \sim 0$ in A, so that by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{C_{s_2} - C_{s_1}} \frac{f(\zeta)}{\zeta - z} d\zeta = f_{s_2}(z) - f_{s_1}(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

□

One consequence is that an analytic function f on A can be written as $f = f_1 + f_2$ where f_1 is analytic in $|z| < R$ and f_2 is analytic in $|z| > r$.

Laurent series are useful for analyzing the behavior of an analytic function near an isolated singularity.

Definition: We say that f has an **isolated singularity** at b if f is analytic in $0 < |z - b| < \epsilon$ for some $\epsilon > 0$ and $f(b)$ is not defined.

(1) If $a_n = 0$ for $n < 0$ then f extends to be analytic at b with $f(b) = a_0$. In this case we say that f has a **removable singularity** at b.

(2) If $a_n = 0$ for $n < n_0$ with $n_0 > 0$ and $a_{n_0} \neq 0$, then we can write

$$f(z) = (z-b)^{n_0} \sum_{n=0}^{\infty} a_{n_0+n}(z-b)^n = a_{n_0}(z-b)^{n_0} + a_{n_0+1}(z-b)^{n_0+1} + \dots.$$

In this case b is called a **zero of order** n_0.

(3) If $a_n = 0$ for $n < -n_0$ with $n_0 > 0$ and $a_{-n_0} \neq 0$ then we can write

$$f(z) = (z-b)^{-n_0} \sum_{n=0}^{\infty} a_{-n_0+n}(z-b)^n = \frac{a_{-n_0}}{(z-b)^{n_0}} + \frac{a_{-n_0+1}}{(z-b)^{n_0-1}} + \dots.$$

In this case b is called a **pole of order** n_0, and $|f(z)| \to \infty$ as $z \to b$.

In each of the above cases there is a unique integer k so that

$$\lim_{z \to b} (z - b)^k f(z)$$

exists and is non-zero, and

$$(z - b)^k f(z)$$

extends to be analytic and non-zero in a neighborhood of b.

(4) If $a_n \neq 0$ for infinitely many negative n, then b is called an **essential singularity**.

If f is analytic in $\{z : |z| > R\}$, then $f(1/z)$ has an isolated singularity at 0, and we say that f has an **isolated singularity at** ∞.

We classify this singularity at ∞ as a zero, pole or essential singularity if $f(1/z)$ has a zero, pole or (respectively) essential singularity at 0.

A non-polynomial entire function has an essential singularity at ∞.

These are called transcendental entire functions. Iterating such functions gives rise to transcendtal dynamics, a sub-field of holomorphic dynamics.

Sullivan's theorem says polynomials can't have wandering domains, but transcendental functions can.

Definition 5.13: A zero or pole is called **simple** if the order is 1.

Definition 5.14: If f is analytic in a region Ω except for isolated poles in Ω then we say that f is **meromorphic in** Ω. A meromorphic function in $\mathbb{C}$ is sometimes just called **meromorphic**.

Rational functions are the meromorphic functions on $\mathbb{S}^2$.

Theorem 5.15: If f is analytic in $U = \{z : 0 < |z - b| < \delta\}$ and if b is an essential singularity for f then $f(U)$ is dense in $\mathbb{C}$.

In other words, every (punctured) neighborhood of an essential singularity has a dense image.

Stronger versions are true: at most one value can be omitted from $f(U)$ (Picard's theorem, Theorem 10.14).

Proof. If not, there is $A \in \mathbb{C}$ and $\epsilon > 0$ so that $|f(z) - A| > \epsilon$ for all $z \in U$. Then

$$\frac{1}{f(z) - A}$$

is analytic and bounded by $1/\epsilon$ on U. By Riemann's theorem, $1/(f(z) - A)$ extends to be analytic in $U \cup \{b\}$.

Thus $f(z) - A$ is meromorphic in $U \cup \{b\}$ and hence f is meromorphic in $U \cup \{b\}$. The Laurent expansion for f then has at most finitely terms with a negative power of $z - b$, contradicting the assumption that b is an essential singularity. $\square$

Section 5.4: The Argument Principle

Theorem 5.16, Argument Principle: *Suppose f is meromorphic in a region Ω with zeros $\{z_j\}$ and poles $\{p_k\}$. Suppose γ is a cycle with $\gamma \sim 0$ in Ω and suppose $\{z_j\} \cap \gamma = \emptyset$ and $\{p_k\} \cap \gamma = \emptyset$. Then*

$$n(f(\gamma), 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, z_j) - \sum_k n(\gamma, p_k). \tag{5.9}$$

In the statement of the argument principle, if f has a zero of order k at z, then z occurs k times in the list $\{z_j\}$, and a similar statement holds for the poles.

If γ is a simple closed curve in Ω, with $n(\gamma, z) = 0$ or $= 1$ for all $z \notin \gamma$, and if γ is homologous to 0 in Ω, then the number of zeros "enclosed" by γ minus the number of poles "enclosed" by γ is equal to the winding number of the image curve $f(\gamma)$ about zero.

Proof. The first equality in (5.9) follows from the change of variables $w = f(z)$.

Note that $\gamma \sim 0$ and $\gamma \subset \Omega$ implies that $n(\gamma, a) = 0$ if a is sufficiently close to $\partial\Omega$. Thus $n(\gamma, z_j) \neq 0$ for only finitely many z_j and for only finitely many p_j because there are no cluster points of $\{z_j\}$ or $\{p_k\}$ in Ω.

This implies that the sums in (5.9) are finite.

Set $\Omega_1 = \Omega \setminus \{z_j : n(\gamma, z_j) = 0\} \cup \{p_k : n(\gamma, p_k) = 0\}$. Then $\gamma \sim 0$ in Ω_1.

If b is a zero or pole of f then we can write

$$f(z) = (z-b)^k g(z)$$

where g is analytic in a neighborhood of b and $g(b) \neq 0$. Then

$$f'(z) = k(z-b)^{k-1} g(z) + (z-b)^k g'(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{k}{z-b} + \frac{g'(z)}{g(z)}.$$

Since $g(b) \neq 0$, g'/g is analytic in a neighborhood of b and hence $(f'/f) - k(z-b)^{-1}$ is analytic near b. Thus

$$\frac{f'(z)}{f(z)} - \sum \frac{1}{z-z_j} + \sum \frac{1}{z-p_k} \tag{5.10}$$

is analytic in Ω_1. By Cauchy's theorem integrating (5.10) over γ gives (5.9). $\square$

Corollary 5.17, Rouché's theorem: *Suppose* γ *is a closed curve in a region* Ω *with* $\gamma \sim 0$ *in* Ω *and* $n(\gamma, z) = 0$ *or* $= 1$ *for all* $z \in \Omega \setminus \gamma$. *If* f *and* g *are analytic in* Ω *and satisfy*

$$|f(z) + g(z)| < |f(z)| + |g(z)| \tag{5.11}$$

for all $z \in \gamma$, *then* f *and* g *have the same number of zeros enclosed by* γ.

Proof. The function $\frac{f}{g}$ is meromorphic in Ω and satisfies

$$\left|\frac{f}{g}+1\right|< \left|\frac{f}{g}\right|+1 \tag{5.12}$$

on γ. By (5.11), $f \neq 0$ and $g \neq 0$ on γ, so that the hypotheses of the argument principle are satisfied.

The left side of (5.12) is the distance from $w = f(z)/g(z)$ to -1. But $|w - (-1)| = |w| + 1$ if and only if $w \in [0, \infty)$.

Thus the assumption (5.11) implies that $\frac{f}{g}(\gamma)$ omits the half-line $[0, \infty)$ and so 0 is in the unbounded component of $\mathbb{C} \setminus \frac{f}{g}(\gamma)$.

Hence $n(\frac{f}{g}(\gamma), 0) = 0$. By the argument principle, the number of zeros of $\frac{f}{g}$ equals the number of poles and so the number of zeros of f equals the number of zeros of g, counting multiplicity. $\square$

Example: How many zeros does $f(z) = z^9 - 2z^6 + z^2 - 8z - 2$ have in $\mathbb{D}$?

How many zeros does it have in $\{|z| < 2\}$?

Example: How many zeros does $f(z) = z^4 - 4z + 5$ have in $\mathbb{D}$?