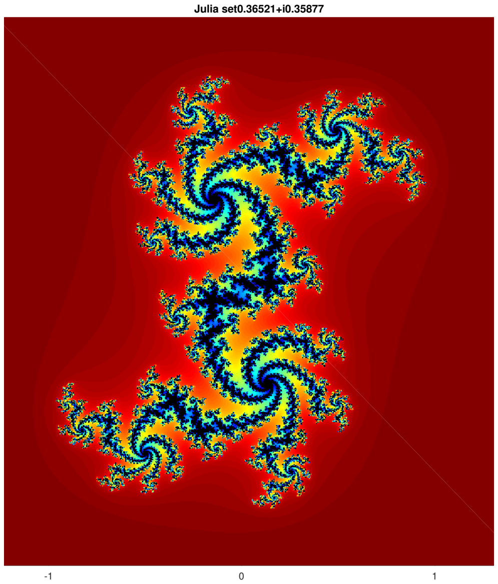
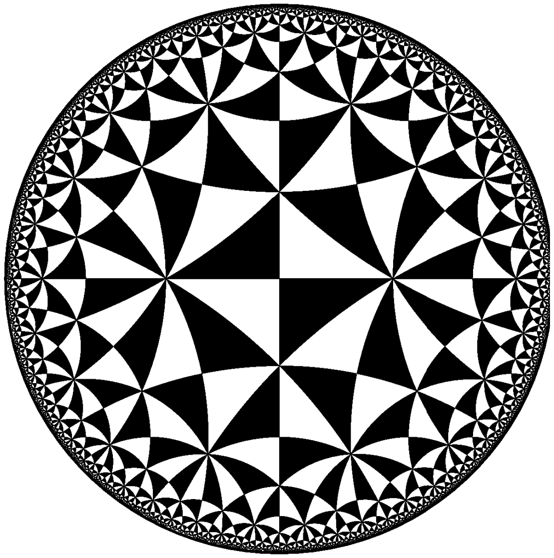
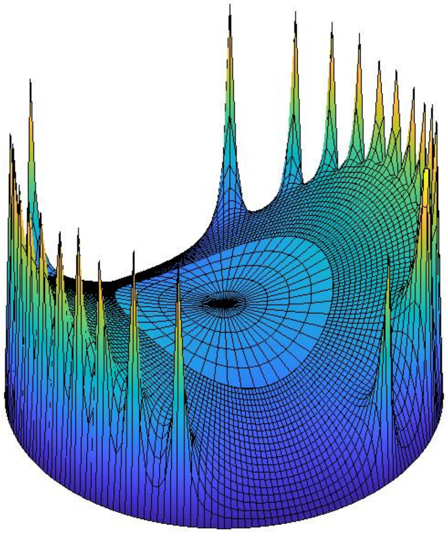
# MAT 536, Spring 2024, Stony Brook University

# Complex Analysis I, Christopher Bishop
# 2024





# Chapter 4: Integration and Approximation

# Section 4.1: Integration on curves

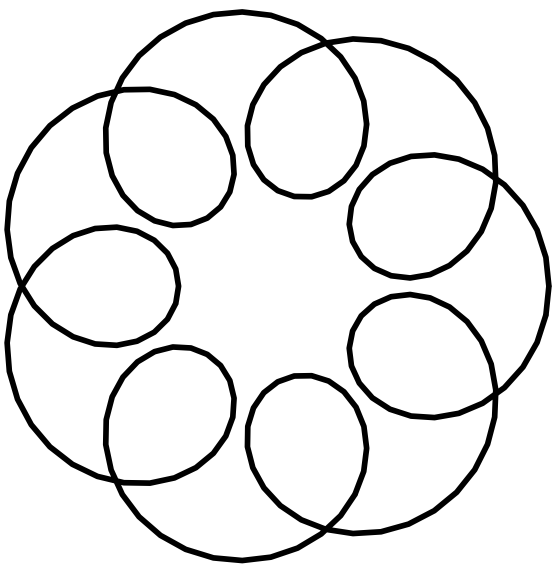
(All definitions, no theorems)

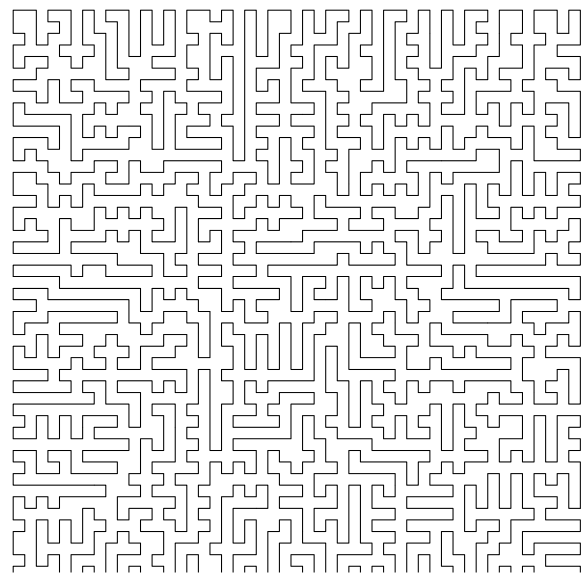
*Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions.*

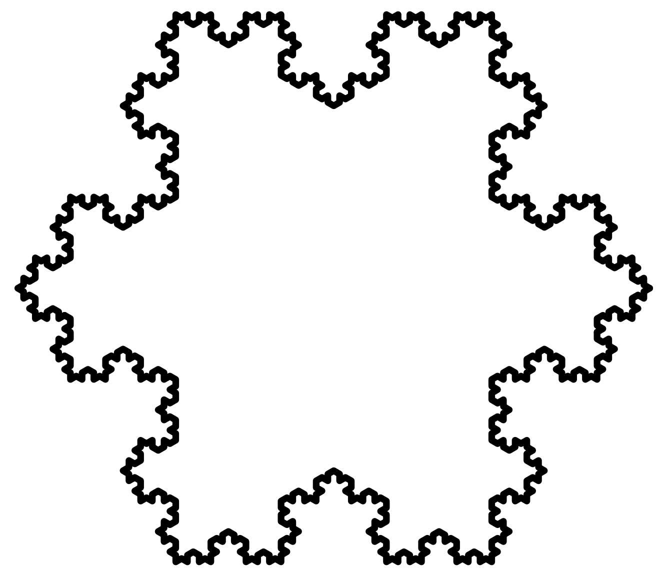
Felix Christian Klein

**Defn 4.1:** A **curve** is a continuous mapping of an interval $I \subset \mathbb{R}$ into $\mathbb{C}$.

The **trace** is the image of the curve. It is a set, whereas the curve is a function.







The trace of a curve can have positive area, or even have interior (Peano curve).

**Definition 4.2:**

(1) A curve $\gamma$ is called an **arc** if it is **one-to-one**.

(2) A curve $\gamma : [a, b] \to \mathbb{C}$ is called **closed** if $\gamma(a) = \gamma(b)$.

(3) A closed curve $\gamma : [a, b] \to \mathbb{C}$ is called **simple** if $\gamma$ restricted to $[a, b)$ is one-to-one.

**Definition 4.3:** A curve $\gamma(t) = x(t) + iy(t)$ is called **piecewise continuously differentiable** if

$$\gamma'(t) = x'(t) + iy'(t)$$

exists and is continuous except for finitely many $t$ and $x'$ and $y'$ have one-sided limits at the exceptional points.

**Examples:** any polygon, any smooth curve, any smooth image of a polygon.

If $\gamma$ is piecewise continuously differentiable then

$$\gamma(t_2) - \gamma(t_1) = (x(t_2) - x(t_1)) + i(y(t_2) - y(t_1)) = \int_{t_1}^{t_2} x'(t)dt + i \int_{t_1}^{t_2} y'(t)dt$$

Note that $\gamma(t_2) - \gamma(t_1)$ corresponds to the vector from $\gamma(t_1)$ to $\gamma(t_2)$, so that

$$\gamma'(t_1) = \lim_{t_2 \to t_1} \frac{\gamma(t_2) - \gamma(t_1)}{t_2 - t_1}$$

is tangent to the curve $\gamma$ at $t_1$, provided $\gamma'(t_1)$ exists.

**Definition 4.4:** A curve $\psi : [c, d] \to \mathbb{C}$ is called a **reparameterization** of a curve $\gamma : [a, b] \to \mathbb{C}$ if there exists a one-to-one, onto, increasing function $\alpha : [a, b] \to [c, d]$ such that $\psi(\alpha(t)) = \gamma(t)$.

**Definition 4.6:** If $\gamma : [a, b] \to \mathbb{C}$ is a curve, then $-\gamma : [-b, -a] \to \mathbb{C}$ is the curve defined by

$$-\gamma(t) = \gamma(-t).$$

**Definition 4.5:** If $\gamma : [a,b] \to \mathbb{C}$ is a piecewise continuously differentiable curve and if $f$ is a continuous complex-valued function defined on (the image of) $\gamma$ then

$$\int_\gamma f(z)dz \equiv \int_a^b f(\gamma(t))\gamma'(t)dt.$$

If $f$ is continuous on a piecewise continuously differentiable curve $\gamma$, then

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$$

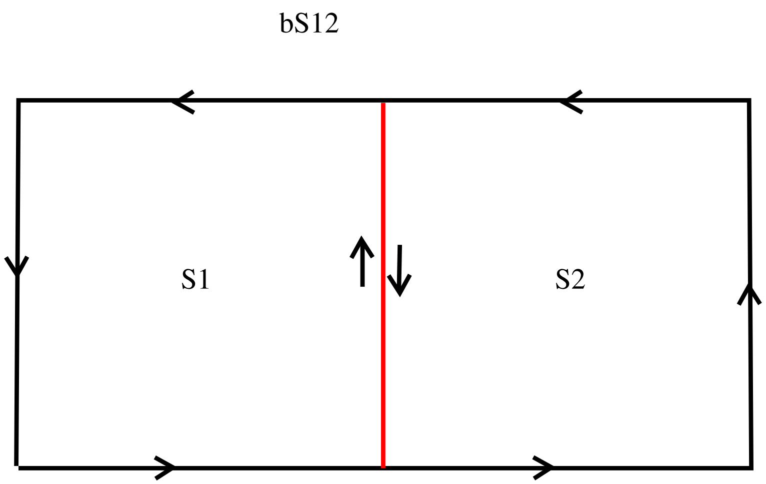
Suppose $\gamma : [a,b] \to \mathbb{C}$ is a piecewise continuously differentiable curve and suppose $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. Set $\gamma(t_j) = z_j$. Then

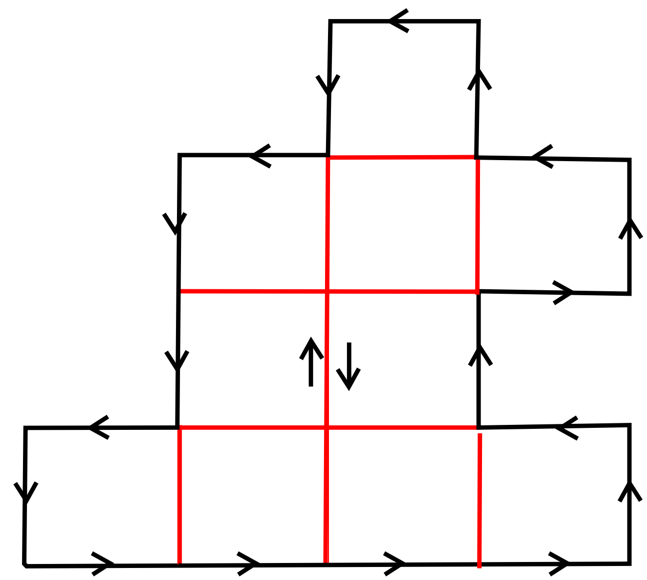
$$
\begin{aligned}
\sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j) &= \sum_{j=0}^{n-1} f(\gamma(t_j))[\gamma(t_{j+1}) - \gamma(t_j)] \\
&\approx \sum_{j=0}^{n-1} f(\gamma(t_j))\gamma'(t_j)[t_{j+1} - t_j].
\end{aligned}
\tag{0.1}
$$

The left side looks like a Riemann sum for $\int_\gamma f(z)dz$ with independent variable $z$ and the last sum is a Riemann sum, using the independent variable $t$, for

$$\int_a^b f(\gamma(t))\gamma'(t)dt.$$

The left side converges to $\int_\gamma f(z)dz$ as the **mesh** $\mu(\{t_j\}) = \max_j(t_{j+1} - t_j)$ of this partition tends to 0.



Using cancelation on overlapping arcs (in opposite directions), we often reduce a complex integral to a sum of simpler ones.

For the purposes of computing integrals it is useful to extend the notion of a curve to allow finite unions of curves.

If $\gamma_1, \ldots, \gamma_n$ are curves defined on $[0,1]$, then we can define $\gamma : [0,n) \to \mathbb{C}$ by $\gamma(t) = \gamma_j(t - j + 1)$ for $j - 1 \le t < j$, $j = 1, \ldots, n$.

If $f$ is continuous on (the image of) each $\gamma_j$, and if each $\gamma_j$ is piecewise continuously differentiable then

$$\int_\gamma f(z)dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)dz.$$

For this reason we define $\sum_j \gamma_j \equiv \gamma$. The associative and commutative laws hold for sums (unions) of curves in this sense.

We do not require the union to be connected.

**Definition 4.7:** A **cycle** $\gamma = \sum_{j=1}^{n} \gamma_j$ is a finite union of closed curves $\gamma_1, \ldots, \gamma_n$.

**Definition 4.8:** If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise continuously differentiable curve and if $f$ is a continuous complex-valued function defined on (the image of) $\gamma$ then we define

$$\int_\gamma f(z)|dz| = \int_a^b f(\gamma(t))|\gamma'(t)|dt.$$

**Definition 4.9:** If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise continuously differentiable curve then the **length** of $\gamma$ is defined to be

$$\ell(\gamma) = |\gamma| = \int_\gamma |dz| = \int_a^b |\gamma'(t)|dt.$$

# Section 4.2: Equivalence of Analytic and Holomorphic

A complex-valued function $f$ is said to be **holomorphic** on an open set $U$ if

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for all $z \in U$ and is continuous on $U$. A complex-valued function $f$ is said to be holomorphic on a set $S$ if it is holomorphic on an open set $U \supset S$.

Some sources do not require continuity of the derivative in the definition of a holomorphic function. It is usually not hard to verify in practice.

**Weaker version of holomorphic, I:**

**Exercise 4.12, Goursat's theorem:** Suppose $f$ has a complex derivative at each point of a region $\Omega$. Prove $f$ is analytic in $\Omega$.

This assigned on Problem Set 4.

Proof uses Morera's theorem (Thm 4.19).

## Weaker version of holomorphic, II:

**Exercise 7.13, Weyl's Lemma:** A continuous function $f$ is **weakly analytic** on a region $\Omega$ provided

$$\int_\Omega f\varphi_{\overline{z}}dA = 0$$

for all compactly supported continuously differentiable functions $\varphi$. Here $dA$ denotes area measure. Prove that a continuous function is weakly analytic if and only if it is analytic.

Here we define

$$\frac{\partial\varphi}{\partial\overline{z}} = \varphi_{\overline{z}} = \frac{1}{2}(\varphi_x + i\varphi_y)$$

- In Chapter 2 we saw analytic functions are holomorphic.

- In particular, polynomials are holomorphic, and rational functions except where the denominator is zero.

- Linear combinations of holomorphic functions are holomorphic.

- By the product rule, products of holomorphic functions are holomorphic.

- By the the chain rule for complex differentiation, the composition of two holomorphic functions is holomorphic.

**Corollary 4.11:** *If $\gamma : [a,b] \to \mathbb{C}$ is a closed, piecewise continuously differentiable curve and if $f$ is holomorphic in a neighborhood of $\gamma$ then*

$$\int_\gamma f'(z)dz = 0.$$

**Cor:** The integral of any polynomial around a closed curve is zero.

*Proof.* If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise continuously differentiable curve and if $f$ is holomorphic on a neighborhood of $\gamma$ then $f \circ \gamma$ is a piecewise continuously differentiable curve and by the chain rule

$$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\gamma'(t),$$

except at finitely many points $t_1, \dots, t_n$.

By the fundamental theorem of calculus,

$$\int_\gamma f'(z)dz = \int_a^b f'(\gamma(t))\gamma'(t)dt = \int_a^b \frac{d}{dt} f(\gamma(t))dt = f(\gamma(b)) - f(\gamma(a)).$$

Because $\gamma$ is closed, $f(\gamma(b)) - f(\gamma(a)) = 0$. $\square$

**Corollary 4.12:** *If* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges in* $B = \{z : |z-z_0| < r\}$ *and if* $\gamma \subset B$ *is a closed, piecewise continuously differentiable curve then*

$$\int_{\gamma} f(z)dz = 0.$$

*Proof.* The series expansion for $f$ converges uniformly on $\gamma$, so the corollary follows by interchanging the order of the integral and the sum. □

Much of this chapter and the next center around extending Corollary 2.3 to larger sets than disks $B$ and more general curves.

If $\gamma$ is a piecewise continuously differentiable closed curve and $a \notin \gamma$ then for $n \neq 1$

$$\int_\gamma \frac{1}{(z-a)^n} dz = 0.$$

because $z^{-n}$ is the derivative of $z^{-n+1}/(1-n)$.

Using partial fraction expansions, in order to integrate a rational function along $\gamma$ we only need to be able to compute $\int_\gamma (z-a)^{-1} dz$ for various values of $a$.

If $r > 0$ set

$$C_r = \{z_0 + re^{it} : 0 \leq t \leq 2\pi\}.$$

**Proposition 4.13:**

$$\frac{1}{2\pi i}\int_{C_r} \frac{1}{z-a} dz = \begin{cases} 1 & \textit{if } |a - z_0| < r \\ 0 & \textit{if } |a - z_0| > r. \end{cases}$$

This is one of the most important calculations in the course.

*Proof.* Suppose $|a - z_0| < r$. Then $C_r'(t) = ire^{it}$ and

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_r} \frac{1}{z-a}dz &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{re^{it} - (a - z_0)} ire^{it} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1 - (\frac{a-z_0}{re^{it}})} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{\infty} \left(\frac{a-z_0}{re^{it}}\right)^n dt \\
&= \sum_{n=0}^{\infty} \frac{(a-z_0)^n}{r^n} \frac{1}{2\pi}\int_0^{2\pi} e^{-int} dt = 1.
\end{aligned}$$

Final integral is zero except when $n = 0$.

Interchanging the order of summation and integration is justified because $|(a - z_0)/(re^{it})| < 1$ implies uniform convergence of the series.

If $|a - z_0| > r$, then write

$$\frac{re^{it}}{re^{it} - (a - z_0)} = \left(\frac{re^{it}}{z_0 - a}\right)\frac{1}{1 - \frac{re^{it}}{a - z_0}} = -\sum_{n=1}^{\infty} \frac{r^n e^{int}}{(a - z_0)^n},$$

so that

$$\begin{aligned} \frac{1}{2\pi i}\int_{C_r} \frac{1}{z - a}dz &= \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{it}}{re^{it} - (a - z_0)}dt \\ &= -\sum_{n=1}^{\infty} \frac{r^n}{(a - z_0)^n}\frac{1}{2\pi}\int_0^{2\pi} e^{int}dt = 0. \end{aligned}$$

□

**Theorem 4.14:** *If $f$ is holomorphic on $D(z_0, r)$ then for $|z - z_0| < r$,*

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

*where $C_r$ is the circle of radius $r$ centered at $z_0$, parameterized in the counter-clockwise direction.*

Interior values of $f$ are determined by the boundary values.

*Proof.* By parameterizing the line segment from $z$ to $\zeta$ we see that

$$f(\zeta) - f(z) = \int_0^1 f'(z + t(\zeta - z))dt.$$

Therefore,

$$\begin{aligned}
\int_{C_r} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta &= \int_{C_r} \int_0^1 f'(z + t(\zeta - z))dtd\zeta \\
&= \int_0^1 \int_{C_r} f'(z + t(\zeta - z))d\zeta dt \\
&= \lim_{\epsilon \to 0} \int_\epsilon^1 \int_{C_r} \frac{d}{d\zeta} f(z + t(\zeta - z))d\zeta \frac{dt}{t} = 0.
\end{aligned}$$

Thus

$$\frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \cdot \frac{1}{2\pi i} \int_{C_r} \frac{d\zeta}{\zeta - z} = f(z),$$

by Proposition 4.13. $\square$

## Holomorphic $=$ Analytic

**Corollary 4.15:** *A complex-valued function $f$ is holomorphic on a region $\Omega$ if and only if $f$ is analytic on $\Omega$. Moreover, the series expansion for $f$ based at $z_0 \in \Omega$ converges on the largest open disk centered at $z_0$ and contained in $\Omega$.*

We already know Analytic $\Rightarrow$ Complex Differentiable $=$ Holomorphic.

*Proof.* Suppose $f$ is holomorphic on $D(z_0, r)$. If $|z - z_0| < r$, then

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{C_r} \left( \sum_{n=0}^{\infty} \frac{1}{(\zeta - z_0)^{n+1}} (z - z_0)^n \right) f(\zeta) d\zeta$$
$$= \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n.$$

Interchanging the order of the summation and integral is justified by the uniform convergence in $\zeta \in C_r$ of the series for $z$ fixed.

Thus $f$ has a convergent power series in $D(x_0, r)$ if $\overline{D(z_0, r)} \subset \Omega$. Hence $f$ is analytic in $\Omega$ (Theorem 2.7). $\square$

**Definition:** A function holomorphic on the whole plane is called **entire**.

**Examples:** polynomials, $e^z$, $\sin(z)$, $\ldots$

**Corollary 4.16, Cauchy's estimate:** *If $f$ is analytic in $\{z : |z - z_0| \leq r\}$ and $C_r(z_0) = \{z_0 + re^{it} : 0 \leq t \leq 2\pi\}$, then*

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \tag{4.4}$$

*and*

$$\left| \frac{f^{(n)}(z_0)}{n!} \right| \leq \frac{\sup_{C_r(z_0)} |f|}{r^n}. \tag{4.5}$$

*Proof.* Equation (4.4) follows from Corollary 2.13, the proof of Corollary 4.15, and the uniqueness theorem, Theorem 2.8.

Inequality (4.5) follows by inserting absolute values into the integral, using $|\int f| \leq \int |f|$, and the fact that $C_r$ has length $2\pi r$. $\square$

**Corollary 4.17:** *If $f$ is analytic and one-to-one in a region $\Omega$ then the inverse of $f$ is analytic on on $f(\Omega)$.*

*Proof.* Since analytic functions are open (Cor 3.3), $f$ has a continuous inverse.

Take $z_0 \in \Omega$ and set $w_0 = f(z_0)$. Then $f(\Omega)$ contains a disk centered at $w_0$. If $w \in f(\Omega)$ tends to $w_0$, then $z = f^{-1}(w)$ tends to $z_0$.

Since $f$ is 1-1, Corollary 3.7 implies $f'(z_0) \neq 0$, so that

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} \to \frac{1}{f'(z_0)}.$$

This proves $f^{-1}$ has a complex derivative at $w_0$ equal to $1/f'(f^{-1}(w_0))$. This derivative is continuous, so that $f^{-1}$ is holomorphic and hence analytic. $\square$

**Theorem 4.19, Morera's Theorem:** *If $f$ is continuous in an open disk $D$ and if*

$$\int_{\partial R} f(\zeta)d\zeta = 0$$

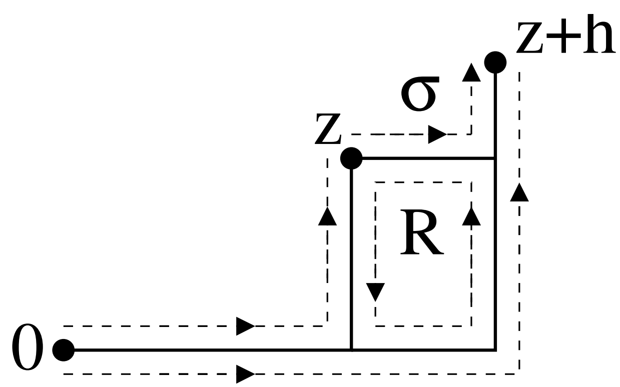
*for all closed rectangles $R \subset D$ with sides parallel to the axes, then $f$ is analytic on $D$.*

Alternate version: $f$ is analytic if integrals around all triangles in $D$ are zero.

*Proof.* We may suppose $D = \mathbb{D}$. Define

$$F(z) = \int_{\gamma_z} f(\zeta) d\zeta,$$

where $\gamma_z$ connects 0 to $z$ by a horizontal and vertical line segment:




If $|h| < 1 - |z|$ then $\gamma_{z+h} = \gamma_z + \sigma + \partial R$ where $\sigma$ is a curve from $z$ to $z + h$ consisting of a horizontal and vertical segment.

By assumption, $\int_{\partial R} f(\zeta)d\zeta = 0$, so that

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(\zeta)d\zeta - \int_{\gamma_z} f(\zeta)d\zeta = \int_\sigma f(\zeta)d\zeta.$$

By the fundamental theorem of calculus, since the identity function has derivative equal to 1, $\int_\sigma d\zeta = z + h - z = h$, and so.

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_\sigma (f(\zeta) - f(z))d\zeta.$$

By (4.2),

$$\left|\frac{1}{h}\int_\sigma (f(\zeta) - f(z))d\zeta\right| \leq \sqrt{2}\sup_{\zeta\in\sigma} |f(\zeta) - f(z)|,$$

because $|\sigma| \leq \sqrt{2}|h|$. Since $f$ is continuous, letting $h \to 0$ proves that $F$ is holomorphic on $D$ with $F' = f$. Thus $f$ is holomorphic on $D$. $\square$

**Corollary:** *If $f$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and continuous on $\mathbb{C}$, then $f$ is holomorphic on $\mathbb{C}$.*

This is an example of a "removability theorem". If $f$ is defined and has some property P on $\Omega \setminus E$, does it also have property P on all of $\Omega$?

Corollary says that a line is removable for continuous, holomorphic functions.

Proof is assigned as exercise on Problem Set 4.

Understanding what kinds of sets are removable in various situations is a long standing research problem. Is related to "rigidity" problems in dynamics.

Several connections to Stony Brook:

- See Removable sets for holomorphic functions by Malik Younsi

- Conformal removability is hard by C. Bishop.

- Non-removabilty of Sierpinski carpets by D. Ntalampekos

- David extension of circle homeomorphisms, welding, mating, and removability by Mikhail Lyubich, Sergei Merenkov, Sabyasachi Mukherjee, Dimitrios Ntalampekos.

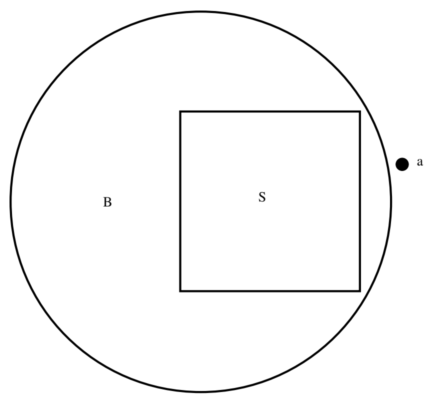
# Section 4.3: Approximation by Rational Functions

**Integration around a square:**

**Proposition 4.20:** *If $S$ is an open square with boundary $\partial S$ parameterized in the counter-clockwise direction then*

$$\frac{1}{2\pi i}\int_{\partial S}\frac{1}{z-a}dz = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{if } a \in \mathbb{C}\setminus\overline{S}. \end{cases}$$

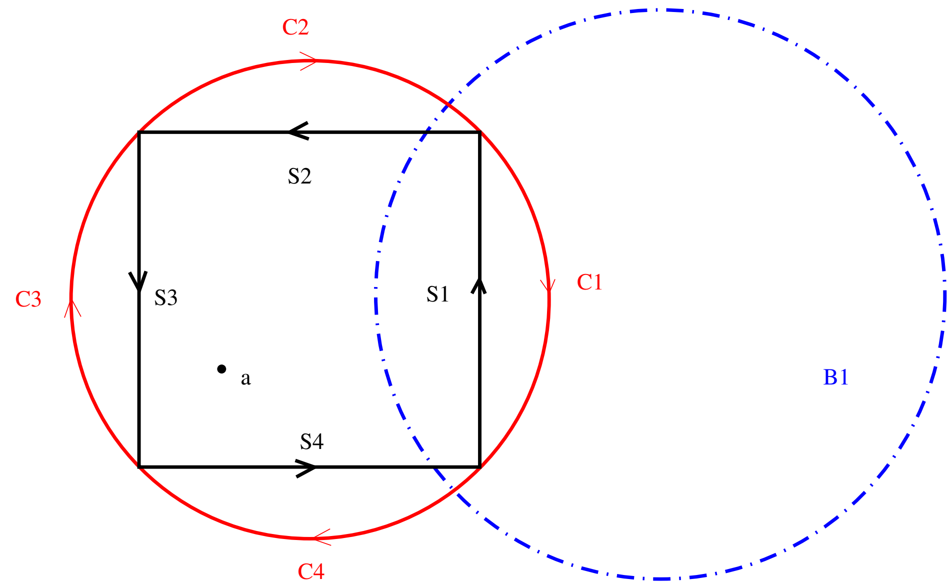
*Proof.* If $a \in \mathbb{C}\setminus\overline{S}$, then we can find a disk $B$ which contains $\overline{S}$ and does not contain $a$. Since $f$ is analytic in this disk, the integral around $S$ is zero.



If $a \in S$, then let $C$ be the circumscribed circle to $\partial S$ parameterized in the *clockwise* direction.

The difference of the two closed curves is a union of four closed curves, each of which is contained in a disk which misses $a$.

The integral of $f$ over each of these is zero, so the integral over the square the circumscribed circle are the same.




$\square$

**Theorem 4.21:** *If $f$ is analytic in a neighborhood of the closure $\overline{S}$ of an open square $S$, then for $z \in S$,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial S} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

*where $\partial S$ is parameterized in the counter-clockwise direction.*

*Proof.* The proof is the same as for Theorem 4.21 (for circles) except that Proposition 4.20 is used instead of Proposition 4.13. $\square$

**Corollary 4.22** *If $f$ is analytic in a neighborhood of the closure $\overline{S}$ of an open square $S$, then*

$$\frac{1}{2\pi i}\int_{\partial S} f(\zeta)d\zeta = 0$$

.

*Proof.* Fix $z \in S$ and apply Theorem 4.21 to $g(\zeta) = f(\zeta)(\zeta - z)$. $\square$

**Theorem 4.23, Runge's Theorem (first version)** *If $f$ is analytic on a compact set $K$ and if $\epsilon > 0$ then there is a rational function $r$ so that*

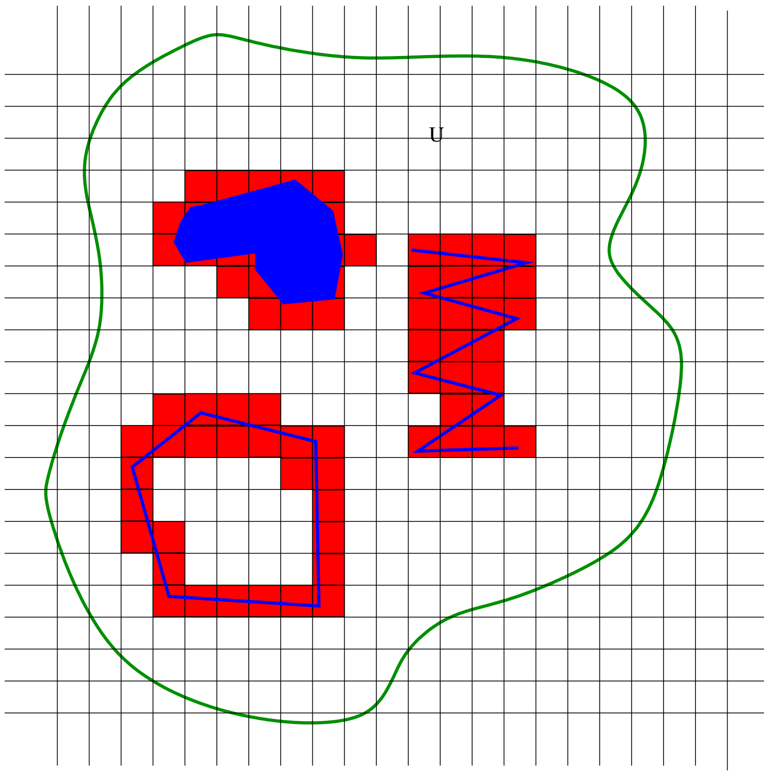
$$\sup_{z \in K} |f(z) - r(z)| < \epsilon.$$

**Idea of proof:** $f$ is given by an integral, and the integral is approximated by Riemann sums. The latter are rational functions.

*Proof.* Suppose $f$ is analytic on $U$ open, with $U \supset K$. Let

$$d = \operatorname{dist}(\partial U, K) = \inf\{|z - w| : z \in \partial U, w \in K\}.$$

Construct a grid of closed squares of side length $d/2$. The union of squares hitting $K$ is contained in $U$ and has a piecewise differentiable boundary $\Gamma$.



$\Gamma$ is a finite union of boundaries of small squares $\{S_j\}$.

If $z \in S_j$ then the integral of $f(\zeta)/(\zeta - z)$ around $\partial S_j$ is $2\pi i f(z)$. Otherwise it is zero. Thus

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) d\zeta}{\zeta - z}.$$

Choose a $z_0 \in K$ and $\delta > 0$ so that if $\{\zeta_k\}$ are $\delta$-spaced on $\Gamma$ then the integral for $z = z_0$ is approximated to within $\epsilon$ by the correspdoning Riemann sum

$$\left| f(z_0) - \frac{1}{2\pi i} \sum_k \frac{f(\zeta_k) \Delta \zeta_k}{\zeta_k - z} \right| < \epsilon.$$

This strict inequality also holds for points in a neighborhood of $z_0$.

By compactness, a finite number of these neighborhoods covers $K$.

Taking the minimum delta, we see that

$$\left| f(z) - \frac{1}{2\pi i} \sum \frac{f(\zeta_k)(\Delta\zeta_k}{\zeta_k - z} \right| < \epsilon.$$

for all $z \in K$ if $\delta$ is small enough. $\square$

**Definition 4.24:** If $r$ is a rational function, by the fundamental theorem of algebra we can write $r(z) = p(z)/q(z)$ where $p$ and $q$ are polynomials with no common zeros. The zeros of $q$ are called the **poles** of the rational function $r$.

**Moving the poles:**

**Lemma 4.25:** *Suppose* $U$ *is open and connected, and suppose* $\partial \in U$. *Then a rational function with poles only in* $U$ *can be uniformly approximated on* $\mathbb{C} \setminus U$ *by a rational function with poles only at* $b$.

*Proof.* Suppose $a, c \in U$ and suppose $|c - a| < \text{dist}(a, \partial U)$. We want to show we can move a pole from $c$ to $a$.

If $z \in \mathbb{C} \setminus U$ then $|z - a| \geq \text{dist}(a, \partial U) > |c - a|$, so that

$$\frac{1}{z-c} = \frac{1}{z-a-(c-a)} = \frac{1}{(z-a)(1-(\frac{c-a}{z-a}))} = \sum_{n=0}^{\infty} \frac{(c-a)^n}{(z-a)^{n+1}}, \tag{4.8}$$

where the sequence of partial sums approximate $1/(z-c)$ uniformly on $\mathbb{C} \setminus U$.

Using products we can approximate $(z-c)^{-n}$ for $n \geq 1$ on $\mathbb{C} \setminus U$. By taking finite linear combinations, we can uniformly approximate on $\mathbb{C} \setminus U$ any rational function with poles at $c$ by rational functions with poles only at $a$.

Next, we link together small moves to make large moves.

Write $c \in R_d$ if every rational function with poles only at $c$ can be uniformly approximated on $\mathbb{C} \setminus U$ by rational functions with poles only at $d$.

This relation is transitive: if $c \in R_d$ and $d \in R_e$ then $c \in R_e$. Set $E = \{a \in U : a \in R_b\}$.

By transitivity and the argument above, if $a \in E$ then $E$ contains a disk centered at $a$ with radius $\operatorname{dist}(a, \partial U)$. Thus $E$ is open.

We claim $E$ is also closed.

If $a_n \in E$ converges to $a_\infty \in U$, then we can choose $n$ so large that for all $z \in \mathbb{C} \setminus U$

$$|z - a_n| \geq |z - a_\infty| - |a_n - a_\infty| \geq \operatorname{dist}(a_\infty, \partial U) - |a_n - a_\infty| > |a_n - a_\infty|.$$

By (4.8), $a_\infty \in R_{a_n}$ and by transitivity, $a_\infty \in E$.

Since $E$ is both open and closed in $U$ and non-empty, we have $E = U$ since $U$ is connected.

Finally, suppose that $r$ is rational with poles only in $U$ and fix $b \in U$.

Each term $1/(z-c)^k$ in the partial fraction expansion of $r$ can be approximated by a rational function with poles only at $b$. Adding the approximations, gives an approximation of $r$ by a rational function with poles only at $b$. $\square$

**Corollary 4.26:** *Suppose $U$ is connected and open and suppose $\{z : |z| > R\} \subset U$ for some $R < \infty$. Then a rational function with poles only in $U$ can be uniformly approximated on $\mathbb{C} \setminus U$ by a polynomial.*

*Proof.* By Lemma 4.25 we need only prove that if $|b| > R$, then a rational function with poles at $b$ can be uniformly approximated by a polynomial on $\mathbb{C} \setminus U$. But

$$\frac{1}{z-b} = \frac{1}{-b(1-z/b)} = -\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{z}{b}\right)^n,$$

where the sum converges uniformly on $|z| \leq R$.

Taking products, We can approximate $(z-b)^{-n}$ for $n \geq 1$ and by taking finite linear combinations, we can approximate any rational function with poles only at $b$ by a polynomial, uniformly on $\{z : |z| \leq R\} \supset \mathbb{C} \setminus U$. $\square$

**Theorem 4.27, Runge's Theorem** *Suppose $K$ is a compact set. Choose one point $a_n$ in each bounded component $U_n$ of $\mathbb{C} \setminus K$. If $f$ is analytic on $K$ and $\epsilon > 0$, then we can find a rational function $r$ with poles only in the set $\{a_n\}$ such that*

$$\sup_{z \in K} |f(z) - r(z)| < \epsilon.$$

*If $\mathbb{C} \setminus K$ has no bounded components, then $r$ can be a polynomial.*

**Corollary 4.28:** *If $f$ is analytic on an open set $\Omega \neq \mathbb{C}$ then there is a sequence of rational functions $r_n$ with poles in $\partial\Omega$ so that $r_n$ converges to $f$ uniformly on compact subsets of $\Omega$.*

The improvement of Corollary 4.28 over Theorem 4.23 is that the poles of $r_n$ are outside of $\Omega$, not just outside $K$.

The corollary says that every analytic function is a limit of rational functions, uniformly on compact subsets.

We shall see shortly that the set of analytic functions on a region is closed under uniform convergence on compact sets.

*Proof.* Set $K_n = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq \frac{1}{n} \text{ and } |z| \leq n\}$.

Then $K_n$ is compact, $\cup K_n = \Omega$ and each bounded component of $\mathbb{C} \setminus K_n$ contains a point of $\partial\Omega$.

To prove the last claim, note that if $U$ is a bounded component of $\mathbb{C} \setminus K_n$ and if $z \in U$, then $|z| < n$ and $|z - \zeta| < 1/n$ for some $\zeta \in \partial\Omega$. Let $L$ be the line segment from $z$ to $\zeta$. If $\alpha \in L$ then $|\alpha - \zeta| < 1/n$, so that $\alpha \notin K_n$.

Thus $L$ is a connected subset of $\mathbb{C} \setminus K_n$, so $L$ must be contained in one component of $\mathbb{C} \setminus K_n$. Because $z \in L \cap U$, we must have $L \subset U$. But then $\zeta \in L \cap \partial\Omega \subset U$.

By Theorem 3.8 we can choose the rational functions approximating $f$ to have poles only in $\partial\Omega$. $\square$

**Lemma 4.30:** *If $G$ is integrable on a piecewise continuously differentiable curve $\gamma$ then*

$$g(z) \equiv \int_\gamma \frac{G(\zeta)}{\zeta - z} d\zeta$$

*is analytic in $\mathbb{C} \setminus \gamma$ and*

$$g'(z) = \int_\gamma \frac{G(\zeta)}{(\zeta - z)^2} d\zeta.$$

*Proof.* Write

$$\begin{aligned}
&\frac{g(z+h)-g(z)}{h} - \int_\gamma \frac{G(\zeta)}{(\zeta-z)^2}d\zeta \\
&\qquad = \frac{1}{h}\int_\gamma \frac{G(\zeta)dz}{\zeta-z-h} - \frac{1}{h}\int_\gamma \frac{G(\zeta)dz}{\zeta-z} - \int_\gamma \frac{G(\zeta)}{(\zeta-z)^2}d\zeta \\
&\qquad = \frac{1}{h}\int_\gamma G(\zeta)\frac{[h(\zeta-z-h)+(\zeta-z)^2-(\zeta-z)(\zeta-z-h)]dz}{(\zeta-z)^2(\zeta-z-h)} \\
&\qquad = \int_\gamma \frac{G(\zeta)h}{(\zeta-z)^2(\zeta-(z+h))}d\zeta,
\end{aligned}$$

which $\to 0$ as $h \to 0$. Thus

$$g'(z) = \int_\gamma \frac{G(\zeta)}{(\zeta-z)^2}d\zeta$$

exists and is continuous on $\mathbb{C}\setminus\gamma$. By Corollary 2.6, $g$ is analytic on $\mathbb{C}\setminus\gamma$. $\square$

**Theorem 4.29, Weierstrass's Theorem:** *Suppose* $\{f_n\}$ *is a collection of analytic functions on a region* $\Omega$ *such that* $f_n \to f$ *uniformly on compact subsets of* $\Omega$*. Then* $f$ *is analytic on* $\Omega$*. Moreover* $f_n' \to f'$ *uniformly on compact subsets of* $\Omega$*.*

*Proof.* Analyticity is a local property, so to prove the first statement we may suppose $D$ is a disk with $\overline{D} \subset \Omega$. By Theorem 4.14 $z \in D$,

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\zeta)}{\zeta - z} d\zeta.$$

Uniform limits of continuous functions are continuous, so $f$ is continuous. Set

$$F(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Then since uniform convergence implies convergence of integrals

$$|f_n(z) - F(z)| \to 0, \text{ for each } z \in D.$$

Thus $F = f$ on $D$ and by Lemma 3.11, $F$ is analytic on $D$.

This proves the first part that $f$ is analytic.

By Theorem 4.14 and Lemma 4.30

$$f_n'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta.$$

and

$$f'(z) = F'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

Again, since $f_n \to f$ uniformly on $\partial D$, we have that $f_n'$ converges uniformly to $f'$ on compact subsets of $D$.

Thus $f_n'$ converges uniformly to $f'$ on closed disks contained in $\Omega$.

Given a compact subset $K$ of $\Omega$, we can cover $K$ by finitely many closed disks contained in $\Omega$ and hence $f_n'$ converges uniformly on $K$ to $f'$. $\square$

**Cor:** *The space $H^\infty(U)$ of bounded holomorphic functions on an open set $U$ is a Banach algebra.*

Even for $U = \mathbb{D}$ this is a very interesting, much studied space.

For $z \in U$, point evaluations $\phi_z : H^\infty(U) \to \mathbb{C}$ by $f \to f(z)$ are bounded, multiplicative linear functionals.

Lennart Carelson's famous "Corona Theorem" says that for $U = \mathbb{D}$ the point evaluations are dense in the (compact) set of all multiplicative functionals.

This is known for some other planar domains, but open in general. Can fail for Riemann surfces

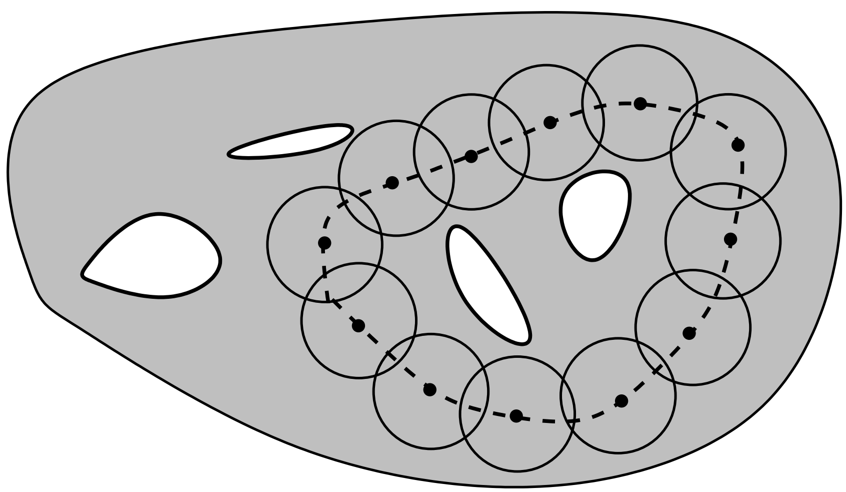
Lennart Carleson, Peter Jones, Don Marshall and myself

At Brown University conference for John Wermer

**Lemma 4.31:** *Suppose* $\Omega$ *is a region and* $\gamma : [0,1] \to \Omega$ *is continuous. Given* $\epsilon < \operatorname{dist}(\gamma, \partial\Omega)$*, we can find a finite partition* $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ *of* $[0,1]$ *so that* $\gamma([t_{j-1}, t_j]) \subset D_j = D(\gamma(t_j), \epsilon)$ *for some* $U_{\alpha_j} \in \mathcal{C}$*,* $j = 1, \ldots, n+1$*.*

*Proof.* Every $\gamma(t)$ is the center of an $\epsilon$ ball inside $\Omega$, so by compactness, a finite number cover the trace. $\square$

Let $\sigma$ be the polygonal curve joining these points.



**Theorem 4.32** *Suppose $\Omega$ is a region and $\gamma : [0,1] \to \mathbb{C}$ is continuous with $\gamma \subset \Omega$. Let $\sigma$ be the polygonal curve defined above. If $f$ is analytic on $\Omega$, define*

$$\int_\gamma f(z)dz = \int_\sigma f(z)dz.$$

*Then this definition of $\int_\gamma f(z)dz$ does not depend on the choice of the polygonal curve $\sigma$ and it agrees with our prior definition if $\gamma$ is piecewise continuously differentiable.*

This says that to prove something about $\int_\gamma f(z)dz$ where $f$ is analytic on a region $\Omega$ and $\gamma \subset \Omega$, it is enough to prove it for all polygonal curves $\gamma$.

*Proof.* If $\gamma$ is piecewise continuously differentiable, then the difference $\gamma - \sigma$ consists of a finite number of closed curves each inside a ball where $f$ is analyic, so the integrals are all zero.

For an arbitrary continuous $\gamma$, suppose $\sigma$ is the polygonal curve associated with a partition $\{t_j\}$ chosen as above. Let $\alpha$ be the polygonal curve associated with a finite refinement $\{s_k\} \supset \{t_j\}$ of the partition $\{t_j\}$.

We can write $\sigma - \alpha = \sum \beta_j$ where each $\beta_j : [0,1] \to \mathbb{C}$ is a closed polygonal curve contained in some $D_j$. By Corollary 4.18 $\int_\alpha f(z)dz = \int_\sigma f(z)dz$.

Given any two partitions satisfying the conditions of the definition of the integral, we can find a common refinement. Thus the definition does not depend on the choice of the partition. $\square$

**Addititonal remarks:** Runge's theorem is still the focus of new research.

Last year my former student, Kiril Lazebink, and I proved:

**Theorem:** If $K$ is compact inside an open $\Omega$, and $f$ is analytic on $\Omega$, then $f$ can be uniformly approximated on $K$ by rational functions that have all their critical values inside $\Omega$.

A critical value is $p(z)$ where $p'(z) = 0$. The orbits of critical values is important in holomorphic dynamics, so knowing the location of critical values is interesting.

Even is situations where a function can be approximated by polynomials, rational approximation may do a much better job.

The best degree $n$ polynomial approximation $p_n$ to $f(x) = |x|$ on $[-1, 1]$ satisfies

$$\sup |f - p_n| \approx \frac{.28016949902386913\ldots}{n}$$

but the best rational approximation $r_n$ of degree $n$ satisfies

$$\sup |f - r_n| \leq 3 \exp(-\sqrt{n}).$$

See Chapter 25 of Trefethen's book Approximation Theory and Approximation Practice.

## Application of Runge's theorem:

**Thm:** *For any function* $f : \mathbb{N} \to \mathbb{C}$ *there is an entire function* $g$ *so that* $\sup_{\mathbb{N}} |f(z) - g(n)| < 1$.

## Application of Runge's theorem:

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*Proof.* Let $p_1(z) = f(1)$ be constant.

In general, apply Runge's theorem to $K_n = \{|z| \leq n - 1\} \cup \{n\}$ (which does not separate the plane) to choose a polynomial $p_n$ so that $|p_n| < 2^{-n}$ on $\{|z| \leq n-1\}$ and $|p_n(n) - f(n) - \sum_{k<n} p_k(n)| < 2^{-n}$.

Then $g = \sum_{k=1}^{\infty} p_k$ converges uniformly on compact sets to a function with the desired properties. $\square$

## Application of Runge's theorem:

**Thm:** *Suppose $S = [0,1]^2$ is a square and $f, g$ are any two entire functions. Then there is a sequence of polynomials $\{p_n\}$ so that $p_n(z) \to g(z)$ for all $z$ in the interior of $S$ and $p_n(z) \to f(z)$ for all $z$ on the boundary of $S$.*

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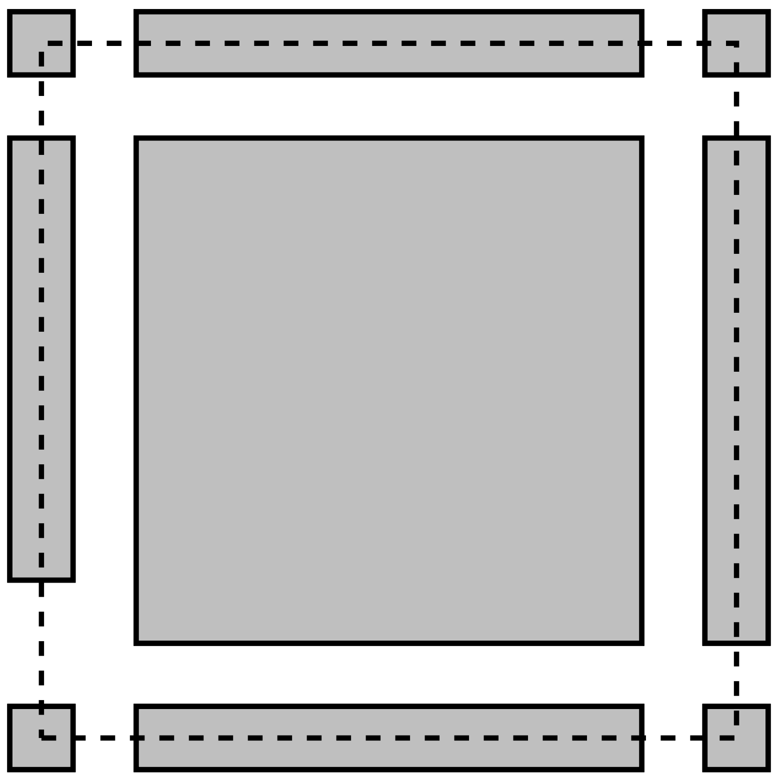

## Application of Runge's theorem:

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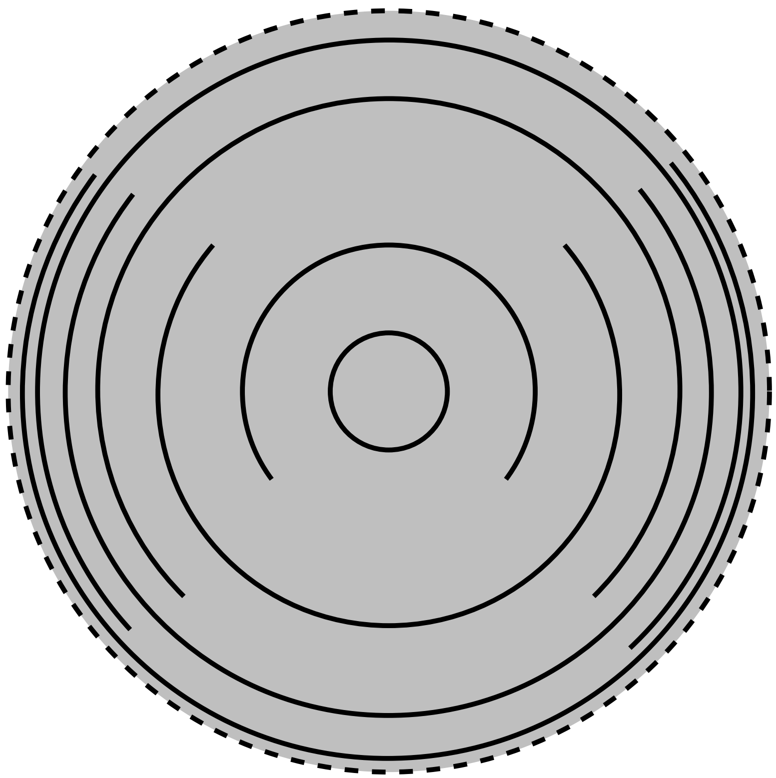

**Application of Runge's theorem:**

**Thm:** it There is an entire function $f$ so that given any other entire function $g$, there is a seqeunce $\{n_k\} \in \mathbb{N}$ so that $f(z+n_k) \to g(z)$ uniformy on compact subsets of the plane.

Proof is assigned on Problem Set 4.