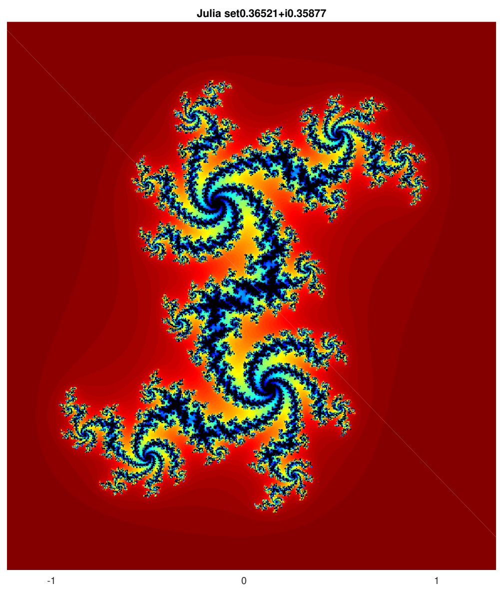
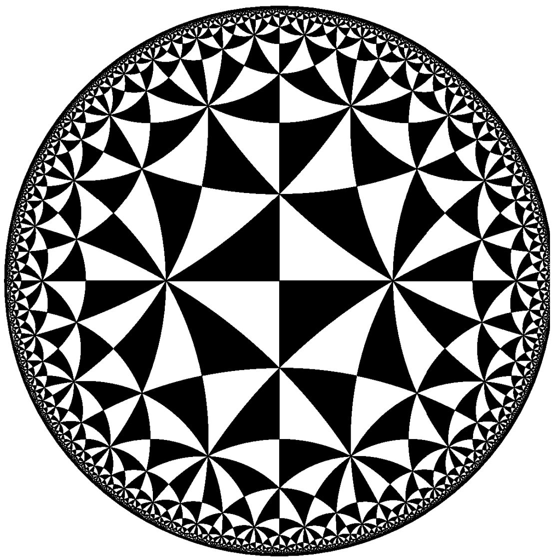
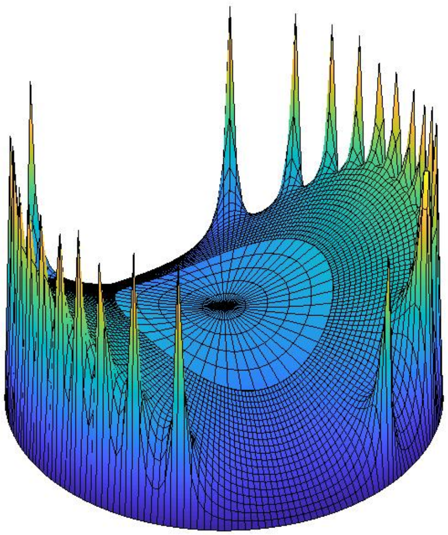
MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop

Wed. Jan 24, 2024







Chapter 2: Analytic functions

Section 2.1: Polynomials

Translation: $z \to z + c$.

Rotation: $z \to \lambda z$, $|\lambda| = 1$.

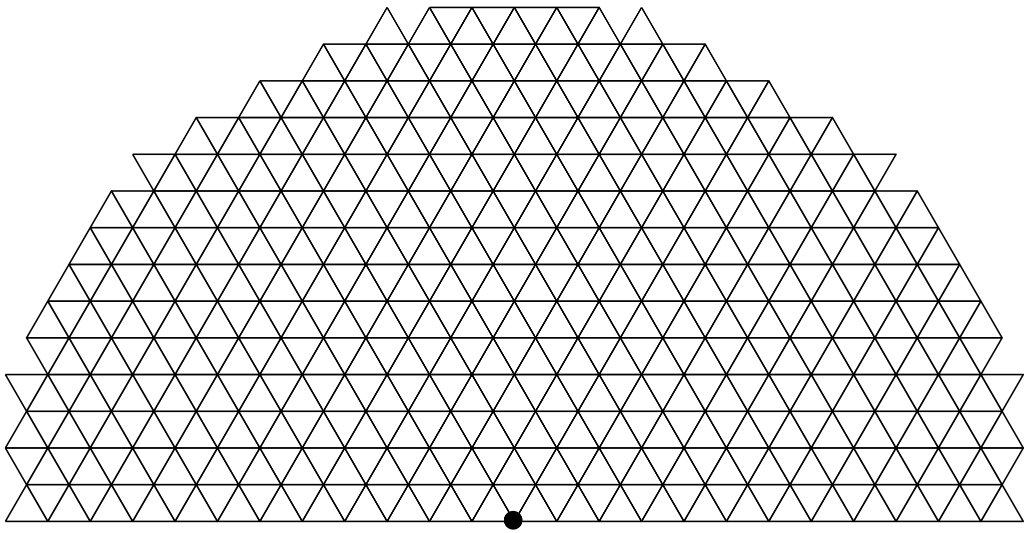
Dilation: $z \to \lambda z$, $\lambda > 0$.

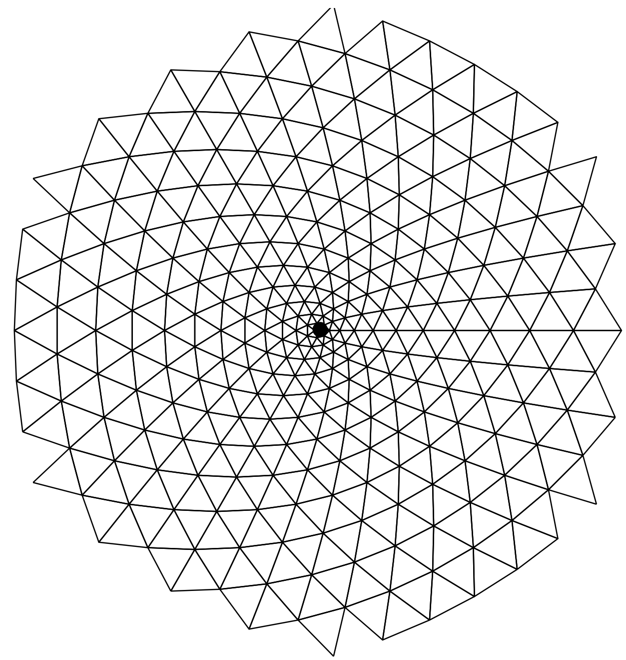
(Complex) linear map: $z \to az + b$, combination of above.

Linear maps = orientation preserving Euclidean similarities

The map $(x, y) \to (2x, y)$ is "real linear" but not "complex linear".

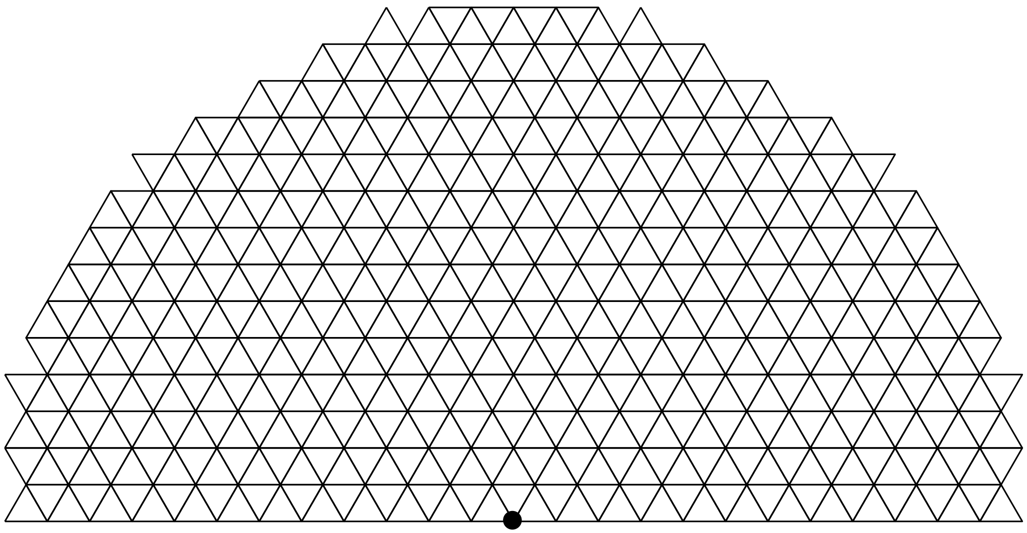
Power functions: $z \to z^n$ or $re^{i\theta} \to r^n e^{in\theta}$.

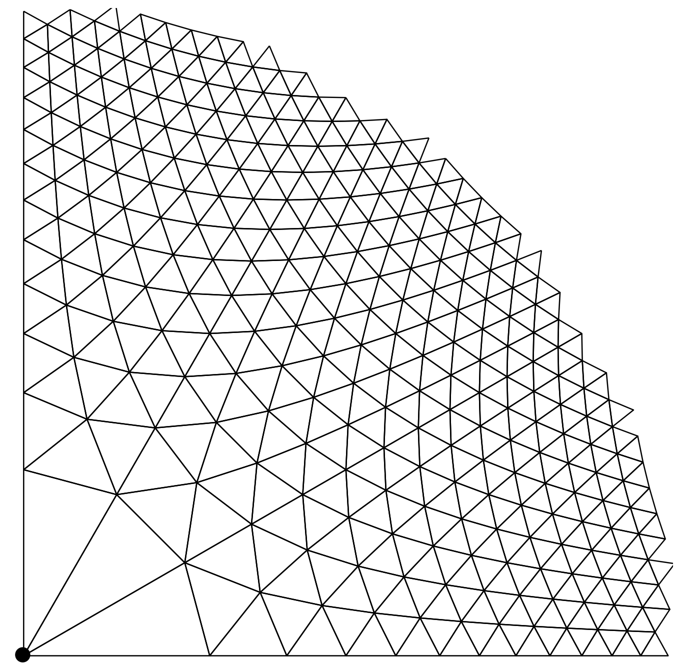




The power map $z \to z^2$

Power functions: $z \to z^n$ or $re^{i\theta} \to r^n e^{in\theta}$.





The power map $z \to z^{1/2}$

However, this map is not defined on whole plane!

Polynomials:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots a_n z^n.$$

Rational functions: $r = p/q$ where p, q are polynomials.

Rational functions for a field.

Lemma: If p is a polynomial, $z_0 \in \mathbb{C}$ and $\theta \in [0, 2\pi)$, then there is a sequence $\{w_n\} \to z_0$ so that $\arg(p(w_n) - p(z)) \to \theta$.

This follows if $p(D(z_0, \epsilon))$ always contains a disk around $p(z_0)$.

We will prove this stronger property later.

Proof. Suppose $z_0 \in \mathbb{C}$. Write $z = (z - z_0) + z_0$, so

$$p(z) = \sum_{k=0}^{n} a_k((z - z_0) + z_0))^n = \sum_{k=0}^{n} b_k(z - z_0)^n,$$

where the coefficients now depend on z_0.

Then $b_0 = p(z_0)$, so

$$p(z) - p(z_0) = \sum_{k=1}^{n} b_k(z - z_0)^n,$$

Let $m \geq 1$ be smallest k so b_k is non-zero. If $|z - z_0|$ is small enough,

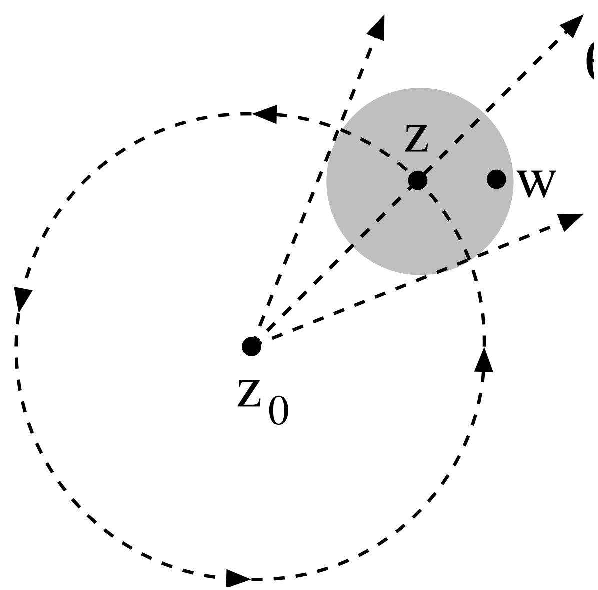
$$p(z) - p(z_0) = (z - z_0)^m [b_m + b_{m+1}(z - z_0) + \dots b_n (z - z_0)^{n-k}]$$
$$= (z - z_0)^m [b_k + o(1)]$$

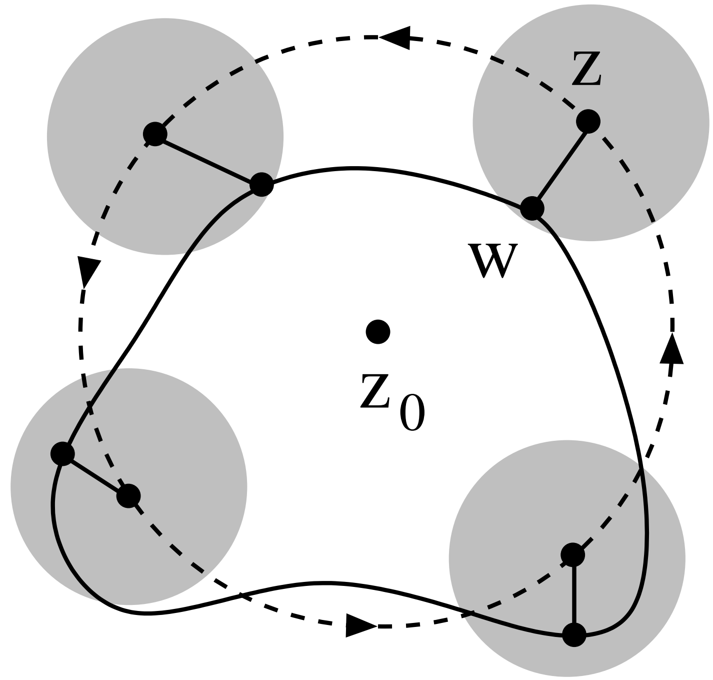
$$|[p(z) - p(z_0)] - b_m (z - z_0)^m| \leq \epsilon |b_n (z - z_0)^m|$$

If we choose w_n so that $\arg b_n (w_n - z_0)^m = \theta$, then

$$|\arg(p(w_n) - p(z_0)) - \theta| < \epsilon.$$

Taking $\epsilon = 1/n$ proves the lemma. □

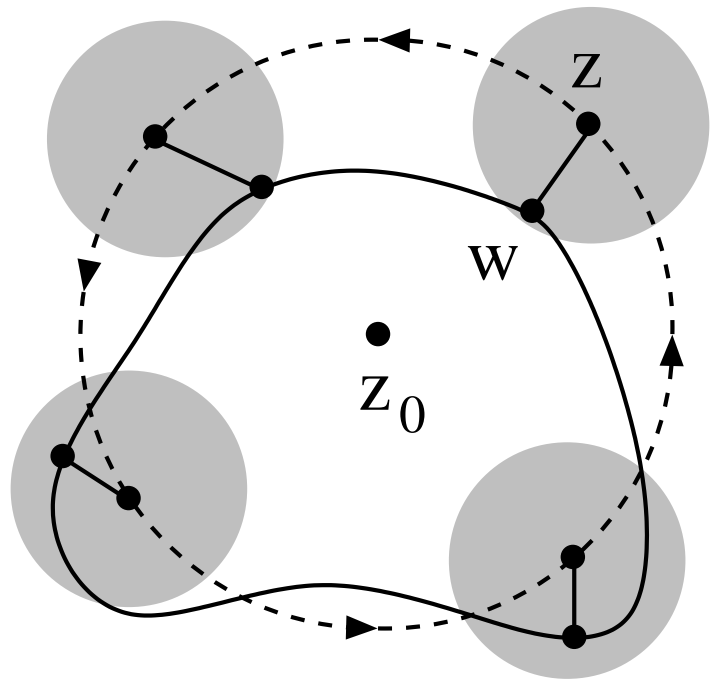




As z moves in circle around z_0, $b_0 + b_m \cdot (z - z_0)^m$ moves in circle around $p(z_0)$.

The point $p(z)$ must follow, always staying on same "side" of b_0. Both paths "go around" the same number of times.

As radius shrinks to zero, so does image of circle. Images must fill in a disk around b_0. How to prove this?



Marshall calls this picture “walking the dog”.

(The leash doesn’t get tangled around the lamp post.)

Section 2.2: The Fundamental Theorem of Algebra and Partial Fractions

Fundamental Theorem of Algebra

Theorem 2.1: *Every non-constant polynomial has a zero.*

Fundamental Theorem of Algebra

Proof. Suppose $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $n \geq 1$, is a polynomial which has no zeros and for which $a_n \neq 0$. We claim that $|p(z)|$ must have a non-zero minimum value on $\mathbb{C}$.

This will follow if $|p(z)| \nearrow \infty$ as $|z| \nearrow \infty$.

Write

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right).$$

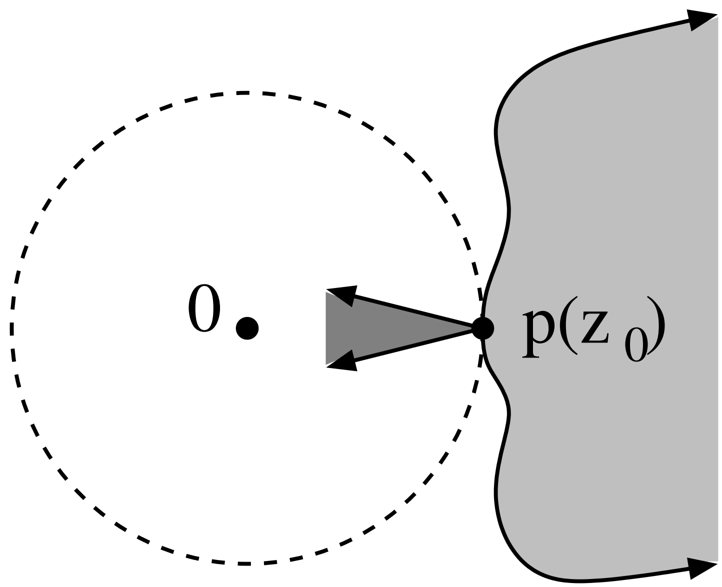
Because $a_n \neq 0$, and because $1/z^k \to 0$, and $|z^n| \to \infty$ as $|z| \to \infty$, we conclude that $|p(z)| \to \infty$ as $|z| \to \infty$.

So if $M = \inf_{\mathbb{C}} |p(z)|$ and if $|p(w_j)| \to M$, then there is an $R < \infty$ so that $|w_j| \leq R$, for all j. Because $\{z : |z| \leq R\}$ is compact and because $|p|$ is continuous, there exists z_0 so that $|p(z_0)| = M$.

This proves the claim that a minimum exists.

Moreover, $M \neq 0$ since p has no zeros. Now by the previous lemma, $p(\mathbb{C})$ contains a closed path about $p(z_0)$ which must intersect the radial segment $[0, p(z_0)]$ and thus there is a point in $p(\mathbb{C})$ strictly closer to the origin.

This is a contradiction, and proves that $|p|$ can't take a non-zero minimum. Thus the minimum is zero. $\square$



Corollary 2.2: *If p is a polynomial of degree $n \geq 1$, then there are complex numbers $z_1, \ldots, z_n$ and a complex constant c so that*

$$p(z) = c \prod_{k=1}^{n} (z - z_k).$$

Proof. First note that

$$z^k - b^k = (z - b)(b^{k-1} + zb^{k-2} + \cdots + z^{k-2}b + z^{k-1}).$$

So if $p(z) = \sum_{k=0}^{n} a_k z^k$ and $p(b) = 0$, then

$$q(z) \equiv \frac{p(z)}{z-b} = \frac{p(z) - p(b)}{z - b} = \sum_{k=1}^{n} a_k (\sum_{j=0}^{k-1} b^{k-1-j} z^j).$$

The coefficient of z^{n-1} is a_n so q is a polynomial of degree $n - 1$.

Repeating this argument n times proves the Corollary. □

Every polynomial is of the form

$$p(z) = C \prod_{k=1}^{n} (z - z_k),$$

where $\{z_k\}$ are the roots of p.

Finding the roots can be challenging. There are formulas for degree $2, 3, 4$.

For higher degrees Able showed there is no algebraic formula for the roots. See Abel-Ruffini Theorem.

Galois theory allows us to decide which polynomials can be solved in radicals.

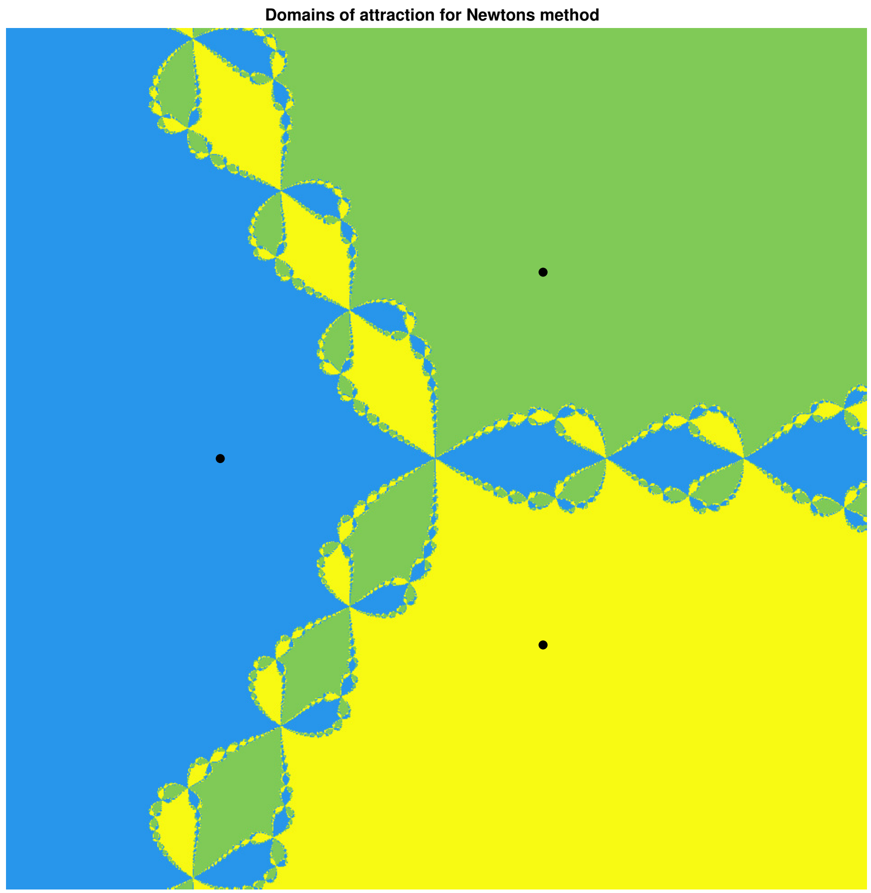
Degree 5 can be solved by more general methods: see Solving the quintic by iteration by Doyle and McMullen.

Newton's method can be used to approximate roots of higher degree polynomials. See How to Find All Roots of Complex Polynomials by Newton's Method.

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

Newton's method works if the initial guess is close enough to a root, but mail fail in general. The preceding paper shows how to choose a finite collection of starting points so that at least one is guaranteed to find a root.

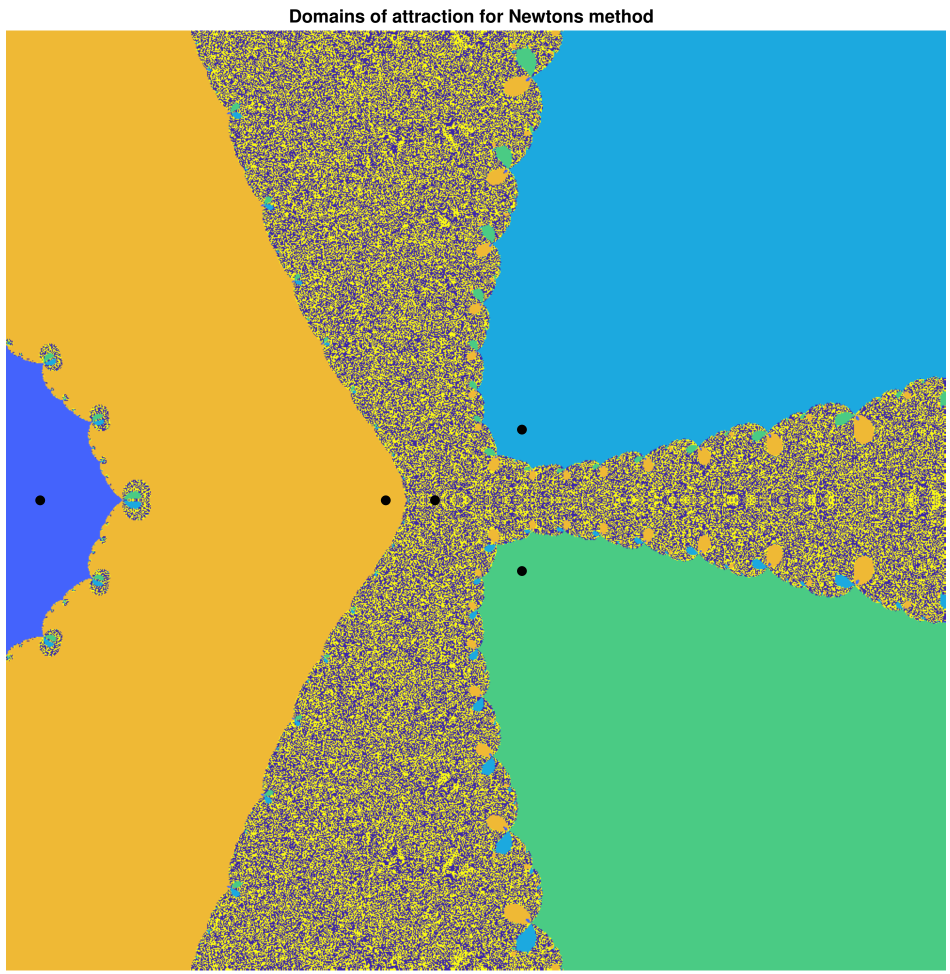
If p is a polynomial, $r(z) = z = p(z)/p'(z)$ is a rational function, so we are just iterating a rational function.



Applying Newton's method to solve $z^3 + 1 = 0$

Colors indicate starting points attracted to each root.

There is a fractal of points not attracted to any root.



Applying Newton's method to solve $2z^5 + 8z^4 + 2z^3 + 3z + 1 = 0$

MATLAB code to plot this picture

Partial fraction expansion: every rational function can be written as

$$r(z) = q(z) + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{c_{k,j}}{(z - z_j)^k}.$$

Proof. See textbook, Corollary 2.3. □

Partial fraction expansions are extremely important in applications and numerical calculations.

For example, see The AAA algorithm for rational approximation.

Section 2.3: Power Series

Geometric Series:

$$\begin{aligned} S_n &= 1 + z + z^2 + \dots z^n \\ zS_n &= z + z^2 + \dots z^{n+1} \\ S_n - zS_n &= 1 - z^{n+1} \end{aligned}$$

$$S_n = \frac{1 - z^{n+1}}{1 - z}$$

Hence for $|z| < 1$,

$$\sum_{n=0}^{\infty} z^n = \lim_{n \to \infty} S_n = \frac{1}{1 - z}.$$

Left side only defined on $\{|z| < 1\}$, but has extension to $\mathbb{C} \setminus \{1\}$.

Example of "analytic continuation".

A similar calculation shows

$$\frac{1}{z-a} = \frac{1}{z-z_0-(a-z_0)} = \frac{1}{-(a-z_0)(1-(\frac{z-z_0}{a-z_0}))}.$$

Substituting

$$w = \frac{z-z_0}{a-z_0}$$

we get for $|w| = |(z-z_0)/(a-z_0)| < 1$,

$$\frac{1}{z-a} = \sum_{n=0}^{\infty} \frac{-1}{(a-z_0)^{n+1}}(z-z_0)^n.$$

This series converges if $|z-z_0| < |a-z_0|$ and diverges if $|z-z_0| \geq |a-z_0|$.

The domain of convergence is an open disk and it is the largest disk centered at z_0 which is contained in the domain of definition of $1/(z-a)$.

Weierstrass M-Test

Theorem 2.4: *If* $|a_n(z-z_0)^n| \leq M_n$ *for* $|z-z_0| \leq r$ *and if* $\sum M_n < \infty$ *then* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges uniformly and absolutely in* $\{z : |z-z_0| \leq r\}$.

Proof. See textbook. ∎

Root Test:

Theorem 2.5; *Suppose* $\sum a_n(z-z_0)^n$ *is a formal power series. Let*

$$R = \liminf_{n\to\infty} |a_n|^{-\frac{1}{n}} = \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}}} \in [0, +\infty].$$

Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$

(1) *converges absolutely in* $\{z : |z-z_0| < R\}$,

(2) *converges uniformly in* $\{z : |z-z_0| \leq r\}$ *for all* $r < R$, *and*

(3) *diverges in* $\{z : |z-z_0| > R\}$.

Proof. See textbook. □

A power series always converges on a disk, possibly zero or infinite radius.

Different behaviors are possible on the boundary

$\sum_{n=0}^{\infty} \frac{z^n}{n}$ $R = 1$, diverges at $z = 1$, converges elsewhere on $\{|z| = 1\}$

$\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ $R = 1$, converges absolutely everywhere on $\{|z| = 1\}$

$\sum_{n=0}^{\infty} nz^n$ $R = 1$, diverges everywhere on $\{|z| = 1\}$

$\sum_{n=0}^{\infty} 2^{n^2} z^n$ $R = 0$, diverges everywhere except $z = 0$

$\sum_{n=0}^{\infty} \frac{z^n}{n!}$ $R = \infty$, converges everywhere in $\mathbb{C}$.

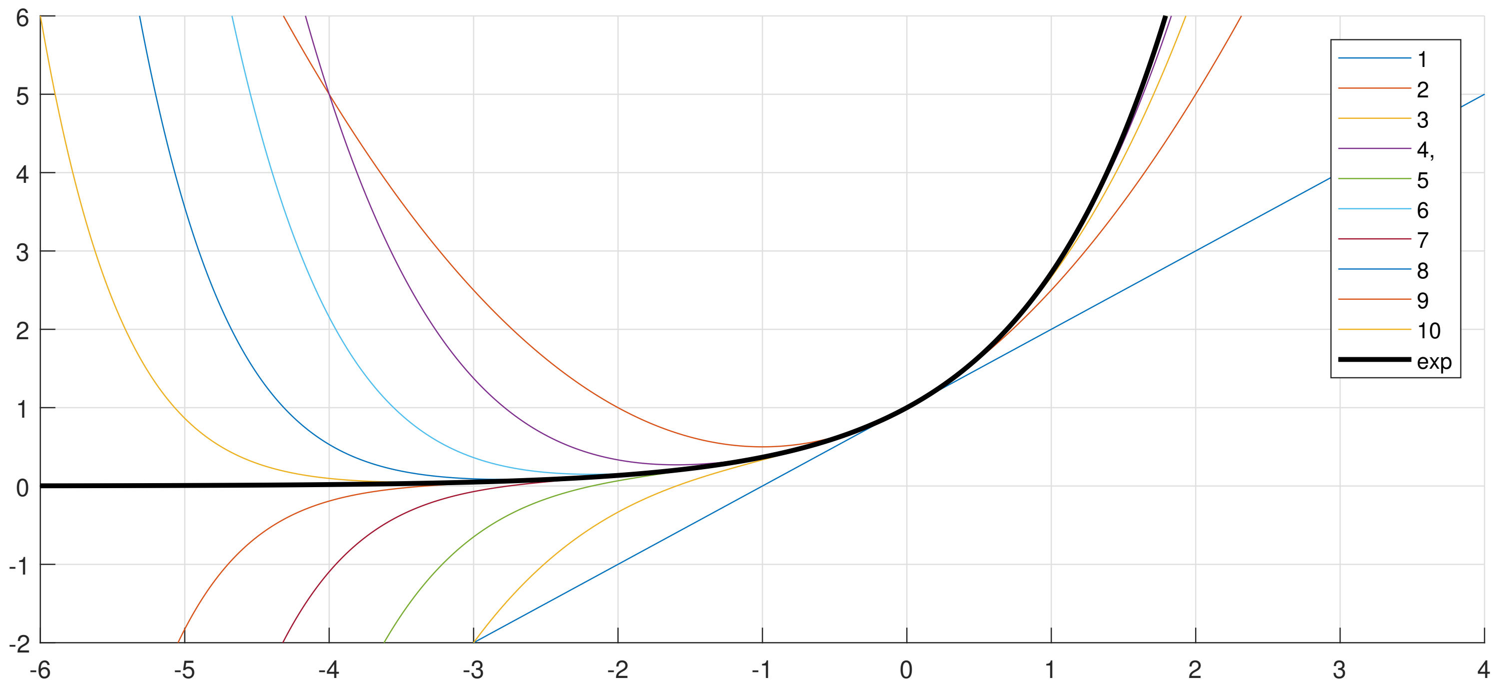
$\sum_{n=0}^{\infty} \frac{z^n}{n}$ $\qquad R = \infty$, converges everywhere in $\mathbb{C}$.

Defn: a function with series converging on the whole plane is called **entire**.

This example converges "uniformly on compact sets".

Defn: Suppose U is open set and $\{f_n\}$ is a sequence of functions on U. We say $\{f_n\}$ converges uniformly on compact sets to f, if for any compact $K \subset U$, the restrictions to K converge uniformly to $f|_K$.

This is very common type of convergence used in complex analysis.



Approximation of $e^x = \sum_{n=0}^{\infty} x^n/n!$ by polynomials = truncations of series.

$$e^{i\theta} = \sum_{n \text{ even}} \frac{(i\theta)^n}{n!} + \sum_{n \text{ odd}} \frac{(i\theta)^n}{n!} = \sum_{n \text{ even}} (-1)^{n/2}\frac{\theta^n}{n!} + i \sum_{n \text{ odd}} (-1)^{(n-1)/2}\frac{\theta^n}{n!} = \cos\theta + i\sin\theta$$

Section 2.4: Analytic Functions

Definition: A function f is **analytic at** z_0 if f has a power series expansion valid in a neighborhood of z_0.

This means that there is an $r > 0$ and a power series $\sum a_n(z - z_0)^n$ which converges in $B = \{z : |z - z_0| < r\}$ and satisfies

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \text{ for all } z \in B.$$

A function f is **analytic on an open set** Ω if f is analytic at each $z_0 \in \Omega$.

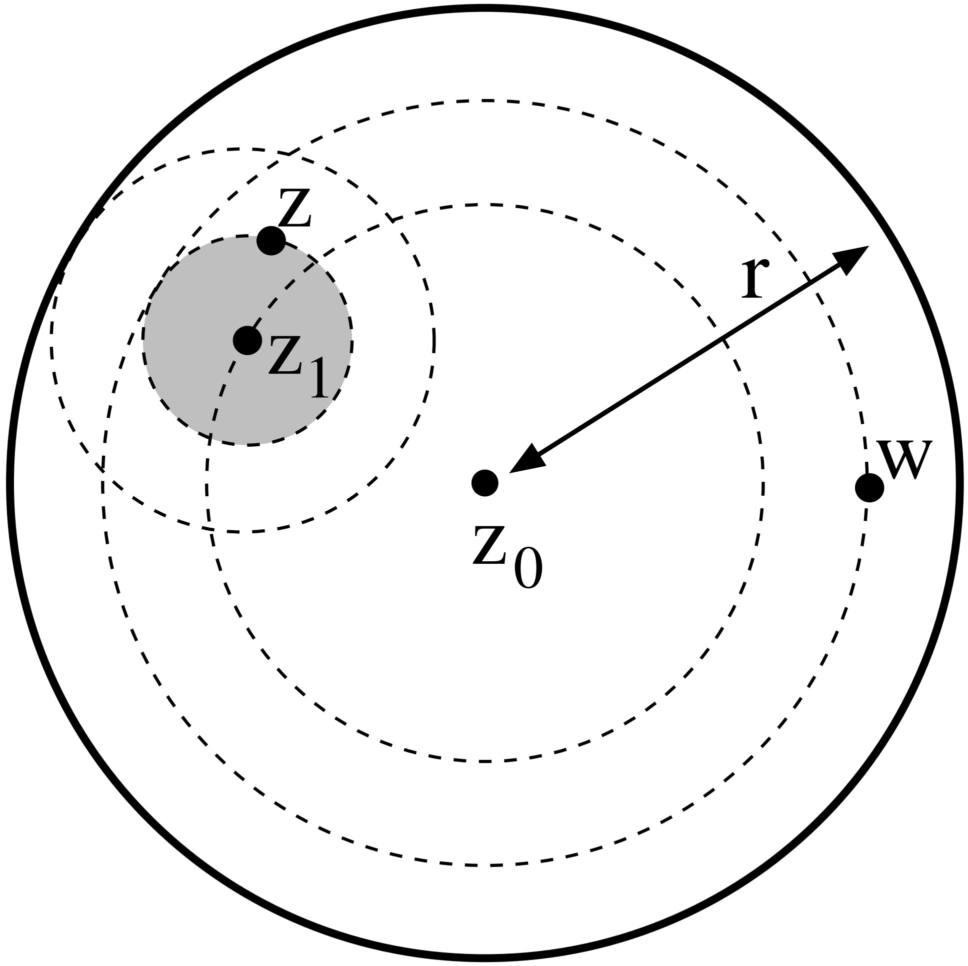
Analytic functions are automatically continuous since they are locally uniform limits of polynomials.

f is analytic on (non-open) E if it analytic on a neighborhood of E.

Being analytic at z requires a power series centered at z. What if z is inside the disk of convergence, but not the center?

Theorem 2.7: *If* $f(z) = \sum a_n(z - z_0)^n$ *converges on* $\{z : |z - z_0| < r\}$ *then* f *is analytic on* $\{z : |z - z_0| < r\}$.

In other words, we can move the center of a power series to a different point inside the disk of convergence.


z
z1
r
w
z0

Proof. Fix z_1 with $|z_1 - z_0| < r$. By the binomial theorem

$$(z - z_0)^n = (z - z_1 + z_1 - z_0)^n = \sum_{k=0}^{n} \binom{n}{k} (z_1 - z_0)^{n-k} (z - z_1)^k.$$

Hence

$$f(z) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a_n \binom{n}{k} (z_1 - z_0)^{n-k} (z - z_1)^k \right]. \tag{4.1}$$

Suppose for the moment, that we can interchange the order of summation, then

$$\sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0)^{n-k} \right] (z - z_1)^k$$

will be the power series expansion for f based at z_1.

To justify this interchange of summation, it suffices to prove absolute convergence of (4.1). By the root test

$$\sum_{n=0}^{\infty} |a_n||w - z_0|^n$$

converges if $|w - z_0| < r$. Set

$$w = |z - z_1| + |z_1 - z_0| + z_0.$$

Then $|w - z_0| = |z - z_1| + |z_1 - z_0| < r$ provided $|z - z_1| < r - |z_1 - z_0|$.

Thus if $|z - z_1| < r - |z_1 - z_0|$ then

$$\begin{aligned}
\infty > \sum_{n=0}^{\infty} |a_n||w - z_0|^n \\
&= \sum_{n=0}^{\infty} |a_n|\big(|z - z_1| + |z_1 - z_0|\big)^n \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} |a_n|\binom{n}{k}|z_1 - z_0|^{n-k}|z - z_1|^k\right]
\end{aligned}$$

as desired. $\square$

Theorem 2.8: *Suppose*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n,$$

for all z such that $|z-z_0| < r$ where $r > 0$. Then $a_n = b_n$ for all n.

In other words, if f has a power series, it is unique.

Proof. Set $c_n = a_n - b_n$. The hypothesis implies that $\sum_{n=0}^{\infty} c_n(z-z_0)^n = 0$, for all z with $|z-z_0| < r$. We need to show that $c_n = 0$ for all n.

Suppose c_m is the first non-zero coefficient. If $0 < |z-z_0| < r$ then

$$(z-z_0)^{-m}\sum_{n=m}^{\infty} c_n(z-z_0)^n = \sum_{k=0}^{\infty} c_{m+k}(z-z_0)^k \equiv F(z).$$

The series for F converges in $0 < |z-z_0| < r$ because we can multiply the terms of the series on the left side by the non-zero number $(z-z_0)^{-m}$ and not affect convergence. By the root test, the series for F converges in a disk and hence in $\{|z-z_0| < r\}$.

Since F is continuous and $c_m \neq 0$, there is a $\delta > 0$ so that if $|z - z_0| < \delta$, then

$$|F(z) - F(z_0)| = |F(z) - c_m| < |c_m|/2.$$

If $F(z) = 0$, then we obtain the contradiction $|-c_m| < |c_m|/2$.

Thus $F(z) \neq 0$ when $|z - z_0| < \delta$. But $(z - z_0)^m = 0$ only when $z = z_0$, and thus

$$\sum_{n=0}^{\infty} c_n(z - z_0)^n = (z - z_0)^m F(z) \neq 0$$

when $0 < |z - z_0| < \delta$, contradicting our assumption on $\sum c_n(z - z_0)^n$. $\square$

Corollary of proof: *If f is analytic at z_0, then for some $\delta > 0$, either $f(z) \neq 0$ when $0 < |z - z_0| < \delta$ or $f(z) = 0$ for all z such that $|z - z_0| < \delta$.*

Defn: If $f(a) = 0$, then a is called a **zero** of f.

Defn: A **region** is a connected open set.

Corollary 2.9: *If f is analytic on a region Ω then either $f \equiv 0$ or the zeros of f are isolated in Ω.*

Zeros can accumulate on boundary, e.g.,

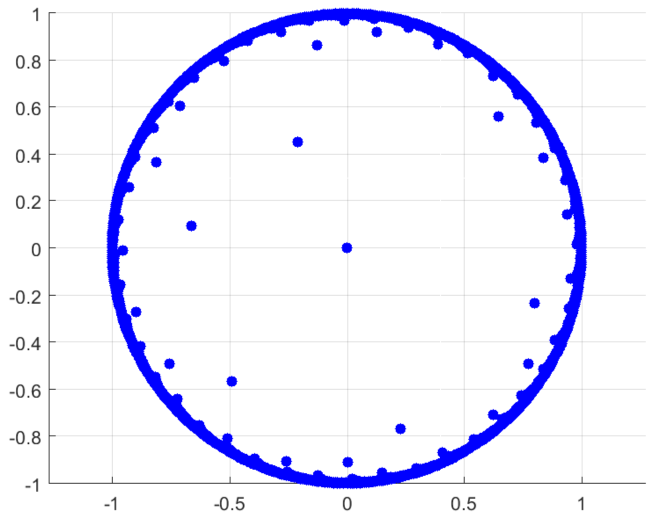
$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \cdot \frac{z_n - z}{1 - \overline{z_n} z}.$$

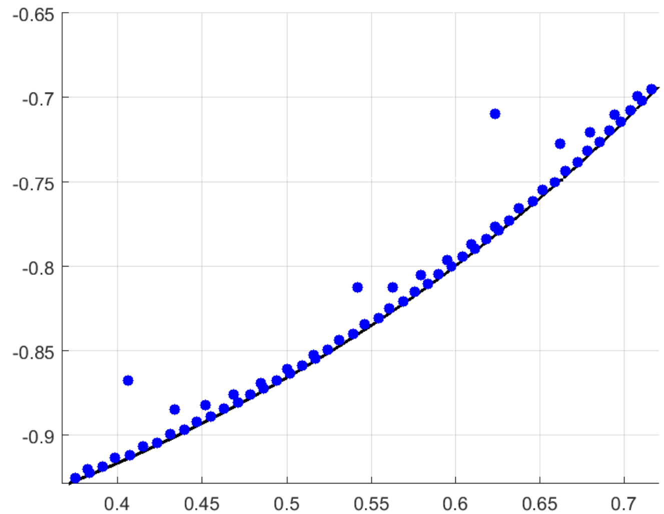
If $\sum (1 - |z_n|) < \infty$, this defines an analytic function on $\mathbb{D} = \{|z| < 1\}$ that has zeros exactly at the points $\{z_n\}$.

This is called a Blaschke product.

B not defined on $\{|z| = 1\}$, but it extends by radial limits to a measurable function there, with $|B| = 1$ almost everywhere (Fatou's theorem).

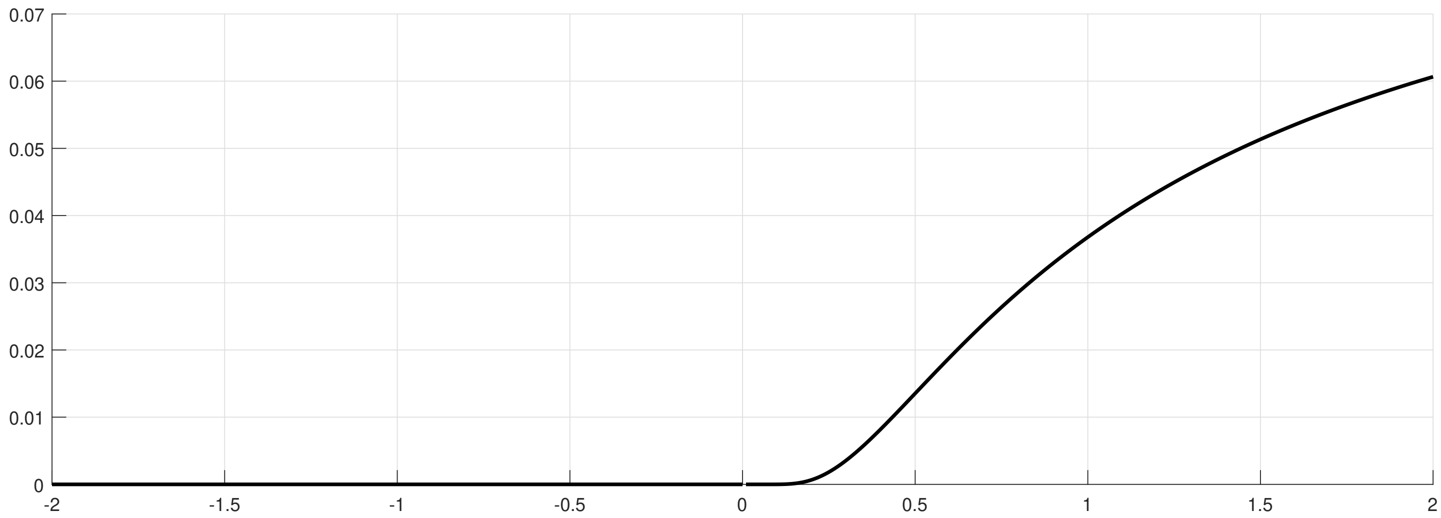
If $z_n = (1 - 1/n^2)e^{in}$, then zeros of B accumulate everywhere on $\mathbb{T}$.





Corollary 2.9 can fail for real-valued C^∞ functions, e.g.,

$$f(x) = \begin{cases} 0, & x \leq 0. \\ \exp(-1/x), & x > 0. \end{cases}$$



Section 2.5: Elementary Operations

Elementary operations preserve analytic functions:

Theorem 2.10: *If f and g are analytic at z_0 then so are*

$$f+g, \qquad f-g, \qquad cf \qquad fg,$$

where c is a constant. If h is analytic at $f(z_0)$ then $(h \circ f)(z) \equiv h(f(z))$ is analytic at z_0.

First three follow from properties of power series.

Need to check $f \cdot g$ and $f \circ g$.

Multiplication:

Proof. Suppose f, g are both analytic on $D = D(z_0, r)$ and

$$f(z) = \sum^{\infty} a_n(z - z_0)^n, \qquad g(z) = \sum^{\infty} b_n(z - z_0)^n,$$

$$f_m(z) = \sum^{m} a_n(z - z_0)^n, \qquad g_m(z) = \sum^{m} b_n(z - z_0)^n,$$

Then $f_m \to f$ and $g_m \to g$ uniformly on compact subsets of D and

$$|a_n|, |b_n| \leq Cs^n$$

for every $s > r$.

Moreover,

$$f_m(z)g_m(z) = \sum_{n=0}^{m} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z - z_0)^n$$

and

$$|\sum_{k=0}^{n} a_k b_{n-k}| \leq \sum_{k=0}^{n} |a_k b_{n-k}| \leq C^2 \sum_{k=0}^{n} s^n \leq C^2 n s^n$$

for all $s > r$. Thus

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z - z_0)^n$$

has radius of convergence at least r, and converges uniformly to fg. Thus fg is analytic. $\square$

Composition:

Proof. Suppose $f(z) = \sum a_n(z - z_0)^n$ is analytic at z_0 and suppose $h(z) = \sum b_n(z - a_0)^n$ is analytic at $a_0 = f(z_0)$.

If the series for f converges in $D = D(z_0, r)$ then for any $s > r$, we have $|a_n| \le Cs^n$. This implies for

$$\sum_{m=1}^{\infty} |a_m||z - z_0|^{m-1}$$

is uniformly bounded if on any disk $D(z_0, t)$ with $t < r$.

Thus, if $|z - z_0| \le t$,

$$|f(z) - a_0| \le \sum_{m=1}^{\infty} |a_m||z - z_0|^m \le |z - z_0| \sum_{m=1}^{\infty} |a_m||z - z_0|^{m-1} \le M|z - z_0|,$$

Therefore,

$$\begin{aligned}
|\sum_{m=0}^{\infty} b_m \left(\sum_{n=1}^{\infty} a_n (z-z_0)^n\right)^m | &\leq \sum_{m=0}^{\infty} |b_m| \left(\sum_{n=1}^{\infty} |a_n||z-z_0|^n\right)^m \\
&\leq \sum_{m=0}^{\infty} |b_m| \left(M|z-z_0|\right)^m \\
&\leq \infty,
\end{aligned}$$

if $|z - z_0|$ is small enough (the radius of convergence of h divided by M).

This proves absolute convergence for the composed series, and thus we can rearrange the doubly-indexed series for the composition so that it is a (convergent) power series. □

Corollary: if f is analytic and non-zero on a disk D, so is $1/f$.

Proof. $1/f$ is f composed with $1/z$, which is analytic on $f(D) \subset \mathbb{C} \setminus \{0\}$. $\square$

Corollary: rational functions are analytic away from their poles.

Corollary: If U is open,let $H^\infty(U)$ be the collection of bounded analytic functions on U. Then $H^\infty(U)$ is a communtative algebra.

The supremum norm makes it a normed algebra. We shall see later is is complete, hence a Banach algebra.

Definition: If f is defined in a neighborhood of z then

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

is called the (complex) derivative of f, provided the limit exists.

- $(z^n)' = nz^{n-1}$.

- $f(z) = \overline{z}$ is not differentiable,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = 1, \qquad \lim_{y \to 0} \frac{f(iy) - f(0)}{iy} = -1.$$

If $f = (x, y) + iv(x, y)$, need $u_x = v_y$ and $u_y = -v_x$ (Cauchy-Riemann equations; more about this later).

- Chain rule, product rule, quotient rule all hold.

Theorem 2.12: *If* $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges in* $D = D(z_0, r)$ *then* $f'(z)$ *exists for all* $z \in D$ *and*

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(z-z_0)^n,$$

for $z \in D$*. Moreover the series for* f' *based at* z_0 *has the same radius of convergence as the series for* f*.*

Proof. If $0 < |h| < r$ then

$$\frac{f(z_0+h)-f(z_0)}{h} - a_1 = \frac{\sum_{n=0}^{\infty} a_n h^n - a_0}{h} - a_1 = \sum_{n=2}^{\infty} a_n h^{n-1} = \sum_{n=1}^{\infty} a_{n+1} h^n.$$

By the root test, the region of convergence for the series $\sum a_{n+1} h^n$ is a disk centered at 0 and hence it converges uniformly in $\{h : |h| \leq r_1\}$, if $r_1 < r$.

In particular, $\sum a_{n+1} h^n$ is continuous at 0 and hence

$$\lim_{h \to 0} \sum_{n=1}^{\infty} a_{n+1} h^n = 0.$$

This proves that $f'(z_0)$ exists and equals a_1.

By Theorem 4.2, f has a power series expansion about each z_1 with $|z_1 - z_0| < r$ given by

$$\sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0)^{n-k}\right](z - z_1)^k$$

Therefore $f'(z_1)$ exists and equals the coefficient of $z - z_1$

$$f'(z_1) = \sum_{n=1}^{\infty} a_n \binom{n}{1} (z_1 - z_0)^{n-1} = \sum_{n=1}^{\infty} a_n n (z_1 - z_0)^{n-1}.$$

By the root test and the fact that $n^{\frac{1}{n}} \to 1$, the series for f' has exactly the same radius of convergence as the series for f. $\square$

Corollary 2.13: *An analytic function f has derivatives of all orders. Moreover if f is equal to a convergent power series on $D = D(z_0, r)$ then the power series is given by*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

for $z \in D$.

Corollary: $\sum_0^\infty nz^{n-1} = 1/(1-z)^2$.

Technical result needed later:

Corollary 2.14: *If* $f(z) = \sum a_n(z - z_0)^n$ *converges in* $D(z_0, r)$ *then*

$$f'(z_0) = \lim_{z,w \to z_0} \frac{f(z) - f(w)}{z - w}.$$

Proof. Set $z = z_0 + h$ and $w = z_0 + k$.

Then for $h - k \neq 0$ and $\epsilon = \max(|h|, |k|) < r$,

$$\frac{f(z_0+h) - f(z_0+k)}{h-k} - a_1 = \sum_{n=2}^{\infty} a_n \frac{h^n - k^n}{h-k} = \sum_{n=2}^{\infty} a_n \sum_{j=0}^{n-1} h^j k^{n-j-1}.$$

But since $|a_n| \leq Cs^n$ for some $s < \infty$,

$$\sum_{n=2}^{\infty} |a_n| \sum_{j=0}^{n-1} |h|^j |k|^{n-j-1} \leq \sum_{n=N}^{M} |a_n| n \epsilon^{n-1} \leq C \sum_{n=2}^{M} n(s\epsilon)^{n-1} \leq \frac{Cs\epsilon}{(1-s\epsilon)^2},$$

tends to zero with ϵ. This proves the limit exists, as desired. □

Corollary 2.15: *If* $f(z) = \sum a_n(z-z_0)^n$ *converges in* $B = \{z : |z-z_0| < r\}$ *then the power series*

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}$$

converges in B *and satisfies*

$$F'(z) = f(z),$$

for $z \in B$.

Some open problem about polynomials:

Smale's mean value problem: if p is a polynomial of degree ≥ 2 and $z \in \mathbb{C}$, prove there is a critical point w of p so that

$$|p(z) - p(w)| \leq |p'(z)| \cdot |z - w|.$$

See Mean value problem.

MLC: Let $\mathcal{M}$ be the set of c's so that $z_0 = 0$, $z_{n+1} = (z_n)^2 + c$ is a bounded sequence. Is $\mathcal{M}$ locally connected?

See Mandelbrot set.

See MATLAB code to draw Mandelbrot set.