MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop
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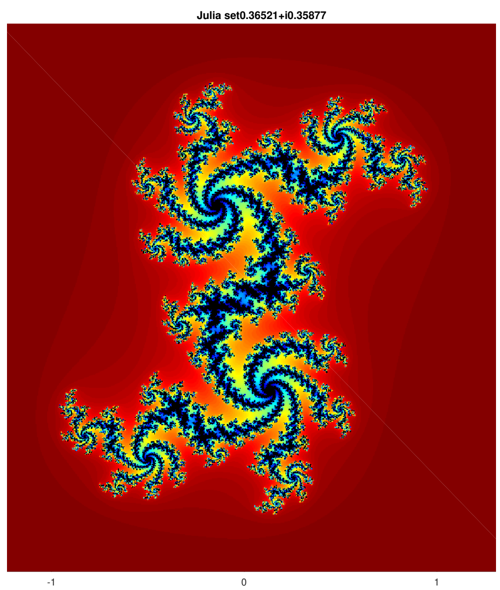

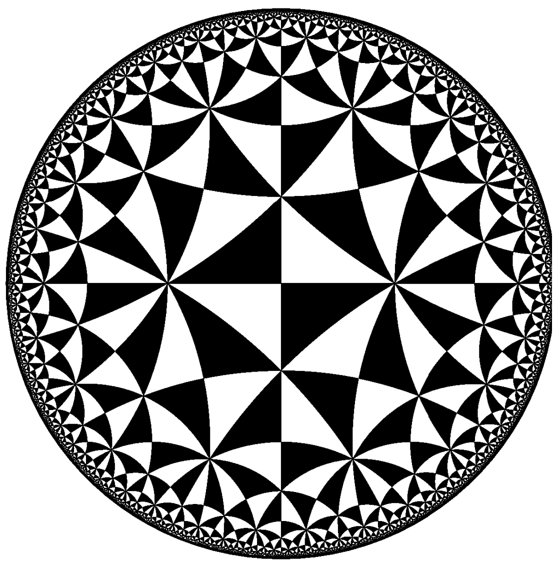

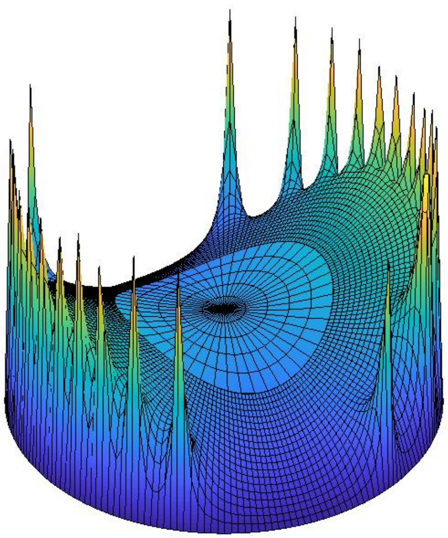

Chapter 15: The Uniformization Theorem

Section 15.2: Green's Function

Suppose W is a Riemann suface and $p \in W$.

The Green's function on W with pole at p_0 is a positive function $G(z, p_0)$ that is harmonic on $W \setminus \{p\}$, has a logarithmic pole at p_0 and tends to zero at ∞.

For example, $\log \frac{1}{|z|}$ is the Green's function for $\mathbb{D}$ with pole at 0.

Some Riemann surfaces have a Green's function; some do not.

Very important distinction. Many different characterizations of two cases.

A Riemann surface has a Green's function iff several other conditions hold.

(1) Brownian motion is recurrent.

(2) Geodesic flow on the unit tangent bundle of W is ergodic.

(3) Poincare series of covering group Γ diverges.

(4) Γ has the Mostow rigidity property (cojugating circle homeomorphisms are Möbius or singular).

(5) Γ has the Bowen's property.

(6) Almost every geodesic ray is recurrent. Equivalently, the set of escaping geodesic rays from a point $p \in W$ has zero (visual) measure.

Let $\mathcal{F}_{p_0}$ be the collection of subharmonic functions v on $W \setminus p_0$ satisfying

$$v = 0 \text{ on } W \setminus K, \text{ for some compact } K \subset W \text{ with } K \neq W, \text{ and} \tag{15.1a}$$

$$\limsup_{p \to p_0} \big(v(p) + \log |z(p)| \big) < \infty. \tag{15.1b}$$

Note that $v \in \mathcal{F}_{p_0}$ is not assumed to be subharmonic at p_0, and indeed it can tend to $+\infty$ as $p \to p_0$. Set

$$g_W(p, p_0) = \sup\{v(p) : v \in \mathcal{F}_{p_0}\}. \tag{15.2}$$

The collection $\mathcal{F}_{p_0}$ is a Perron family on $W \setminus \{p_0\}$, so one of the following two cases holds by Harnack's theorem:

Case 1: $g_W(p, p_0)$ is harmonic in $W \setminus \{p_0\}$, or

Case 2: $g_W(p, p_0) = +\infty$ for all $p \in W \setminus \{p_0\}$.

In the first case, $g_W(p, p_0)$ is called **Green's function on** W with pole (or logarithmic singularity) at p_0.

In 2nd case, Green's function with pole at p_0 does not exist on W.

Lemma 15.4: *Suppose* $p_0 \in W$ *and suppose* $z : U \to \mathbb{D}$ *is a coordinate function such that* $z(p_0) = 0$. *If* $g_W(p, p_0)$ *exists, then*

$$g_W(p, p_0) > 0 \text{ for } p \in W \setminus \{p_0\}, \tag{15.3}$$

$$g_W(p, p_0) + \log|z(p)| \text{ extends to be harmonic in } U. \tag{15.4}$$

Proof. The function

$$v_0(p) = \begin{cases} -\log|z(p)| & \text{for } p \in U \\ 0 & \text{for } p \in W \setminus U \end{cases}$$

is in $\mathcal{F}_{p_0}$. Hence $g_W(p, p_0) \geq 0$ and $g_W(p, p_0) > 0$ if $p \in U$.

By the maximum principle applied to $-g$ in $W \setminus \{p_0\}$, $G_W > 0$.

If $v \in \mathcal{F}_{p_0}$ then by Lindelöf's maximum principle

$$\sup_{U \setminus \{p_0\}} \big(v + \log|z|\big) = \sup_{\partial U} v \leq \sup_{\partial U} g_W < \infty.$$

Taking the supremum over $v \in \mathcal{F}_{p_0}$, we obtain

$$g_W + \log|z| \leq \sup_{\partial U} g_W < \infty$$

in $U \setminus \{p_0\}$. We also have that for $p \in U \setminus \{p_0\}$¡

$$g_W + \log|z| \geq v_0 + \log|z| = 0. \tag{15.5}$$

Thus $g_W + \log|z|$ is bounded and harmonic in $U \setminus \{p_0\}$.

Using the Poisson integral formula on $z_\alpha(U)$ we can find a bounded harmonic function on U which agrees with $g_W + \log|z|$ on ∂U. By Lindelöf's maximum principle, $g_W + \log|z|$ extends to be harmonic in U. $\square$

Green's function for disk:

The Green's function for the unit disk $\mathbb{D}$ is given by

$$g_{\mathbb{D}}(z,a) = \log\left|\frac{1-\overline{a}z}{z-a}\right|.$$

Proof: If $g(z) = \log\frac{|1-\overline{a}z|}{|z-a|}$ then by Lindelöf's maximum principle, each candidate subharmonic function v in the Perron family $\mathcal{F}_a$ is bounded by g.

Moreover $\max(g-\epsilon, 0) \in \mathcal{F}_a$, when $\epsilon > 0$. Letting $\epsilon \to 0$, we conclude $g = g_{\mathbb{D}}(z,a)$.

Note $g_{\mathbb{D}}(z,w) = g_d isk(w,z)$. This is true in general.

Green's function for simpy connected domain:

If $\Omega \subset \mathbb{C}$ is simply connected and $f : \Omega \to \mathbb{D}$ is conformal with $f(w) = 0$ then

$$G_\Omega(z, w) = -\log |f(z)|.$$

Thus $f = \exp(-G - i\widetilde{G})$.

Theorem 15. 5: *Suppose W_0 is a Riemann surface and suppose U_0 is a coordinate disk whose closure is compact in W_0. Set $W = W_0 \setminus \overline{U_0}$. Then $g_W(p, p_0)$ exists for all $p, p_0 \in W$ with $p \neq p_0$.*

Proof. Fix $p_0 \in W$ and let $U \subset W$ be a coordinate disk (containing p_0) with compact closure and with coordinate function $z : U \to \mathbb{D}$ and $z(p_0) = 0$.

To prove that g_W exists, we show that the family $\mathcal{F}_{p_0}$ is bounded above.

Fix r, with $0 < r < 1$, and set $rU = \{p \in W : |z(p)| < r\}$. If $v \in \mathcal{F}_{p_0}$ then by Lindelöf's maximum principle, for all $p \in U$ we have

$$v(p) + \log|z(p)| \le \max_{q \in \partial U}\big(v(q) + \log|z(q)|\big) = \max_{q \in \partial U} v(q).$$

Thus

$$\max_{p \in \partial rU} v(p) + \log r \le \max_{p \in \partial U} v(p). \tag{15.6}$$

Let $\mathcal{F}$ denote the collection of functions u which are subharmonic on $W \setminus \overline{rU}$ with $u = 0$ on $W_0 \setminus K$ for some compact set $K \subset W_0$, which can depend on u, and such that

$$\limsup_{p \to \zeta} u(p) \leq 1 \quad \text{and} \quad \limsup_{p \to \alpha} u(p) \leq 0$$

for $\zeta \in \partial rU$ and $\alpha \in \partial U_0$.

Applying the maximum principle on the interior of $K \setminus (U_0 \cup rU)$ we obtain $u \leq 1$ on $W \setminus \overline{rU}$. Because $\mathcal{F}$ is a Perron family,

$$\omega(p) = \sup\{u(p) : u \in \mathcal{F}\}$$

is harmonic in $W \setminus \overline{rU}$.

We can construct a local barrier at each point of $\partial U_0 \cup \partial rU$ for the region $W \setminus \overline{rU}$ by transporting the problem to a region in $\mathbb{D}$ via a coordinate map.

Thus by exactly the same argument used for the Dirichlet problem, the harmonic function ω extends to be continuous at each point of ∂U_0 and each point of $\partial(rU)$ so that $\omega(p) = 0$ for $p \in \partial U_0$ and $\omega(p) = 1$ for $p \in \partial(rU)$.

In particular, this implies ω is not constant. Moreover $0 \leq \omega \leq 1$ because $0 \in \mathcal{F}$ and each candidate $u \in \mathcal{F}$ is bounded by 1.

Since ω is not constant, the maximum principle implies $0 < \omega(p) < 1$ for $p \in W \setminus \overline{rU}$.

ω is called the **harmonic measure** of $\partial(rU)$ in the region $W \setminus \overline{rU}$.

For $v \in \mathcal{F}_{p_0}$, by the maximum principle we have that

$$v(p) \leq \left(\max_{\partial rU} v \right) \omega(p)$$

for $p \in W \setminus \overline{rU}$ by the maximum principle applied to $v - (\max_{\partial rU} v)w$. So

$$\max_{\partial U} v \leq \left(\max_{\partial rU} v \right) \max_{p \in \partial U} \omega(p) \leq \left(\max_{\partial rU} v \right) (1 - \delta) \tag{15.7}$$

for some $\delta > 0$.

Adding inequalities (15.6) and (15.7) yields

$$\delta \max_{\partial rU} v \leq \log \frac{1}{r}$$

for all $v \in \mathcal{F}$, with δ is independent of v.

This implies that Case 2 does not hold and hence Green's function exists. $\square$

Theorem 15.6: *Suppose W is a Riemann surface for which g_W with pole at $p_0 \in W$ exists. Let W^* be the simply-connected universal covering surface of W, let $\pi : W^* \to W$ be the universal covering map and suppose $\pi(p_0^*) = p_0$. Then g_{W^*} with pole at p_0^* exists and satisfies*

$$g_W(\pi(p^*), \pi(p_0^*)) = \sum_{q^*:\pi(q^*)=\pi(p_0^*)} g_{W^*}(p^*, q^*). \tag{15.8}$$

Thus we can recover Green's function on a Riemann surface from Green's function on the universal cover by summing over deck transformations.

Proof. Suppose $q_1^*, \ldots, q_n^*$ are distinct points in W^* with $\pi(q_j^*) = p_0 = \pi(p_0^*)$.

Suppose $v_j \in \mathcal{F}_{q_j^*}$, the Perron family for the construction of $g_{W^*}(\cdot, q_j^*)$. So $v_j = 0$ off K_j^*, a compact subset of W^* and

$$\limsup_{p^* \to q_j^*} \big(v_j(p^*) + \log|z \circ \pi(p^*)|\big) < \infty,$$

where z is a coordinate chart on W with $z(p_0) = 0$.

Recall $g_W(p, p_0) + \log|z(p)|$ extends to be finite and continuous at p_0, and so

$$\lim_{p^* \to q_j^*} g_W(\pi(p^*), p_0) + \log|z(\pi(p^*))|$$

exists and is finite because $\pi(q_j^*) = p_0$. Thus

$$\left(\sum_{j=1}^{n} v_j(p^*)\right) - g_W(\pi(p^*), p_0)$$

is bounded in a neighborhood of each q_j^*, for $j = 1, \ldots, n$, and ≤ 0 off $K^* = \cup_j K_j^*$, by (15.3).

By Lindelöf's maximum principle on the interior of $K^* \setminus \{q_1^*, \ldots, q_n^*\}$, it is bounded above by 0. Taking the supremum over all such v_j we conclude that $g_{W^*}(p^*, q_j^*)$ exists and

$$\left(\sum_{j=1}^{n} g_{W^*}(p^*, q_j^*)\right) - g_W(\pi(p^*), p_0) \leq 0.$$

Taking the supremum over all such finite sums we have

$$S(p^*) \equiv \sum_{q^*:\pi(q^*)=p_0} g_{W^*}(p^*, q^*) \leq g_W(\pi(p^*), p_0).$$

Moreover, as a supremum of finite sums of positive harmonic functions, $S(p^*)$ is harmonic on $W^* \setminus \pi^{-1}(p_0)$ by Harnack's principle. By (15.4)

$$S(p^*) + \log |z(\pi(p^*))| \tag{2.9}$$

extends to be harmonic in a neighborhood of each $q^* \in \pi^{-1}(p_0)$.

Now take $v \in \mathcal{F}_{p_0}$, the Perron family used to construct $g_W(p, p_0)$. Let U^* be a coordinate disk containing p_0^*, so that $z \circ \pi$ is a coordinate map of U^* onto $\mathbb{D}$, vanishing at p_0^*.

Then by (15.9) and Lindelöf's maximum principle

$$v(\pi(p^*)) - S(p^*) \leq 0$$

for $p^* \in U^*$.

Taking the supremum over all such v we obtain

$$g_W(\pi(p^*), p_0) \leq S(p^*)$$

and thus (15.8) holds on $U^* \setminus p_0^*$ and therefore on W^*. $\square$

Corollary 15.7: *Suppose W is a Riemann surface for which Green's function g_W with pole at p exists, for some $p \in W$, and suppose $W^* = \mathbb{D}$. Then g_W with pole at q exists for all $q \in W$ and*

$$g_W(p,q) = g_W(q,p). \tag{15.10}$$

We will remove the hypothesis that $W^* = \mathbb{D}$ in Section 3. See Corollary 3.2.

This symmetry is amazing and very useful.

Proof. As noted earlier $g_{\mathbb{D}}(a,b) = -\log|(a-b)/(1-\overline{b}a)|$.

If τ is an LFT of the disk onto the disk, then by an elementary computation $g_{\mathbb{D}}(a,\tau(b)) = g_{\mathbb{D}}(\tau^{-1}(a),b)$ for all $a,b \in \mathbb{D}$ with $a \neq \tau(b)$.

Let $\mathbb{G}$ denote the group of deck transformations, a group of LFTs τ mapping $\mathbb{D}$ onto $\mathbb{D}$ and satisfying $\pi \circ \tau = \pi$. Moreover if $\pi(q^*) = \pi(p^*)$ then there is a $\tau \in \mathbb{G}$ such that $\tau(p^*) = q^*$, by Lemma 14.14.

Suppose $g(p, p_0)$ exists for some $p_0 \in W$ and all $p \neq p_0$.

Choose $p_0^* \in \mathbb{D}$ so that $\pi(p_0^*) = p_0$.

By Theorem 15.6 for $p* \notin \pi^{-1}(p_0)$

$$
\begin{aligned}
g_W(\pi(p^*), \pi(p_0^*)) &= \sum_{\tau \in \mathbb{G}} -\log\left|\frac{p^* - \tau(p_0^*)}{1 - \overline{\tau(p_0^*)}p^*}\right| && (15.11) \\
&= \sum_{\tau \in \mathbb{G}} -\log\left|\frac{\tau^{-1}(p^*) - p_0^*}{1 - \overline{\tau^{-1}(p^*)}p_0^*}\right|. && (15.12)
\end{aligned}
$$

Fix $p^* \in \mathbb{D} \setminus \pi^{-1}(p_0)$. Each term in the sum

$$S(q^*) = \sum_{\tau \in \mathbb{G}} -\log\left|\frac{\tau^{-1}(p^*) - q^*}{1 - \overline{\tau^{-1}(p^*)}q^*}\right|$$

is a positive harmonic function of $q^* \in \mathbb{D} \setminus \tau^{-1}(p^*)$.

Since the sum of these positive harmonic functions converges when $q^* = p_0^*$, the function S is harmonic in $\mathbb{D} \setminus \{\tau^{-1}(p^*) : \tau \in \mathbb{G}\}$, by Harnack's theorem.

If $v \in \mathcal{F}_p$, the Perron family for $g_W(q,p)$ where $p = \pi(p^*)$, then by Lindelöf's maximum principle $v \leq S \circ \pi^{-1}$.

Taking the supremum over all $v \in \mathcal{F}_p$ we conclude that $g_W(q,p)$ exists and $g_W(q,p) \leq S \circ \pi^{-1}(q)$, for all $q \neq p$. Thus

$$g_W(\pi(p_0^*), \pi(p^*)) \leq S(p_0^*) = g_W(\pi(p^*), \pi(p_0^*)).$$

Reversing the roles of p_0^* and p^* proves (15.10) for $q = p_0$ and all $p \neq p_0$.

Because Green's function g_W with pole at p then exists for every $p \in W$, (15.10) must hold for all p and q. $\square$

Section 15.3: Simply Connected Riemann Surfaces

Theorem 15.8, Uniformization, Case 1: *If W is a simply-connected Riemann surface then the following are equivalent:*

$$g_W(p, p_0) \text{ exists for some } p_0 \in W \tag{15.13}$$

$$g_W(p, p_0) \text{ exists for all } p_0 \in W, \tag{15.14}$$

$$\text{There is a one-to-one analytic map } \varphi \text{ of } W \text{ onto } \mathbb{D}. \tag{15.15}$$

Moreover if g_W exists, then

$$g_W(p_1, p_0) = g_W(p_0, p_1), \tag{15.16}$$

and $g_W(p, p_0) = -\log|\varphi(p)|$, where $\varphi(p_0) = 0$.

Paul Koebe

Proved uniformization theorem in 1907.

Koebe was considered a conceited and disagreeable man with a reputation for picking up the ideas of younger people and, because he was so quick, being able to finalise and publish them first He was, nevertheless, an outstanding mathematician. – Constance Reid

Proof. First suppose (15.15) holds.

Then there is a one-to-one analytic map φ of W onto $\mathbb{D}$ and let $p_0 \in W$. By composing φ with an LFT, we can assume that $\varphi(p_0) = 0$.

If $v \in \mathcal{F}_{p_0}$ then $v = 0$ off a compact set K, so by Lindelöf's maximum principle applied on the interior of $K \setminus \{p_0\}$ we have on W

$$v + \log|\varphi| \leq 0,$$

Taking the supremum over all such v shows that $g_W(p, p_0) < \infty$ and therefore (15.14) holds. Clearly (15.14) implies (15.13).

Now suppose (15.13) holds.

By (15.4), $g + \log|z - p_0|$ is harmonic at p_0, so there is an analytic function f defined on a coordinate disk U containing p_0 so that

$$\mathrm{Re} f(p) = g_W(p, p_0) + \log|z(p)|$$

for $p \in U$.

Hence the function $\varphi(p) = ze^{-f(p)}$ is analytic in U and satisfies $|\varphi(p)| = e^{-g_W(p,p_0)}$ and $\varphi(p_0) = 0$.

On any coordinate disk U_α with $p_0 \notin U_\alpha$, $g_W(p, p_0)$ is the real part of an analytic function because it is harmonic.

The difference of two analytic functions with the same real part is constant, so φ on U can be analytically continued along all curves in W beginning at p_0.

By the monodromy theorem there is a function φ, analytic on W, such that

$$|\varphi(p)| = e^{-g_W(p,p_0)} < 1.$$

We claim that φ is one-to-one.

If $\varphi(p) = \varphi(p_0) = 0$, then $p = p_0$ because $g_W(p, p_0)$ is finite for $p \neq p_0$. Let $p_1 \in W$, with $p_1 \neq p_0$. Then by (15.3), $|\varphi(p_1)| < 1$ and

$$\varphi_1 \equiv \frac{\varphi - \varphi(p_1)}{1 - \overline{\varphi(p_1)}\varphi}$$

is analytic on W and $|\varphi_1| < 1$.

If $v \in \mathcal{F}_{p_1}$, then by (15.1) and Lindelöf's maximum principle, as argued above,

$$v + \log|\varphi_1| \leq 0.$$

Taking the supremum over all such v, we conclude that $g_W(p, p_1)$ exists and that

$$g_W(p, p_1) + \log|\varphi_1| \leq 0. \tag{15.7}$$

Taking the supremum over all such v, we conclude that $g_W(p, p_1)$ exists and that

$$g_W(p, p_1) + \log|\varphi_1| \leq 0. \tag{3.5}$$

Setting $p = p_0$ in (15.17) gives

$$g_W(p_0, p_1) \leq -\log|\varphi_1(p_0)| = -\log|\varphi(p_1)| = g_W(p_1, p_0).$$

Switching the roles of p_0 and p_1 gives (15.16)

Moreover equality holds in (15.17) at $p = p_0$ so that by the maximum principle $g_W(p, p_1) = -\log|\varphi_1(p)|$ for all $p \in W \setminus \{p_1\}$.

Now if $\varphi(p_2) = \varphi(p_1)$, then by the definition of φ_1, $\varphi_1(p_2) = 0$. Thus $g_W(p_2, p_1) = \infty$ and $p_2 = p_1$.

Therefore φ is one-to-one.

The image $\varphi(W) \subset \mathbb{D}$ is simply-connected, for if $\gamma \subset \varphi(W)$ is a closed curve then $\varphi^{-1}(\gamma) \subset W$ is closed and therefore homotopic to a constant curve.

Applying the map φ to the homotopy gives a homotopy in $\varphi(W)$ of γ to a constant curve.

If $\varphi(W) \neq \mathbb{D}$ then by the Riemann mapping theorem we can find a one-to-one analytic map ψ of $\varphi(W)$ onto $\mathbb{D}$ with $\psi(0) = 0$.

The map $\psi \circ \varphi$ is then a one-to-one analytic map of W onto $\mathbb{D}$, with $\psi \circ \varphi(p_0) = 0$, proving (15.15). $\square$

The map φ_1 in the proof above is actually onto, as can be seen by applying the "onto" argument in the proof of the Riemann mapping theorem.

Corollary 15.9: *Suppose W is a Riemann surface for which Green's function g_W with pole at p exists, for some $p \in W$. Then g_W with pole at q exists for all $q \in W$ and*

$$g_W(p, q) = g_W(q, p). \tag{15.18}$$

Proof. If W is a Riemann surface such that g_W with pole at some $p \in W$ exists, then g_{W^*} exists by Theorem 15.6.

But then by Theorem 15.8, W^* is conformally equivalent to $\mathbb{D}$.

Applying Exercise 15.4(a) and Corollary 15.7 yields the corollary. □

Exercise 15.4(a): If φ is a one-to-one analytic map of a Riemann surface W_1 onto a Riemann surface W_2 then Green's function on W_1 exists if and only if Green's function on W_2 exists. Moreover $g_{W_2}(\varphi(p), \varphi(p_0)) = g_{W_1}(p, p_0)$.

Before proving the uniformization theorem when there is no Green's function, we need a technical lemma, proving existence of the dipole Green's function.

The dipole Green's function has two logarithmic poles with opposite signs, e.g.,

$$\log\left|\frac{z-a}{z-b}\right|$$

on the plane. This has two opposite poles and tends to 0 at infinity.

The next lemma says that a dipole Green's function always exists.

For surfaces with Green's function this is easy: take $G(z,p)-G(z,q)$ for $p \neq q$.

Lemma 15.11: *Suppose* W *is a Riemann surface and for* $j = 1, 2$, *suppose that* $z_j : U_j \to \mathbb{D}$ *are coordinate functions with coordinate disks* U_j *satisfying* $\overline{U_1} \cap \overline{U_2} = \emptyset$, *and* $z_j(p_j) = 0$. *Then there is a function* $G(p) \equiv G(p, p_1, p_2)$, *harmonic in* $p \in W \setminus \{p_1, p_2\}$ *such that*

$$G + \log|z_1| \text{ extends to be harmonic in } U_1, \tag{15.19}$$

$$G - \log|z_2| \text{ extends to be harmonic in } U_2, \tag{15.20}$$

and

$$\sup_{p \in W \setminus (U_1 \cup U_2)} |G(p)| < \infty. \tag{15.21}$$

Proof. As noted above, we may assume W has no Green'sfunction.

The idea of the proof is to remove a small disk from W, giving a surface with Green's function, and therefore with a dipole Green's function. Then let the radius of the removed disk decrease to zero, and prove the dipole Green's functions have a limit (this is the tricky part).

We consider the difference

$$g(p, p_1) - g(p, p_2) = [g(p, p_1) - g(p_2, p_1)] - [g(p, p_2) - g(p_1, p_2)],$$

and show these two terms stay bounded as the disk shrinks.

Suppose z_0 is a coordinate function with coordinate chart U_0 such that $\overline{U_0} \cap \overline{U_j} = \emptyset$ for $j = 1, 2$.

Let p_0 be the point in U_0 such that $z_0(p_0) = 0$.

Set $tU_0 = \{p \in W : |z_0(p)| < t\}$ and set $W_t = W \setminus tU_0$.

By Theorem 15.5 (omitted disk implies Green's function exists), $g_{W_t}(p, p_1)$ exists for all $p, p_1 \in W_t$ with $p \neq p_1$.

Fix r, $0 < r < 1$, and set $rU_1 = \{p \in W : |z_1(p)| < r\}$.

By the maximum principle

$$g_{W_t}(p, p_1) \leq M_1(t) \equiv \max_{q \in \partial rU_1} g_{W_t}(q, p_1), \tag{15.22}$$

for all $p \in W_t \setminus rU_1$, because the same bound holds for all candidates in the Perron family defining g_{W_t}.

The growth estimate (15.6) shows that

$$M_1(t) \le \max_{p\in\partial U_1} g_{W_t}(p,p_1) + \log\frac{1}{r}. \tag{15.23}$$

By (15.22), $u_t(p) \equiv M_1(t) - g_{W_t}(p,p_1)$ is a positive harmonic function in $W_t \backslash rU_1$ and by (15.23) there exists $q \in \partial U_1$ with $u_t(q) \le \log\frac{1}{r}$.

Riemann surfaces are pathwise connected so let γ be a curve in $W \setminus (U_1 \cup U_2)$ connecting ∂U_1 to ∂U_2 which does not pass through p_0.

Then for $t \leq t_0$, $K = \partial U_1 \cup \overline{U_2} \cup \gamma \subset W_t \setminus \overline{rU_1}$ is compact and connected.

By Harnack's inequality there is a constant $C < \infty$ depending on K and r but not on t so that for all $p \in K$ and $t \leq t_0$

$$0 \leq u_t(p) \leq C,$$

and

$$|g_{W_t}(p, p_1) - g_{W_t}(p_2, p_1)| = |u_t(p_2) - u_t(p)| \leq 2C.$$

Likewise, if $K' = \partial U_2 \cup \overline{U_1} \cup \gamma$ there is a constant $C < \infty$ so that

$$|g_{W_t}(p, p_2) - g_{W_t}(p_1, p_2)| \leq C,$$

for all $p \in K'$ and $t \leq t_0$.

By Corollary 3.2, $g_{W_t}(p_1, p_2) = g_{W_t}(p_2, p_1)$ and so the function

$$\begin{aligned} G_t(p, p_1, p_2) &\equiv g_{W_t}(p, p_1) - g_{W_t}(p, p_2) \\ &= (g_{W_t}(p, p_1) - g_{W_t}(p_2, p_1)) - (g_{W_t}(p, p_2) - g_{W_t}(p_1, p_2)) \end{aligned}$$

is harmonic in $W_t \setminus \{p_1, p_2\}$ and satisfies

$$|G_t(p, p_1, p_2)| \leq C,$$

for all $p \in K \cap K' \supset \partial U_1 \cup \partial U_2$ and some finite C independent of t.

If $v \in \mathcal{F}_{p_1}$, the Perron family for $g_{W_t}(p, p_1)$, then $v = 0$ off a compact subset of W_t and $g_{W_t} > 0$ so that by the maximum principle

$$\begin{aligned}\sup_{W_t \setminus U_1} [v(p) - g_{W_t}(p, p_2)] &\leq \max(0, \sup_{\partial U_1} [v(p) - g_{W_t}(p, p_2)]) \\ &\leq \max(0, \sup_{\partial U_1} [g_{W_t}(p, p_1) - g_{W_t}(p, p_2)]) \leq C.\end{aligned}$$

Taking the supremum over all such v yields

$$\sup_{p \in W_t \setminus U_1} G_t(p, p_1, p_2) \leq C.$$

Similarly

$$\inf_{p \in W_t \setminus U_2} G_t(p, p_1, p_2) = - \sup_{p \in W_t \setminus U_2} -G_t(p, p_1, p_2) \geq -C,$$

and so

$$|G_t(p, p_1, p_2)| \leq C$$

for all $p \in W_t \setminus \{U_1 \cup U_2\}$.

The function $G_t + \log|z_1|$ extends to be harmonic in U_1, so by the maximum principle, we have that

$$\sup_{U_1} |G_t + \log|z_1|| = \sup_{\partial U_1} |G_t + \log|z_1|| = \sup_{\underline{\mathrm{U}}_1} |G_t| \leq C.$$

Similarly

$$\sup_{U_2} |G_t - \log|z_2|| = \sup_{\partial U_2} |G_t - \log|z_2|| = \sup_{\underline{\mathrm{U}}_2} |G_t| \leq C.$$

By normal families, there exists a sequence $t_n \to 0$ so that G_{t_n} converges uniformly on compact subsets of $W \setminus \{p_0, p_1, p_2\}$ to a function $G(p, p_1, p_2)$ harmonic on $W \setminus \{p_0, p_1, p_2\}$ satisifying (15.19), (15.20) and (15.21). The function $G(p, p_1, p_2)$ extends to be harmonic at p_0 because it is bounded in a punctured neighborhood of p_0.

Indeed, we can transfer this problem to the unit disk via the coordinate map, then use the Poisson integral formula to create a bounded harmonic function on the disk with the same values on $\partial\mathbb{D}$.

By Lindelöf's maximum principle, this is the harmonic extension. $\square$

Theorem 15.10, Uniformization, Case 2 *Suppose W is a simply-connected Riemann surface for which Green's function does not exist. If W is compact, then there is a one-to-one analytic map of W onto $\mathbb{C}^*$. If W is not compact, there is a one-to-one analytic map of W onto $\mathbb{C}$.*

Proof. We may suppose that $g_W(p, p_1)$ does not exist for all $p, p_1 \in W$.

Because W is simply-connected we can apply the monodromy theorem to obtain a meromorphic function φ_1 defined on W such that

$$|\varphi_1(p)| = e^{-G(p,p_1,p_2)},$$

where G is the dipole Green's function from Lemma 15.11.

Note that φ_1 has a simple zero at p_1, a simple pole at p_2 and no other zeros or poles.

Let us prove φ_1 is one-to-one.

If $p_0 \in W \setminus \{p_1, p_2\}$, then $\varphi_1(p_0) \neq 0, \infty$. Let φ_0 be the meromorphic function on W such that

$$|\varphi_0(p)| = e^{-G(p,p_0,p_2)}$$

and consider the function

$$H(p) = \frac{\varphi_1(p) - \varphi_1(p_0)}{\varphi_0(p)}.$$

Then H is analytic on W because its poles at p_2 cancel and because φ_0 has a simple zero at p_0.

By (15.21) and the analyticity of H, $|H|$ is bounded on W.

But if $v \in \mathcal{F}_{p_1}$, the Perron family used to construct $g_W(p, p_1)$, then by Lindelöf's maximum principle

$$v(p) + \log\left|\frac{H(p) - H(p_1)}{2\sup_W |H|}\right| \leq 0.$$

Because $g_W(p, p_1)$ does not exist, $\sup\{v(p) : v \in \mathcal{F}_{p_1}\} \equiv +\infty$ for every $p \in W \setminus \{p_1\}$, and therefore

$$H(p) \equiv H(p_1) = -\varphi_1(p_0)/\varphi_0(p_1) \neq 0, \infty.$$

Since $H \neq 0$, we conclude that $\varphi_1(p) \neq \varphi_1(p_0)$, unless $\varphi_0(p) = 0$. But if $\varphi_0(p) = 0$, then $p = p_0$. Thus φ_1 is one-to-one on $W \setminus \{p_1, p_2\}$.

But the only zero of φ_1 is p_1 and the only pole of φ_1 is p_2, so that φ_1 is one-to-one on W.

We have shown that φ_1 is a one-to-one analytic map from W to a simply-connected region $\varphi_1(W) \subset \mathbb{C}^*$.

If $\mathbb{C}^* \setminus \varphi_1(W)$ contains more than one point, then by the Riemann mapping theorem, there is a one-to-one analytic map of $\varphi_1(W)$, and hence of W, onto $\mathbb{D}$.

Since we assumed that g_W does not exist, this contradicts Theorem 15.8.

Thus $\mathbb{C}^* \setminus \varphi_1(W)$ contains at most one point, and the last two statements of Theorem 3.3 are now obvious. $\square$

Theorem, 15.12, The Uniformization Theorem: *Suppose W is a simply-connected Riemann surface.*

(1) *If Green's function exists for W, then there is a one-to-one analytic map of W onto $\mathbb{D}$.*

(2) *If W is compact, then there is a one-to-one analytic map of W onto $\mathbb{C}^*$.*

(3) *If W is not compact and if Green's function does not exist for W, then there is a one-to-one analytic map of W onto $\mathbb{C}$.*

Corollary 15.13, Rado: *Every Riemann surface satisfies the second axiom of countability.*

Proof. The universal covering map π sends a countable base on the universal covering surface W to a countable base on W. $\square$

Section 15.4: Classification of all Riemann Surfaces

Theorem 15.14: *If $U = \mathbb{C}^*$, $\mathbb{C}$, or $\mathbb{D}$ and if $\mathbb{G}$ is a properly discontinuous group of LFTs of U onto U, then $U/\mathbb{G}$ is a Riemann surface. A function f is analytic, meromorphic, harmonic, or subharmonic on $U/\mathbb{G}$ if and only if there is a function h defined on U which is (respectively) analytic, meromorphic, harmonic, or subharmonic on U satisfying $h \circ \tau = h$ for all $\tau \in \mathbb{G}$ and $h = f \circ \pi$ where $\pi : U \to U/\mathbb{G}$ is the quotient map. Every Rieman surface is conformally equivalent to $U/\mathbb{G}$ for some such U and $\mathbb{G}$.*

The only Riemann surface covered by the $\mathbb{C}^*$ is $\mathbb{C}^*$ (Proposition 16.2).

The only surfaces covered by $\mathbb{C}$ are $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$, and tori (Propososisition 16.3).

Any other Riemann surface is covered by the disk $\mathbb{D}$.