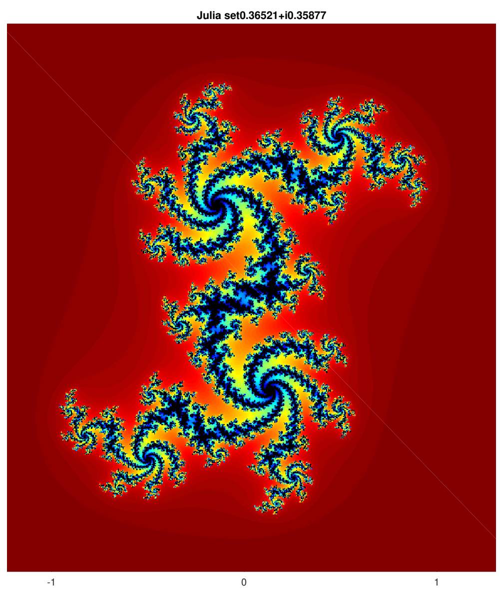
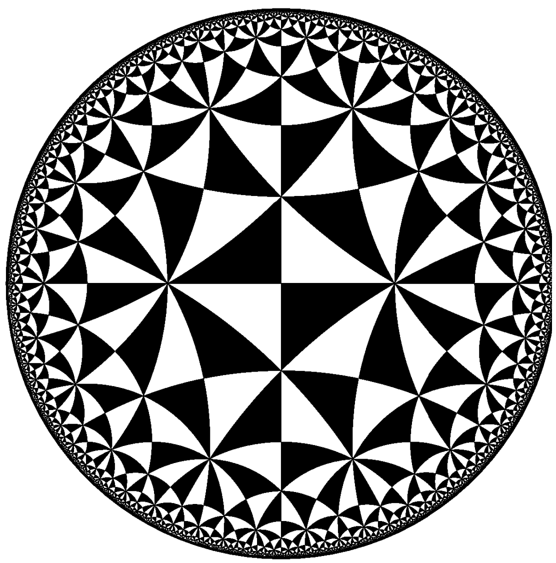
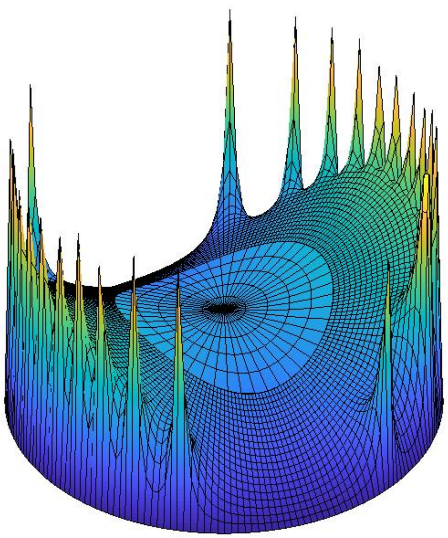
MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop
2024







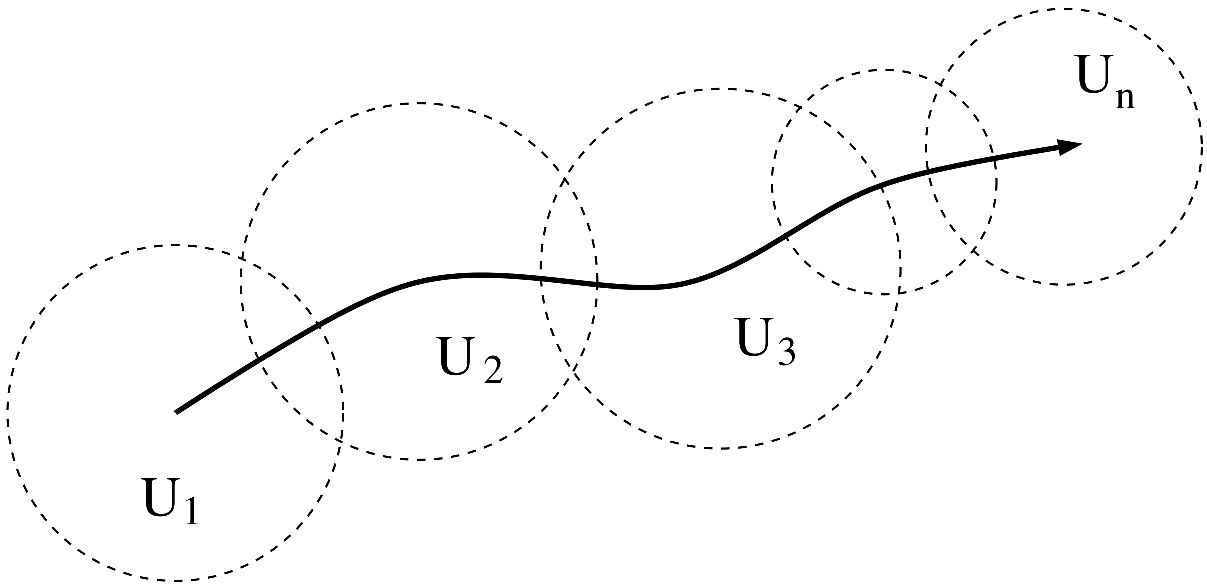
Chapter 14: Riemann surfaces

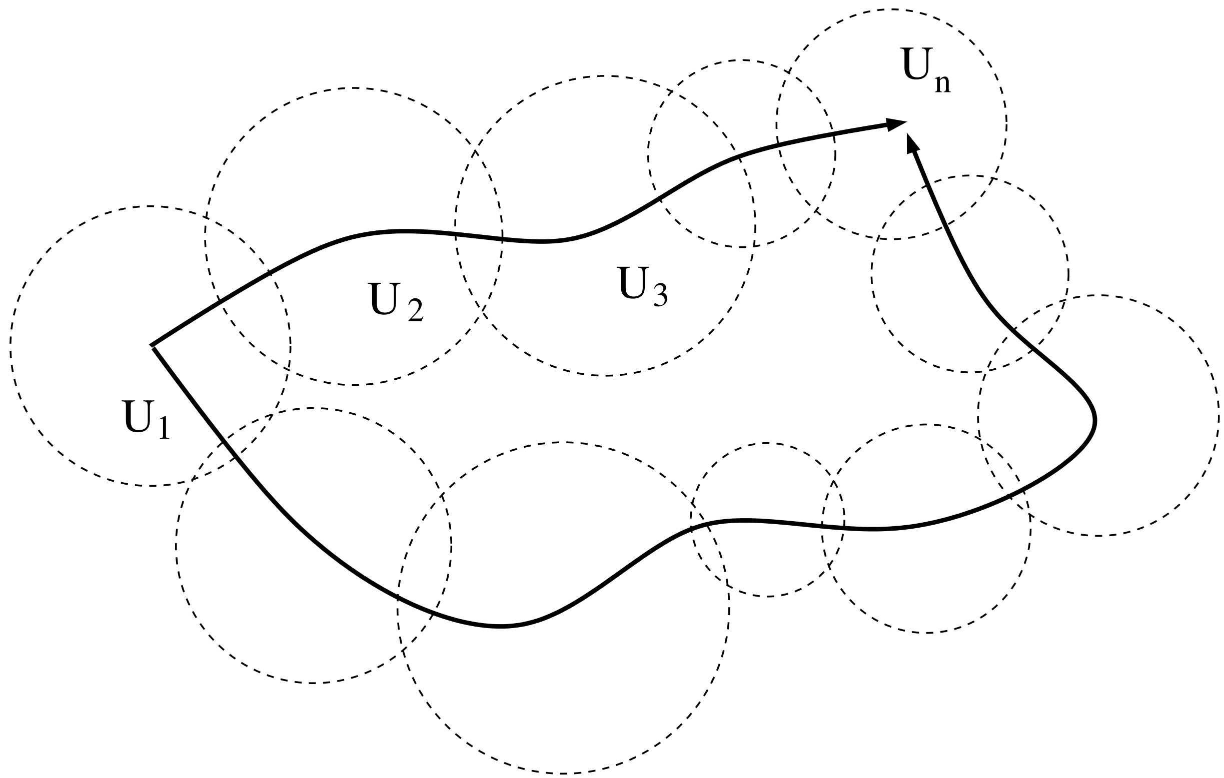
Section 14.1: Analytic Continuation and Monodromy

Definition 14.1: Suppose $\{f_j\}_1^n$ are analytic on U_j, and $f_j = f_{j+1}$ on $U_j \cap U_{j+1}$, $j = 0, \ldots, n-1$.

Then f_n is called a **direct analytic continuation** of f_0 to U_n.

Definition 14.2: If $\gamma : [0,1] \to \mathbb{C}$ is a curve and if f_0 is analytic in a neighborhood of $\gamma(0)$, then an **analytic continuation of f_0 along γ** is a finite sequence $f_1, \ldots, f_n$ of functions where $0 = t_0 < t_1 < ... < t_{n+1} = 1$ is a partition of $[0,1]$ and f_j is defined and analytic in a neighborhood of $\gamma([t_j, t_{j+1}])$, $j = 0, \ldots, n$ such that $f_j = f_{j+1}$ in a neighborhood of $\gamma(t_{j+1})$, $j = 0, \ldots, n-1$.





Do different paths give the same function? Sometimes.

Sometimes no: If we compute $\int \frac{dz}{z}$ we get a local branch of $\log z$ in any disk not containing 0.

If we take $\log 1 = 0$ and continue counterclockwise around $\mathbb{T}$ we get $2\pi i$ when we return to 1, not 0. If analytically continue clockwise around $\mathbb{T}$ we get $-2\pi i$.

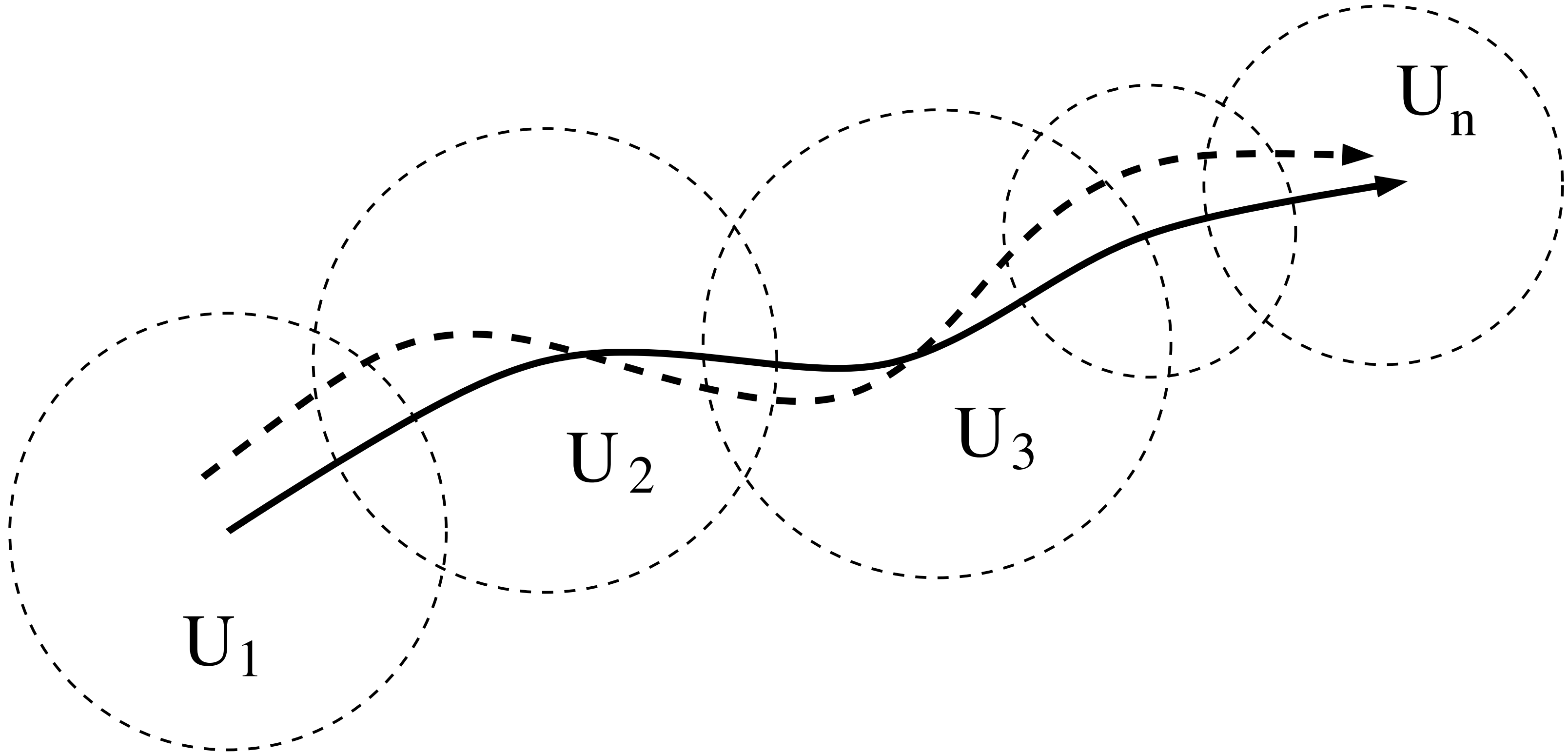
In this case we get different values depending on path.

If $f_1, \ldots, f_n$ is an analytic continuation of f_0 for the partition $0 = t_0 < t_1 \cdots < t_{n+1} = 1$, then we can refine the partition by choosing s with $t_j < s < t_{j+1}$ and using the function f_j on $\gamma([t_j, s])$ and on $\gamma([s, t_{j+1}])$.

So if $g_1, \ldots, g_m$ is another analytic continuation along γ of f_0, then we can choose a common refinement so that the two sequences of analytic functions are defined on the same partition $0 = u_0 < u_1 < \cdots < u_{k+1} = 1$.

But $f_1 = f_0 = g_1$ in a neighborhood of $\gamma(u_1)$, so by the uniqueness theorem $f_1 = g_1$ on an neighborhood of $\gamma([u_1, u_2])$, and by induction $g_j = f_j$ on a neighborhood of $\gamma([u_j, u_{j+1}])$.

In this sense, analytic continuation along a curve is unique.



Close paths give the same analytic continuation.

Suppose $f_1, f_2, \ldots, f_n$ is an analytic continuation of f_0 along γ with partition $0 = t_0 < \cdots < t_{n+1} = 1$.

We can choose $\epsilon > 0$ so that if σ is another curve such that $|\sigma(t) - \gamma(t)| < \epsilon$, for all $0 \leq t \leq 1$, then $f_1, f_2, \ldots, f_n$ is an analytic continuation of f_0 along σ.

Indeed if $\epsilon > 0$ is sufficiently small then f_j is defined and analytic in a neighborhood of $\sigma([t_j, t_{j+1}])$ and $f_j = f_{j+1}$ in a neighborhood of $\sigma(t_{j+1})$, $j = 0, \ldots, n$.

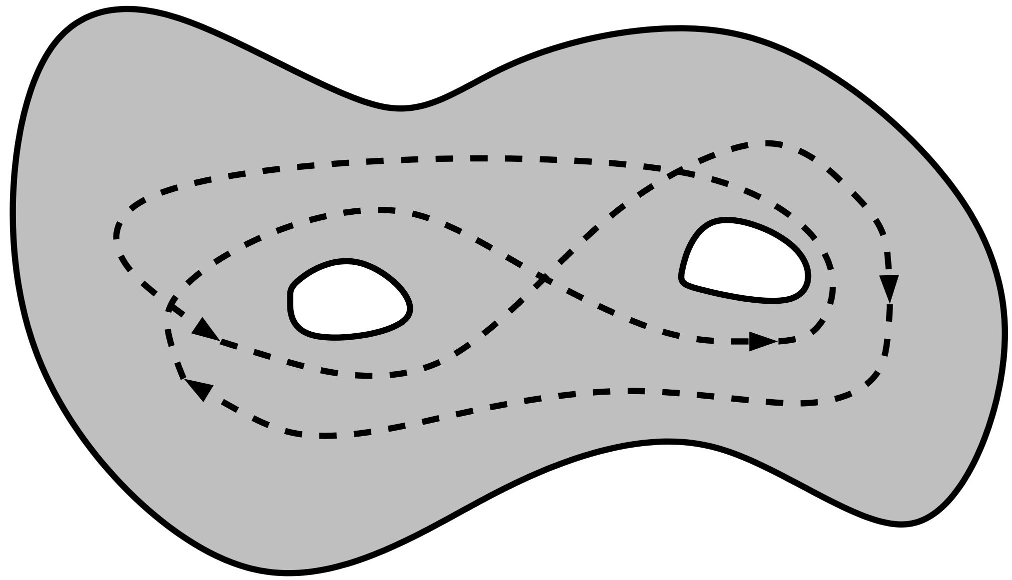
Suppose γ_0 and γ_1 are curves in a region Ω that begin at b and end at c.

We say γ_0 is **homotopic in** Ω to γ_1 if there exists a collection of curves $\gamma_s : [0,1] \to \Omega$, $0 < s < 1$, so that $\gamma_s(t)$, as a function of (t,s), is uniformly continuous on the closed unit square $[0,1] \times [0,1]$, with $\gamma_s(0) = b$ and $\gamma_s(1) = c$, $0 \le s \le 1$. If γ_0 is homotopic to γ_1, we write $\gamma_0 \approx \gamma_1$.

The function $H(t,s) \equiv \gamma_s(t)$ is called a **homotopy in** Ω from γ_0 to γ_1.

Reflexive: $\gamma_1 \approx \gamma_0$ using γ_{1-s}.

Transitive: if $\gamma_1 \approx \gamma_2$ and $\gamma_2 \approx \gamma_3$ then $\gamma_1 \approx \gamma_3$.



Homology $\neq$ Homotopy

This curve is homologous to zero, but not homotopic to zero

Homology is "Abelianization" of homotopy group (quotient by commutator subgroup)

Lemma 14.3: *If* $\gamma : [0,1] \to \Omega$ *is a curve in a region* Ω *then there is an* $\epsilon > 0$*, depending on the region* Ω *and the curve* γ*, such that if* $\sigma : [0,1] \to \Omega$ *is a curve with* $|\gamma(t) - \sigma(t)| < \epsilon$ *for all* $t \in [0,1]$ *and* $\sigma(0) = \gamma(0)$ *and* $\sigma(1) = \gamma(1)$*, then* $\sigma \approx \gamma$ *and* $\gamma - \sigma \sim 0$*.*

Recall: $\approx$ is homotopy and $\sim$ is homologous.

Proof. Choose disks $B_j \subset \Omega$ and a partition $0 = t_0 < t_1 \cdots < t_n < t_{n+1} = 1$ so that $\gamma_j = \gamma([t_j, t_{j+1}]) \subset B_j$. If $\epsilon > 0$ is sufficiently small then $\sigma_j = \sigma([t_j, t_{j+1}]) \subset B_j$, for each j. Then

$$\gamma_s(t) = (1 - s)\gamma(t) + s\sigma(t)$$

is a homotopy in Ω from γ to σ.

Let $L_j \subset B_j \cap B_{j-1}$ denote the line segment from $\gamma(t_j)$ to $\sigma(t_j)$, and set $L_0 = \{\gamma(0)\}$ and $L_{n+1} = \{\gamma(1)\}$. Then $\alpha_j \equiv \gamma_j + L_{j+1} - \sigma_j - L_j$ is a closed curve contained in $B_j \subset \Omega$ and hence is homologous to 0 in Ω. Thus $\gamma - \sigma = \sum_{j=0}^{n} \alpha_j$ is also homologous to 0. □

Corollary 14.4: *If* $\gamma_0 \approx \gamma_1$ *in a region* Ω *then* $\gamma_0 - \gamma_1 \sim 0$ *in* Ω.

Proof. If $\gamma_s(t)$ is a homotopy of γ_0 to γ_1 then we can cover $[0, 1]$ with finitely many open intervals J_k so that if $r, s \in J_k$ then $\gamma_r - \gamma_s \sim 0$ by Lemma 14.3. Thus $\gamma_0 - \gamma_1 \sim 0$ by transitivity. □

Theorem 14.5: *A region $\Omega \subset \mathbb{C}$ is simply-connected if and only if every closed curve contained in Ω is homotopic to a constant curve.*

Proof. If γ is homotopic to a constant curve then $\gamma \sim 0$ by Corollary 14.4. So if all curves in Ω are homotopic to constant curves then by Theorem 5.7, Ω is simply-connected.

Conversely if Ω is simply-connected and if γ is a closed curve in $\Omega \neq \mathbb{C}$ beginning and ending at z_0 and if f is a conformal map of Ω onto $\mathbb{D}$ with $f(z_0) = 0$ then $f(\gamma)$ is a closed curve in $\mathbb{D}$ beginning and ending at 0.

But then $\gamma_s(t) = f^{-1}(sf(\gamma(t)))$ is a homotopy of γ to the constant curve z_0. If $\Omega = \mathbb{C}$, then we can use $z - z_0$ instead of f. □

Theorem 14.6: *Suppose* $\gamma_s(t)$, $0 \leq s,t \leq 1$, *is a homotopy from* γ_0 *to* γ_1 *in a region* Ω. *Suppose* f_0 *is analytic in a neighborhood of* $b = \gamma_0(0) = \gamma_1(0)$ *and suppose* f_0 *can be analytically continued along each* γ_s. *Then the analytic continuation of* f_0 *along* γ_0 *agrees with the analytic continuation of* f_0 *along* γ_1 *in a neighborhood of* $c = \gamma_0(1) = \gamma_1(1)$.

Proof. The analytic continuation of f_0 along each γ_s is unique, $0 \le s \le 1$.

For each $s \in [0,1]$, the analytic continuation of f_0 along γ_s agrees with the analytic continuation of f_0 along γ_u in a neighborhood U_s of c if $|u-s| < \epsilon$ for some $\epsilon = \epsilon(s)$.

By compactness, we can cover $[0,1]$ with finitely many such open intervals $(s_j - \epsilon_j, s_j + \epsilon_j)$, for $1 \le j \le m$.

Then the analytic continuations of f_0 along each γ_s agree on $\cap_{j=1}^m U_{s_j}$. $\square$

Corollary 12.7, The Monodromy Theorem *Suppose* Ω *is simply-connected and suppose* f_0 *is analytic in a neighborhood of* $b \in \Omega$*. If* f_0 *can be analytically continued along all curves in* Ω *beginning at* b *then there is an analytic function* f *on* Ω *so that* $f = f_0$ *in a neighborhood of* b*.*

Proof. If $c \in \Omega$ and if γ_0 is a curve in Ω from b to c, let f_n be the analytic continuation of f_0 along γ_0 to a neighborhood of c, and define $f(c) = f_n(c)$.

If γ_0 and γ_1 are curves in Ω beginning at b and ending at c then $\gamma_0 \approx \gamma_1$ by Theorem 14.5 and Exercise 14.3e.

So by Theorem 14.6 the definition of $f(c)$ does not depend on the choice of the curve γ_0. Thus $f = f_n$ in a neighborhood of c, so that f is analytic at c. Thus f is defined and analytic in Ω and $f = f_0$ in a neighborhood of 0. $\square$

The monodromy theorem can be used to give another proof that a harmonic function u on a simply-connected region Ω is the real part of an analytic function.

If f is analytic on a ball $B \subset \Omega$ with $\mathrm{Re} f = u$ on B, then f can be continued along all curves in Ω.

By the monodromy theorem, because Ω is simply-connected, there is an analytic function f on all of Ω with $\mathrm{Re} f = u$.

The monodromy theorem canbe used to find a global inverse of an analytic function with f' never zero. Suppose f is analytic in Ω with $f' \neq 0$ on Ω.

If $c \in \Omega$, then there is a function g analytic in a neighborhood of $f(c)$ so that $g(f(z)) = z$ in a neighborhood of c.

If g can be analytically continued along all curves in $f(\Omega)$ and if $f(\Omega)$ is simply-connected, then by the monodromy theorem there is a function G which is analytic on $f(\Omega)$ satisfying $G(f(z)) = z$ for $z \in \Omega$.

Analytic continuation really only depended upon the continuity of the functions and the uniqueness theorem on disks, so that the monodromy theorem holds for much more general classes of functions.

For example, if two harmonic functions agree on a small disk in a region, then they agree on the entire region. So if we replace "analytic" with "harmonic" or "meromorphic" in our definition of continuation along a curve and in the statement of the monodromy theorem, then the theorem remains true.

Section 14.2: Riemann Surfaces and Universal Covers

Definition 14.8: A **Riemann surface** is a connected Hausdorff space W, together with a collection of open subsets $U_\alpha \subset W$ and functions $z_\alpha : U_\alpha \to \mathbb{C}$ such that

(1) $W = \cup U_\alpha$

(2) z_α is a homeomorphism of U_α onto the unit disk $\mathbb{D}$, and

(3) if $U_\alpha \cap U_\beta \neq \emptyset$ then $z_\beta \circ z_\alpha^{-1}$ is analytic on $z_\alpha(U_\alpha \cap U_\beta)$.

A Riemann surface W is pathwise connected, since the set of points that can be connected to p_0 is both open and closed for each $p_0 \in W$.

A function $f : W \to \mathbb{C}$ is called analytic if for every coordinate function z_α, the function $f \circ z_\alpha^{-1}$ is analytic on $\mathbb{D}$. Harmonic, subharmonic, and meromorphic functions on W are defined in a similar way.

Differentiation presents a problem, since if z_α is a coordinate map, the derivative of $f \circ z_\alpha^{-1}$ will depend on the choice of z_α.

However, if both f and g are analytic on a Riemann surface then $f' \circ z_\alpha^{-1}(z)/g' \circ z_\alpha^{-1}(z)$ does not depend on the choice of z_α by the chain rule.

This is why it is important to use diffrerential forms on surfaces, but we will not pursue forms here.

We think of two Riemann surfaces as equivalent if there is a holomoprhic homeomorphism between them. Also called "conformally equivalent".

For example, any two bounded, simply connected planar domains are equivalent by the Riemann mapping theorem.

The disk and plane are not equivalent by Liouville's theorem: if there was an analytic map $f : \mathbb{R}^2 \to \mathbb{D}$, it would have to be constant.

A Riemann surface W is called **simply-connected** if every closed curve in W is homotopic to a constant curve.

The disk, plane and 2-sphere are distinct simply connected Riemann surfaces.

The uniformization theorem says that these are the only simply connected Riemann surfaces, up to conformal equivalence.

Example, planar domains: every planar domain is a Riemann surface.

Indeed, every open subset of a Riemann surface is another Riemann surface.

Example 14.9, The two sphere: Use two charts $\mathbb{S}^2 \setminus \{\infty\}$ and $\mathbb{S}^2 \setminus \{0\}$

Example: The torus. Identify opposite sides of a parelleogram, say with corners at $0, 1, \omega, 1+\omega$ for each $\omega \in \mathbb{H}$.

Different choices of ω can give different Riemann surfaces, but sometimes same. It is understood exactly which ones are distinct. Different choices of ω can give different Riemann surfaces, but sometimes same. It is understood exactly which ones are distinct. This is the beginning of Teichmüller theory.

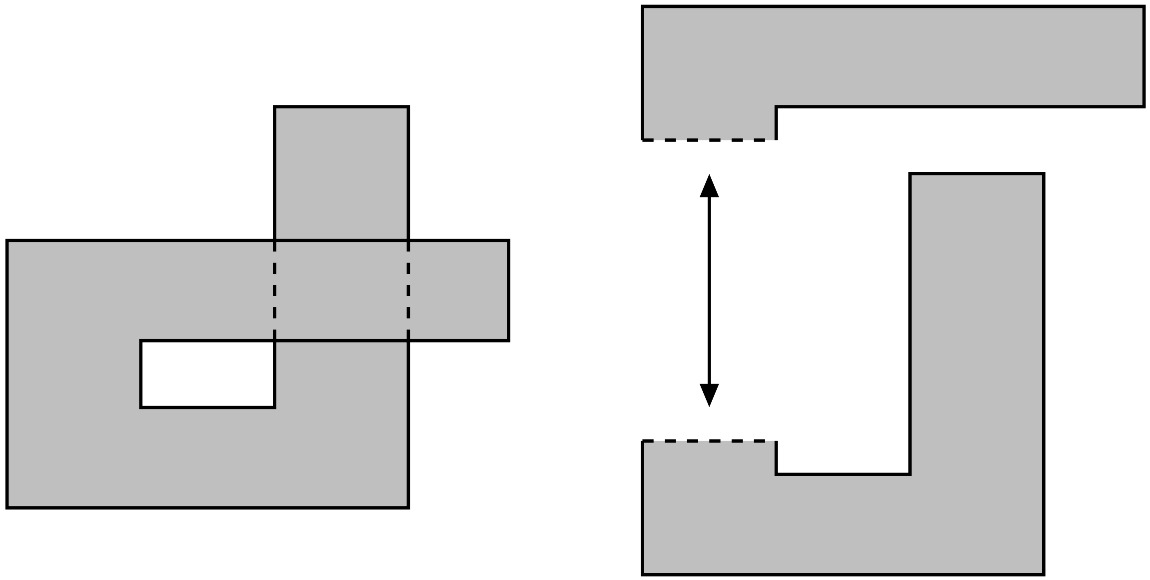
There are uncountably many conformally different tori. Same for all compact surfaces of higher genus.

Example 14.11, Riemann surface of an analytic function

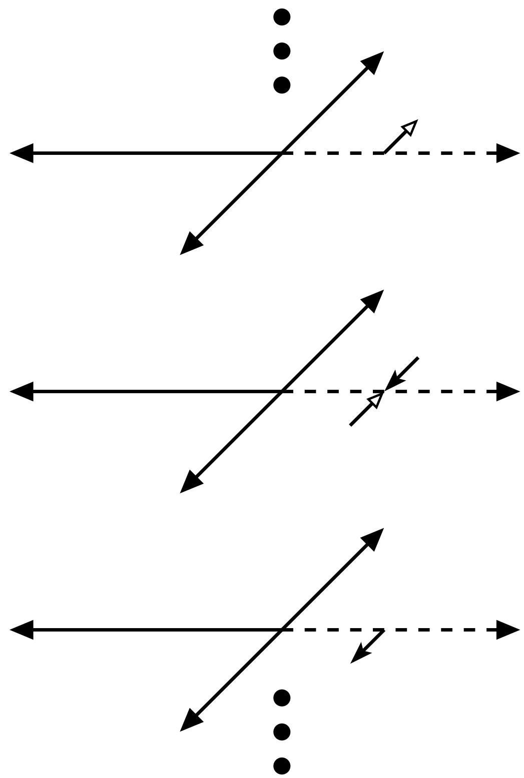
If f is analytic on a region $\Omega \subset \mathbb{C}$ with $f' \neq 0$ on Ω, then we can construct a Riemann surface Ω_f associated with f by declaring charts to be the images $f(B(z,r))$ of disks $B(z,r)$ on which f is one-to-one. The associated chart maps are f^{-1} composed with a linear map of $B(z,r)$ onto $\mathbb{D}$.

More formally, we write $\Omega = \cup_{j=1}^{\infty} B_j$ where f is one-to-one on each B_j. Set $U = \coprod_{j=1}^{\infty} f(B_j)$, the disjoint union of the sets $f(B_j)$. We then identify $w \in f(B_i)$ and $w \in f(B_j)$ if and only if $w = f(z)$ for some $z \in B_i \cap B_j$. In other words, we identify the copies of $f(B_i \cap B_j)$ in the two images $f(B_i)$ and $f(B_j)$. The corresponding quotient space Ω_f is a Riemann surface.

The function f can be viewed as a one-to-one map of Ω onto Ω_f, and f^{-1} becomes a well-defined function on Ω_f.

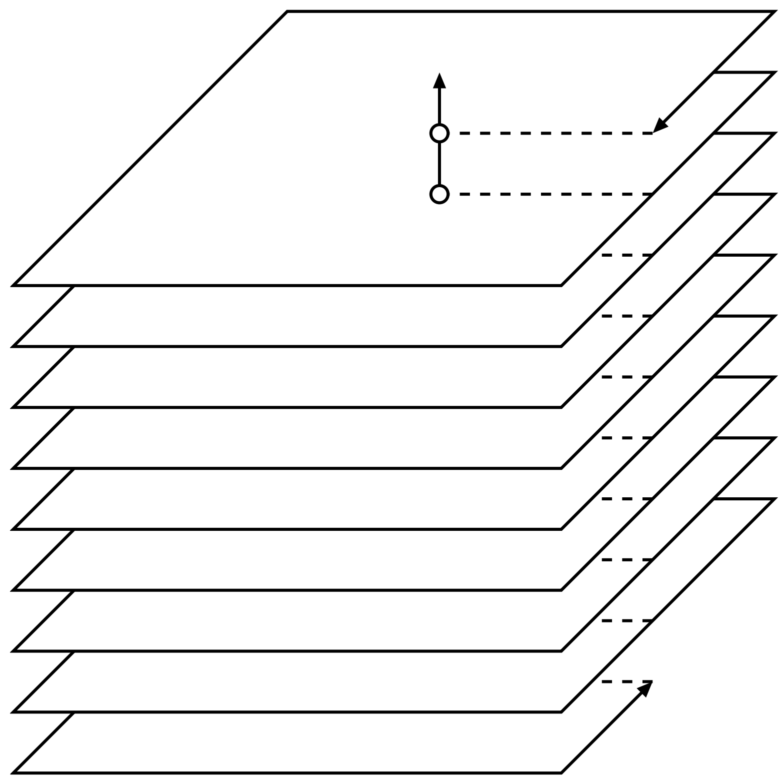


Can construct Riemann surfaces by identifying planar regions along boundary arcs via similarities.



Identitfy countably many copies of $\mathbb{C} \setminus [0, \infty)$ along $(0, \infty)$.

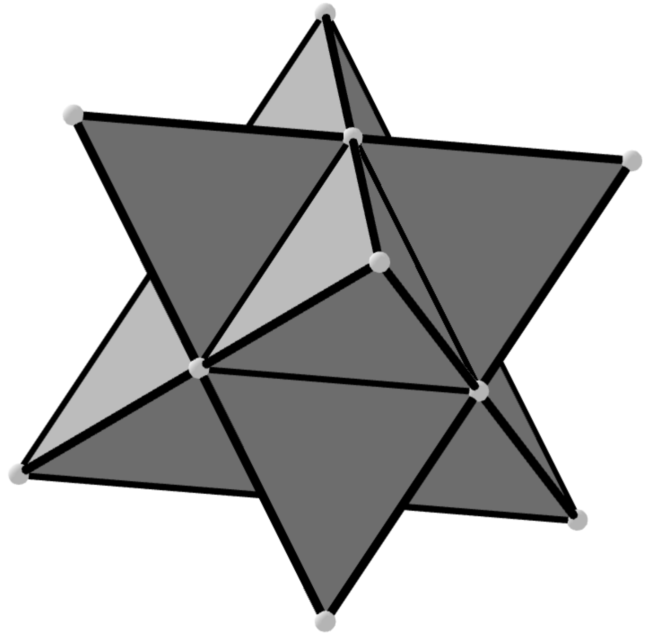
This is the surface of $\exp(z)$. Then $\log(z)$ is well defined from this surface to $\mathbb{C}$.



Identitfy countably many copies of $\mathbb{C} \setminus [0, \infty)$ along $(0, \infty)$.

This is the surface of $\exp(z)$. Then $\log(z)$ is well defined from this surface to $\mathbb{C}$.

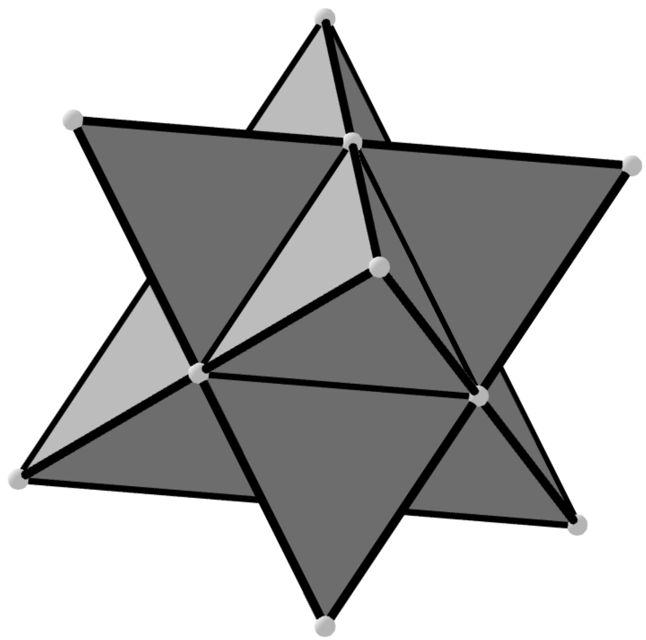
This is also universal cover of $\mathbb{C} \setminus \{0\}$.

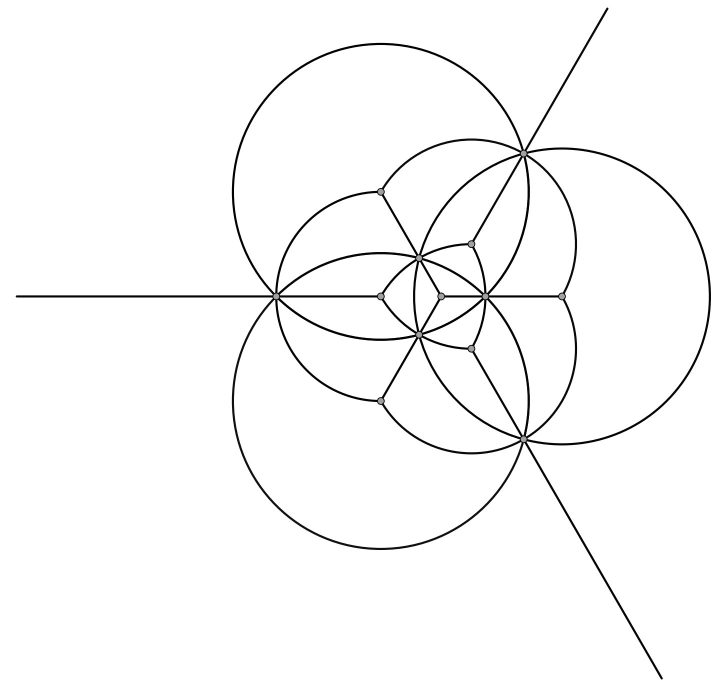


Identify equilateral triangles along boundaries to give Riemann surface.

Topologically this example is a 2-sphere. Uniformization therorem implies it is conformally equivalent to round 2-sphere.

We can get other topological surfaces, but only countably many can occur using fnitely many triangles.





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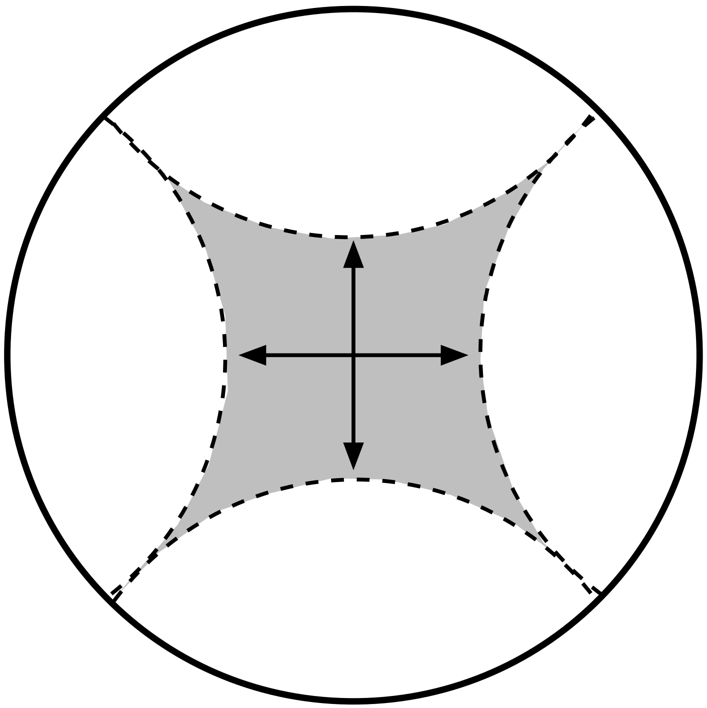
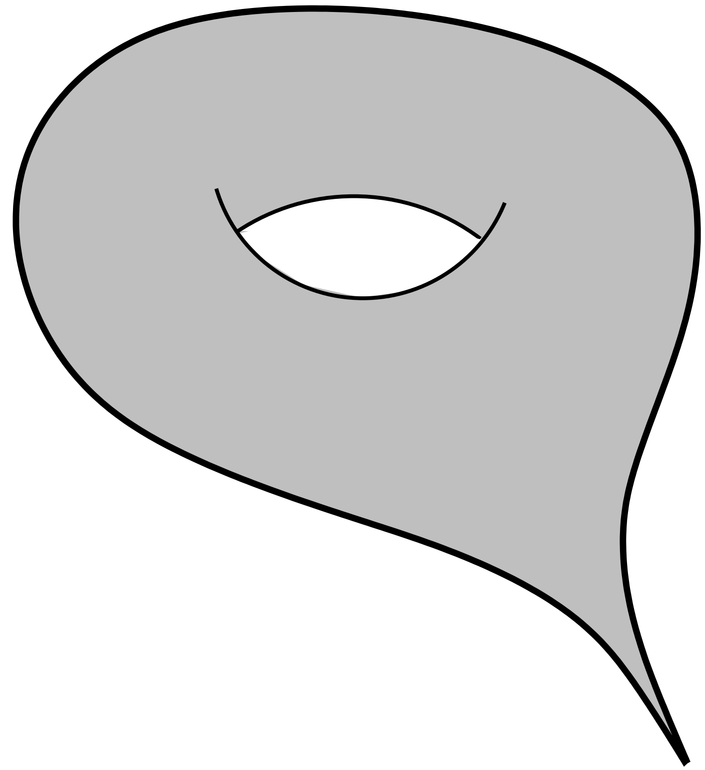
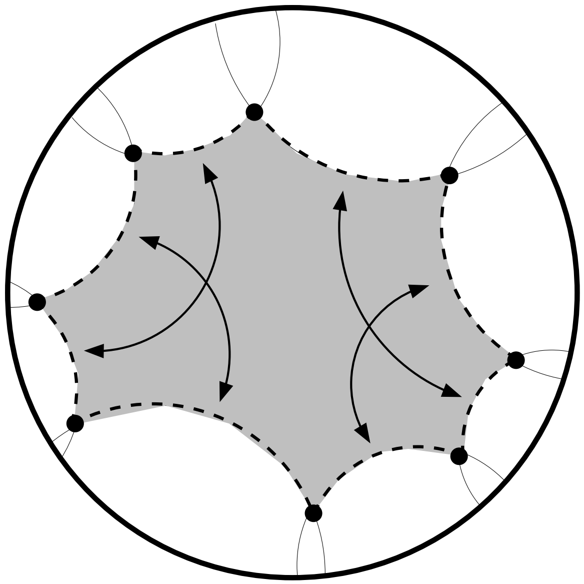


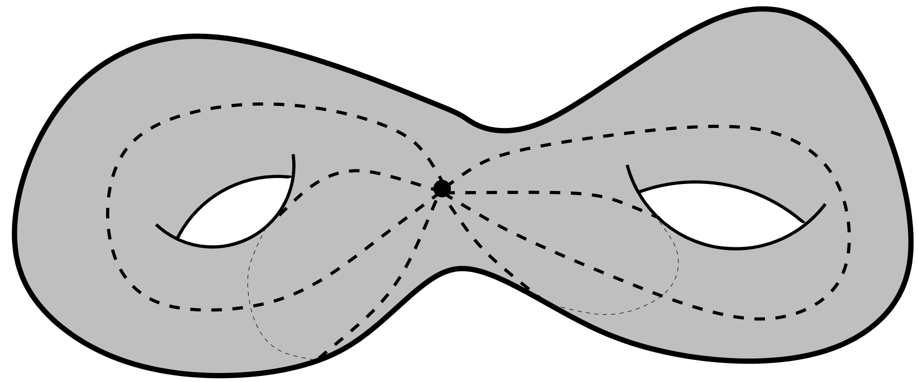


We can identify sides of a hyperbolic polygon via Möbius transformations to get a surface.

This example gives a torus with a puncture (not a compact surface).

Fundamental polygon is not compact in unit disk.





This example gives a genus 2 compact surface.

A theorem of Poincaré give a criterion for a polygon and set of side pairing maps to give a Riemann surface.

Angles at identitfied vertices must sum to 2π.

Defn: A smooth affine algebraic curve is

$$X = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$$

where f is a polynomial such at each point $p \in X$ either

$$\frac{\partial f}{\partial x}(p) \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial y}(p) \neq 0,$$

Implicit function theorem covers X by charts where either x or y are the maps to complex plane.

Examples:

Hyperelliptic curves, $\qquad y^2 = (x - a_1) \ldots (x - a_n).$

Fermat curves, $\qquad x^n + y^n = 1$

The examples on the previous slide are special because coefficients are integers.

If we allow algebraic coefficients, only countably many compact surfaces occur.

Belyi's Theorem (1979): say that this collection is exactly the same as compact surfaces built from equilateral triangles.

Foundation of Grothendieck's theory of *dessins d'enfants* (children's drawings) that links complex analysis, combinatorics and number theory.

Universal covering space:

Suppose Ω is a Riemann surface. Fix a point $b \in \Omega$. Let $[\gamma]$ be the equivalence class under homotopy of a curve $\gamma \subset \Omega$. Let Ω^* be the collection of equivalence classes

$$\Omega^* = \{[\gamma] : \gamma \text{ is a curve in } \Omega \text{ with } \gamma(0) = b\}.$$

Define a projection map $\pi : \Omega^* \to \Omega$ by $\pi([\gamma^*]) = \gamma(1)$.

Ω^* is the **universal cover** of Ω and π is the **covering map**.

We can make Ω^* into a Riemann surface by describing coordinate maps and charts. If $c \in \Omega$, let B be a topological disk in Ω. Let γ be a curve in Ω from b to c. For any point $d \in B$, let σ_d be a curve in B from c to d. Let

$$B^* = \{[\gamma\sigma_d] : d \in B\} \tag{14.3}$$

be the equivalence classes of all curves $\gamma\sigma_d$, for all $d \in B$. Since all curves in B beginning at c and ending at d are homotopic, $[\gamma\sigma_d]$ does not depend on the choice of σ_d.

By the definition of π, $\pi([\gamma\sigma_d]) = d$ so that π is a one-to-one map of B^* onto B. Note that if $\gamma_1 \approx \gamma$, then we obtain the same set of equivalence classes B^* using γ_1 instead of γ. See Exercise 14.3c.

Thus $B^* \subset \pi^{-1}(B)$ is uniquely determined by the equivalence class of γ, namely a point in B^*, and the disk B. Set $z_{B^*} = (\pi - c)/r$.

Then we can give a topology on Ω^* by declaring each set B^* to be open, for all disks $B(c, r) \subset \Omega$ and all equivalence classes $[\gamma]$ of curves $\gamma \subset \Omega$ from b to c.

Indeed the "disks" B^* form a basis for this topology. Equivalently we give Ω^* the topology required to make each z_{B^*} a homeomorphism.

Theorem 14.12: *Suppose $\Omega \subset \mathbb{C}$ is a region. Then*

(1) *The surface Ω^* is a simply-connected Riemann surface with coordinate functions z_{B^*} and charts B^*.*

(2) *If B_1^* and B_2^* are coordinate charts with $\pi(B_1^*) = \pi(B_2^*)$ then either $B_1^* \cap B_2^* = \emptyset$ or $B_1^* = B_2^*$.*

(3) *If $\gamma \subset \Omega$ is a curve beginning at b and if $b^* \in \Omega^*$ with $\pi(b^*) = b$, then there is a unique curve $\gamma^* \subset \Omega^*$, called a* **lift** *of γ, beginning at b^* with $\pi(\gamma^*) = \gamma$.*

(4) *A curve $\gamma^* \subset \Omega^*$ is closed if and only if $\gamma = \pi(\gamma^*)$ is homotopic to a constant curve in Ω.*

I believe covering surfaces and the universal covering space is in the Top-Geo core courses, so I won't go through the proof here. See Marshall's textbook, or a more general disucssion in Munkre's *Topology* book.

Section 14.3: Deck Transformations

Let π be a universal covering map of W^* onto W. Fix a point $b \in W$ and write W^* as the collection of equivalence classes of curves beginning at b. If σ is a closed curve beginning and ending at b, we define a map $M_{[\sigma]} : W^* \to W^*$ by

$$M_{[\sigma]}([\gamma]) = [\sigma\gamma].$$

Note that this map is one-to-one and onto and does not depend on the choice of the curve in $[\sigma]$.

If B^* is a coordinate disk centered at $[\gamma]$ then $M_{[\sigma]}(B^*)$ is a coordinate disk centered at $[\sigma\gamma]$ with the same projection as B^*.

So the map $M_{[\sigma]}$ is a homemorphism of W^* onto W^*. These maps $M_{[\sigma]}$ are called the **deck transformations**.

The deck transformations form a group $\mathbb{G}$ under composition. If $b^* = [\{b\}]$ is the equivalence class of the constant curve curve $\{b\}$, then M_{b^*} is the identity map in this group $\mathbb{G}$.

The group $\mathbb{G}$ gives an equivalence relation on W^*, where $p^* \sim q^*$ if and only if there is a deck transformation M with $M(p^*) = q^*$. The quotient space $W^*/\mathbb{G}$ with the quotient topology is a Riemann surface.

The coordinate charts on $W^*/\mathbb{G}$ are just the images by the quotient map of the coordinate charts on W^*. Lemma 3.1 says that the map π induces a one-to-one map of $W^*/\mathbb{G}$ onto W.

As a group, $\mathbb{G}$ is isomorphic to the group of equivalence classes under homotopy of all closed curves in W beginning at b, which is called the **fundamental group of W at b.**

For this reason, the set of deck transformations is sometimes also called the fundamental group of W.

Lemma 14.14: *If $p^*, q^* \in W^*$, then $\pi(p^*) = \pi(q^*)$ if and only if there is a deck transformation M with $M(p^*) = q^*$.*

Proof. Recall that $\pi([\gamma]) = \gamma(1)$. If $p^* = [\gamma], q^* = [\alpha] \in W^*$ have the same projection $\gamma(1) = \alpha(1)$, then setting $\sigma = \alpha\gamma^{-1}$ we have that σ is a closed curve beginning at b and $\sigma\gamma \approx \alpha$ so that $M_{[\sigma]}([\gamma]) = [\alpha]$.

Conversely if $\sigma \subset W$ is a closed curve beginning and ending at b then $\pi([\sigma\gamma]) = \gamma(1) = \pi([\gamma])$, so that $M_{[\sigma]}([\gamma])$ and $[\gamma]$ have the same projection. □

Definition 14.5: If W_1 and W_2 are Riemann surfaces and if f is a map of W_1 into W_2, then we say that f is **analytic** provided

$$w_\beta \circ f \circ z_\alpha^{-1}$$

is analytic for each coordinate function z_α on W_1 and w_β on W_2 wherever it is defined.

The covering map from the universal cover to a Riemann surface is analytic.

If f is meromorphic on region $W \subset \mathbb{C}$, then f can also be viewed as a map into the extended plane $\mathbb{C}^*$, or via stereographic projection into the Riemann sphere $\mathbb{S}^2$.

Such maps f are then analytic in the sense above, so some care must be taken when speaking of analytic functions on a Riemann surface that both the domain and range surfaces are understood.

A **holomorphic** function on a Riemann surface is a analytic map to the plane.

A **meromorphic** function on a Riemann surface is a analytic map to the 2-sphere.

Meromorphic functions on the 2-sphere are the rational maps.

Mereomorphic functions on a surface X form a field.

For a compact surface given as zero set of $P(X, Y)$ where P is irreducible, this field is quotient of $R(X, Y)$ by the ideal generated by P.

The Riemann-Roch theorem computes the dimension of the space of meromorphic functions on a surface with prescribed zeros and poles.

Corollary 14.16: *The deck transformations form a group of one-to-one analytic maps of W^* onto W^* with the property that each $p^* \in W^*$ has a neighborhood B^* so that $M(B^*) \cap B^* = \emptyset$ for all deck transformation M not equal to the identity map. The projection map $\pi : W^* \to W$ induces a one-to-one analytic map of $W^*/\mathbb{G}$ onto W.*

Proof follows from proof of Theorem 14.12.

Such a group action is called **properly discontinuous**.

This is stronger than saying the group is discrete, i.e., the identity element is isolated. A discrete group of LFTs on $\mathbb{D}$ is called a **Fuchsian group**.