MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop
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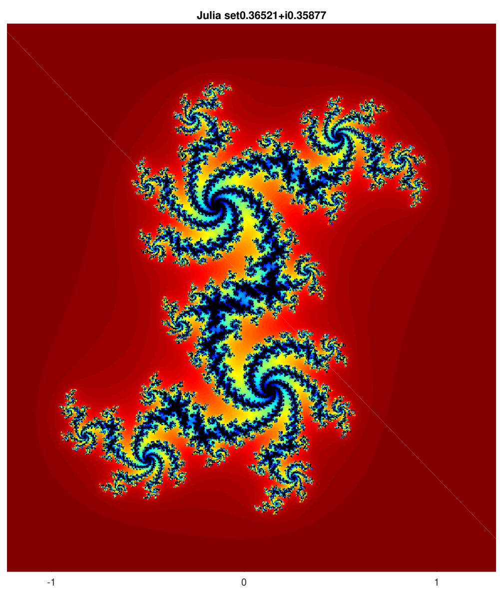
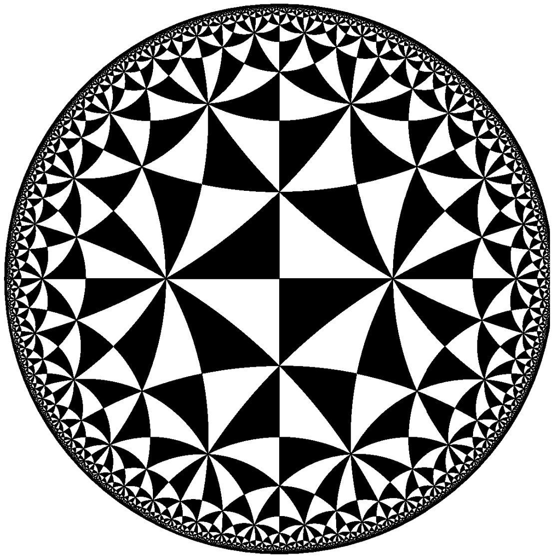
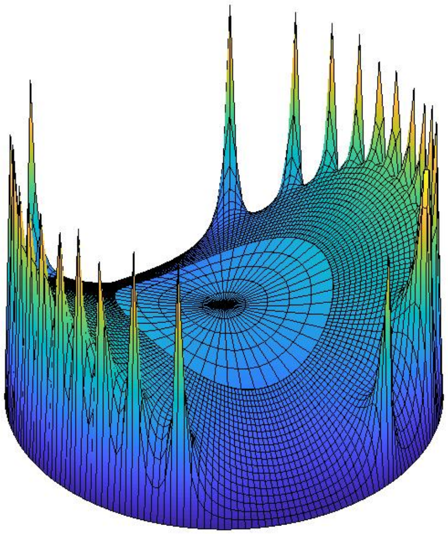






Chapter 13: The Dirichlet problem

Section 13.1: The Perron Process

Let $C(\partial\Omega)$ denote the set of continuous functions on the boundary of a region Ω. The **Dirichlet problem** on Ω for a function $f \in C(\partial\Omega)$ is to find a harmonic function u on Ω which is continuous on $\overline{\Omega}$ and equal to f on $\partial\Omega$.

If u exists then it is unique by the maximum principle, but it is not always possible to solve the Dirichlet problem.

If $f = 0$ on $\partial\mathbb{D}$ and $f(0) = 1$, then $f \in C(\partial\Omega)$ where $\Omega = \mathbb{D} \setminus \{0\}$. But by Lindelöf's maximum principle if u is harmonic and bounded on Ω with $u = 0$ on ∂D then $u(z) = 0$ for all $z \in \Omega$. Thus u extends to be continuous at 0, but $u(0) \neq f(0)$.

Recall: a subharmonic function v on Ω is continuous as a map of Ω into $[-\infty, +\infty)$ and satisfies the mean-value inequality on sufficiently small circles.

Also recall that:

(1) A subharmonic function v on a region Ω satisfies the maximum principle: if there exists $z_0 \in \Omega$ such that $v(z_0) = \sup_{z \in \Omega} v(z)$ then v is constant on Ω.

(2) If v_1, v_2 are subharmonic then $v_1 + v_2$ and $\max(v_1, v_2)$ are subharmonic.

(3) If v is subharmonic on a region Ω and $v > -\infty$ on ∂D where D is a disk with $\overline{D} \subset \Omega$, then the function v_D which equals v on $\Omega \setminus D$ and equals the Poisson integral of v on D is also subharmonic.

In other words, if $D = \{z : |z - c| < r\}$, we can replace v on D by

$$v_D(z) = \int_0^{2\pi} \frac{1 - |(z-c)/r|^2}{|e^{it} - (z-c)/r|^2} v(c + re^{it}) \frac{dt}{2\pi},$$

and still be subharmonic.

Definition 13.1: A family $\mathcal{F}$ of subharmonic functions on a region Ω is called a **Perron family** if it satisfies:

(1) if $v_1, v_2 \in \mathcal{F}$ then $\max(v_1, v_2) \in \mathcal{F}$, and

(2) if $v \in \mathcal{F}$ and if D is a disk with $\overline{D} \subset \Omega$ with $v > -\infty$ on ∂D, then $v_D \in \mathcal{F}$,

(3) for each $z \in \Omega$ there exists $v \in \mathcal{F}$ such that $v(z) > -\infty$.

Definition 13.2: If $\mathcal{F}$ is a Perron family on a region Ω then we define

$$u_{\mathcal{F}}(z) \equiv \sup_{v \in \mathcal{F}} v(z).$$

Theorem 13.3: *If $\mathcal{F}$ is a Perron family on a region Ω then $u_{\mathcal{F}}$ is harmonic on Ω or $u_{\mathcal{F}}(z) = +\infty$ for all $z \in \Omega$.*

Proof. Fix $z_0 \in \Omega$. Find $v_1, v_2, \dots \in \mathcal{F}$ such that $\lim_j v_j(z_0) = u_{\mathcal{F}}(z_0)$. Set

$$v_j' = \max(v_1, v_2, \dots, v_j).$$

By (1) and induction, $v_j' \in \mathcal{F}$, $v_j' \leq v_{j+1}'$ and $\lim v_j'(z_0) = u_{\mathcal{F}}(z_0)$. Suppose D is a disk with $\overline{D} \subset \Omega$.

By (3), the continuity of subharmonic functions, and (1), we may suppose that v_j' continuous and $v_j' > -\infty$ on ∂D.

Let v_j'' equal v_j' on $\Omega \setminus D$ and equal the Poisson integral of v_j' on D.

Then by (2), $v_j'' \in \mathcal{F}$. Moreover $v_j' \leq v_j''$ by the maximum principle.

The Poisson integral on D of the non-negative function $v_{j+1}' - v_j'$ is non-negative so that $v_j'' \leq v_{j+1}''$.

Set $V = \lim_j v_j''$.

By Harnack's principle, either $V \equiv +\infty$ in D or V is harmonic in D. Note also that $V(z_0) = u_{\mathcal{F}}(z_0)$ because of the choice of v_j and the maximality of $u_{\mathcal{F}}(z_0)$.

Now take $z_1 \in D$, $w_j \in \mathcal{F}$, with $\lim_j w_j(z_1) = u_{\mathcal{F}}(z_1)$. Set

$$w_j' = \max(v_j'', w_1, w_2, \ldots, w_j)$$

and let $w_j'' \in \mathcal{F}$ equal w_j' on $\Omega \setminus D$ and equal the Poisson integral of w_j' on D.

As before $w_j' \leq w_j'' \leq w_{j+1}''$. Set $W = \lim_j w_j''$.

As before either W is harmonic in D or $W \equiv +\infty$ in D, and $W(z_1) = u_{\mathcal{F}}(z_1)$. Because $v_j'' \leq w_j' \leq w_j''$, we must have that $V \leq W$.

But also

$$u_{\mathcal{F}}(z_0) = V(z_0) \leq W(z_0) \leq u_{\mathcal{F}}(z_0)$$

so that $V(z_0) = W(z_0)$. If $u_{\mathcal{F}}(z_0) < \infty$ then $V - W$ is harmonic on D and achieves its maximum value 0 on D at z_0.

By the maximum principle, it must equal 0 on D.

Because z_1 was arbitrary in D, there are two possibilities, either $u_{\mathcal{F}} \equiv +\infty$ on D or $u_{\mathcal{F}} = V$ on D and hence is harmonic on D.

Since $\overline{D}$ was an arbitrary closed disk in Ω, $\{z : u_{\mathcal{F}}(z) = +\infty\}$ is then both closed and open in Ω. Since Ω is connected, either $u_{\mathcal{F}} \equiv +\infty$ on Ω or $u_{\mathcal{F}}$ is harmonic on Ω. $\square$

Notice that the proof of Theorem 1.3 only used local properties of the family $\mathcal{F}$, the mean-value property and the maximum principle on small disks contained in Ω. We will use this observation in the proof of the uniformization theorem (Chapter 15).

If $\Omega \subset \mathbb{C}^*$ is a region, and if f is a real-valued function defined on $\partial\Omega$ (a compact subset of $\mathbb{C}^*$) with $|f| \leq M < \infty$ on $\partial\Omega$, set

$$\mathcal{F}_f = \{v \text{ subharmonic on } \Omega : \limsup_{z \in \Omega \to \zeta} v(z) \leq f(\zeta), \text{ for all } \zeta \in \partial\Omega\}.$$

Then (3) holds since the constant function $-M \in \mathcal{F}$. So $\mathcal{F}_f$ is a Perron family.

Moreover each $v \in \mathcal{F}$ is bounded by M by the maximum principle. Thus

$$u_f(z) \equiv \sup_{v \in \mathcal{F}_f} v(z)$$

is harmonic in Ω.

The function u_f is called the **Perron solution to the Dirichlet problem** on Ω for the function f. It is a natural candidate for a harmonic function in Ω which equals f on $\partial\Omega$.

Does it equal f on the boundary? Yes, under certain conditions.

Section 13.2: Local Barriers

Definition 14.4: If $\Omega \subset \mathbb{C}^*$ is a region and if $\zeta_0 \in \partial\Omega$ then b is called a **local barrier at ζ_0 for the region** Ω provided

(1) b is defined and subharmonic on $\Omega \cap D$ for some open disk D with $\zeta_0 \in D$.

(2) $b(z) < 0$ for $z \in \Omega \cap D$, and

(3) $\lim_{z \in \Omega \to \zeta_0} b(z) = 0$.

If b is a local barrier for the region Ω at ζ_0, defined on $\Omega \cap D$, then it is also a local barrier on $\Omega \cap D_1$ for any smaller disk D_1 with $\zeta_0 \in D_1 \subset D$.

Definition 13.5: If there exists a local barrier at $\zeta_0 \in \partial\Omega$ then ζ_0 is called a **regular point of** $\partial\Omega$. Otherwise $\zeta_0 \in \partial\Omega$ is called an **irregular point of** $\partial\Omega$. If every $\zeta \in \partial\Omega$ is regular, then Ω is called a **regular region**.

On a simply connected domain $G(z) = \log|f(z)|$, where $f : \Omega \to \mathbb{D}$ is a Riemann map, is a barrier at every boundary point. Thus simply connected domains are regular.

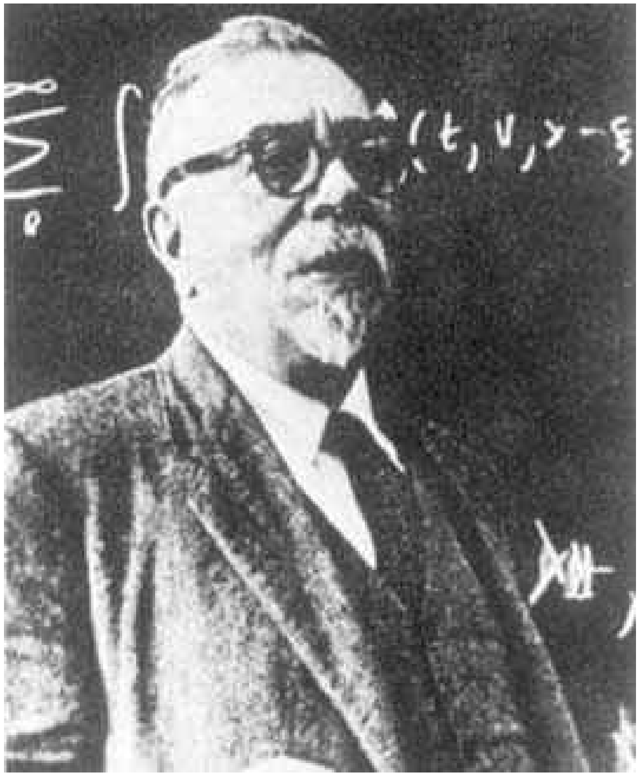
Isolated boundary points are not regular.

There is a known characterization of regular points due to Norbert Wiener, but we will only deal with some easier sufficient conditions.

Definition: a closed set X is uniformly perfect if there is an $M < \infty$, so that for all $0 < r < \text{diam}(X)$ and $x \in$, there is a y with $r \leq |x - y| \leq Mr$.

For example, the middle thirds Cantor set is uniformly perfect. So are most common fractals, such as self-simlar sets and polynomial Julia sets.

Wiener's criteria implies that if $\partial\Omega$ is unformly perfect, then every point is regular.



Norbert Wiener

Wiener entered college at 11 and graduate school at 14. He made many contributions, including a rigorous version of Brownian motion.

"He spoke many languages, but was not easy to understand in any of them." – Hans Freudenthal

Theorem 13.6: *Suppose $\Omega \subset \mathbb{C}^*$ is a region and suppose $\zeta_0 \in \partial\Omega$. If $C \subset \partial\Omega$ is closed in $\mathbb{C}^*$ and connected, with $\zeta_0 \in C$ but $C \neq \{\zeta_0\}$, then ζ_0 is regular.*

One approach: take a single closed connected component C. Then $\mathbb{C}^* \setminus C$ is simply connected, so has a Riemann map to disk. Take $G = \log|f|$ as barrier.

Proof. We start with a topological claim: that each component Ω_1 of $\mathbb{C}^* \setminus C$ is simply-connected.

To prove this, we may assume $\infty \in C$. Let γ be any closed polygonal curve contained in Ω_1. Then $n(\gamma, \zeta) = 0$ for all $\zeta \in C$ because C is connected and unbounded, and hence contained in the unbounded component of the complement of γ.

If $\beta \notin \Omega_1$ and $n(\gamma, \beta) \neq 0$, let L be a line segment from β to some $\zeta \in C$. Then L must intersect $\partial\Omega_1$ before it intersects γ because $\gamma \subset \Omega_1$.

Because $\partial\Omega_1 \subset C$, this intersection gives a point $\zeta_1 \in C$ with $n(\gamma, \zeta_1) = n(\gamma, \beta) \neq 0$. This contradiction proves that every closed polygonal curve in Ω_1 is homologous to 0, so Ω_1 is simply-connected, proving the claim.

If $\zeta_2 \in C$ with $\zeta_2 \neq \zeta_0$ then

$$f(z) = \log\left(\frac{z-\zeta_0}{z-\zeta_2}\right)$$

is analytic on each component Ω_1 of $\mathbb{C}^* \setminus C$ and therefore on Ω.

Then $b = \mathrm{Re}(1/f(z))$ is a local barrier at ζ_0, defined on $\Omega \cap D$ where $D = \{z : |(z-\zeta_0)/(z-\zeta_2)| < 1/2\}$. $\square$

We want to show that the Perron solution u with boundary data f has the correct boundary values wherever a local barrier exists.

To prove this we need a couple of lemmas first.

Lemma 13.8: *$u_f \leq -u_{-f}$ on Ω.*

Proof. If $v \in \mathcal{F}_f$, and if $w \in \mathcal{F}_{-f}$, then

$$\limsup_{z \in \Omega \to \zeta \in \partial\Omega} v(z) + w(z) \leq f(\zeta) + (-f(\zeta)) = 0.$$

By the maximum principle, $v + w \leq 0$ on Ω.

Hence the supremum over all $v \in \mathcal{F}_f$ and $w \in \mathcal{F}_{-f}$ is $u_f + u_{-f} \leq 0$. $\square$

Lemma 13.9, Bouligand: *Suppose* ζ_0 *is a regular point of* $\partial\Omega$ *and suppose* b *is a local barrier at* ζ_0 *defined on* $\Omega \cap B(\zeta_0, \epsilon)$. *Given* $0 < \delta < \epsilon$, *there exists* b_δ *which is subharmonic on all of* Ω *satisfying*

(1) $b_\delta < 0$ *on* Ω

(2) $b_\delta \equiv -1$ *on* $\Omega \setminus B(\zeta_0, \epsilon)$

(3) $\liminf_{z \in \Omega \to \zeta_0} b_\delta(z) \geq -\delta$.

Proof. Set $B = B(\zeta_0, r)$ for some $r < \epsilon$. Then

$$\partial B \cap \Omega = \bigcup_{k=1}^{\infty} I_k,$$

where each I_k is an open arc.

Choose $n < \infty$ and compact subarcs $J_k \subset I_k$, $k = 1, \ldots, n$ so that if $K = \cup_{k=1}^{n} J_k$ and $L = (\partial B \cap \Omega) \setminus K$, then the total length of L is less than $\pi \delta r$.

Let $\omega = PI_B(\chi_L)$ be the Poisson integral of the characteristic function of L on B. Then ω is harmonic on B and $0 < \omega < 1$ on B.

Because L is open, Schwarz's Theorem (Thm 7.5) implies that $\omega(z) \to 1$ as $z \in B \to \eta \in L$.

Because b is continuous and $b < 0$ on $\Omega \cap B(\zeta_0, \epsilon)$ and because $K \subset \Omega \cap B(\zeta_0, \epsilon)$ is compact there exists $-m < 0$ so that $b \leq -m$ on K. So if $\eta \in K \cup L$ then

$$\limsup_{z \in B \to \eta} \left(\frac{b}{m} - w \right) \leq -1. \tag{13.3}$$

Set

$$b_\delta(z) = \begin{cases} \max\left(2\left(\frac{b(z)}{m} - \omega(z)\right), -1\right) & \text{on } \Omega \cap B \\ -1 & \text{on } \Omega \setminus B \end{cases}$$

By (13.3) $b_\delta \equiv -1$ in a neighborhood of $K \cup L$ in Ω.

Because $2(\frac{b}{m} - \omega)$ is subharmonic on $B \cap \Omega$, we have that b_δ is continuous and subharmonic in Ω. Moreover $b_\delta < 0$ on Ω.

Finally note that $\omega(\zeta_0) = \int_L dt/(2\pi r) = |L|/(2\pi r) < \delta/2$.

Thus $\liminf_{z \in \Omega \to \zeta_0} b_\delta(z) = -2\omega(\zeta_0) \geq -2\delta/2 = -\delta$. $\square$

Theorem 13.7: *Suppose* $\zeta_0 \in \partial\Omega$ *is regular. If* f *is a real-valued function defined on* $\partial\Omega$ *with* $|f| \le M < \infty$ *on* $\partial\Omega$ *and if* f *is continuous at* ζ_0 *then*

$$\lim_{z \in \Omega \to \zeta_0} u_f(z) = f(\zeta_0),$$

where u_f *is the Perron solution for the function* f*.*

Corollary 13.8: *If* Ω *is a regular region and if* $f \in C(\partial\Omega)$ *then* u_f *is harmonic in* Ω *and extends to be continuous on* $\overline{\Omega}$ *with* $u_f = f$ *on* $\partial\Omega$*.*

Proof. Choose $\epsilon > 0$ so that $|f(\zeta) - f(\zeta_0)| < \delta$ for $\zeta \in \overline{B} \cap \partial\Omega$, where $B = B(\zeta_0, \epsilon)$. Let b_δ be the function produced in Bouligand's Lemma. Set

$$v(z) = f(\zeta_0) - \delta + (M + f(\zeta_0))b_\delta(z).$$

Then v is subharmonic on Ω and for $\zeta \in \partial\Omega \cap \overline{B}$

$$\limsup_{z\in\Omega\to\zeta} v(z) \leq f(\zeta_0) - \delta \leq f(\zeta),$$

because $M + f(\zeta_0) \geq 0$ and $b_\delta < 0$.

If $\zeta \in \partial\Omega \setminus \overline{B}$ then

$$\limsup_{z\in\Omega\to\zeta} v(z) = f(\zeta_0) - \delta - (M + f(\zeta_0)) \leq f(\zeta),$$

because $b_\delta = -1$ on $\Omega \setminus B$.

Thus $v \in \mathcal{F}_f$ by (13.1).

By Lemma 13.10(iii)

$$\liminf_{z\in\Omega\to\zeta_0} u_f(z) \geq \liminf_{z\in\Omega\to\zeta_0} v(z) \geq f(\zeta_0) - \delta + (M + f(\zeta_0))(-\delta).$$

Since $\delta > 0$ is arbitrary, $\liminf_{z\in\Omega\to\zeta_0} u_f(z) \geq f(\zeta_0)$.

Replacing f with $-f$ we also have that $\liminf_{z\to\zeta_0} u_{-f}(z) \geq -f(\zeta_0)$. By Lemma13.9

$$\limsup_{z\in\Omega\to\zeta_0} u_f(z) \leq \limsup_{z\in\Omega\to\zeta_0} -u_{-f}(z) = -\liminf_{z\in\Omega\to\zeta_0} u_{-f}(z) \leq f(\zeta_0). \quad \square$$

Section 13.1: The Riemann Mapping Theorem (again)

Theorem 13.11, Riemann Mapping Theorem *If* $\Omega \subset \mathbb{C}$ *is a simply-connected region not equal to all of* $\mathbb{C}$*, then there exists a conformal map* φ *of* Ω *onto* $\mathbb{D}$*.*

Proof. As in the previous proof (via normal families) we may assume Ω is bounded and $0 \in \Omega$. Define $f(\zeta) = \log |\zeta|$ and note this is in $C(\partial\Omega)$.

Let u_f be the Perron solution to the Dirichlet problem for the boundary function f. By our results so far, u_f is harmonic on Ω and extends to be continuous on $\overline{\Omega}$ and equal to f on $\partial\Omega$.

By Theorem 7.10, there is an analytic function g on Ω such that $\text{Re}g = u_f$.

Set $\varphi(z) = ze^{-g(z)}$. Then φ is analytic in Ω, $\varphi(0) = 0$, and $\varphi(z) \neq 0$ if $z \neq 0$.

Moreover $|\varphi(z)| = e^{\log|z| - u_f(z)} \to 1$ as $z \in \Omega \to \partial\Omega$. Thus φ maps Ω into $\mathbb{D}$ by the maximum principle.

We want to show φ is 1-1 and onto the disk.

Fix $\epsilon > 0$ and let $K_\epsilon = \varphi^{-1}(|w| \leq 1 - \epsilon)$. Because $|\varphi| \to 1$ as $z \to \partial\Omega$, the set K_ϵ is a compact subset of Ω.

As in the proof of Runge's Theorem IV.3.4, we can construct a closed curve $\gamma \subset \Omega$ which winds once around each point of K_ϵ.

The winding number $n(\varphi(\gamma), w)$ is constant in each component of $\mathbb{C} \setminus \varphi(\gamma)$ and $|\varphi| > 1 - \epsilon$ on γ so that if $|w| < 1 - \epsilon$ then $n(\varphi(\gamma), w) = n(\varphi(\gamma), 0)$.

By the argument principle the number of zeros of $\varphi - w$ must equal the number of zeros of φ. But by construction $\varphi = 0$ only at one point, namely 0.

Letting $\epsilon \to 0$, we conclude that each value in $|w| < 1$ is attained exactly once. Thus φ maps Ω one-to-one and onto $\mathbb{D}$. $\square$

Definition: A region Ω is **doubly connected** if $\mathbb{C}^* \setminus \Omega = E_1 \cup E_2$ where E_1 and E_2 are disjoint, connected and closed in $\mathbb{C}^*$.

Theorem 13.12: *If Ω is doubly connected then there is a conformal map f of Ω onto an annulus $A = \{z : r_1 < |z| < r_2\}$, for some $0 \leq r_1 < r_2 \leq \infty$.*

Proof. By applying the Riemann mapping theorem twice we can assume $\Omega \subset \mathbb{D}$ s bounded by $\mathbb{T}$ and an anaytic curve γ.

Let $W = \log(\Omega)$. This is an apprioximate vertical strip with the imaginary axis as one side and is 2π-periodic vertically. By the Riemann mapping theorem it can be mapped by f to a true vertical strip $S = \{x + ui : s < x < 0\}$ for some negative number s, and so that ∞ maps to ∞.

Suppose $ic = f(2\pi i + z_0) - f(z_0)$. Then $g(z) = 2\pi f(z)/c$ is a conformal map of W onto a vertical strip so that $g(z+2\pi i) = g(z)+2\pi i$. (This uses uniquness of normalized Riemann map.)

Then $\exp(g(z))$ is $2\pi i$ periodic in W and defines a conformal map from Ω to a round annulus. $\square$

Koebe circle domain theorem says that any finitely connected domain can be conformally mapped to one bounded by circle or points.

A new proof is given in Karyn Lundberg's Stony Brook 2005 thesis.

The famous Koebe conjecture says this is true for all planar domains.

It was proven for domains with countably many boundary components by He and Schramm. See Fixed Points, Koebe Uniformization and Circle Packings by Zheng-Xu He and Oded Schramm, 1993.

This is still a very active area of research, e.g., see Removability, rigidity of circle domains and Koebe's conjecture by M. Younsi, 2016 and Exhaustions of circle domains by D. Ntalampekos and K. Rajala, 2023.

If we replace circles by slits, the question is known. It is a theorem that every planar domain can be mapped to a domain where every boundary component is either a point or a horizontal slit.

Other combinations are known, e.g., every boundary component can be a point or a radial slit.

The slits case is easier because he desired map can be characterized as the extremal solution minimizing a certain energy integral, and a solution exists by compactness by a normal families argument. We do not have any analogous formulation of the problem for circle domains.

The Koebe–Andreev–Thurston (KAT) (planar) circle packing theorem states that for every planar graph G with n vertices, there is a corresponding packing of n disks (with mutually disjoint interiors) in the plane, whose contact graph is isomorphic to G.

Follows from Koebe circle domain theorem in finitely connected case.

Circle packings are a way of devloping an discrete analog of holomorphic function, e.g. see website of Ken Stephenson