MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop
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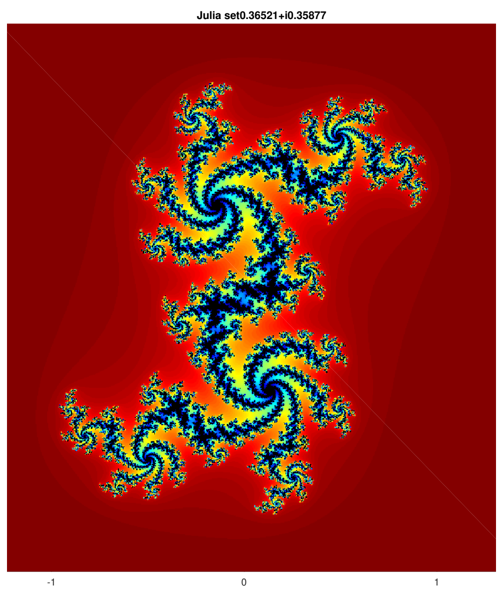

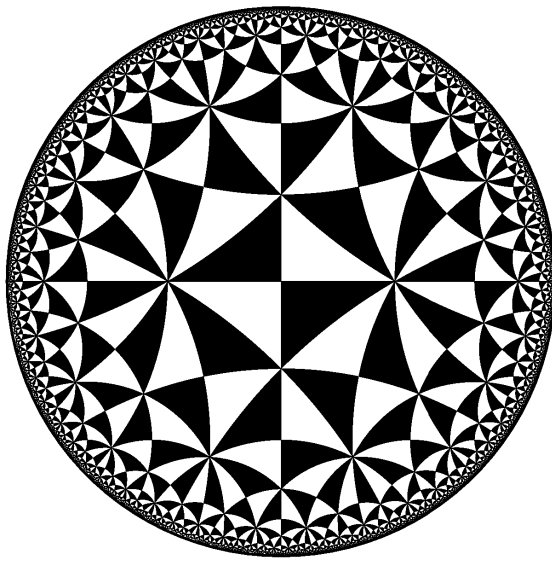

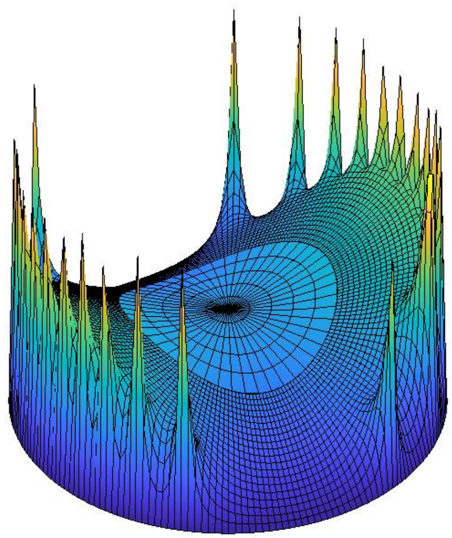

Chapter 12: Conformal Maps to Jordan Regions

Section 12.2: Janiszewski's Lemma

A **polygonal curve** in $\mathbb{C}^*$ is a curve consisting of finitely many line segments or half-lines.

Suppose U is an open subset of the extended plane $\mathbb{C}^*$. If $a \in U$, set

$$U_a = \{z \in U : \text{ there exists a polygonal curve } \gamma \subset U \text{ from } a \text{ to } z\}.$$

The set U_a has the following properties:

(1) If $z \in U_a$ then there is a simple polygonal curve from a to z.

(2) $U_a \cap U_b = \emptyset$ or $U_a = U_b$.

(3) The set U_a is open.

(4) The set $U \setminus U_a$ is open in U.

(5) There are at most countably many distinct sets U_a.

(6) If $E \subset U$ is connected then E is contained in one U_a.

(7) Each component U_a of U has boundary ∂U_a contained in $\mathbb{C}^* \setminus U$.

(8) If F is a homeomorphism of $\mathbb{C}^*$ then U_a is a component of U if and only if $F(U_a)$ is a component of $F(U)$.

The sets U_a are called the **components** of U.

Definition12.4: A closed set $E \subset \mathbb{C}^*$ **separates** points $a, b \notin E$ if a and b belong to distinct components of $\mathbb{C}^* \setminus E$.

Lemma 12.5, Janiszewski: *Suppose K_1 and K_2 are compact subsets of $\mathbb{C}$ such that $K_1 \cap K_2$ is connected and $0 \notin K_1 \cup K_2$. If K_1 does not separate 0 and ∞ and if K_2 does not separate 0 and ∞ then $K_1 \cup K_2$ does not separate 0 and ∞.*

Janiszewski's lemma follows from the next two lemmas.

Lemma 12.6: *If a compact set E separates* 0 *and* ∞ *then we cannot define* $\log z$ *to be analytic in a neighborhood of* E*.*

In other words, there is no function g which is analytic in a neighborhood of E satisfying $e^{g(z)} = z$, if E separates 0 and ∞

We recall some facts from Chapter 5.

A closed curve $\gamma \subset \Omega$ is **homologous to** 0 **in** Ω if $n(\gamma, a) = 0$ for all $a \notin \Omega$.

A region $\Omega \subset \mathbb{C}^*$ is called **simply-connected** if $\mathbb{C}^* \setminus \Omega$ is connected in $\mathbb{C}^*$.

Theorem 5.7: *A region* $\Omega \subset \mathbb{C}$ *is simply-connected if and only if every cycle in* Ω *is homologous to* 0 *in* Ω*. If* Ω *is not simply-connected then we can find a simple closed polygonal curve contained in* Ω *which is not homologous to* 0*.*

Lemma 12.6: *If a compact set E separates 0 and ∞ then we cannot define $\log z$ to be analytic in a neighborhood of E.*

Proof of Lemma 12.6. Suppose g is analytic in an open set $W \supset E$ and $e^{g(z)} = z$. Then $0, \infty \in \mathbb{C}^* \setminus W \subset \mathbb{C}^* \setminus E$.

Since 0 and ∞ belong to distinct components of $\mathbb{C}^* \setminus E$, by Theorem 5.7 we can find a polygonal curve $\sigma \subset W$ so that $n(\sigma, 0) = 1$.

But by the chain rule, $g'(z) = 1/z$, so that $2\pi i = \int_\sigma 1/z \, dz = \int_\sigma g'(z) dz = 0$ by the fundamental theorem of calculus. This contradiction proves the lemma. $\square$

Lemma 12.7: *Suppose* K_1 *and* K_2 *are compact sets such that* $K_1 \cap K_2$ *is connected and* $0, \infty \notin K_1 \cup K_2$*. If* K_1 *does not separate* 0 *and* ∞ *and if* K_2 *does not separate* 0 *and* ∞ *then we can define* $\log z$ *to be analytic in a neighborhood of* $K_1 \cup K_2$*.*

Lemma 12.7: *Suppose* K_1 *and* K_2 *are compact sets such that* $K_1 \cap K_2$ *is connected and* $0, \infty \notin K_1 \cup K_2$. *If* K_1 *does not separate* 0 *and* ∞ *and if* K_2 *does not separate* 0 *and* ∞ *then we can define* $\log z$ *to be analytic in a neighborhood of* $K_1 \cup K_2$.

Proof. By hypothesis, we can find simple polygonal curves σ_1 and σ_2 connecting 0 to ∞ with $\sigma_j \cap K_j = \emptyset$, $j = 1, 2$. Then

$$(\sigma_1 \cup \sigma_2) \cap (K_1 \cap K_2) = \emptyset.$$

If E is connected and contained in an open set U, then E is contained in a single component of U.

Therefore, the connected set $K_1 \cap K_2$ is contained in one component U of $\mathbb{C}^* \setminus (\sigma_1 \cup \sigma_2)$.

Since $\mathbb{C}^* \setminus \sigma_j$ is simply-connected, $j = 1, 2$, we can find functions f_j analytic on $\mathbb{C}^* \setminus \sigma_j$ so that $e^{f_j(z)} = z$ on $\mathbb{C}^* \setminus \sigma_j$. Then $e^{f_1 - f_2} = 1$ on $\mathbb{C}^* \setminus (\sigma_1 \cup \sigma_2)$.

Thus on each component of $\mathbb{C}^* \setminus (\sigma_1 \cup \sigma_2)$, $f_1 - f_2$ is a constant $2\pi k i$ for some integer k. We may then add a constant to f_2 so that $f_1 = f_2$ on U.

Because $K_1 \setminus U$ and $K_2 \setminus U$ are disjoint compact sets, we can find open sets V_j so that

$$K_j \setminus U \subset V_j \subset \overline{V_j} \subset \mathbb{C}^* \setminus \sigma_j,$$

for $j = 1, 2$ and $V_1 \cap V_2 = \emptyset$. Set

$$f(z) = \begin{cases} f_1(z) & \text{for } z \in V_1 \cup U \\ f_2(z) & \text{for } z \in V_2 \cup U \end{cases}$$

Then f is analytic on $V_1 \cup V_2 \cup U$, an open neighborhood of $K_1 \cup K_2$, and $e^{f(z)} = z$. □

Janiszewski's lemma follows immediaely.

Corollary 12.3: *If* J *is a Jordan arc in* $\mathbb{C}^*$ *then* $\mathbb{C}^* \setminus J$ *is open and connected.*

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Proof. Since LFTs are homeomorphisms of $\mathbb{C}^*$, it suffices to show that if $0, \infty \notin J$ then J does not separate 0 and ∞.

Write $J = \cup_1^n J_k$ where J_k are subarcs of J such that $J_k \cap J_{k-1}$ is a single point. We may choose each J_k so small that for each k, there is a half line from 0 to ∞ contained in $\mathbb{C}^* \setminus J_k$ and hence no J_k separate 0 and ∞.

But $J_1 \cap J_2$ a single point, and hence connected, so that by Janiszewski's lemma the arc $J_1 \cup J_2$ does not separate 0 and ∞.

The intersection of this arc with J_3 again is a single point, and hence $J_1 \cup J_2 \cup J_3$ does not separate 0 and ∞ by Janiszewski's lemma. By induction J does not separate 0 and ∞. $\square$

Section 12.3: Jordan Curve Theorem

Theorem 12.9, Jordan Curve Theorem: *If $J \subset \mathbb{C}^*$ is a (closed) Jordan curve then $\mathbb{C}^* \setminus J$ has exactly two components, each of which is simply-connected. Moreover, J is the boundary of each component.*

The proof is divided into four lemmas.

Lemma 12.10: If J is a (closed) Jordan curve, and if U is a component of $\mathbb{C}^* \setminus J$ then $\partial U = J$.

Proof. Suppose $\zeta \in J$ and $z_0 \in U$. Because J is the homeomorphic image of the unit circle, given $n < \infty$ we can find Jordan arcs $J_n, J_n' \subset J$ with $J_n \cup J_n' = J$, $\zeta \in J_n$, $\zeta \notin J_n'$ and $J_n \subset D_n$, where $D_n = \{z : |z - \zeta| < 1/n\}$.

By Corollary 12.8 (arcs don't separate plane) J_n' does not separate ζ and z_0. So there is a polygonal curve σ_n from z_0 to ζ such that $\sigma_n \cap J_n' = \emptyset$.

Let z_n be the first intersection of σ_n with ∂D_n. The subarc $\alpha_n \subset \sigma_n$ from z_0 to z_n does not intersect J'_n and does not intersect $J_n \subset D_n$, and hence does not intersect J.

Since α_n is connected, we have $\alpha_n \subset U$, and hence $z_n \in U$. But $\lim_n z_n = \zeta \notin U$, so $\zeta \in \partial U$. This shows that $J \subset \partial U$.

If $\zeta \in \partial U$, then ζ does not belong to any component of $\mathbb{C}^* \setminus J$ since the components are open. Thus $\zeta \in J$, and we've shown $J = \partial U$. $\square$

Take $\zeta_0 \in \mathbb{C}^* \setminus J$. Then there is a straight line segment $[\zeta_0, \zeta_1]$ with $\zeta_1 \in J$ and $[\zeta_0, \zeta_1) \cap J = \emptyset$.

Morever we can choose another line segment $[\zeta_0, \zeta_2]$ with $\zeta_2 \in J$ and $[\zeta_0, \zeta_2) \cap J = \emptyset$ and $[\zeta_0, \zeta_1] \cap [\zeta_0, \zeta_2] = \{\zeta_0\}$. For otherwise J would be contained in a half line from ζ_1 to ∞ which is impossible.

Write $J = J_1 \cup J_2$ where J_j are Jordan arcs with $J_1 \cap J_2 = \{\zeta_1, \zeta_2\}$. Switching ζ_1 and ζ_2 if necessary,

$$\sigma = J_1 \cup [\zeta_1, \zeta_0] \cup [\zeta_0, \zeta_2]$$

is a (closed) Jordan curve. In other words, σ is the modification of J found by replacing J_2 with the union of two intervals.

Lemma 12.11: *The set* $\mathbb{C}^* \setminus \sigma$ *has exactly two components.*

Proof. Suppose that D_0 is an open disk centered at ζ_0 with $J \cap D_0 = \emptyset$. Then $D_0 \setminus ([\zeta_1, \zeta_0] \cup [\zeta_0, \zeta_2])$ consists of two connected open circular sectors.

each sector must be contained in a component of $\mathbb{C}^* \setminus \sigma$ and so by Lemma 12.10 ($\partial U = J$), there can be at most two components in $\mathbb{C}^* \setminus \sigma$.

Take z_1 and z_2 in distinct sectors of $D_0 \setminus ([\zeta_1, \zeta_0] \cup [\zeta_0, \zeta_2])$. Then $(\sigma \setminus D_0) \cup \partial D_0$ does not separate z_1 and z_2.

Moreover $\sigma \cap ((\sigma \setminus D_0) \cup \partial D_0) = \sigma \setminus D_0$, which is connected.

So if σ also does not separate z_1 and z_2 then by Janiszewski's lemma, $E = \sigma \cup (\sigma \setminus D_0) \cup \partial D_0$ does not separate z_1 and z_2.

But clearly $\partial D_0 \cup [\zeta_1, \zeta_0] \cup [\zeta_0, \zeta_2] \subset E$ does separate. This contradiction proves that z_1 and z_2 are in distinct components of $\mathbb{C}^* \setminus \sigma$, proving the lemma. $\square$

Lemma 12.11: *The set $\mathbb{C}^* \setminus J$ has at least two components.*

Proof. Take $\zeta \in J_1 \setminus \{\zeta_1, \zeta_2\}$ and define

$$\alpha = J_2 \cup [\zeta_2, \zeta_0] \cup [\zeta_0, \zeta_1].$$

Let D_ζ be a disk centered at ζ such that $D_\zeta \cap \alpha = \emptyset$.

By Lemma 12.10 ($\partial U = J$) $\zeta \in \partial G_1 \cap \partial G_2$ where G_1 and G_2 are the two components of the complement of σ.

Take $w_1 \in G_1 \cap D_\zeta$ and $w_2 \in G_2 \cap D_\zeta$. The points w_1 and w_2 are separated by σ, but not by α. Note that $J \cap \alpha = J_2$ is connected so if J also does not separate w_1 and w_2 then by Janiszewski's lemma, $J \cup \alpha$ does not separate w_1 and w_2.

But this is a contradiction since $\sigma \subset J \cup \alpha$. We conclude that J must separate w_1 and w_2 and hence $\mathbb{C}^* \setminus J$ has at least two components. $\square$

Lemma 12.13: *The set* $\mathbb{C}^* \setminus J$ *has no more than two components.*

Proof. Suppose H_1 and H_2 are the components of $\mathbb{C}^* \setminus J$ containing w_1 and w_2 from the proof of Lemma 12.12. If $\mathbb{C}^* \setminus J$ has another component H_3, then by Lemma 12.10, we can find $w_3 \in H_3 \cap D_\zeta$.

But $w_3 \notin \sigma$ and hence $w_3 \in G_1$ or $w_3 \in G_2$. If $w_3 \in G_1$ then w_1 and w_3 are not separated by σ, nor by α. But $\sigma \cap \alpha = [\zeta_1, \zeta_0] \cup [\zeta_0, \zeta_2]$ is connected, so that by Janiszewski's lemma w_1 and w_3 are not separated by $\sigma \cup \alpha$.

But this contradicts the assumption that $J \subset \sigma \cup \alpha$ separates w_1 and w_3. A similar contradiction is obtained if $w_3 \in G_2$, proving the lemma. $\square$

Proof of the Jordan Curve Theorem:

The theorem now follows from Lemmas 12.10, 12.12 and 12.13 and the observation that the complement in $\mathbb{C}^*$ of one component of $\mathbb{C}^* \setminus J$ is equal to the closure of the other component and hence connected. So each component is simply-connected. $\square$

Jordan's orginal proof has been critized as not being sufficiently rigorous, but his proof is defended by Tom Hales in Jordan's proof of the Jordan curve therorem

Hales is a proponent of automated proof checkers, and he has verifified the proof of the Jordan curve theorem on a computer. It is one of the few resuts that has had its proof formally checked. See The Jordan curve theorem, formally and informally by T.Hales.

Hales is also well known for his work on the Langslands program and for his proof of the centuries old Kepler conjecture, that the "obvious" packing of spheres in 3-space is the optimal one.

Section 12.4: Carathéodory's Theorem

Definition: A **Jordan region** is simply-connected region in $\mathbb{C}^*$ whose boundary is a Jordan curve.

Theorem 12.14, Carathéodory-Tohorst Theorem: *If φ is a conformal map of $\mathbb{D}$ onto a Jordan region Ω, then φ extends to be a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. In particular $\varphi(e^{it})$ is a parametrization of $\partial\Omega$.*

Although usually called “Carathéodory’s theorem, the result actually appears in the 1917 Bonn thesis of Marie Torhorst, a student of Carathéodory. For a discussion of the history, see On prime ends and local connectivity by Lasse Rempe. Torhorst did not become an academic mathematician, but eventually became Minister of Education for the state of Thüringen in communist East Germany following WWII.

The proof we give uses the “length-area” method, which is the begining of the study of extremal length, a extremely important tool in complex function theory, quasiconformal mappings, analysis on metric spaces and holomorphic dynamics.

Proof. Using an LFT, we may suppose Ω is bounded. First we show φ has a continuous extension at each $\zeta \in \partial\mathbb{D}$.

Let $0 < \delta < 1$ and set $\gamma_\delta = \mathbb{D} \cap \{z : |z - \zeta| = \delta\}$.

The idea of the proof is that the image curve $\varphi(\gamma_\delta)$ cuts off a region U_δ whose closure shrinks to the point ζ as $\delta \to 0$.

It is not hard to show that the area of U_δ decreases to 0, but we need more.

We claim that the diameter of the boundary of U_δ, and hence the diameter of U_δ, tends to 0.

The curve $\varphi(\gamma_\delta)$ is an analytic Jordan arc with length

$$L(\delta) = \int_{\gamma_\delta} |\varphi'(z)||dz|.$$

By the Cauchy-Schwarz inequality

$$L^2(\delta) \leq \left(\int_{\gamma_\delta} 1^2 |dz| \right) \cdot \left(\int_{\gamma_\delta} |\varphi'(z)|^2 |dz| \right) \leq \pi\delta \int_{\gamma_\delta} |\varphi'(z)|^2 |dz|,$$

so that for $r < 1$,

$$\begin{aligned} \int_0^r \frac{L^2(\delta)}{\delta} d\delta &\leq \pi \int \int_{\mathbb{D} \cap B(\zeta, r)} |\varphi'(z)|^2 dx\, dy \\ &= \pi \cdot \text{area}\, (\varphi(\mathbb{D} \cap B(\zeta, r)) < \infty \end{aligned}$$

since Ω is bounded.

Therefore there is a decreasing sequence $\delta_n \to 0$ such that $L(\delta_n) \to 0$. When $L(\delta_n) < \infty$, the curve $\varphi(\gamma_{\delta_n})$ has endpoints α_n, β_n, and both of these endpoints must lie on $\partial\Omega$, because φ is proper. Furthermore,

$$|\alpha_n - \beta_n| \leq L(\delta_n) \to 0. \tag{12.2}$$

Let γ be a homeomorphism of $\partial\mathbb{D}$ onto $\partial\Omega$. Write $\alpha_n = \gamma(\zeta_n)$ and $\beta_n = \gamma(\psi_n)$.

Because γ is uniformly continuous, given $\epsilon > 0$ there is a $\delta > 0$ so that if $|\zeta_n - \psi_n| < \delta$ then for ζ in the smaller arc of $\partial\mathbb{D}$ between ζ_n and ψ_n, we have $|\gamma(\zeta) - \gamma(\zeta_n)| < \epsilon$.

But γ^{-1} is also uniformly continuous so there exists $\eta > 0$ so that if $|\alpha_n - \beta_n| < \eta$ then $|\zeta_n - \psi_n| < \delta$.

Thus if σ_n is the closed subarc of $\partial\Omega$ of smallest diameter with endpoints α_n and β_n, then by (12.2)

$$\operatorname{diam}(\sigma_n) \to 0.$$

By the Jordan curve theorem, the curve $\sigma_n \cup \varphi(\gamma_{\delta_n})$ divides the plane into two (connected, open) regions, and one of these regions, say U_n, is bounded.

The unbounded component V of the complement of $\partial\Omega$ is connected so if $z \in U_n \cap V$ then there is a polygonal arc γ_z from z to ∞ contained in V.

Because $\varphi(\mathbb{D}) \cap V = \emptyset$ and $\sigma_n \subset \partial\Omega$, γ_z does not meet ∂U_n, contradicting the boundedness of U_n. Thus $U_n \cap V = \emptyset$. By the Jordan curve theorem applied to $\partial\Omega$, $U_n \subset \Omega$.

Since

$$\operatorname{diam}(\partial U_n) = \operatorname{diam}\big(\sigma_n \cup \varphi(\gamma_{\delta_n})\big) \to 0,$$

we conclude that

$$\operatorname{diam}(U_n) \to 0. \tag{12.3}$$

Set $D_n = \mathbb{D} \cap \{z : |z - \zeta| < \delta_n\}$. Then $\varphi(D_n)$ and $\varphi(\mathbb{D} \setminus \overline{D}_n)$ are connected sets which do not intersect $\varphi(\gamma_{\delta_n})$. Since φ maps onto Ω, either $\varphi(D_n) = U_n$ or $\varphi(\mathbb{D} \setminus \overline{D}_n) = U_n$.

But $\operatorname{diam}(\varphi(\mathbb{D} \setminus D_n)) \geq \operatorname{diam}(\varphi(B(0, 1/2)) > 0$ and $\operatorname{diam}(U_n) \to 0$ by (4.3) so that $\varphi(D_n) = U_n$ for n sufficiently large.

Since δ_n is decreasing, $\varphi(D_{n+1}) \subset \varphi(D_n)$ and so $\bigcap \overline{\varphi(D_n)}$ consists of a single point. Thus φ has a continuous extension to $\mathbb{D} \cup \{\zeta\}$.

Let φ also denote the extension $\varphi : \overline{\mathbb{D}} \to \overline{\Omega}$. If $z_n \in \overline{\mathbb{D}}$ converges to $\zeta \in \partial\mathbb{D}$ then we can find $z_n' \in \mathbb{D} \to \zeta$ so that $\varphi(z_n) - \varphi(z_n') \to 0$.

By the continuity of φ at ζ, we must have $\varphi(z_n) \to \varphi(\zeta)$, and conclude that φ is continuous on $\overline{\mathbb{D}}$.

Final part of proof is to show φ is a homeomorphism.

Enough to show φ 1-1 on boundary, since any continuous, 1-1 map on a compact set is a homeomorphism.

Because $\varphi(\mathbb{D}) = \Omega$, φ maps $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. To show φ is one-to-one, suppose $\varphi(\zeta_1) = \varphi(\zeta_2)$ but $\zeta_1 \neq \zeta_2$.

Because φ is proper, $\varphi(\partial\mathbb{D}) \subset \partial\Omega$ and so we can assume $\zeta_j \in \partial\mathbb{D}$, $j = 1, 2$. The Jordan curve

$$\left\{\varphi(r\zeta_1) : 0 \leq r \leq 1\right\} \cup \left\{\varphi(r\zeta_2) : 0 \leq r \leq 1\right\}$$

bounds a bounded region W.

Arguing exactly as above replacing U_n, D_n and $\mathbb{D} \setminus \overline{D_n}$ with W and the two components of

$$\mathbb{D} \setminus \Big(\{r\zeta_1 : 0 \leq r \leq 1\} \cup \{r\zeta_2 : 0 \leq r \leq 1\} \Big)$$

we conclude that $W \subset \Omega$ and $\varphi^{-1}(W)$ must be one of these two components.

Because $\varphi(\partial\mathbb{D}) \subset \partial\Omega$ and φ is proper on $\varphi^{-1}(W)$, we conclude that

$$\varphi(\partial\mathbb{D} \cap \partial\varphi^{-1}(W)) \subset \partial\Omega \cap \partial W = \{\varphi(\zeta_1)\}.$$

Thus φ is constant on an arc of $\partial\mathbb{D}$. It follows that $\varphi - \varphi(\zeta_1) \equiv 0$, by the Schwarz Reflection principle.

This contradiction shows φ must be one-to-one. $\square$

Corollary 12.15: *If* $h : \partial\mathbb{D} \to \mathbb{C}$ *is a homeomorphism then* h *extends to be a homeomorphism of* $\mathbb{C}$ *onto* $\mathbb{C}$*.*

Corollary 12.16: *If* $J \subset \mathbb{C}$ *is a closed Jordan curve then* J *can be oriented so that* $n(J, z) = 1$ *for* z *in the bounded component of the complement of* J *and* $n(J, z) = 0$ *for* z *in the unbounded component of the complement of* J*.*

Proofs are given in the textbook.

Boundary continuity of conformal maps, allows us to push forward Lebesgue measure on $\mathbb{T}$ to probability measures on Jordan curves, called "harmonic measure". Harmonic measures can be defined in other ways (Brownian motion, Green's functions, potential theory,...), but Jordan domains in the plane are the easiest case, because of the connection to complex analysis and conformal maps.

If Ω is simply connected and $\partial\Omega$ is locally connected, then any conformal map $\mathbb{D} \to \Omega$ extends continuously to the boundary of $\mathbb{D}$.

There are examples with continuous extension everywhere except one point.

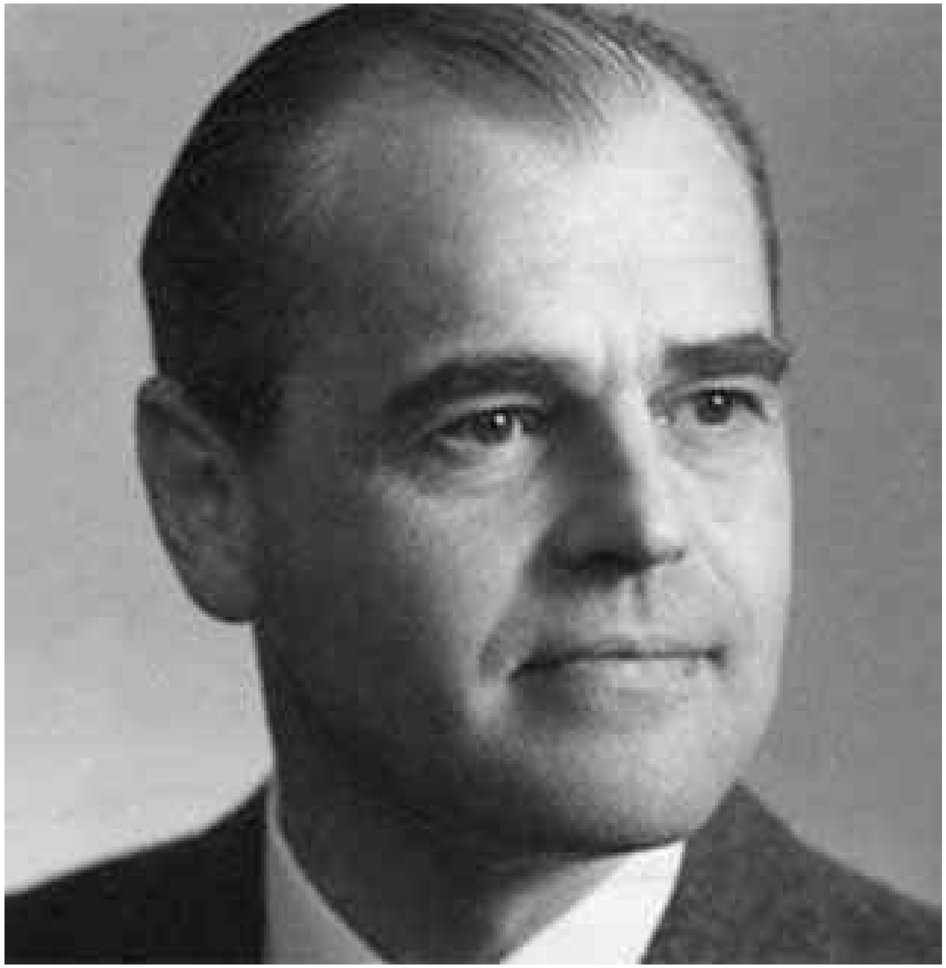
There are examples where there is no continuous extension anywhere.

Fatou's theorem says that any bounded analytic function on $\mathbb{D}$ has radial limits almost everywhere on the circle.

A conformal map onto a bounded domain is an example.

Arne Beurling proved a stronger version: a conformal map has radial limits except on a set of zero logarithmic capacity.

Zero log-capacity sets are very small: zero length, even zero Hausdorff dimension. The middle thirds Cantor has positive log-capacity.



Arne Beurling (1908-1986)

We can define conformal maps f, g to either side of a Jordan curve (the map onto the ubounded component will have a pole), and then $h = g^{-1} \circ f$ is a circle homeomorphism, called a conformal welding. All smooth circle homeomoprhisms arise in this way, but not all homeomorphims. It is an open problem to characterize which circle homeomorphims are conformal weldings. See Conformal Welding and Koebe's Theorem, by C. Bishop.