MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop
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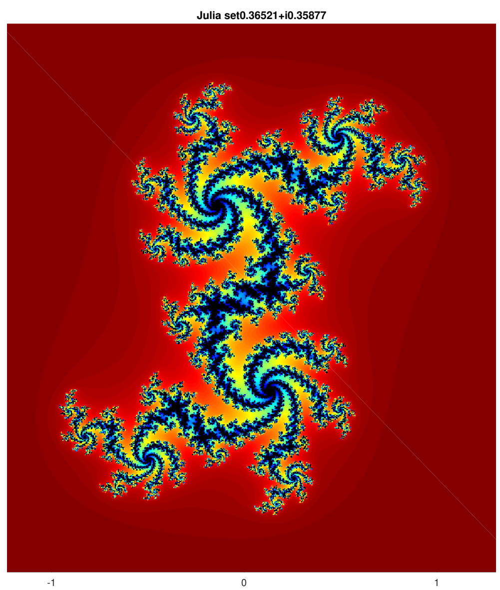

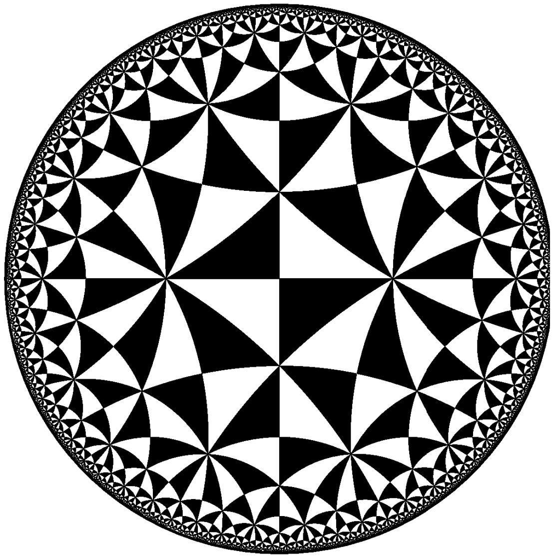

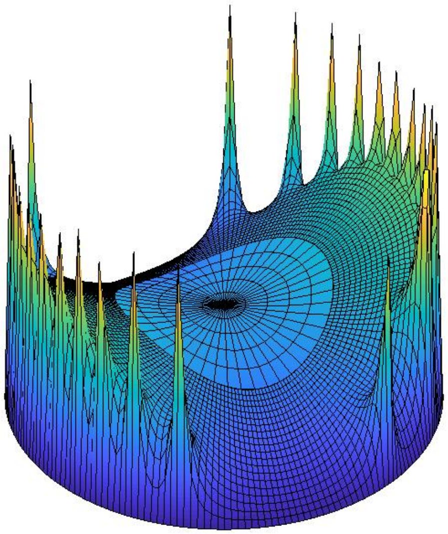

Chapter 10: Series and Products

Section 10.1: Mittag-Lefferler's Theorem

Suppose f is meromorphic. Near a pole b we have the Laurent expansion,

$$f(z) = \frac{c_n}{(z-b)^n} + \frac{c_{n-1}}{(z-b)^{n-1}} + \cdots + \frac{c_1}{(z-b)} + a_0 + a_1(z-b) + a_2(z-b)^2 + \ldots .$$

The sum of the first n terms

$$S_b(z) = \frac{c_n}{(z-b)^n} + \frac{c_{n-1}}{(z-b)^{n-1}} + \cdots + \frac{c_1}{(z-b)}$$

is called the **singular part of f at b**.

If f is rational, then by a partial fraction expansion

$$f(z) = \sum_{k=1}^{m} S_{b_k}(z) + p(z),$$

where p is a polynomial and $\{b_k\}$ are the poles of f.

If f is meromorphic in a region Ω with only finitely many poles $\{b_k\}$ and singular parts S_{b_k}, $k = 1, \ldots, m$, then

$$f(z) = \sum_{k=1}^{m} S_{b_k}(z) + g(z),$$

where g is analytic in Ω.

Theorem 11.1, Mittag-Leffler's theorem: *Suppose* $b_k \in \Omega \to \partial\Omega$, *with* $b_k \neq b_j$ *if* $k \neq j$. *Set*

$$S_k(z) = \sum_{j=1}^{n_k} \frac{c_{j,k}}{(z - b_k)^j}$$

where each n_k *is a positive integer and* $c_{j,k} \in \mathbb{C}$. *Then there is a function meromorphic in* Ω *with singular parts* S_k *at* b_k, $k = 1, 2, \ldots,$ *and no other singular parts in* Ω.

If $\Omega = \mathbb{C}$ we interpret the first hypothesis of Theorem 1.1 to be that $|b_k| \to \infty$.

Proof. Let

$$K_n = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geq \frac{1}{n} \ \text{ and } \ |z| \leq n\}.$$

K_n is a compact subset of Ω such that each bounded component of $\mathbb{C} \setminus K_n$ contains a point of $\partial\Omega$ and $K_n \subset K_{n+1} \subset \cup K_n = \Omega$.

Because $b_k \to \partial\Omega$ each K_n contains only finitely many b_k.

By Runge's Theorem there is a rational f_n with poles in $\mathbb{C} \setminus \Omega$ so that

$$\Big| \{ \sum_{b_k \in K_{n+1} \setminus K_n} S_k(z) \} - f_n(z) \Big| < 2^{-n}$$

for all $z \in K_n$.

Then for each $m = 1, 2, \ldots$

$$\sum_{n \geq m} \Big(\{ \sum_{b_k \in K_{n+1} \setminus K_n} S_k(z) \} - f_n(z) \Big)$$

converges uniformly on K_m to an analytic function on K_m by the Weierstrass M-test and Weierstrass's Theorem IV.3.10. Set

$$f(z) = \sum_{b_k \in K_1} S_k(z) + \sum_{n=1}^{\infty} \Big(\{ \sum_{b_k \in K_{n+1} \setminus K_n} S_k(z) \} - f_n(z) \Big). \tag{11.1}$$

Then f is a well-defined analytic function on $\Omega \setminus \{b_k\}$ and $f - S_k$ has a removable singularity at b_k for each $k = 1, 2, \ldots$. □

Example 11.5, Weierstrass $\mathcal{P}$ function:

Suppose $w_1, w_2 \in \mathbb{C} \setminus \{0\}$ with w_1/w_2 not real. In other words w_1 and w_2 are not on the same line through the origin. There is no non-constant entire function f satisfying $f(z+w_1) = f(z+w_2) = f(z)$ for all z, by Liouville's theorem.

But there are meromorphic functions with this property. The Weierstrass $\mathcal{P}$ function is defined by

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - mw_1 - nw_2)^2} - \frac{1}{(mw_1 + nw_2)^2} \right) \tag{11.8}$$

where the sum is taken over all pairs of integers except $(0,0)$.

To prove convergence of this sum, we first observe that there is a $\delta > 0$ so that $|mw_1 + nw_2| \geq \delta$ unless $m = n = 0$, for if $|m_j w_1 + n_j w_2| \to 0$, then

$$|\frac{w_1}{w_2} + \frac{n_j}{m_j}| \to 0,$$

contradicting the assumption that w_1/w_2 is not real.

Thus $\{\zeta_{m,n} = mw_1 + nw_2 : m, n \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the integers, forms a lattice of points in $\mathbb{C}$ with no two points closer than δ.

If we place a disk of radius $\delta/2$ centered at each point of the lattice, then the disks are disjoint.

The area of the annulus $k \le |\zeta| \le k+1$ is $(2k+1)\pi$ so there are at most Ck lattice points in this annulus, for some constant C depending on δ.

For $|z| < R$, we split the sum in (11.8) into a finite sum of terms with $|\zeta_{m,n}| \leq 2R$ and the sum of terms with $|\zeta_{m,n}| > 2R$.

Note that if $|z| < R$ and $|\zeta| > 2R$, then

$$\left|\frac{1}{(z-\zeta)^2} - \frac{1}{\zeta^2}\right| = \left|\frac{2z\zeta - z^2}{\zeta^2(z-\zeta)^2}\right| \leq \frac{R(2|\zeta| + R)}{|\zeta|^2|\zeta/2|^2} \leq \frac{10R}{|\zeta|^3}.$$

We conclude that for $K \geq R$ and $|z| < R$

$$\sum_{|\zeta_{m,n}|>2K} \left|\frac{1}{(z-\zeta_{m,n})^2} - \frac{1}{\zeta_{m,n}^2}\right| = \sum_{k=2K}^{\infty} \sum_{k<|\zeta_{m,n}|\leq k+1} \left|\frac{1}{(z-\zeta_{m,n})^2} - \frac{1}{\zeta_{m,n}^2}\right|$$

$$\leq \sum_{k=2K}^{\infty} Ck\frac{10K}{k^3} < \infty.$$

By Weierstrass's theorem, the Weierstrass $\mathcal{P}$ function is meromorphic in $\mathbb{C}$ with singular part $S(z) = 1/(z - mw_1 - nw_2)^2$ at $mw_1 + nw_2$ and no other poles.

Next we show that $\mathcal{P}(z+w_1)=\mathcal{P}(z)$. By Weierstrass's theorem

$$\mathcal{P}'(z) = -\frac{2}{z^3} - \sum_{\zeta_{m,n\neq 0}} \frac{2}{(z-\zeta_{m,n})^3}.$$

By the same estimate, this series converges absolutely so that we can rearrange the terms, obtaining $\mathcal{P}'(z+w_1)=\mathcal{P}'(z)$, and hence $\mathcal{P}(z+w_1)-\mathcal{P}(z)$ is a constant.

The series for $\mathcal{P}$ is even, so $\mathcal{P}(z+w_1)=\mathcal{P}(z)$ when $z=-w_1/2$, and thus $\mathcal{P}(z+w_1)=\mathcal{P}(z)$ for all z.

A similar argument shows that $\mathcal{P}(z+w_2)=\mathcal{P}(z)$.

Section 11.2: Weierstrass Products

A few facts from calculus: Proposition 11.6: *Suppose* $p_j \in \mathbb{C} \setminus \{0\}$. *Then* $\prod_{j=1}^{n} p_j$ *converges to a non-zero complex number* P *as* $n \to \infty$ *if and only if*

$$\sum_{j=1}^{\infty} \log p_j$$

converges to a complex number S *where* $\log p_j$ *is defined so that* $-\pi < \arg p_j \leq \pi$. *Moreover, if convergence holds then* $P = e^S$ *and* $\lim p_j = 1$.

Definition 11.7: If p_j are non-zero complex numbers then we say $\prod_{j=1}^{\infty} p_j$ **converges absolutely** if $\sum |\log p_j|$ converges.

Lemma 11.8: *If p_j are nonzero complex numbers then $\prod_{j=1}^{\infty} p_j$ converges absolutely if and only if*

$$\sum_{j=1}^{\infty} |p_j - 1|$$

converges.

Definition 11.9: Suppose $\{f_j\}$ are analytic on a region Ω. We say that $\prod_{j=1}^{\infty} f_j(z)$ **converges** on Ω if

$$\lim_{n\to\infty} \prod_{j=0}^{n} f_j(z)$$

converges uniformly on compact subsets of Ω to a function f which is not identically equal to 0.

Theorem 11.10, Weierstrass: *Suppose* Ω *is a bounded region. If* $\{b_j\} \subset \Omega$ *with* $b_j \to \partial\Omega$*, and if* n_j *are positive integers, then there exists an analytic function* f *on* Ω *such that* f *has a zero of order exactly* n_j *at* b_j*,* $j = 1, 2, \ldots,$ *and no other zeros in* Ω*.*

Proof. Let

$$K_n = \{z \in \Omega : |z - w| \geq \frac{1}{n} \text{ for all } w \in \partial\Omega\}.$$

Then as in the proof of Corollary IV.3.9, K_n is a compact subset of $\mathbb{C}$ such that each component of $\mathbb{C} \setminus K_n$ contains a point of $\partial\Omega$ and $K_n \subset K_{n+1}$.

Choose $a_j \in \partial\Omega$ so that

$$\operatorname{dist}(b_j, \partial\Omega) = |b_j - a_j|.$$

If $b_j \notin K_n$ then the line segment from b_j to a_j does not intersect K_n. Thus we can define $\log((z - b_j)/(z - a_j))$ so as to be analytic in $\mathbb{C} \setminus K_n$.

Each K_n contains at most finitely many b_k because $b_k \to \partial\Omega$.

By Runge's Theorem we can find a rational r_n with poles in $\mathbb{C} \setminus \Omega$ so that

$$\left|\Big\{ \sum_{b_k \in K_{n+1} \setminus K_n} n_k \log\left(\frac{z-b_k}{z-a_k}\right)\Big\} - r_n(z)\right| < 2^{-n}, \tag{2.1}$$

for all $z \in K_n$. Then

$$\sum_{n \geq m} \Big(\Big\{ \sum_{b_k \in K_{n+1} \setminus K_n} n_k \log\left(\frac{z-b_k}{z-a_k}\right)\Big\} - r_n(z)\Big)$$

converges uniformly on K_m to an analytic function on K_m. Set

$$f(z) = \prod_{b_k \in K_1} \left(\frac{z-b_k}{z-a_k}\right)^{n_k} \prod_{n=1}^{\infty} \Big(\prod_{b_k \in K_{n+1} \setminus K_n} \left(\frac{z-b_k}{z-a_k}\right)^{n_k} \Big) e^{-r_n(z)}.$$

Then f is a well-defined analytic function on Ω with a zero of order n_k at b_k, $k = 1, 2, \ldots$, and no other zeros. $\square$

Corollary 11.13: *If Ω is a region then there is a function f analytic on Ω such that f does not extend to be analytic in any larger region.*

Proof. Take a sequence $\{a_n\} \subset \Omega \to \partial\Omega$ such that $\partial\Omega \subset \overline{\{a_n\}}$.

By the Weierstrass product theorem we can find f analytic on Ω, with $f(a_n) = 0$ but f not identically zero.

If f extends to be analytic in a neighborhood of $b \in \partial\Omega$ then the zeros of the extended function would not be isolated. $\square$

In several complex variables, a similar result is not true.

If $B_r = \{(z, w) : |z|^2 + |w|^2 < r^2\}$, then any function which is analytic on $B_2 \setminus B_1$ extends to be analytic on B_2.

Corollary 11.14: *Suppose* Ω *is a region and* $a_n \to \partial\Omega$, *with* $a_n \neq a_m$ *when* $n \neq m$, *and suppose* $\{c_n\}$ *are complex number. Then there exists* f *analytic on* Ω *such that*

$$f(a_n) = c_n, n = 1, 2, \ldots.$$

Results like this are usually called interpolation theorems.

Proof. By the Weierstrass product theorem, we can find G analytic on Ω with a simple zero at each a_n. Let

$$d_n = \lim_{z \to a_n} \frac{G(z)}{z - a_n} = G'(a_n).$$

Since the zero of G at a_n is simple, $d_n \neq 0$. By Mittag-Leffler's theorem we can find F meromorphic on Ω with singular part

$$S_n(z) = \frac{c_n/d_n}{z - a_n}$$

at a_n and no other poles in $\mathbb{C}$.

Then $f(z) = F(z)G(z)$ is analytic on $\Omega \setminus \{a_n\}$ and

$$\lim_{z \to a_n} F(z)G(z) = \lim_{z \to a_n} (z - a_n)F(z)\frac{G(z)}{z - a_n} = \frac{c_n}{d_n} d_n = c_n.$$

Thus the singularity of f at each a_n is removable and f extends to be analytic on Ω with $f(a_n) = c_n$, $n = 1, 2, \ldots$. □

Corollary 11.15: *If f is meromorphic in Ω then there are functions g and h, analytic on Ω, such that*

$$f = \frac{g}{h}.$$

Proof. Let $\{a_n\}$ be the poles of f, where the list is written such that a pole of order k occurs k times in this list.

By the Weierstrass product theorem, there is a function h analytic on Ω with zeros $\{a_n\}$ and no other zeros. Then

$g = fh$ is analytic on $\Omega \setminus \{a_n\}$ with removable singularity at each a_n.

Thus g extends to be analytic in Ω and $f = g/h$. $\square$

Section 11.3: Blaschke Products

Theorem 11.17, Jensen's theorem: *Suppose* f *is meromorphic on* $|z| \leq R$ *with zeros* $a_1, \ldots, a_n$ *and poles* $b_1, \ldots, b_m$*. Suppose also that* 0 *is not a zero or a pole of* f*. Then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(Re^{it})| dt = \log |f(0)| + \sum \log \frac{R}{|a_k|} - \sum \log \frac{R}{|b_j|}. \qquad (11.11)$$

Proof. Replacing $f(z)$ by $f(Rz)$, we may assume $R = 1$.

First suppose that f has no poles or zeros on $|z| = 1$. Write

$$f(z) = \frac{\prod_k \frac{z-a_k}{1-\overline{a_k}z}}{\prod_j \frac{z-b_j}{1-\overline{b_j}z}} g(z),$$

where g is analytic on $\mathbb{D}$ and has no zeros on $|z| \le 1$. Then

$$|f(0)| = \frac{\prod_j |a_j|}{\prod_k |b_k|} |g(0)|$$

and $\log|f(e^{it})| = \log|g(e^{it})| = \mathrm{Re} \log g(e^{it})$, where $\log g(z)$ is analytic on $\mathbb{D}$.

Note that if $z = e^{it}$ then $dz/(iz) = dt$. So by Cauchy's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{it})|dt = \mathrm{Re}\frac{1}{2\pi}\int_{-\pi}^{\pi} \log g(e^{it})dt$$
$$= \mathrm{Re}\log g(0) = \log|f(0)| - \sum_j \log|a_j| + \sum_k \log|b_k|.$$

Thus (11.11) holds if f has no zeros or poles on $|z| = R$.

We may suppose f has no poles in $|z| \leq R$ by Corollary 11.15.

Then $v_M = \max(-M, \log|f|)$ is subharmonic on Ω, for $M < \infty$.

If $r \leq R$ then using the Poisson integral formula, find u_M harmonic on $|z| < r$ with $u_M = v_M$ on $|z| = r$.

Then $v_M - u_M$ is subharmonic and ≤ 0 in $|z| < r$ by the maximum principle.

Thus for $0 < s < r$

$$\int_0^{2\pi} v_M(se^{it})dt \leq \int_0^{2\pi} u_M(se^{it})dt = 2\pi u_M(0) = \int_0^{2\pi} v_M(re^{it})dt.$$

Letting $M \to \infty$ shows that the left side of (11.11) is non-decreasing when f has no poles. But the right side of (11.11) is continuous in R. Because equality holds when f has no zeros on $|z| = R$ and the zeros of f are isolated, (11.11) follows. □

Corollary 11.18: *Suppose f is analytic in $\mathbb{D}$, $f \not\equiv 0$, with zeros $\{a_n\}$, and suppose*

$$\sup_{r<1} \int_{-\pi}^{\pi} \log |f(re^{it})| dt < \infty.$$

Then

$$\sum_{n=0}^{\infty} (1 - |a_n|) < \infty.$$

Proof. Since f has only finitely many zeros at 0, we may divide them out and suppose $f(0) \neq 0$.

Then by Jensen's formula

$$\sum_{n=0}^{\infty} \log \frac{1}{|a_n|} < \infty.$$

But $\lim_{x \to 1} \frac{\log x}{x-1} = 1$, so that $\sum \log \frac{1}{|a_n|} < \infty$ if and only if $\sum (1 - |a_n|) < \infty$. □

Theorem 11.19: *If* $\{a_n\} \subset \mathbb{D}$ *such that* $\sum(1 - |a_n|) < \infty$ *then*

$$B(z) = \prod_{n=0}^{\infty} \frac{|a_n|}{-a_n} \frac{z - a_n}{1 - \overline{a_n} z}$$

converges uniformly and absolutely on compact subsets of $\mathbb{D}$*, where we define the convergence factor* $|a_n|/(-a_n)$ *to be equal to* 1 *if* $a_n = 0$*. The function* B *is analytic on* $\mathbb{D}$*, is bounded by 1 and has zero set exactly equal to* $\{a_n\}$*.*

The function in the theorem is called a **Blaschke product**.

Proof. WLOG $a_n \neq 0$ for all n. Then for $|z| \leq r$

$$\left| \frac{|a_n|}{-a_n} \left(\frac{z - a_n}{1 - \overline{a_n} z} \right) - 1 \right| = \frac{(1 - |a_n|) \big| |a_n| z + a_n \big|}{|a_n (1 - \overline{a_n} z)|} \leq \frac{2(1 - |a_n|)}{(\inf |a_n|)(1 - r)}.$$

By Weierstrass's M-test and Lemma 11.8, B converges uniformly and absolutely on $|z| \leq r$.

Note that the partial products for B are all bounded by 1 and analytic on $\mathbb{D}$. $\square$

Theorem 11.20: *If f is bounded and analytic on $\mathbb{D}$ with zero set $\{a_n\}$ (counting multiplicity) and if B is the Blaschke product with zero set $\{a_n\}$ then*

$$f(z) = B(z)e^{g(z)}$$

for some analytic function g with

Proof. WLOG, $\sup_{\mathbb{D}} |f| = 1$. Write

$$B_N(z) = \prod_{n=1}^{N} \frac{|a_n|}{-a_n} \frac{z - a_n}{1 - \overline{a_n} z}.$$

By Schwarz's lemma $|f(z)/B_N(z)| \leq 1$. Choose r_n so that $B \neq 0$ on $|z| = r_n$.

Then f/B_N converges uniformly to f/B on $|z| = r_n$ and by the maximum principle the convergence is uniform on $|z| \leq r_n$.

Now let $r_n \to 1$. Thus $h = f/B$ is bounded by 1 on $\mathbb{D}$ and non-vanishing.

This implies $g = \log h$ can be defined as an analytic function on $\mathbb{D}$. By construction, $\sup_{\mathbb{D}} |h| \leq \sup_{\mathbb{D}} |f| = 1$, but also $|B| \leq 1$, so $\sup_{\mathbb{D}} |h| = \sup_{\mathbb{D}} |f|$. $\square$

It is difficult to overestimate the importance of Blaschke products in function theory on $\mathbb{D}$.

We shall not prove this, but $|B|$ has radial limits equal to 1 almost everywhere on the circle.

Blaschke products are a special case of inner functions: bounded holomorphic functions F on $\mathbb{D}$ so that $|F|$ has radial limit 1 almost everywhere on $\mathbb{T}$.

A bounded analytic function on $\mathbb{D}$ can be factored as $F = B \cdot D \cdot G$ where

$B =$ Blaschke product
$S =$ Inner function with no zeros in $\mathbb{D}$
$G =$ Outer function

Outer means $\log |G|$ is Poisson extension of radial limits of $\log |G|$ on $\mathbb{T}$.

$L^\infty(\mathbb{T})$ = bounded measurable functions on unit circle.

$H^\infty(\mathbb{T})$ = subalgebra of L^∞ whose Poisson extensions are holomorphic.

A **Douglas algebra** is a closed algebra of $L^\infty(\mathbb{T})$ that contains $H^\infty(\mathbb{T})$.

Chang-Marshall theorem: *Any Douglas algebra A is generated by H^∞ and the complex conjugates of some collection of Blaschke products.*

Marshall's theorem: *H^∞ is generated by the Blaschke products.*

Corollary: *The unit ball of H^∞ is the closed convex hull of the Blaschke products.*

Douglas-Rudin-Jones theorem: *If u is measurable on $\mathbb{T}$ and $|u| = 1$ almost everywhere, then for every $\epsilon > 0$ are there Blaschke products B_1 and B_2 so that $\|u - B_1/B_2\|_\infty < \epsilon$.*

Frostman's theorem: *Suppose f is an inner function, i.e., it is bounded and holomorphic on the unit disk and $|f|$ has radial boundary values 1 almost everywhere. Then*

$$\frac{f(z) - a}{1 - \overline{a} f(z)}$$

is a Blaschke product for almost every $a \in \mathbb{D}$ (actually, except for a set of zero logarithmic capacity).