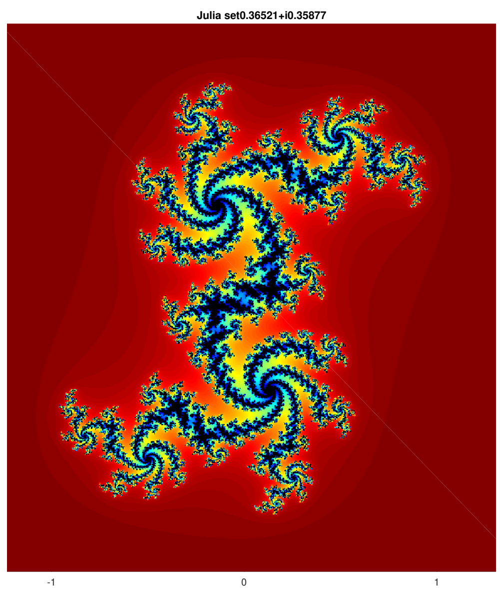
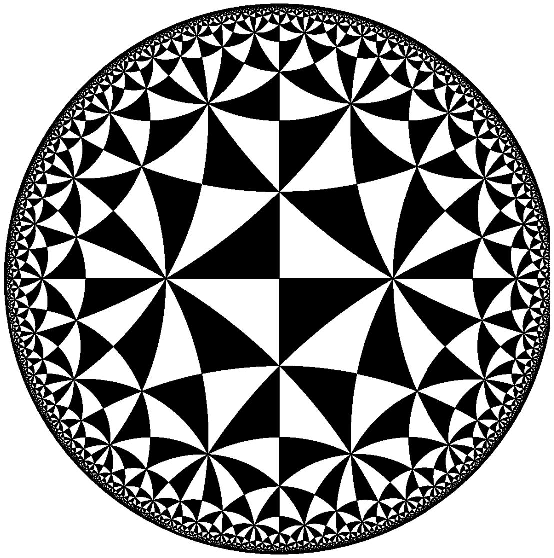
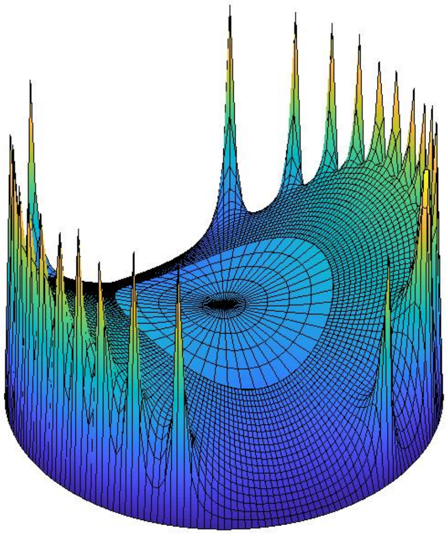
MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop
2024



Chapter 10: Normal Families

Section 10.1: Normality and Equicontinuity

Normality is about compactness of analytic maps: given a certain family, can we extract a convergent subsequence?

Compactness for families of functions is often given by Arzela-Ascoli theorem.

Normality theorems give conditions under which Arzela-Ascoli applies.

Definition 10.1: A collection, or family, $\mathcal{F}$ of continuous functions on a region $\Omega \subset \mathbb{C}$ is said to be **normal on Ω** provided every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of Ω.

- The family $\mathcal{F}_1 = \{f_c(z) = z + c : |c| < 1\}$ is normal in $\mathbb{C}$ but not countable.

- The family $\mathcal{F}_2 = \{z^n : n = 0, 1, \ldots\}$ is normal in $\mathbb{D}$ but the only limit function, the zero function, is not in $\mathcal{F}_2$.

- The sequence z^n converges uniformly on each compact subset of $\mathbb{D}$, but does not converge uniformly on $\mathbb{D}$.

- The family $\mathcal{F}_3 = \{g_n\}$, where $g_n \equiv 1$ if n is even and $g_n \equiv 0$ if n is odd, is normal but the sequence $\{g_n\}$ does not converge.

Lemma 10.2 *Suppose Ω is a region and suppose $\Omega = \cup_{j=1}^{\infty} \Delta_j$ where $\Delta_j \subset \Omega$ are closed disks. A family of continuous functions $\mathcal{F}$ is normal on Ω if and only if for each j, every sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on Δ_j.*

Proof. The only if part follows by definition.

Suppose that for each j, every sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on Δ_j. Suppose $\{f_n\} \subset \mathcal{F}$. Then there is a subsequence $\{f_n^{(1)}\} \subset \{f_n\}$ such that $\{f_n^{(1)}\}$ converges uniformly on Δ_1.

Likewise there is a sequence $\{f_n^{(2)}\} \subset \{f_n^{(1)}\}$ which converges uniformly on Δ_2, and indeed there is a sequence $\{f_n^{(k)}\} \subset \{f_n^{(k-1)}\}$ which converges uniformly on Δ_k.

Then the "diagonal" sequence

$$\{f_k^{(k)}\}$$

converges uniformly on each Δ_j, since it is a subsequence of $\{f_n^{(k)}\}$ for each k. A compact set $K \subset \Omega$ can be covered by finitely many Δ_j, so the diagonal sequence converges uniformly on K. $\square$

Define a metric on the space $C(\Omega)$ of continuous functions on a region Ω as follows. Write $\Omega = \cup_{j=1}^{\infty} \Delta_j$, where each Δ_j is a closed disk with $\Delta_j \subset \Omega$. If $f, g \in C(\Omega)$, set

$$\rho_j(f,g) = \sup_{z \in \Delta_j} \frac{|f(z) - g(z)|}{1 + |f(z) - g(z)|}, \qquad \rho(f,g) = \sum_{j=1}^{\infty} 2^{-j} \rho_j(f,g).$$

Then ρ is a metric:

(1) If $\rho(f, g) = 0$, then $f = g$ on each Δ_j and hence $f = g$ on Ω.

(2) $\rho(f, g) = \rho(g, f)$ for $f, g \in C(\Omega)$,

(3) $\rho(f, g) \leq \rho(f, h) + \rho(h, g)$ for $f, g, h \in C(\Omega)$.

The triangle inequality follows from the observations that $\frac{x}{1+x} = 1 - \frac{1}{1+x}$ is increasing for $x \geq 0$ and if a and b are non-negative numbers then

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Proposition 10.3: *A sequence* $\{f_n\} \subset C(\Omega)$ *converges uniformly on compact subsets of* Ω *to* $f \in C(\Omega)$ *if and only if*

$$\lim_n \rho(f_n, f) = 0.$$

- In other words, the space $C(\Omega)$ with the topology of uniform convergence on compact subsets is a metric space.

- Compactness and sequential compactness are the same for metric spaces.

- A family is normal if and only if its closure is compact in this topology.

Proof. If $\rho(f_n, f) \to 0$, then $\rho_j(f_n, f) \to 0$ for each j. This implies f_n converges uniformly to f on Δ_j for each j, because $a/(1+a) < b$ is the same as $a < b/(1-b)$, for $a > 0$ and $0 < b < 1$.

Since each compact set $K \subset \Omega$ can be covered by finitely many Δ_j, the sequence f_n converges uniformly on K. Conversely if f_n converges to f uniformly on each Δ_j, then given $\epsilon > 0$, choose n_j so that for $z \in \Delta_j$

$$|f_n(z) - f(z)| < \frac{\epsilon}{2}, \text{ whenever } n \geq n_j,$$

for $j = 1, 2, 3, \ldots$.

Choose m so that

$$\sum_{j=m}^{\infty} 2^{-j} < \frac{\epsilon}{2}.$$

Then for $n \geq \max\{n_1, \dots n_{m-1}\}$

$$\rho(f_n, f) \leq \sum_{j=1}^{m-1} 2^{-j} \frac{\epsilon}{2} + \sum_{j=m}^{\infty} 2^{-j} < \epsilon. \quad \square$$

Definition 10.4: A family of functions $\mathcal{F}$ defined on a set $E \subset \mathbb{C}$ is

(1) **equicontinuous at $\boldsymbol{w \in E}$** if for each $\epsilon > 0$ there exist a $\delta > 0$ so that if $z \in E$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$.

(2) **equicontinuous on $\boldsymbol{E}$** if it is equicontinuous at each $w \in E$.

(3) **uniformly equicontinuous on $\boldsymbol{E}$** if for each $\epsilon > 0$ there exists a $\delta > 0$ so that if $z, w \in E$ with $|z - w| < \delta$ then $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$.

Theorem 10.5, Arzela-Ascoli Theorem: *A family $\mathcal{F}$ of continuous functions is normal on a region $\Omega \subset \mathbb{C}$ if and only if*

(1) *$\mathcal{F}$ is equicontinuous on Ω, and*

(2) *there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.*

This result is proven usually proven in MAT 532 (in Chapter 4 of Folland's book). If you have not had this class, refer to the proof in Marshall's textbook.

We could similarly consider families of continuous functions with values in a complete metric space. The Arzela-Ascoli theorem holds in this context with the same proof, replacing $|f(z) - f(w)|$ with the metric distance between $f(z)$ and $f(w)$.

For example we could consider continuous functions with values in the Riemann sphere using the chordal distance between any two points on the sphere.

Equivalently, we can consider functions with values in the extended plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ with the chordal metric

$$\chi(\alpha,\beta) = \begin{cases} \frac{2|\alpha-\beta|}{\sqrt{1+|\alpha|^2}\sqrt{1+|\beta|^2}} & \text{if } \alpha, \beta \in \mathbb{C} \\ \frac{2}{\sqrt{1+|\alpha|^2}} & \text{if } \beta = \infty. \end{cases}$$

Then to say that a function f is "continuous" at z_0 with $f(z_0) = \infty$ means that for all $R < \infty$ there is a $\delta > 0$ so that $|f(z)| > R$ for all z with $|z - z_0| < \delta$.

Definition 10.6: A family $\mathcal{F}$ of continuous functions is said to be **locally bounded** on Ω if for each $w \in \Omega$ there is a $\delta > 0$ and $M < \infty$ so that if $|z - w| < \delta$ then $|f(z)| \leq M$ for all $f \in \mathcal{F}$.

Theorem 10.7: *The following are equivalent for a family $\mathcal{F}$ of analytic functions on a region Ω.*

(1) *$\mathcal{F}$ is normal on Ω.*

(2) *$\mathcal{F}$ is locally bounded on Ω.*

(3) *$\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is locally bounded on Ω and there is a $z_0 \in \Omega$ so that $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.*

Proof. Suppose $\mathcal{F}$ is normal. If $w \in \Omega$ then $|f(z) - f(w)| < 1$ for z in a sufficiently small ball centered at w, for all $f \in \mathcal{F}$, because $\mathcal{F}$ is equicontinuous at w. Thus $\mathcal{F}$ is locally bounded.

Now suppose $\mathcal{F}$ is locally bounded on Ω. If $|f| \leq M$ on a closed disk $\overline{B}(z_1, r) \subset \Omega$ centered at z_1 with radius $r > 0$, then by Cauchy's estimate $|f'(z)| \leq 4M/r^2$ on $B(z_1, r/2)$. It follows that $\mathcal{F}'$ is locally bounded, and (3) holds.

Finally, if (3) holds and $z_1 \in \Omega$, then $|f'(z)| \leq L < \infty$ for z in a disk Δ centered at z_1. Integrating f' along a line segment from z_1 to $z \in \Delta$, we have $|f(z) - f(z_1)| \leq L|z - z_1|$ for all $f \in \mathcal{F}$. So $\mathcal{F}$ is equicontinuous at z_1. By the Arzela-Ascoli theorem, $\mathcal{F}$ is normal. $\square$

Lemma 10.8: *If $\{f_n\}$ is a sequence of meromorphic functions which converges uniformly on compact subsets of a region $\Omega \subset \mathbb{C}$ in the chordal metric then the limit function is either meromorphic on Ω or identically equal to ∞.*

Proof. Note that

$$\chi(\alpha, \beta) = \chi(\frac{1}{\alpha}, \frac{1}{\beta}). \tag{1.7}$$

If $\{f_n\}$ is a sequence of meromorphic functions which converges uniformly on compact subsets of Ω in the χ metric then the limit function f is continuous as a map into the extended plane $\mathbb{C}^*$.

If $|f(z_0)| < \infty$ then f is bounded in a neighborhood of z_0, and hence f_n converges to f uniformly in the Euclidean metric in a neighborhood of z_0. By Weierstrass's theorem, f is analytic in a neighborhood of z_0.

If $f(z_0) = \infty$, then $1/f_n$ is bounded in a neighborhood of z_0, for n sufficiently large, and hence extends to be analytic by Riemann's removable singularity theorem. By (1.7) and Weierstrass's theorem, $1/f$ is analytic in a neighborhood of z_0. By the identity theorem, if $1/f$ is not identically 0 in a neighborhood of z_0, then the zero of $1/f$ at z_0 is isolated and hence f has an isolated pole at z_0.

The set of non-isolated zeros of $1/f$ is then both open and closed in Ω. Because Ω is connected, either f is identically equal to ∞ or f is analytic except for isolated poles. This proves Lemma 1.8. $\square$

Definition 10.9: If f is meromorphic on a region $\Omega \subset \mathbb{C}$ then

$$f^{\#}(z) = \lim_{w \to z} \frac{\chi(f(z), f(w))}{|z - w|}$$

is called the **spherical derivative** of f.

If z is not a pole of f then

$$f^{\#}(z) = \frac{2|f'(z)|}{1 + |f(z)|^2}.$$

By (10.7), $(1/f)^{\#} = f^{\#}$ so that $f^{\#}(z)$ is finite and continuous at each $z \in \Omega$.

The **spherical distance** $d(p^*, q^*)$ between $p^*, q^* \in \mathbb{S}^2$ is the arc-length of the shortest curve on the sphere containing p^* and q^*. The quantity $f^\#$ is called the spherical derivative because

$$f^\#(z) = \lim_{w \to z} \frac{d(f(z)^*, f(w)^*)}{|z - w|},$$

where $f(z)^*$ denotes the stereographic projection of $f(z)$.

Theorem 10.10, Marty's Theorem: *A family $\mathcal{F}$ of meromorphic functions on a region $\Omega \subset \mathbb{C}$ is normal in the chordal metric if and only if $\mathcal{F}^{\#} = \{f^{\#} : f \in \mathcal{F}\}$ is locally bounded.*

Proof. Suppose $\mathcal{F}$ is normal in the chordal metric on Ω and suppose the spherical derivatives are not bounded in any neighborhood of z_0.

Then there is a sequence $f_n \in \mathcal{F}$ and $z_n \to z_0$ so that $f_n^{\#}(z_n) \to \infty$. Taking a subsequence, we may suppose that f_n converges uniformly on compact subsets of Ω in the chordal metric to a meromorphic function f, or to ∞, by normality and Lemma 10.8

If $f(z_0) \neq \infty$ then f is bounded in the Euclidean metric in a neighborhood N_{z_0} of z_0. Because f_n converges to f in the chordal metric, f_n must also be bounded in N_{z_0} and thus f_n converges to f in the Euclidean metric on N_{z_0}. By Weierstrass's theorem, f_n' converges to f' uniformly on compact subsets of N_{z_0} and thus $f_n^{\#}$ converges uniformly on compact subsets of N_{z_0} to $f^{\#}$.

This contradicts the unboundedness of $f_n^{\#}(z_n)$. If $f(z_0) = \infty$, then we can apply this same argument to $1/f$ and $1/f_n$ using (1.7).

Conversely suppose the spherical derivatives are bounded by M in a disk Δ centered at z_0. If $z, w \in \Delta$, let $z_j = z + (j/n)(w - z), 0 \le j \le n$.

Then for n large,

$$\chi(f(z), f(w)) \le \sum_{j=1}^{n} \chi(f(z_j), f(z_{j-1})) \approx \sum_{j=1}^{n} f^{\#}(z_j)|z_j - z_{j-1}| \le M|z - w|.$$

Thus $\mathcal{F}$ is equicontinuous, with the chordal metric, on Δ. By the version of the Arzela-Ascoli theorem with the chordal metric, $\mathcal{F}$ is normal on Δ, and by the chordal version of Lemma 10.2, normal on Ω. $\square$

Section 8.2: Hurwitz's Theorem

Theorem 8.8, Hurwitz: *Suppose* $\{g_n\}_{n=1}^{\infty}$ *is a sequence of analytic functions on a region* Ω *and suppose* $g_n(z) \neq 0$ *for all* $z \in \Omega$ *and all* n*. If* g_n *converges uniformly to* g *on compact subsets of* Ω*, then either* g *is identically zero in* Ω *or* g *is never equal to* 0 *in* Ω*.*

Proof. The limit function g is analytic on Ω by Weierstrass's theorem. In particular, if g is not identically zero, then the zeros of g are isolated in Ω.

Moreover, again by Weierstrass's theorem, g_n' converges to g' uniformly on compact subsets of Ω. If $\Delta \subset \Omega$ is a closed disk with $g \neq 0$ on $\partial\Delta$, then g_n'/g_n converges to g'/g uniformly on $\partial\Delta$.

By the argument principle, for n sufficiently large, the number of zeros of g_n in Δ is the same as the number of zeros of g in Δ. Because g_n is never zero, the theorem follows. $\square$

Corollary 8.9, Hurwitz's Theorem: *If* $\{g_n\}_1^\infty$ *is a sequence of one-to-one and analytic functions on a region* Ω *and if* g_n *converges to* g *uniformly on compact subsets of* Ω *then either* g *is one-to-one and analytic on* Ω *or* g *is constant.*

Proof. Fix $w \in \Omega$ and apply Theorem 8.8 to $g - g(w)$ on $\Omega \setminus \{w\}$. □

Section 10.1: Riemann Mapping Theorem

Theorem 10.11, Riemann Mapping Theorem *Suppose* $\Omega \subset \mathbb{C}$ *is simply-connected and* $\Omega \neq \mathbb{C}$*. Then there exists a one-to-one analytic map* f *of* Ω *onto* $\mathbb{D} = \{z : |z| < 1\}$*. If* $z_0 \in \Omega$ *then there is a unique such map with* $f(z_0) = 0$ *and* $f'(z_0) > 0$*.*

Idea of proof:

- Show there is a conformal map of Ω into $\mathbb{D}$ so that $f(z_0) = 0$ and $f'(z_0) > 0$.
- Among all such maps, choose one maximizing $f'(z_0)$. (uses normality)
- Prove this map is 1-1 and onto $\mathbb{D}$.

Proof. Fix $z_0 \in \Omega$. Set

$$\mathcal{F} = \{f : f \text{ is one-to-one, analytic, } |f| < 1 \text{ on } \Omega, f(z_0) = 0, f'(z_0) > 0\}.$$

If Ω is bounded, there is a linear map taking Ω to $\mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) > 0$. (shrink Ω to diameter $< 1/2$, then translate and rotated as needed).

If Ω is unbounded, translate it so it does not contain zero. The $\sqrt{z}$ is defined on Ω, and if $D \subset \Omega$ is a disk, then $W = \sqrt{D}$ is an open set in the image, and $-W$ is an open set in the complement. Choosing $a \in W$ and $\delta = \text{dist}(a, \partial W)$, we then have $|1/(z+a)| < 1/\delta$ on W. Thus Ω can be conformally mapped to a bounded domain.

We are now back in the previous case.

By Theorem 10.7, $\mathcal{F}$ is normal. Let $\{f_n\} \subset \mathcal{F}$ such that

$$\lim_{n\to\infty} f_n'(z_0) = M = \sup\{f'(z_0) : f \in \mathcal{F}\}. \tag{10.8}$$

Replacing f_n with a subsequence, we may suppose that f_n converges uniformly on compact subsets of Ω.

By Weierstrass's theorem, the limit function f is analytic and $\{f_n'\}$ converges to f'. Thus $f'(z_0) = M$. In particular f is not constant, and $M < \infty$.

By Hurwitz's Theorem (Corollary 8.9), f is one-to-one. Also $f(z_0) = \lim f_n(z_0) = 0$, so that $f \in \mathcal{F}$.

Next we show that f must map Ω onto $\mathbb{D}$.

Suppose there is a point $\zeta_0 \in \mathbb{D}$ such that $f \neq \zeta_0$ on Ω. Then

$$g_1(z) = \frac{f(z) - \zeta_0}{1 - \overline{\zeta_0} f(z)} \equiv T_1 \circ f(z)$$

is a non-vanishing function on the simply-connected region Ω and hence it has an analytic square root which is, again, one-to-one.

Set $g_2(z) = \sqrt{g_1(z)}$ and

$$g(z) = \frac{g_2(z) - g_2(z_0)}{1 - \overline{g_2(z_0)} g_2(z)} \equiv T_2 \circ g_2(z).$$

Then $\lambda g(z) \in \mathcal{F}$, where $\lambda = |g'(z_0)|/g'(z_0)$. Because T_1, T_2 are LFTs of $\mathbb{D} \to \mathbb{D}$,

$$\varphi = T_1^{-1} \circ S \circ T_2^{-1},$$

where $S(z) = z^2$, is a two-to-one analytic map (counting multiplicity) of $\mathbb{D}$ onto $\mathbb{D}$ with $\varphi(0) = 0$, and

$$f(z) = \varphi \circ g(z).$$

By Schwarz's lemma (or direct computation) $|\varphi'(0)| < 1$, so that

$$f'(z_0) = |f'(z_0)| = |\varphi'(0)||g'(z_0)| < |g'(z_0)| = \lambda g'(z_0),$$

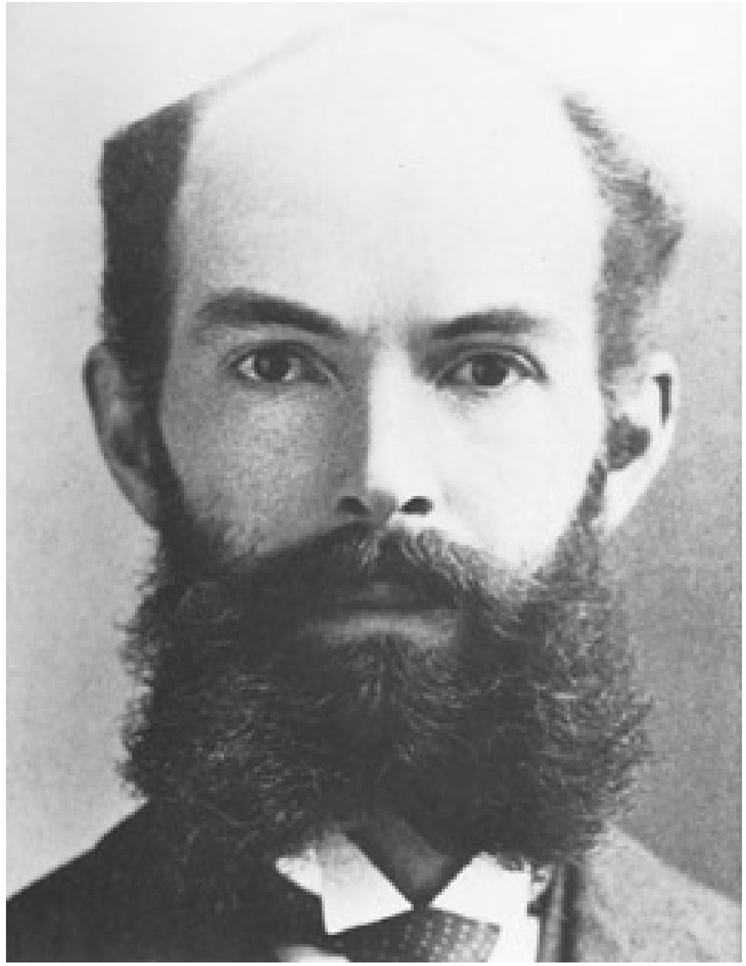
contradicting the maximality of $f'(z_0)$, and hence f maps $\mathbb{D}$ onto $\mathbb{D}$.

To prove uniqueness, suppose f, g are two conformal maps $\mathbb{D} \to \Omega$ with $f(0) = g(0) = z_0$ and $f'(0)g'(0) > 0$.

Then $h = g^{-1} \circ f$ is a 1-1, onto analytic map from $\mathbb{D}$ to itself with $h(0) = 0$ and $h'(0) = 1$. By the Schwarz lemma, h is the identity so $f = g$. $\square$

Georg Friedrich Bernhard Riemann

Stated RMT in 1851



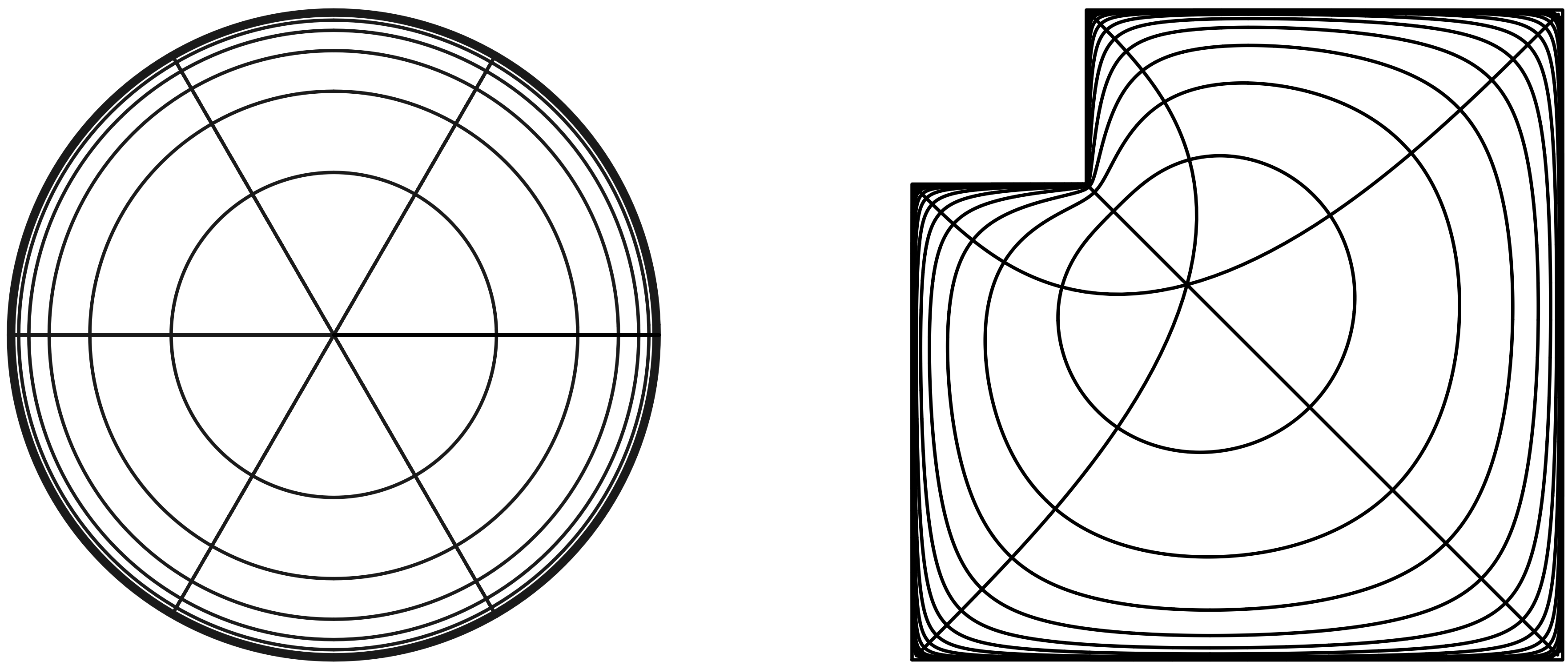
William Fogg Osgood

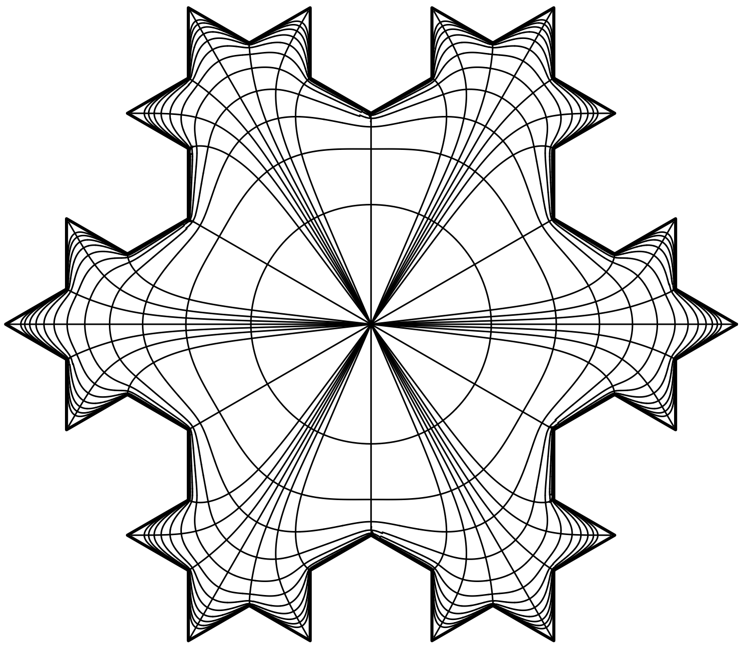
First proof of RMT, Trans. AMS, vol. 1, 1900

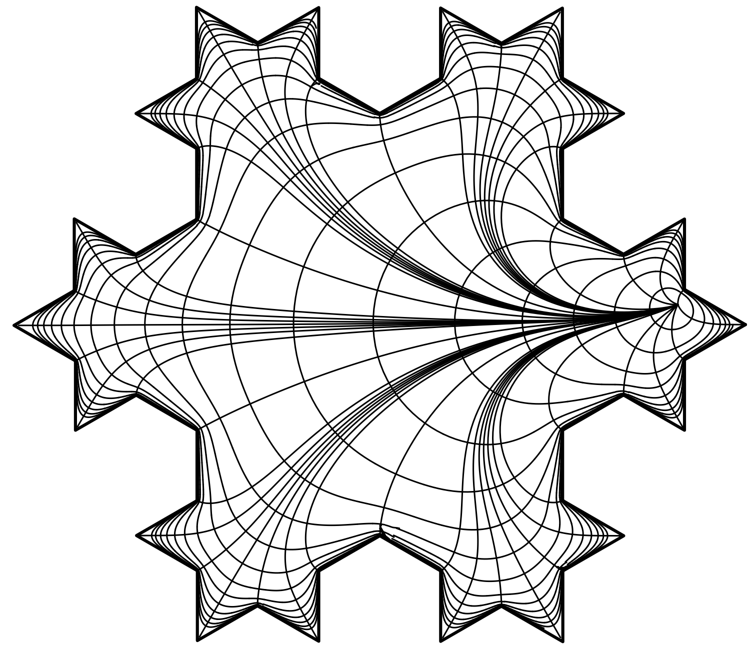
Harvard 1866, Math Faculty 1890-1933, Chair 1918-22

The proof of Osgood represented, in my opinion, the "coming of age" of mathematics in America. Until then, numerous American mathematicians had gone to Europe for their doctorates, or for other advanced study, as indeed did Osgood. But the mathematical productivity in this country in quality lagged behind that of Europe, and no American before 1900 had reached the heights that Osgood then reached.

J.L. Walsh, "History of the Riemann mapping theorem", Amer. Math. Monthly, 1973.







The compactness proof of Riemann's theorem seems non-constructive, but it does describe an algorithm (Koebe's method):

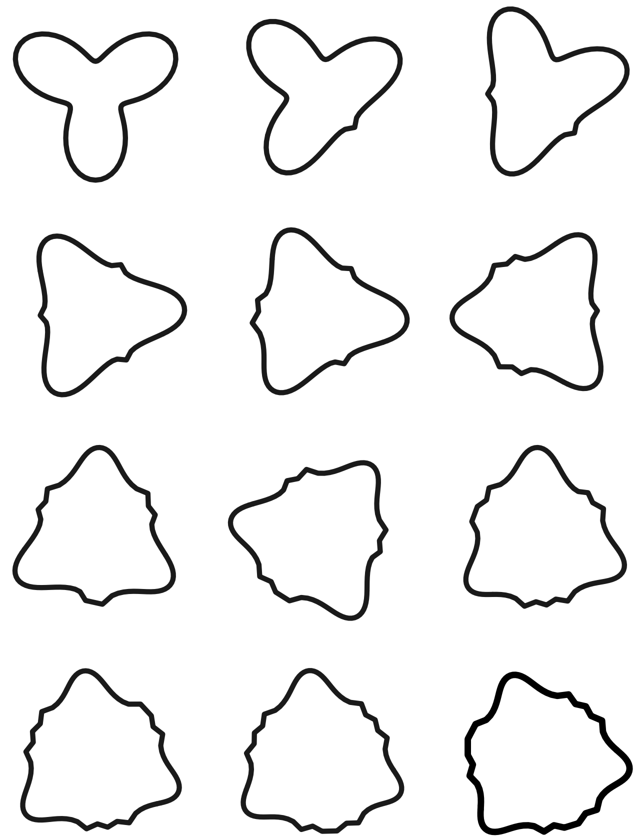
(1) Find a linear map $f : \Omega \to \Omega_0 \subset \mathbb{D}$ with z_0 mapping to 0.

(2) Assuming Ω_n has been defined, find point w on $\partial\Omega_n$ closest to 0.

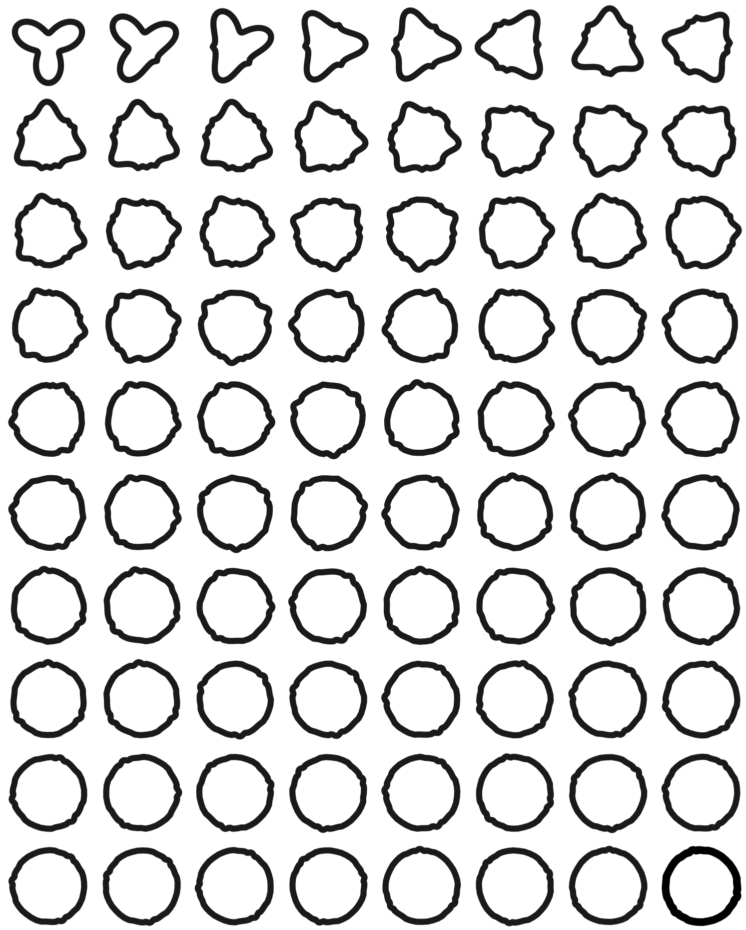
(3) Choose LFTs $\tau, \sigma : \mathbb{D} \to \mathbb{D}$ so that $\tau(w) = 0$ and $\sigma(\sqrt{\tau(f(z_0))}) = 0$.

(4) Let $\Omega_{n+1} = \sigma(\sqrt{\tau(\Omega_n)})$.

(5) Repeat steps until point w is within specified distance of unit circle.



On the top left is a subdomain of the disk whose boundary is parameterized by $\gamma(t) = e^{it}\frac{1}{3}(3+\sin(t)))$. This is a polygon with 100 vertices defined by the points $t = k/100$, $k = 1, \ldots, 100$. The next 11 figures show the first 11 iterations of Koebe's method. The next figure show more iterations.



This shows the first 80 iterations of Koebe's method for the same domain.

Definition: We say that a region Ω is **symmetric about** $\mathbb{R}$ provided $\overline{z} \in \Omega$ if and only if $z \in \Omega$. If Ω is symmetric about $\mathbb{R}$ and if f is analytic on Ω then $\overline{f(\overline{z})}$ is analytic on Ω.

Proposition 8.14: *If f is a conformal map of a simply-connected symmetric region Ω onto $\mathbb{D}$ such that $f(x_0) = 0$ and $f'(x_0) > 0$ for some $x_0 \in \mathbb{R} \cap \Omega$ then $f(z) = \overline{f(\overline{z})}$.*

Proof. Apply the uniqueness conclusion in the Riemann mapping theorem to the functions f and $\overline{f(\overline{z})}$. □

8.15 Theorem, Schwarz Reflection Principle : *Suppose* Ω *is a region which is symmetric about* $\mathbb{R}$. *Set* $\Omega^+ = \Omega \cap \mathbb{H}$ *and* $\Omega^- = \Omega \cap (\mathbb{C} \setminus \overline{\mathbb{H}})$. *If* v *is harmonic on* Ω^+, *continuous on* $\Omega^+ \cup (\Omega \cap \mathbb{R})$ *and equal to* 0 *on* $\Omega \cap \mathbb{R}$ *then the function defined by*

$$V(z) = \begin{cases} v(z) \ \textit{for } z \in \Omega \setminus \Omega^- \\ -v(\overline{z}) \ \textit{for } z \in \Omega^- \end{cases}$$

is harmonic on Ω. *If also* $v(z) = \mathrm{Im}\ \mathrm{f(z)}$ *where* f *is analytic on* $\mathbb{H} \cap \Omega$ *then the function*

$$g(z) = \begin{cases} f(z) \ \textit{for } z \in \Omega^+ \\ \overline{f(\overline{z})} \ \textit{for } z \in \Omega^- \end{cases}$$

extends to be analytic in Ω.

Proof. The extended function V is continuous on Ω.

To prove the first claim, we need only prove that V has the mean-value property for small circles centered on $\Omega \cap \mathbb{R}$. But for $x_0 \in \mathbb{R} \cap \Omega$, $V(x_0 + re^{it}) = -V(x_0 + re^{-it})$ so that the mean-value over a circle centered at x_0 contained in Ω is zero, the value of V at x_0. Thus V is harmonic in Ω.

Suppose now that $v = \text{Im f}$ where f is analytic on $\mathbb{H} \cap \Omega$. If D is a disk contained in Ω and centered on $\mathbb{R}$, then $V = \text{Im h}$ for some analytic function h on D. It is uniquely defined by requiring that $h = f$ on $\Omega^+ \cap D$. Since $f(z) = \overline{f(\overline{z})}$ $h(z) = \overline{h(\overline{z})}$ on D, and so $h = g$ on $D \cap \Omega \setminus \mathbb{R}$. Thus h provides the unique analytic extension of g to all of D, and g extends to be analytic on Ω. $\square$

Corollary 8.16: *If the function f in the Schwarz reflection principle is also one-to-one in Ω^+ with* $\mathrm{Im\ f} > 0$ *on Ω^+, then its extension g is also one-to-one on Ω.*

Proof. By definition g is one-to-one on Ω^- with $\mathrm{Im\ g} < 0$ on Ω^-. So if $g(z_1) = g(z_2)$ then $z_1, z_2 \in \mathbb{R}$. Since g is open, it maps a small disk centered at z_j, $j = 1, 2$, onto a neighborhood of $g(z_1) = g(z_2)$.

If $z_1 \neq z_2$ then there are two points $\zeta_1, \zeta_2 \in \Omega^+$ near z_1, z_2 (respectively) with $g(\zeta_1) = g(\zeta_2)$, contradicting the assumption that f is one-to-one on Ω^+. $\square$

Theorem 8.19, Schwarz-Christoffel: *Suppose* Ω *is a bounded simply-connected region whose positively oriented boundary* $\partial\Omega$ *is a polygon with vertices* $v_1, ..., v_n$*. Suppose the tangent direction on* $\partial\Omega$ *increases by* $\pi\alpha_j$ *at* v_j*,* $-1 < \alpha_j < 1$*. Then there exists* $x_1 < x_2 < \cdots < x_n$ *and constants* c_1, c_2 *so that*

$$f(z) = c_1 \int_{\gamma_z} \prod_{j=1}^{n} (\zeta - x_j)^{-\alpha_j} d\zeta + c_2$$

is a conformal map of $\mathbb{H}$ *onto* Ω*, where the integral is along any curve* γ_z *in* $\mathbb{H}$ *from* i *to* z*.*

Elwin Bruno Christtoffel

Hermann Amandus Schwarz

- The exponents $\{\alpha_j\}$ are known from the target polygon, but the $\{x_j\}$ are not.

- The points are the preimages of the vertices under the conformal map.

- Finding these points numerically is challenging: there are several heuristics that work in practice, but are not proven to work, e.g., SC-Toolbox program by T. Driscoll.

- A provably correct algorithm is given in the paper Conformal mapping in linear time and explained in the recorded lecture Fast conformal mapping via computational and hyperbolic geometry.

Section 10.1: Zalcman, Montel and Picard

Theorem 10.12, Zalcman's Lemma *A family $\mathcal{F}$ of meromorphic functions on a region Ω is not normal in the chordal metric if and only if there exists a sequence $\{z_n\}$ converging to $z_\infty \in \Omega$, a sequence of positive numbers ρ_n converging to 0, and a sequence $\{f_n\} \subset \mathcal{F}$ such that*

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta) \tag{10.9}$$

converges uniformly in the chordal metric on compact subsets of $\mathbb{C}$ to a nonconstant function g which is meromorphic in all of $\mathbb{C}$. Moreover, if $\mathcal{F}$ is not normal then $\{z_n\}$ and $\{\rho_n\}$ can be chosen so that

$$g^\#(\zeta) \leq g^\#(0) = 1, \tag{10.10}$$

for all $\zeta \in \mathbb{C}$.

Larry Zalcman.

For example the family $\mathcal{F} = \{z^n\}$ is not normal on $2\mathbb{D} = \{z : |z| < 2\}$.

Set $z_n = 1$ and $\rho_n = 1/n$. Then for $\zeta \in \mathbb{C}$

$$f_n(z_n + \rho_n\zeta) = (1 + \frac{\zeta}{n})^n \to e^\zeta,$$

as $n \to \infty$, because

$$\frac{n\log(1+\zeta/n)}{\zeta} = \frac{\log(1+\zeta/n) - 0}{\zeta/n}$$

is the difference quotient for the derivative of $\log(1 + z)$ at $z = 0$.

Proof. One direction of Zalcman's Lemma is easy.

If $\mathcal{F}$ is normal then any sequence $\{f_n\}$ contains a convergent subsequence, which we can relabel as $\{f_n\}$, and call the limit function f. If $z_n \to z_\infty \in \Omega$ and $\rho_n \to 0$, then

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta) \to f(z_\infty),$$

for each $\zeta \in \mathbb{C}$, as $n \to \infty$ since the family $\{f_n\}$ is uniformly equicontinuous in a neighborhood of z_∞. Thus the limit of g_n is constant.

Conversely, suppose $\mathcal{F}$ is not normal. By Marty's theorem, there exists $w_n \to w_\infty \in \Omega$ and $f_n \in \mathcal{F}$ such that the spherical derivatives satisfy

$$f_n^{\#}(w_n) \to \infty.$$

WLOG we may suppose $w_\infty = 0$ and $\{|z| \le r\} \subset \Omega$. Then

$$M_n = \max_{|z|\le r}(r - |z|) f_n^{\#}(z) = (r - |z_n|) f_n^{\#}(z_n),$$

for some $|z_n| < r$, since $f_n^{\#}$ is continuous.

Also, $M_n \to \infty$, since $w_n \to 0$. Then

$$g_n(\zeta) = f_n(z_n + \zeta/f_n^{\#}(z_n)),$$

is defined on $\{|\zeta| \le M_n\}$ because

$$|z_n + \zeta/f_n^{\#}(z_n)| \le |z_n| + M_n/f_n^{\#}(z_n) = |z_n| + r - |z_n| = r.$$

Fix $\zeta \in \mathbb{C}$. Then for $|\zeta| < M_n$,

$$
\begin{aligned}
g_n^{\#}(\zeta) &= \frac{f_n^{\#}(z_n + \zeta / f_n^{\#}(z_n))}{f_n^{\#}(z_n)} \\
&= \left(f_n^{\#}(z_n + \zeta / f_n^{\#}(z_n))\right) \left(\frac{1}{f_n^{\#}(z_n)}\right) \\
&\leq \left(\frac{M_n}{r - |z_n + \zeta / f_n^{\#}(z_n)|}\right) \left(\frac{r - |z_n|}{M_n}\right) \\
&\leq \frac{r - |z_n|}{r - |z_n| - |\zeta| / f_n^{\#}(z_n)} \\
&\leq \frac{1}{1 - |\zeta| / ((r - |z_n|) f_n^{\#}(z_n))} \\
&= \frac{1}{1 - |\zeta| / M_n} \to 1,
\end{aligned}
$$

as $n \to \infty$, since $M_n \to \infty$.

By Marty's Theorem 10.10, the family $\{g_n\}$ contains a convergent subsequence in the chordal metric.

Relabeling the subsequence, (10.9) holds with $\rho_n = 1/f_n^{\#}(z_n)$:

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta) \tag{10.9}$$

By calculation on previous slide, the limit function g satisfies the inequality and equality in (10.10)

$$g^{\#}(\zeta) \leq g^{\#}(0) = 1, \tag{10.10}$$

and is meromorphic by Lemma 1.8.

Because $g^{\#}(0) = 1$, g is nonconstant. Because $\{z_n\}$ is contained in a compact subset of Ω, we can arrange that $z_n \to z_\infty \in \Omega$ by taking subsequences. □

Theorem 10.13, Montel's Theorem: *A family $\mathcal{F}$ of meromorphic functions on a region Ω that omits three distinct fixed values $a, b, c \in \mathbb{C}^*$ is normal in the chordal metric.*

Proof. Normality is local by the chordal version of Lemma 10.2, so we may assume $\Omega = \mathbb{D}$. An LFT and its inverse are uniformly continuous in the chordal metric.

So we may suppose $a = 0$, $b = 1$ and $c = \infty$ by composing with an appropriate LFT. Without loss of generality, we may assume that $\mathcal{F}$ is the family of all analytic functions on $\mathbb{D}$ which omit the values 0 and 1.

Consider functions that omit zero and the 2^k-th roots of unity:

$$\mathcal{F}_m = \{f \text{ analytic on } \mathbb{D} : f \neq 0 \text{ and } f \neq e^{2\pi i k 2^{-m}}, k = 1, \ldots, 2^m\}.$$

Then

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots .$$

If $f \in \mathcal{F}_m$ then f is analytic and $f \neq 0$, so that we can define $f^{\frac{1}{2}}$ so as to be analytic. Moreover $f^{\frac{1}{2}} \in \mathcal{F}_{m+1}$.

If $\mathcal{F}$ is not normal then there exists a sequence $\{f_n\} \subset \mathcal{F}$ with no convergent subsequence. Moreover, $\{f_n^{\frac{1}{2}}\}$ is then a sequence in $\mathcal{F}_1$ with no convergent subsequence.

By induction, each $\mathcal{F}_m$ is not normal. Thus for each m we can construct a limit function h_m as in Zalcman's lemma. The functions h_m are entire by Exercise 3 and nonconstant since $h_m^{\#}(0) = 1$.

By (10.10) and Marty's theorem $\{h_m\}$ is a normal family. If h is a limit of a subsequence, uniformly on compact subsets of $\mathbb{C}$ in the chordal metric, then h is entire by Exercise 3 and nonconstant since $h^{\#}(0) = 1$.

By Hurwitz's theorem, h omits the 2^m roots of 1 for each m. These points are dense in the unit circle. Since $h(\mathbb{C})$ is connected and open, either $h(\mathbb{C}) \subset \mathbb{D}$ or $h(\mathbb{C}) \subset \mathbb{C} \setminus \mathbb{D}$. Thus either h or $1/h$ is bounded.

By Liouville's theorem, h must be constant, which contradicts $h^{\#}(0) = 1$, and hence $\mathcal{F}$ is normal. $\square$

Theorem 10.14, Picard's Great Theorem *If f is meromorphic in $\Omega = \{z : 0 < |z - z_0| < \delta\}$, and if f omits three (distinct) values in $\mathbb{C}^*$, then f extends to be meromorphic in $\Omega \cup \{z_0\}$.*

• An equivalent formulation of Picard's great theorem is that an analytic function omits at most one complex number in every neighborhood of an essential singularity.

• $f(z) = e^{1/z}$ does omit the values 0 and ∞ in every neighborhood of the essential singularity 0, so that Picard's theorem is the strongest possible statement.

• The weaker statement that a non-constant entire function can omit at most one complex number is usually called **Picard's little theorem**.

See Emile Picard

Proof. As before, we may assume $a = 0$, $b = 1$, $c = \infty$, and $z_0 = 0$. Let $\epsilon_n \to 0$.

If f omits 0 and 1 in a punctured neighborhood of 0, then by Montel's theorem the family $\{f(\epsilon_n z)\}$ is normal in the chordal metric on compact subsets of $\mathbb{C} \setminus \{0\}$.

Relabelling a subsequence, we may suppose $f(\epsilon_n z)$ converges uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$ to a function g analytic on $\mathbb{C} \setminus \{0\}$ or to $g \equiv \infty$ by Lemma 10.8 and Hurwitz's theorem.

Lemma 10.8: *If $\{f_n\}$ is a sequence of meromorphic functions which converges uniformly on compact subsets of $\Omega \subset \mathbb{C}$ in the chordal metric, then the limit function is either meromorphic on Ω or identically equal to ∞.*

If g is analytic, then $|g(z)| \leq M < \infty$ on $|z| = 1$ and hence $|f| \leq M + 1$ on $|z| = \epsilon_n$ for $n \geq n_0$. But then by the maximum principle, $|f| \leq M + 1$ on $\epsilon_{n+1} < |z| < \epsilon_n$ for $n \geq n_0$.

Thus $|f| \leq M+1$ on $0 < |z| < \epsilon_{n_0}$, so that by Riemann's theorem on removable singularities f is analytic in a neighborhood of 0.

If $g \equiv \infty$, we can apply a similar argument to $1/f(\epsilon_n z)$ to conclude that $1/f$ is analytic at 0 and hence f is meromorphic in a neighborhood of 0. $\square$

Theorem: *There is a holomorphic covering map from* $\mathbb{D}$ *to* $\mathbb{C} \setminus \{0, 1\}$

Proof. Let

$$\Omega = \{z = x + iy : y > 0, 0 < x < 1, |z - \frac{1}{2}| > \frac{1}{2}\} \subset \mathbb{H}.$$

This is simply connected and hence can be conformally mapped to $\mathbb{H}$ with $0, 1, \infty$ each fixed. We can then use Schwarz reflection to extend the map across the sides of Ω. Every such reflection of Ω stays in $\mathbb{H}$ maps to either the lower or upper half-planes. Continuing this forever gives a covering map from a simply connected subdomain U of $\mathbb{H}$ to W. Since U is simply connected and not the whole plane (it is a subset of $\mathbb{H}$) it is conformally equivalent to $\mathbb{D}$ and hence a covering $q : \mathbb{D} \to W$ exists. $\square$

Picard's little theorem, 2nd proof: *If f is a non-constant entire function, then $E = \mathbb{C} \setminus f(\mathbb{C})$ contains at most one point.*

Proof. If E contains two points $\{a, b\}$, then using the theory of covering surfaces (from Top/Geo core course) the covering map $p : \mathbb{D} \to \mathbb{C} \setminus \{a, b\}$, f can be lifted to a holomorphic map $f : \mathbb{C} \to \mathbb{D}$. By Liouville's theorem, the lift is constant and hence so must f. $\square$

Zalcman's Lemma has been used in many situations with the heuristic "Bloch's Principle" that if $\mathcal{P}$ is a property of meromorphic functions then

$$\{f \text{ meromorphic on } \Omega : f \text{ has } \mathcal{P}\}$$

is normal on Ω if and only if no non-constant meromorphic function on $\mathbb{C}$ has $\mathcal{P}$. For example, $\mathcal{P}$ could be "omits three values".

For a precise statement, see Bloch's principle by W. Bergweiler.

See André Bloch. He did much of his work in psychiatric hospital, where he was committed after murdering his brother, aunt and uncle, stabbing them at a family dinner in 1917.

The Bloch space is also named for him. These are holomorphic maps $\mathbb{D} \to \mathbb{C}$ that are Lipschitz from the hyperbolic to the Euclidean metric.

Normal families can be used to prove results like:

Theorem 10.15, Koebe: *There is a* $K > 0$ *so that if* f *is analytic and one-to-one on* $\mathbb{D}$ *with* $f(0) = 0$ *and* $f'(0) = 1$*, then* $f(\mathbb{D}) \supset \{z : |z| < K\}$.

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Theorem 10.16, Landau: *There is a constant* $L > 0$ *so that if* f *is analytic on* $\mathbb{D}$ *with* $f'(0) = 1$ *then* $f(\mathbb{D})$ *contains a disk of radius* L.

Theorem 1.17, Bloch: *There is a constant* $B > 0$ *so that if* f *is analytic on* $\mathbb{D}$ *with* $f'(0) = 1$ *then there is a region* $\Omega \subset \mathbb{D}$ *such that* f *is one-to-one and analytic on* Ω *and* $f(\Omega)$ *is a disk of radius* B.

In Koebe's theorem the optimal value is $1/4$.

The other two optimal constants are unknown, although in both cases there are upper and lower bounds that are only about 10% apart, and conjectures for the optimal values have been made.

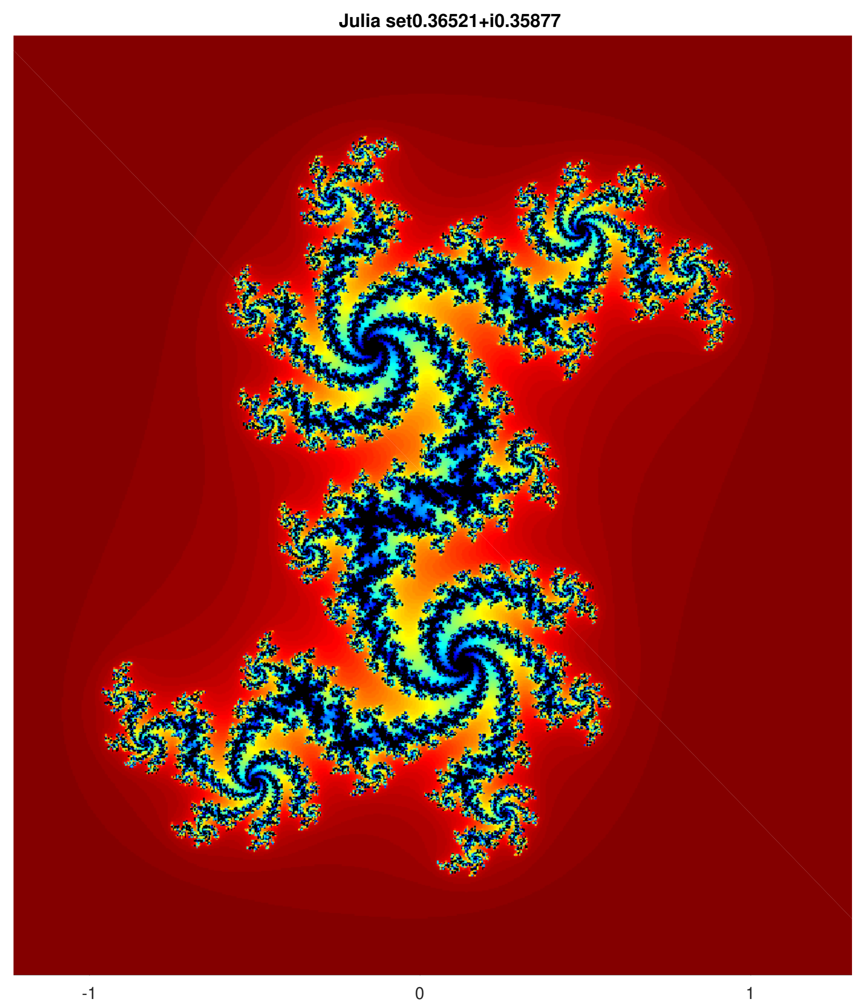
Let f^n denote f composed with itself n times.

Given an analtyic map $f : \mathbb{C} \to \mathbb{C}$, the **Fatou set** is the union of open disks D so that $\{f^n\}$ is a normal family on D.

The **Julia set** is the complement of the Fatou set.

Pierre Fatou

Gaston Julia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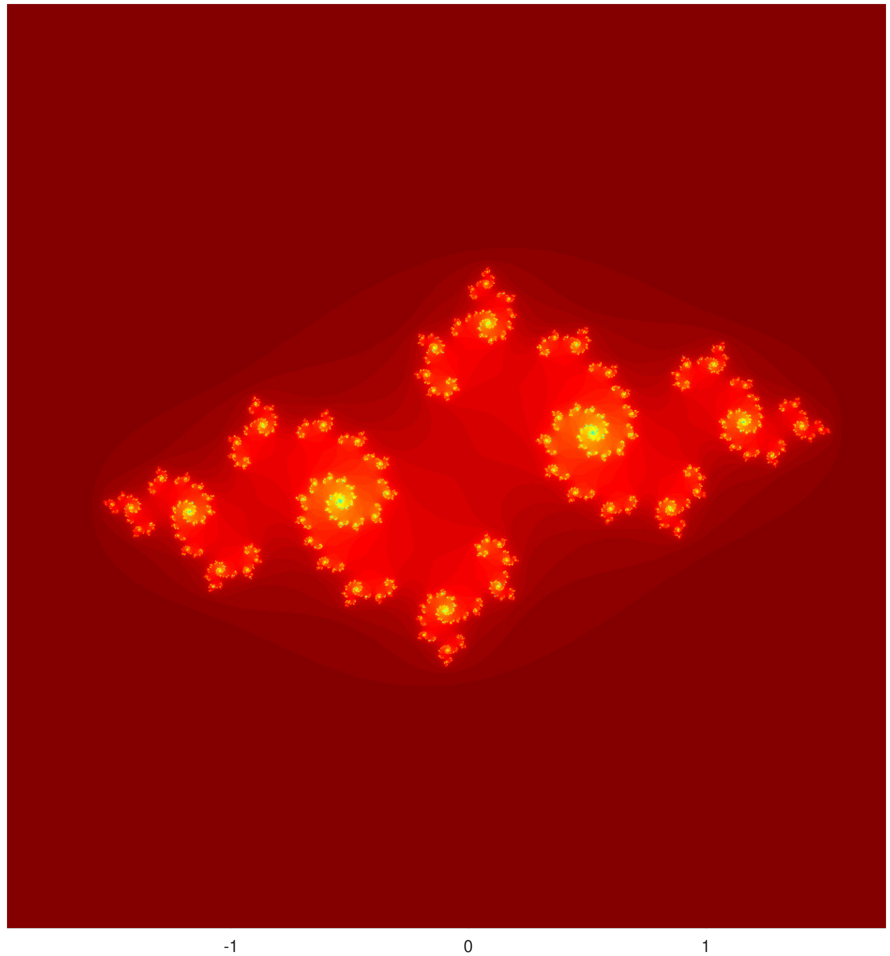
Julia set-0.79474+i0.30649



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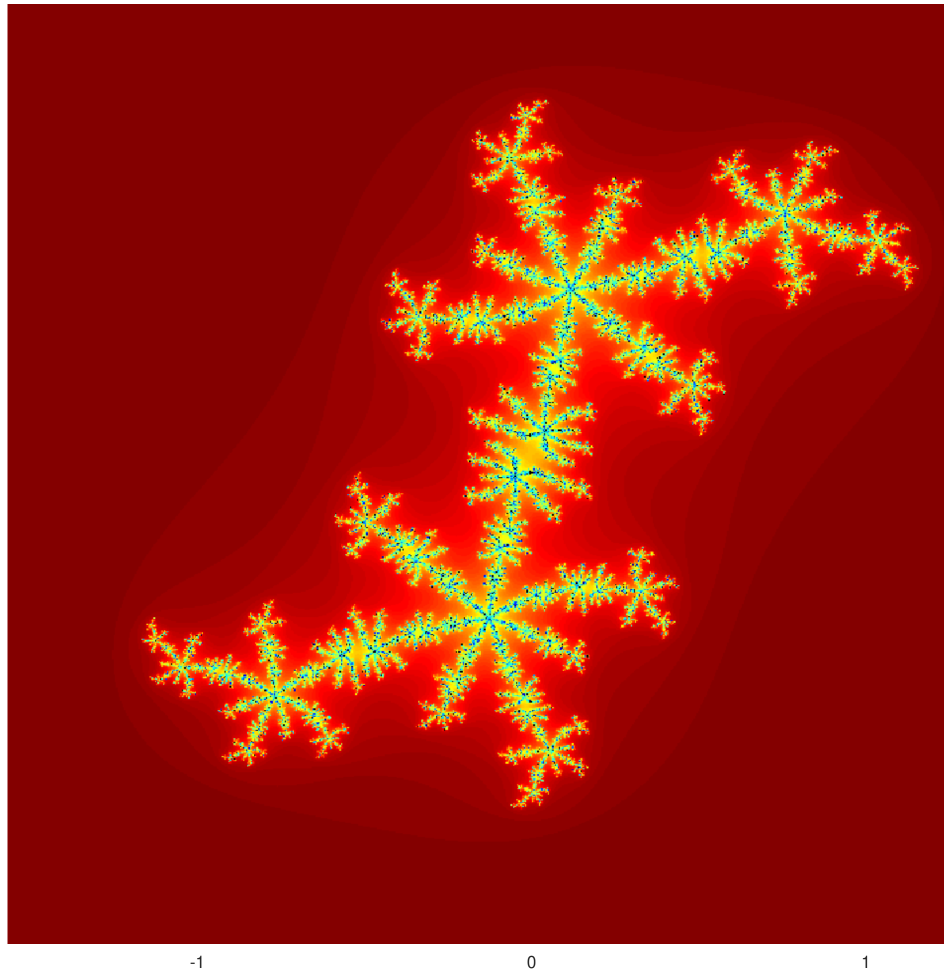

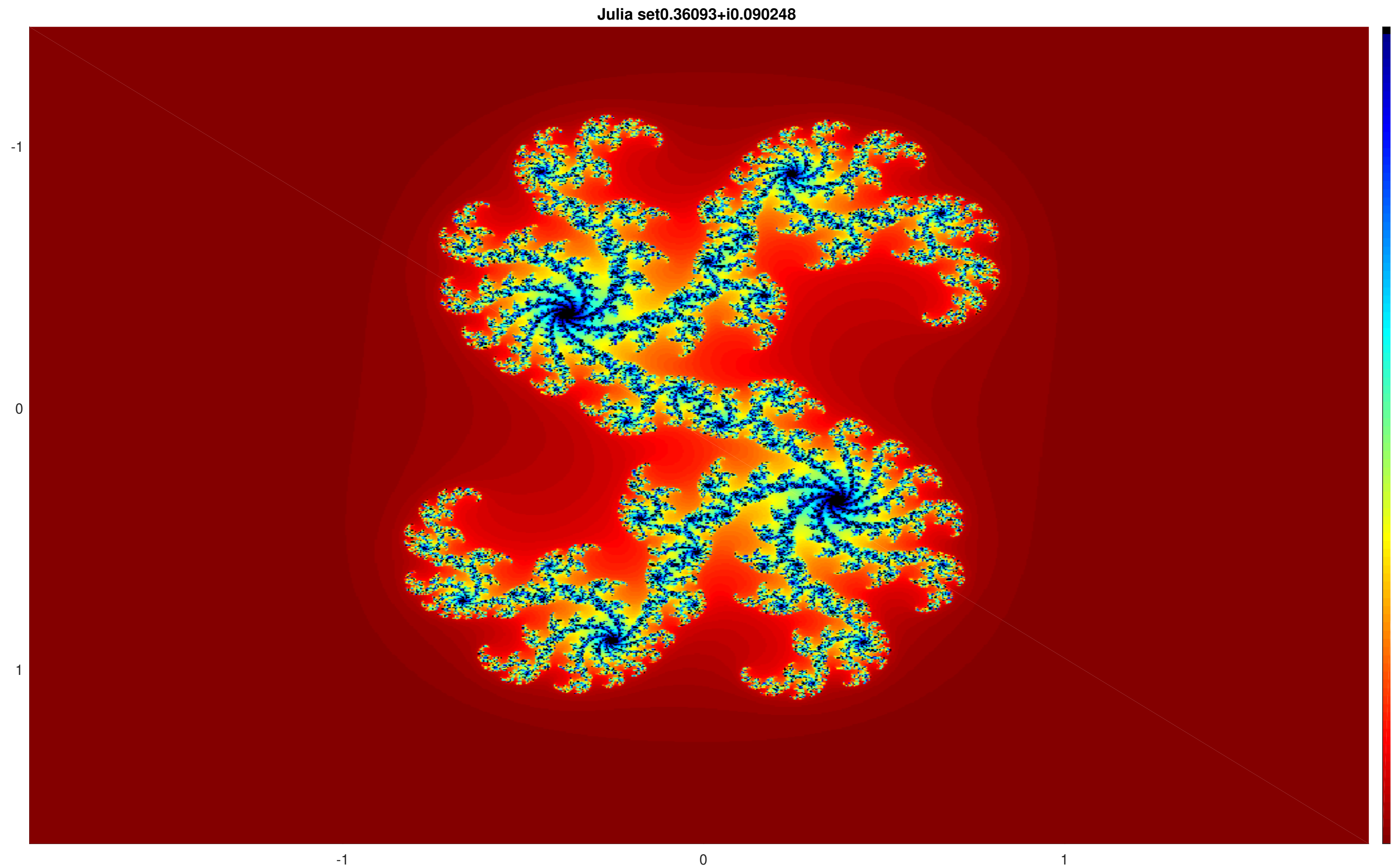
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-1
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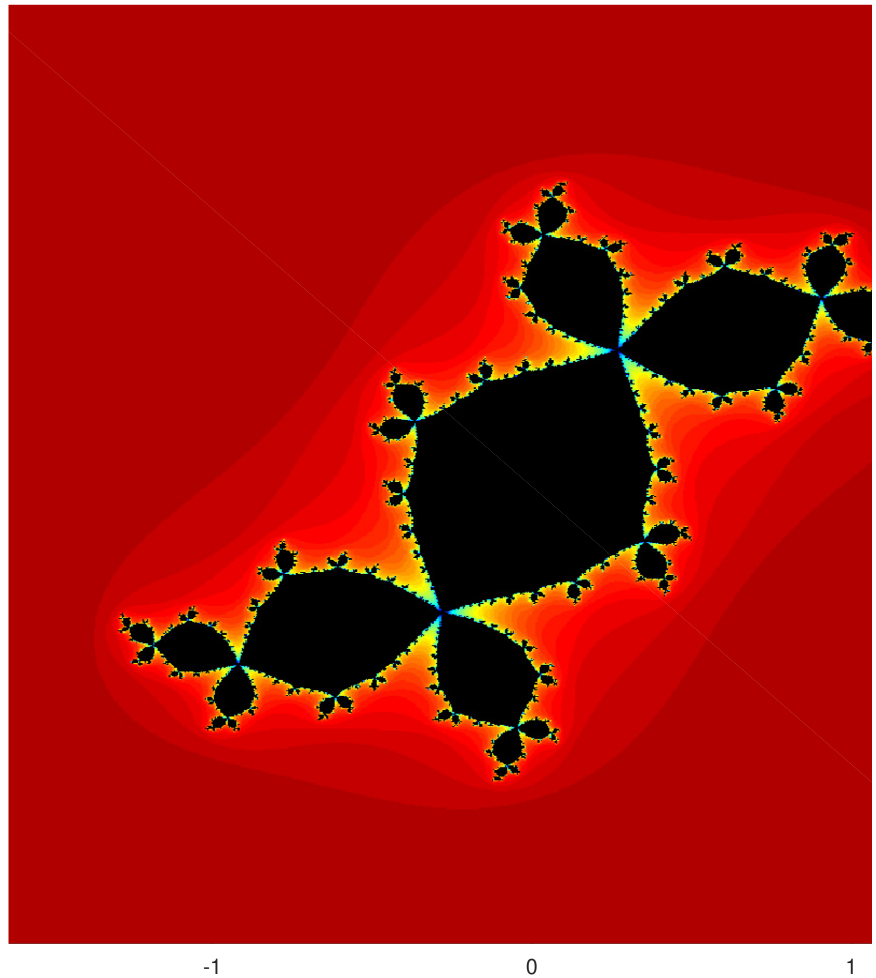
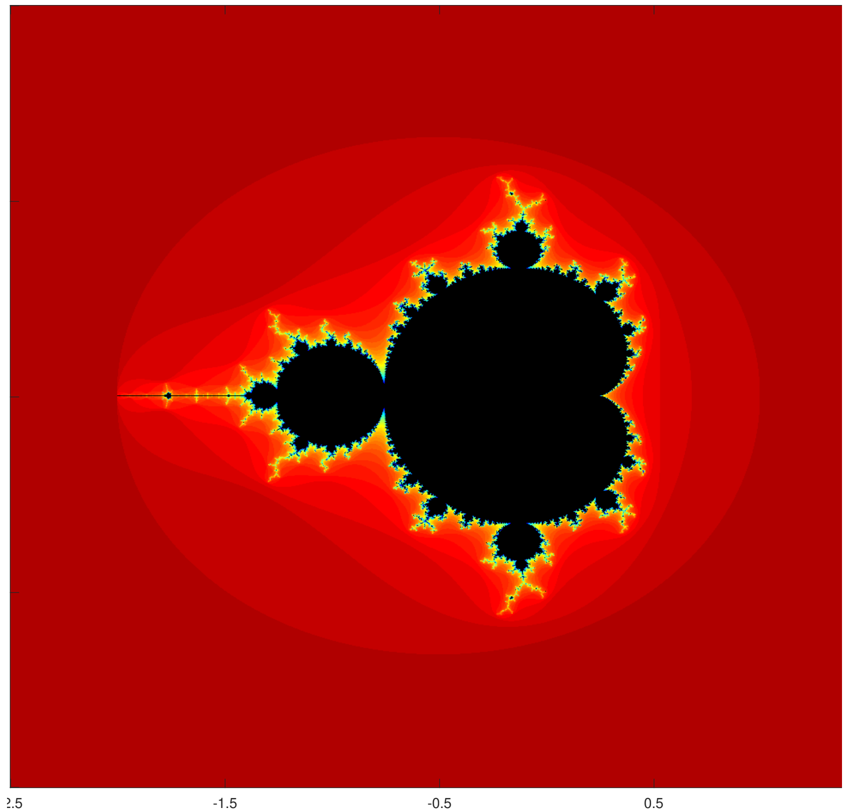


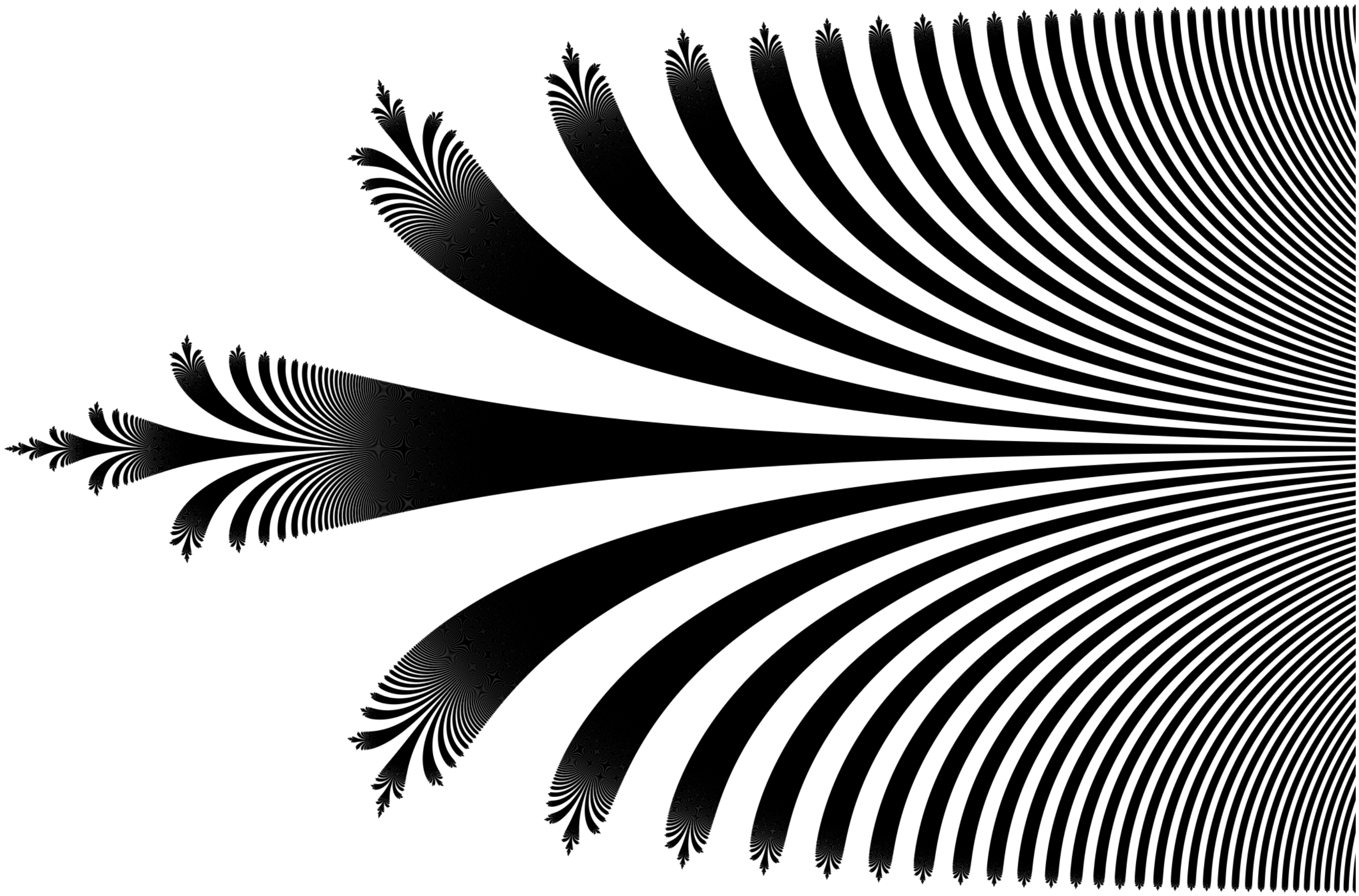


The Mandelbort set

= set of parameters c so that Julia set of $z^2 + c$ is connected.

= set of parameters c so that oribit of 0 is bounded.

Benoit Mandelbrot (1924-2010)

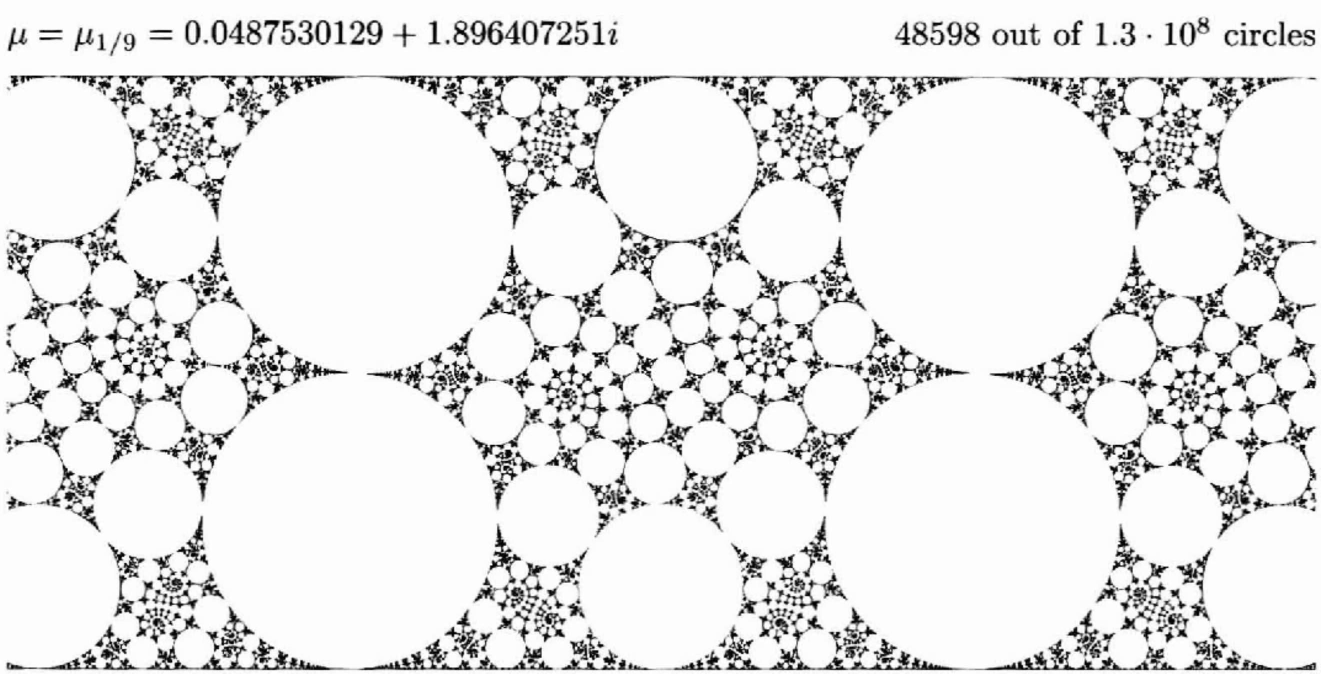


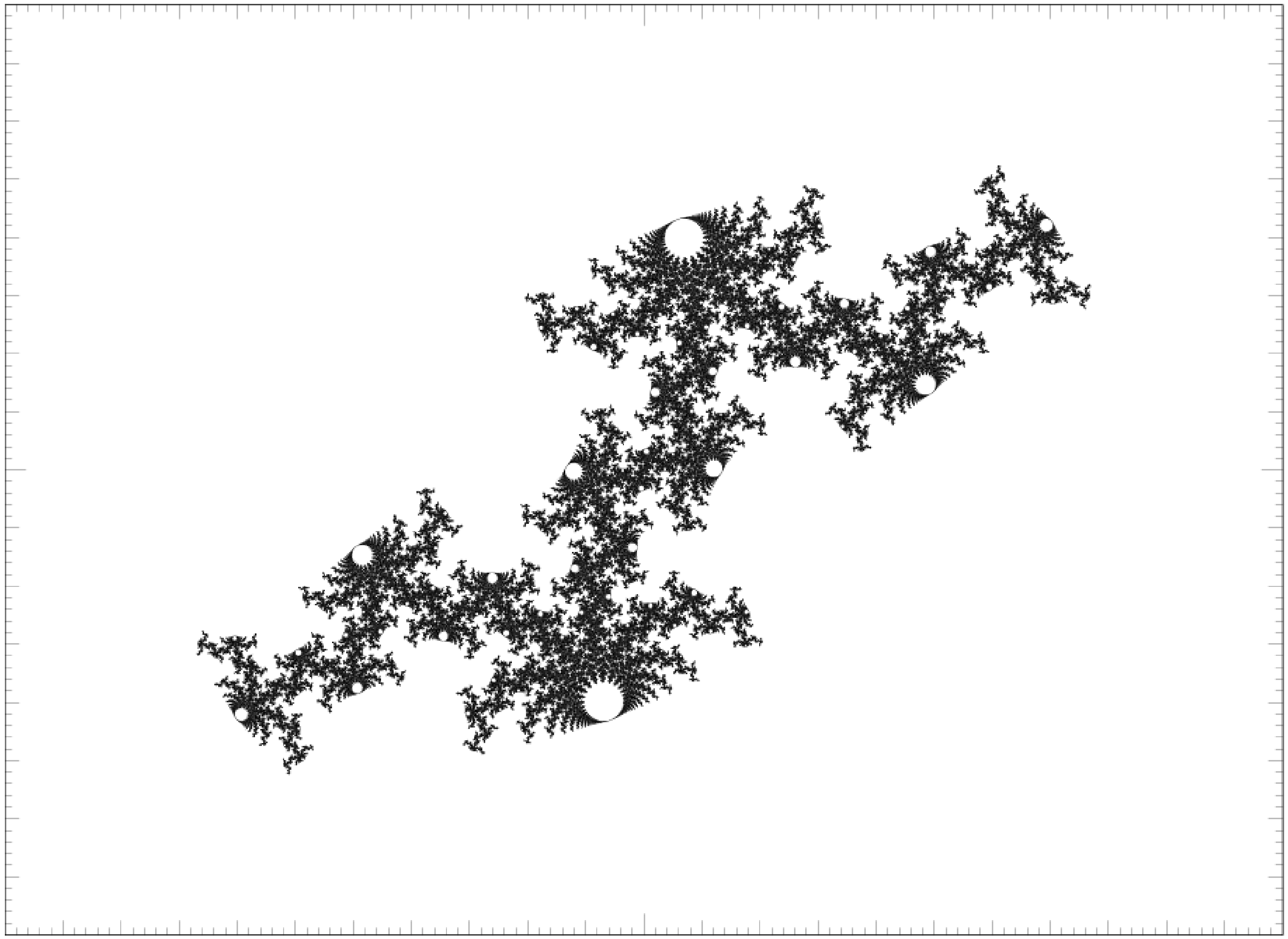
$\mathcal{J}((e^z - 1)/2)$, courtesy of Arnaud Chéritat

Given a discrete group of linear fractional transformations acting on the plane, the **ordinary set** is the union of disk where group is a normal family.

The complement is called the **limit set**.

These sets are closely related to hyperbolic 3-manifolds.





A limit set of dimension 2. The geometrically inifinite case.