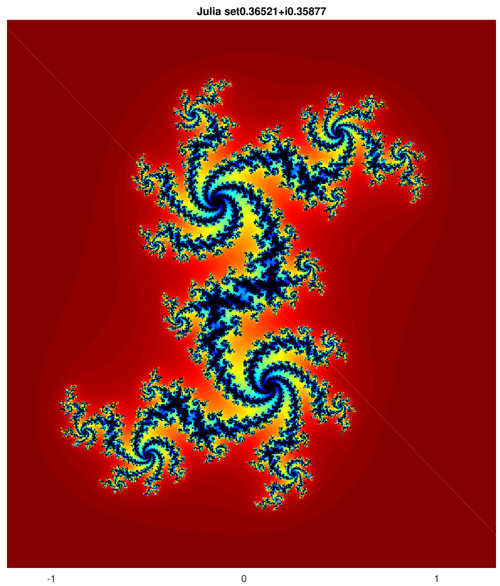
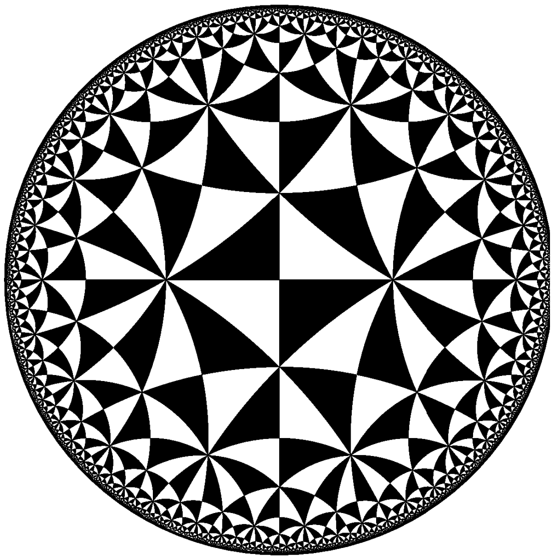
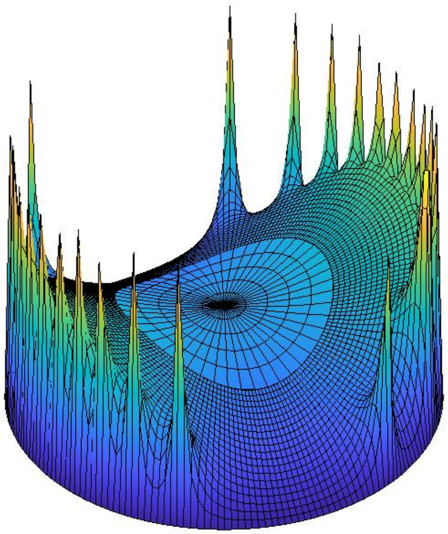
MAT 536, Spring 2024, Stony Brook University

Complex Analysis I, Christopher Bishop

Mon. Jan 22, 2024







MAT 536 is the core course in one complex variable for first year PhD students.

- Complex numbers, arithmetic, geometry
- Definitions of holomorphic functions (many equivalent definitions)
- Basic properties: maximum principle, Cauchy's theorem,...
- Normal families, Riemann mapping theorem
- Harmonic functions, Dirichlet problem, Perron method
- Riemann surfaces, uniformization theorem

This class is mostly independent of other core courses, but we will use a few facts from them, e.g., the Arzela-Ascoli theorem (normal families) and theory of covering spaces (Riemann surfaces).

There will be weekly problem sets, a midterm and a final exam. Each count of 1/3 of grade.

Problem sets due in class on Mondays. Grader is Alex Rodriguez.

Tentative midterm date = Wed, March 20 (Wednesday after Spring break)

Final exam = Friday May 10: Final Exam 11:15am-1:45pm

Grades will be posted on Brightspace.

Class webpage

Instructor: Christopher Bishop

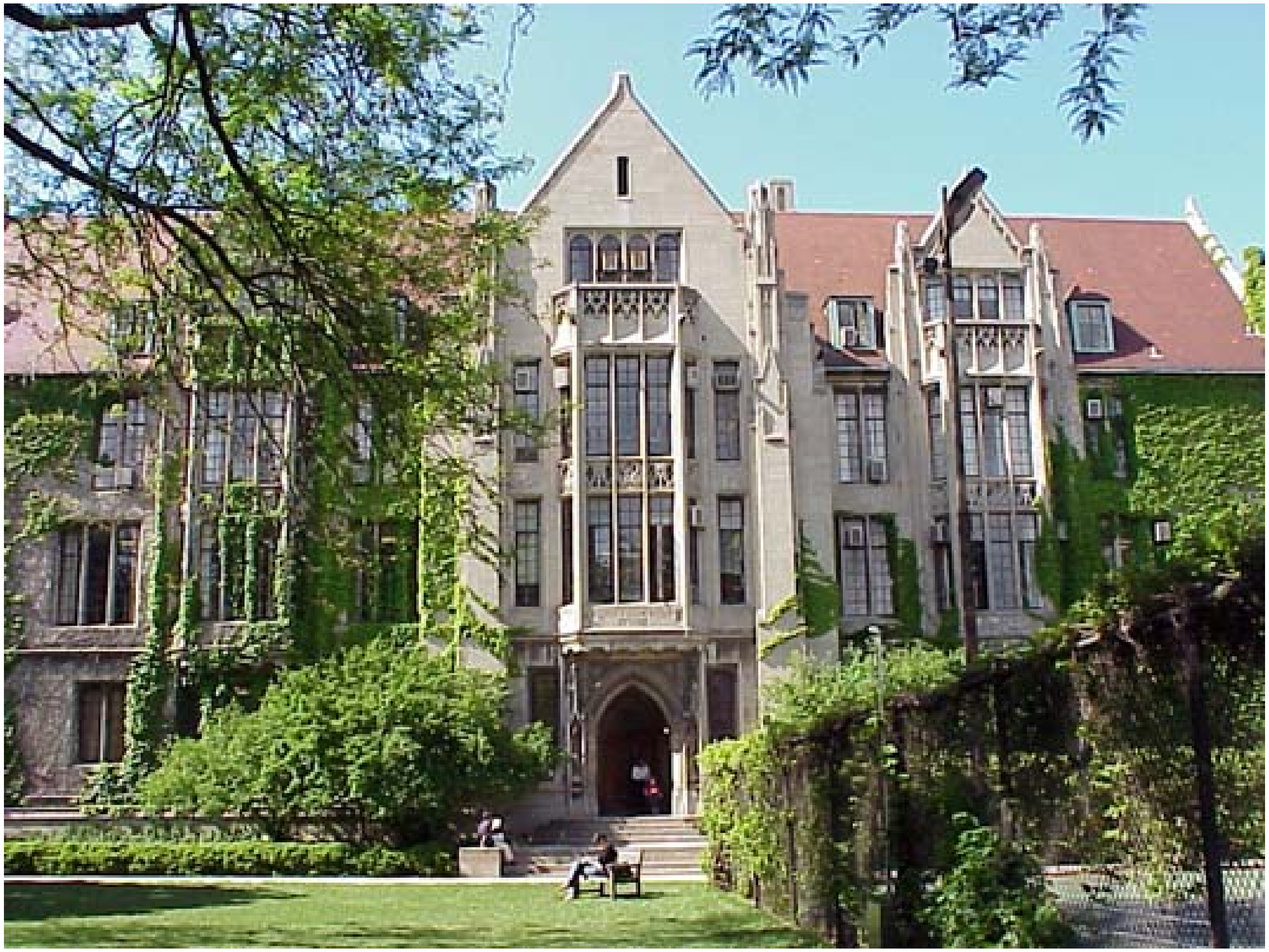
My expertise is real and complex analysis: conformal maps, quasiconformal maps, Fuchsian and Kleinian groups, hyperbolic geometry, analysis on fractals, holomorphic dynamics, computational geometry.

Math Dept, Michigan State

Churchill College, Cambridge

New Math Dept, Cambridge

Old Math Dept, Cambridge



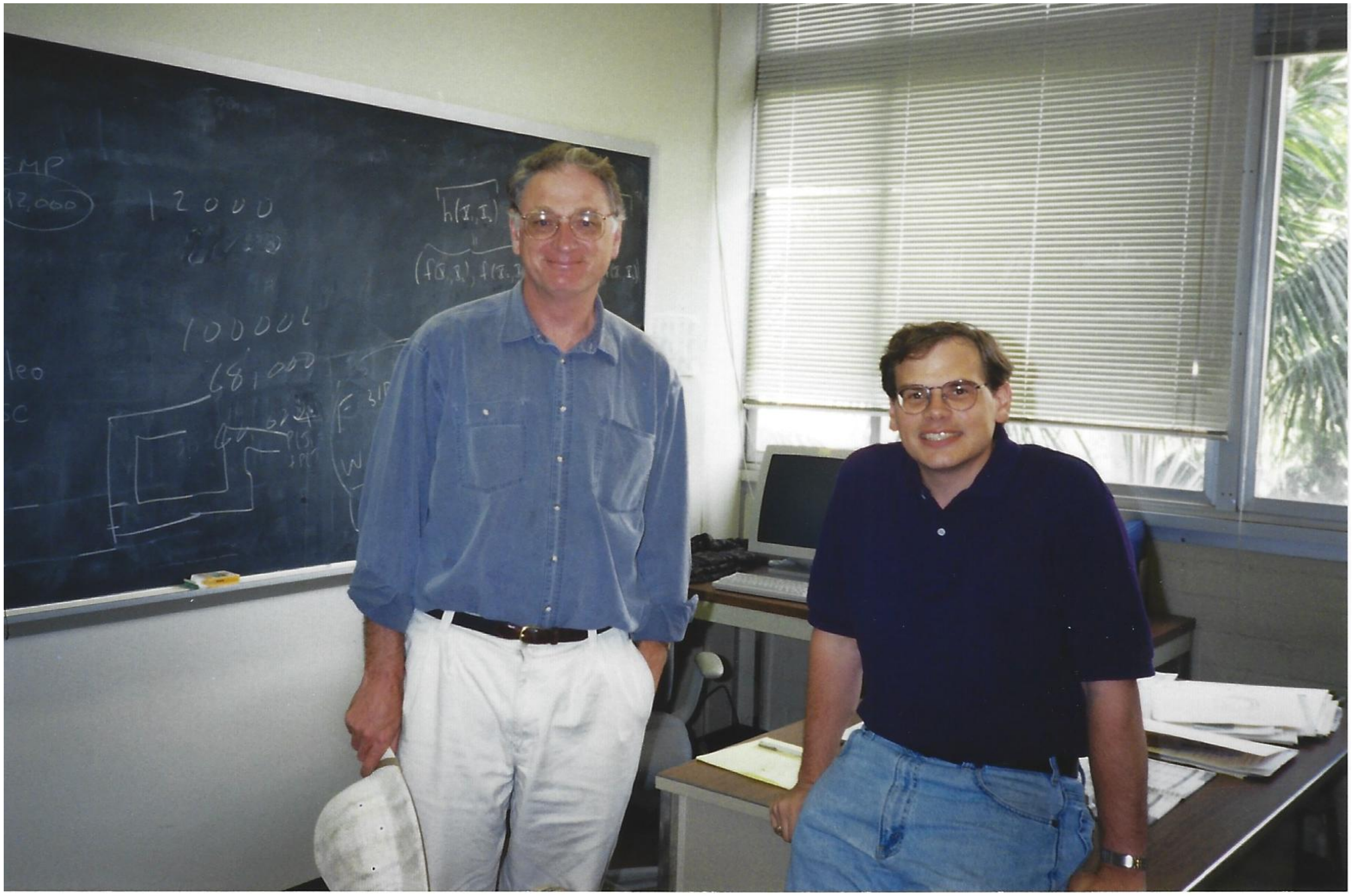
Math Dept, Eckhart Hall, U Chicago

Math Dept, Leet Oliver Memorial, Yale

My advisor, Peter W. Jones

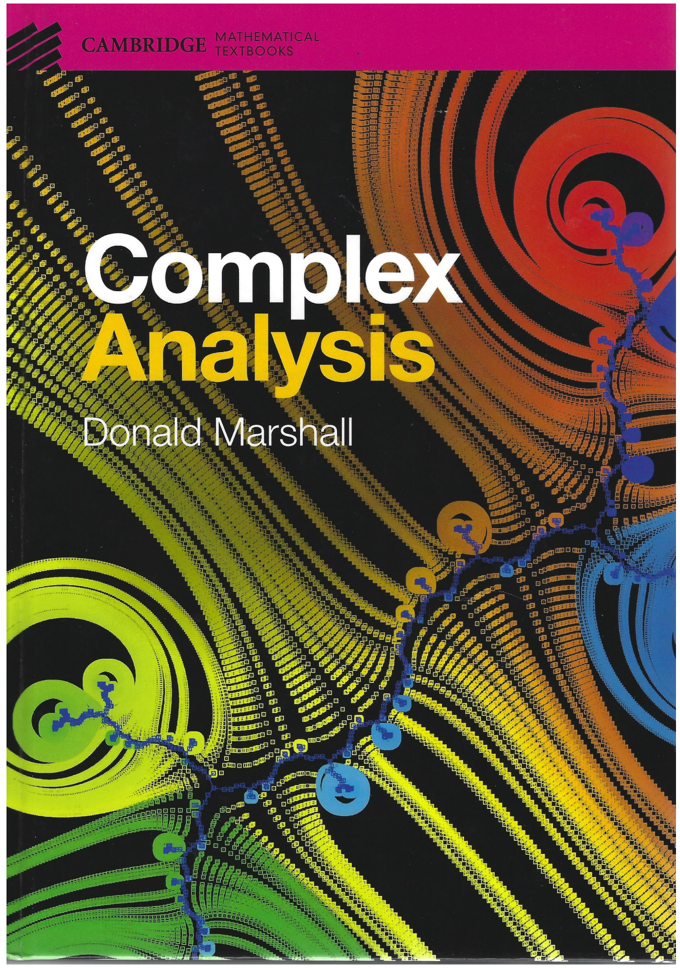
MSRI, Berkeley

Math Dept, UCLA



My post-doc mentor, John Garnett


MATHEMATICS
PHYSICS
UMBILIC TORUS



Complex Analysis by Don Marshall

I expect to cover Chapters 1,2,3,4,5,7,10,12,13,14,15.

Perhaps some others if time permits.

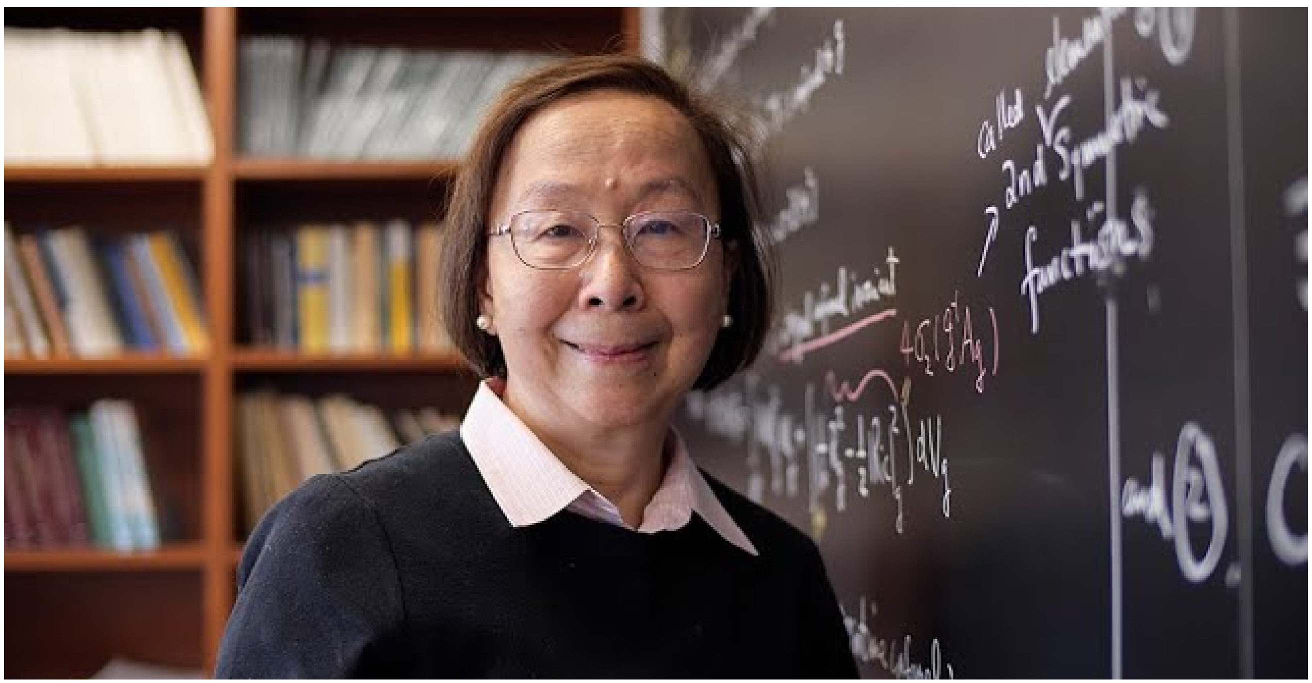
Don Marshall

Chang-Marshall Theorem: characterizes all closed sub-algebras of $L^\infty(\mathbb{T})$ that contain $H^\infty(\mathbb{T})$.

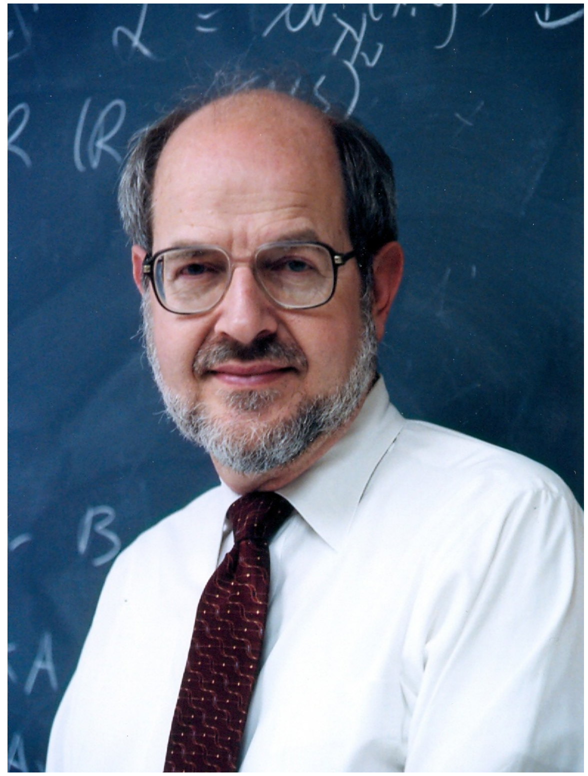
$L^\infty(\mathbb{T})$ = bounded, Lebesgue measurable functions on unit circle.

$H^\infty(\mathbb{T})$ = a.e. radial limits of bounded holomorphic functions on unit disk.

Closed algebras between H^∞ and L^∞ are called **Douglas algebras**.



Sun-Yun Alice Chang



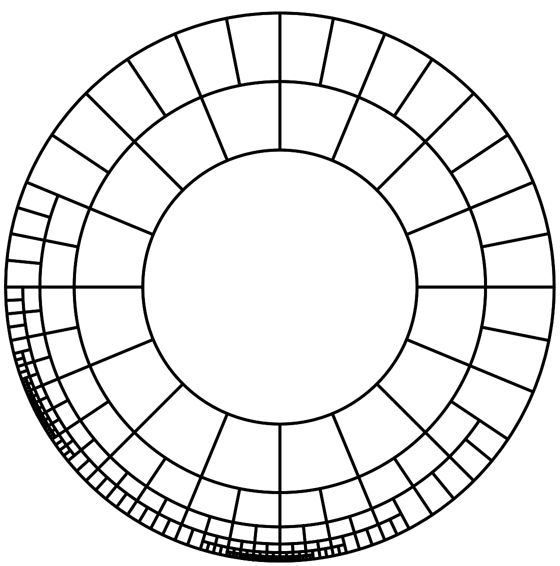
Ron Douglas

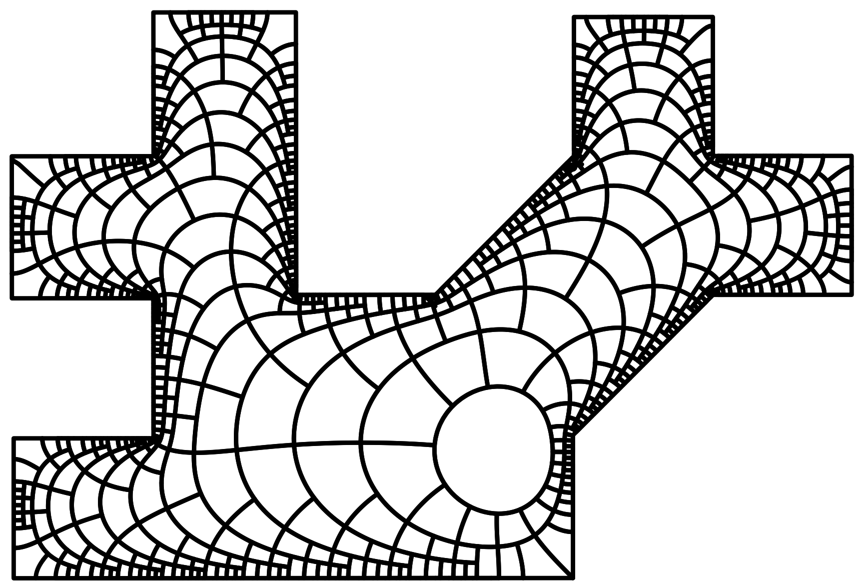
Former Stony Brook preofessor, for whom Douglas algebras are named

His son, Michael Douglas, was also a professor here (SCGP, string theory)

Marshall's Zipper algorithm: efficient way to compute 1-1 holomorphic maps from disk to a polygon.

Numerical implementation of the Riemann mapping theorem.





Holomorphic functions: four basic approaches

- has a power series expansion around each point $\sum_0^\infty a_n z^n$ (Weierstrass)
- complex derivative exists $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ (Cauchy)
- locally rigid mappings between regions (Riemann)
- uniform limits of rational functions (Runge)

We will prove these (and many other definitions) are equivalent.

Cauchy's viewpoint is most common. Marshall's text follows Weierstrass.

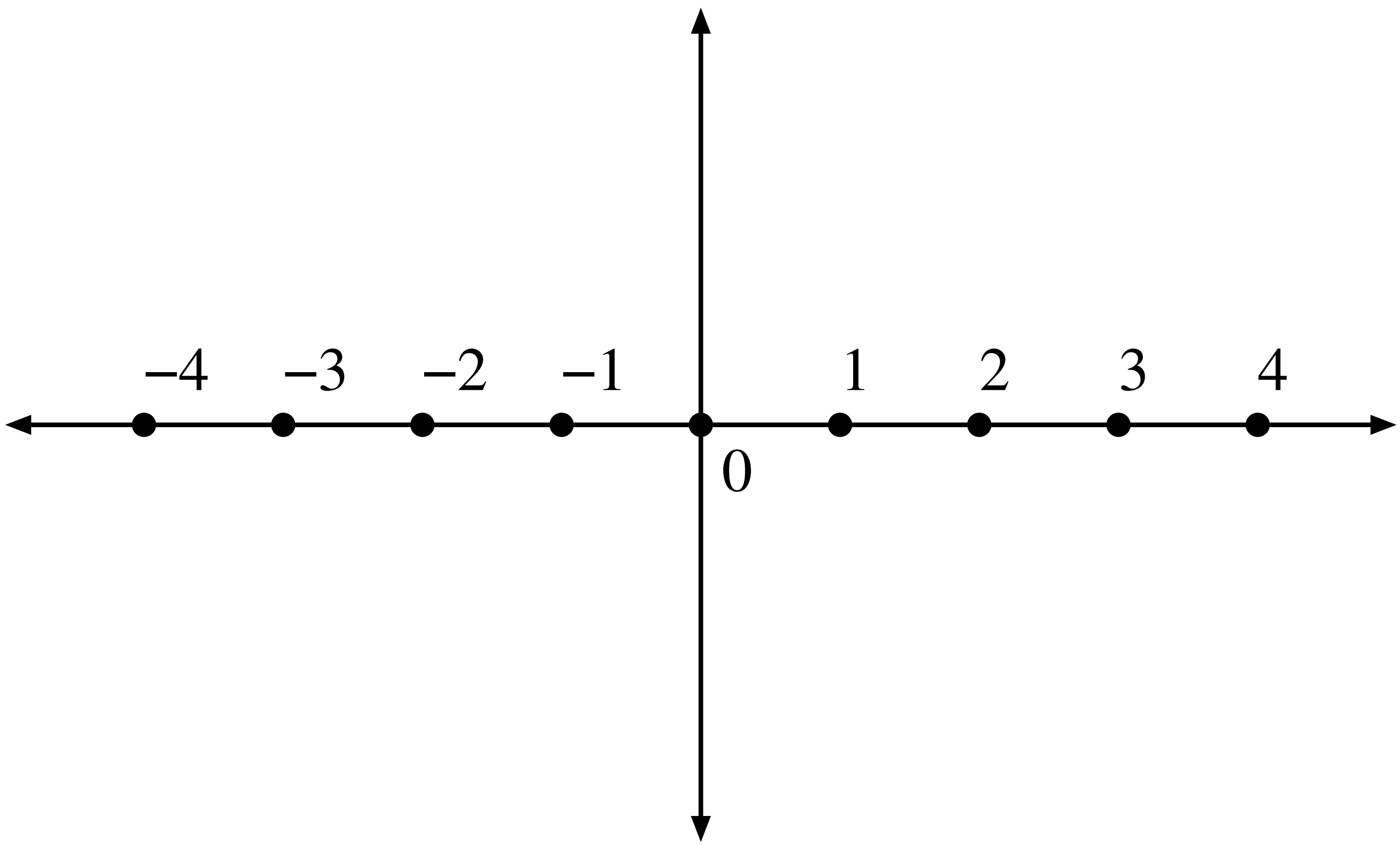
Karl Theodor Wilhelm Weierstrass

Augustin Louis Cauchy

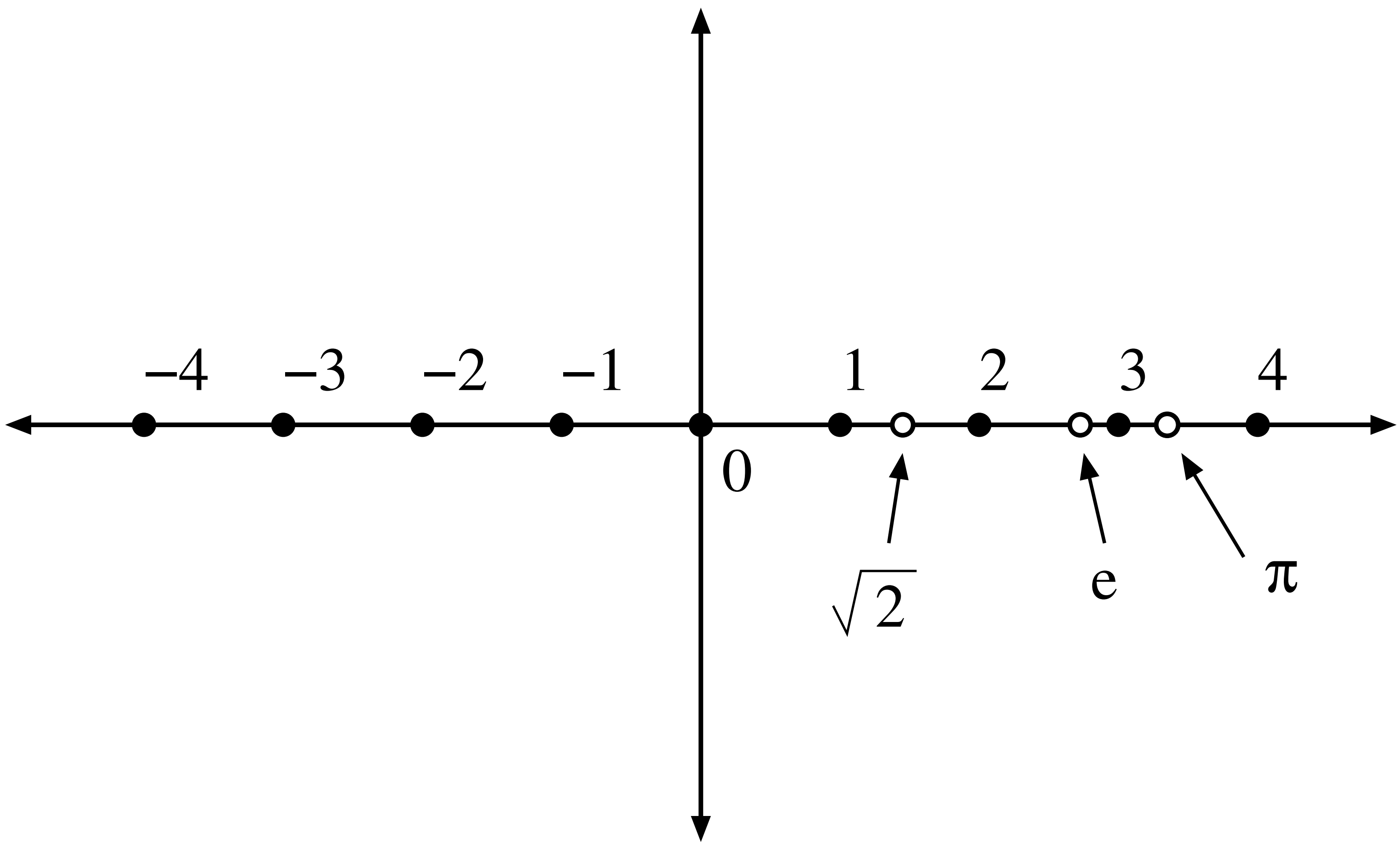
Georg Friedrich Bernhard Riemann

Carl David Tolmé Runge

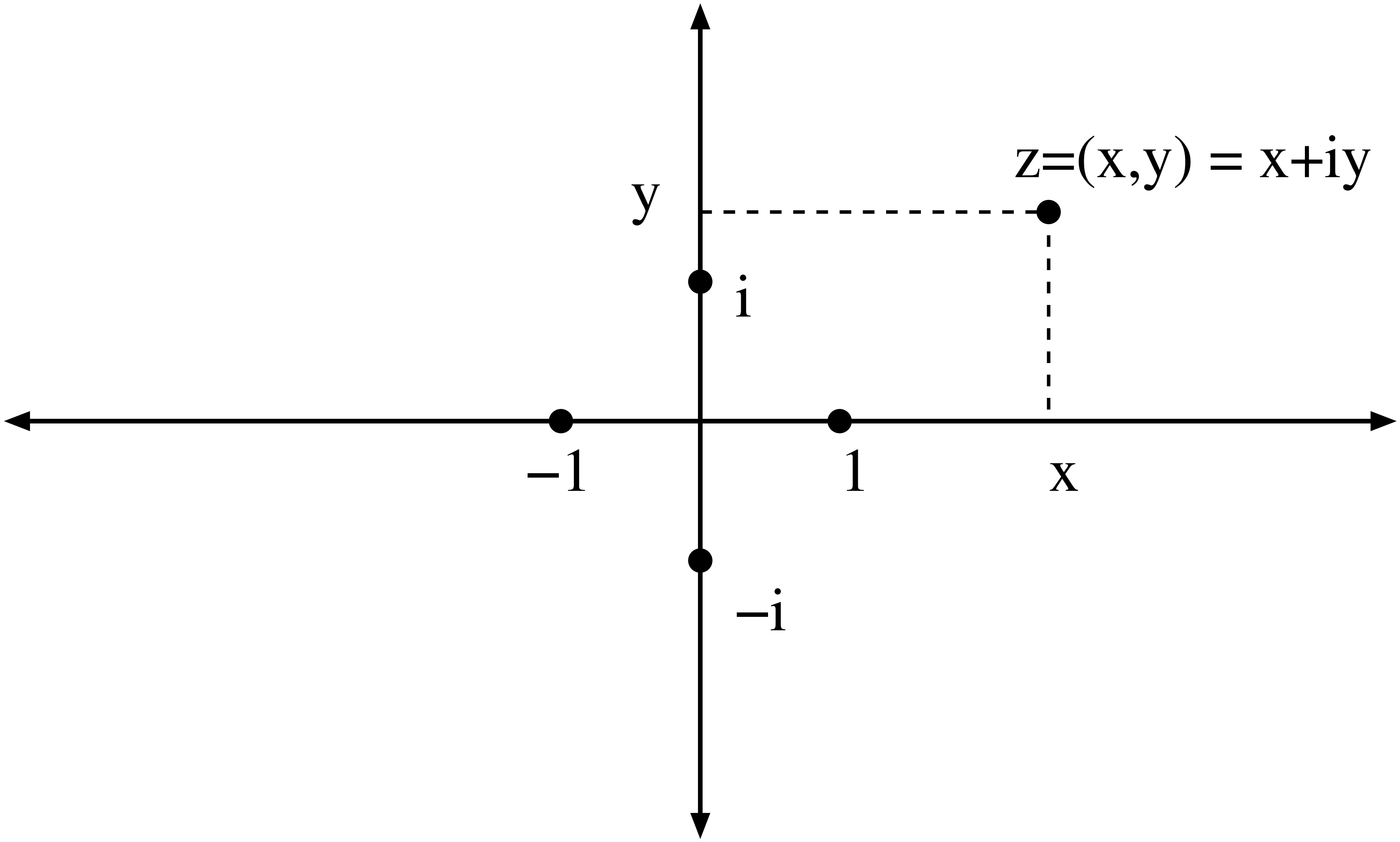
Chapter 1: Preliminaries



The integers

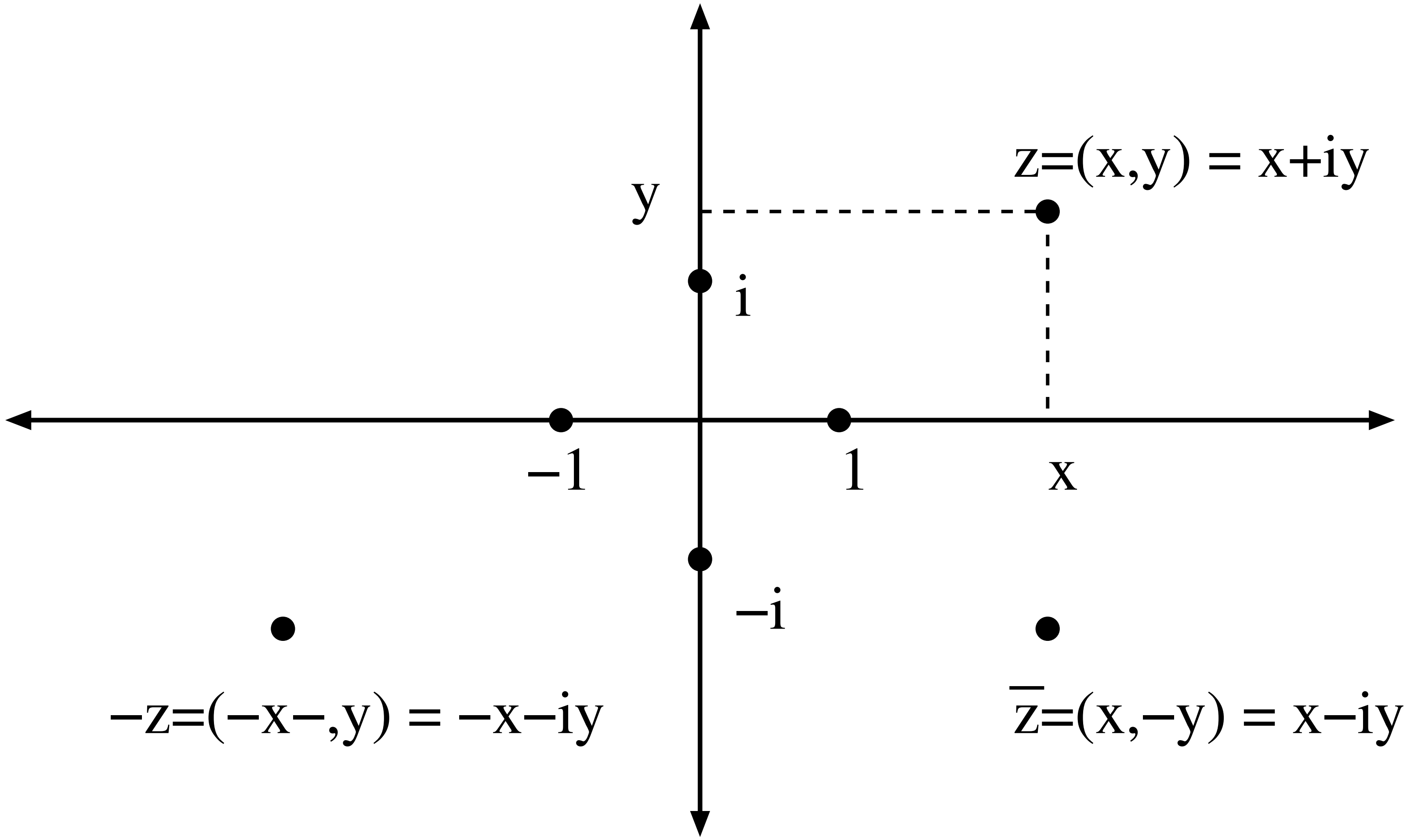


The real numbers

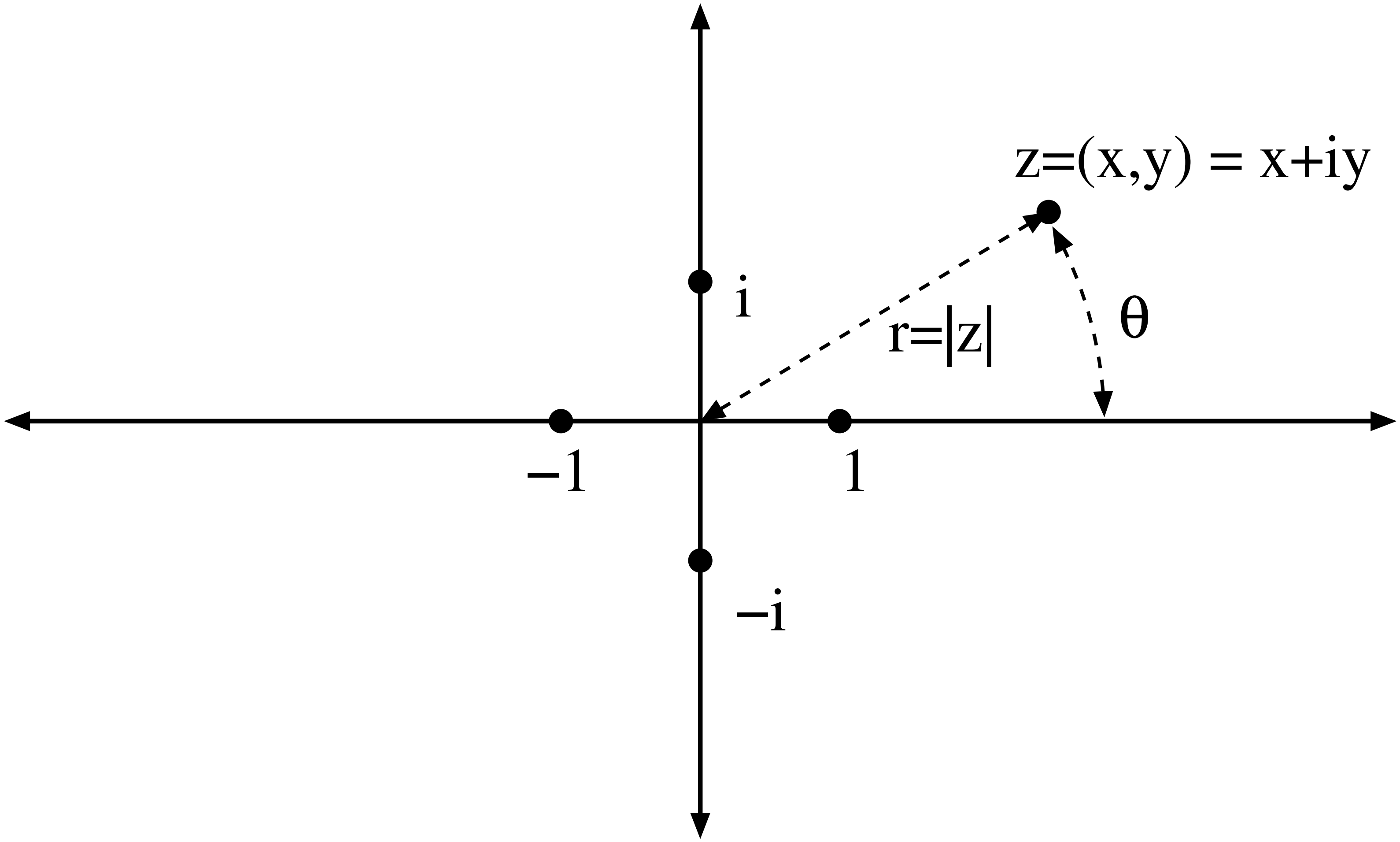


The complex numbers

$\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$, $|z| = \sqrt{x^2 + y^2}$



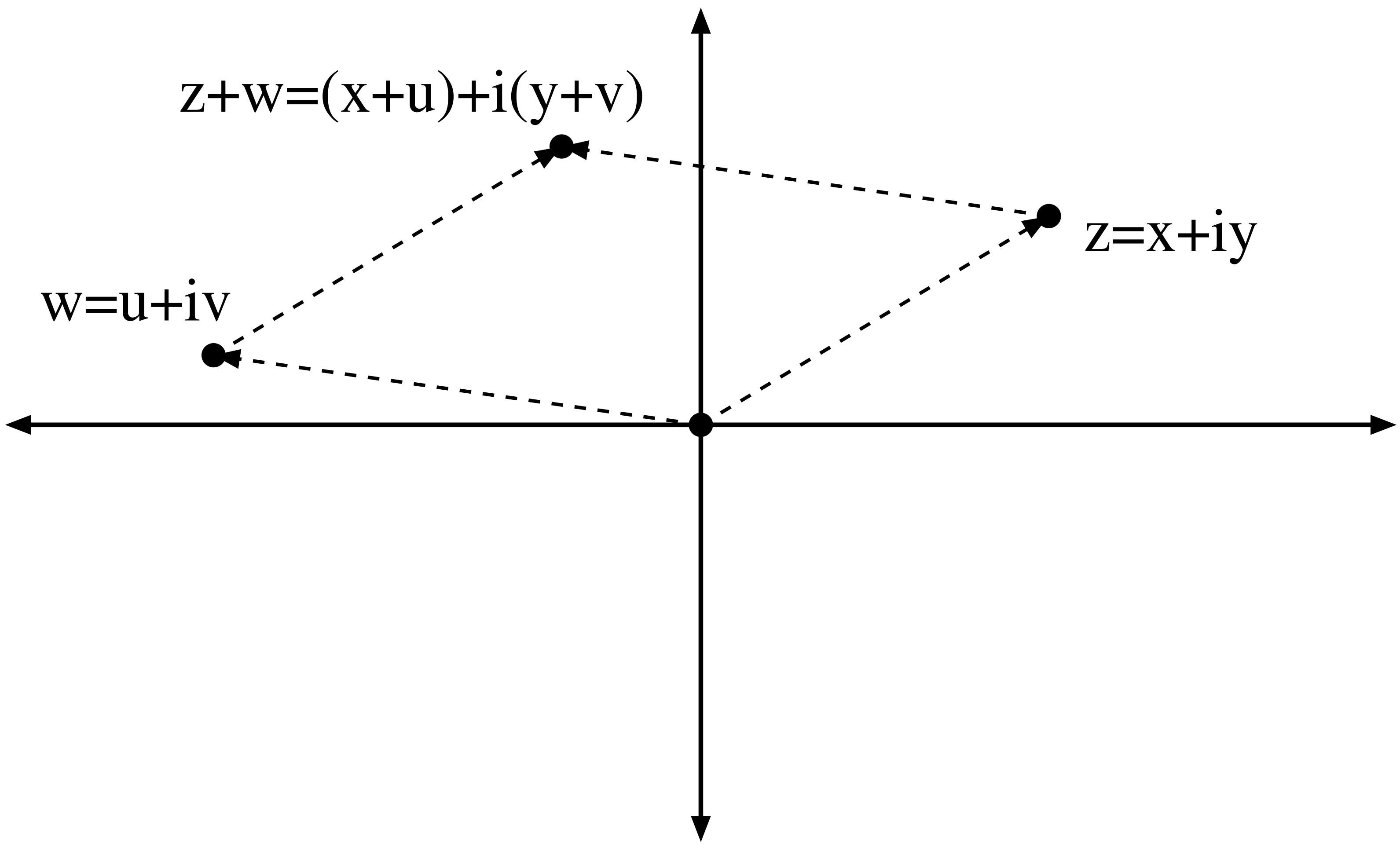
Negatives and complex conjugate



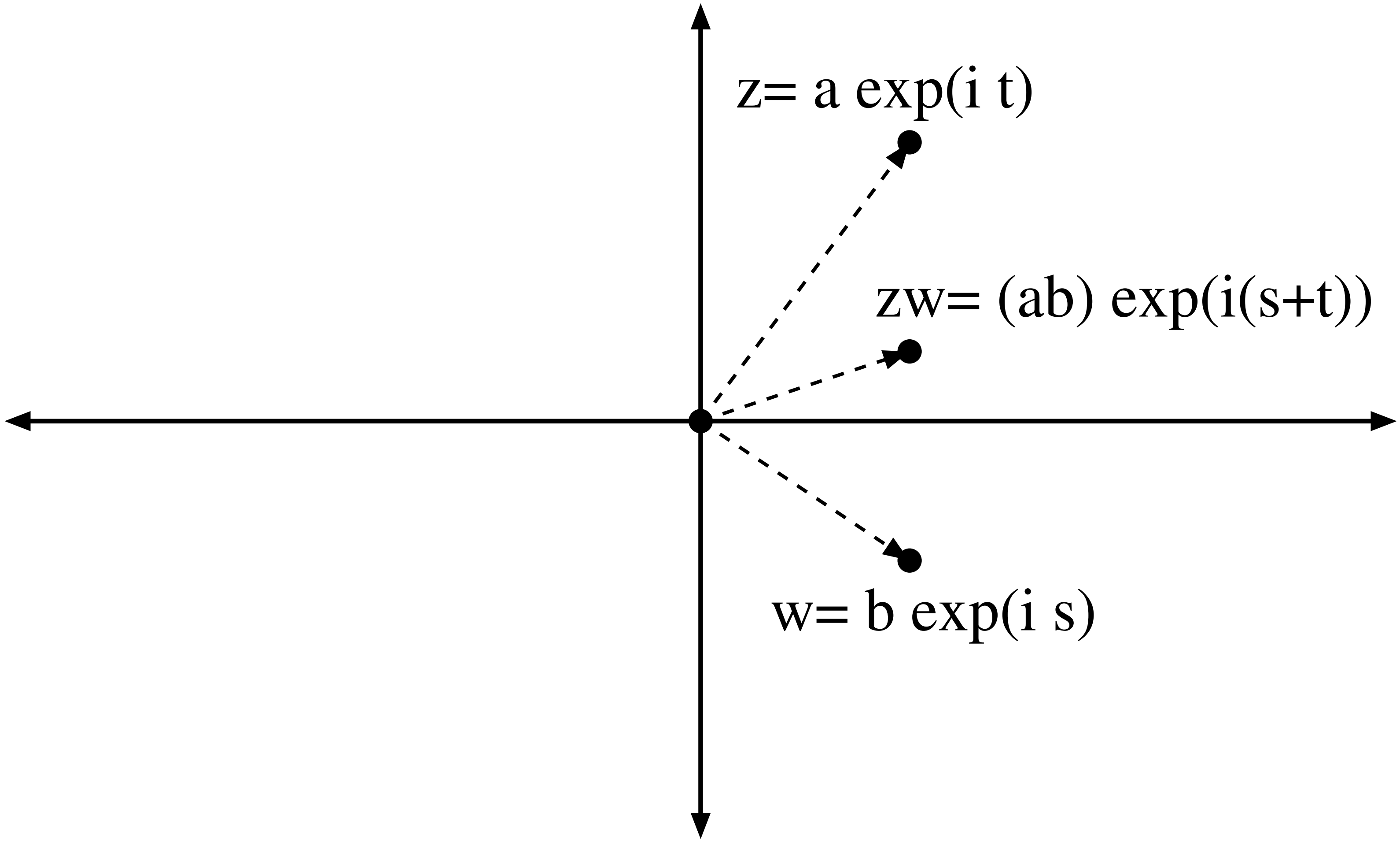
Polar co-ordinates $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$.

We write $z = re^{i\theta}$. Will explain notation later.

$|z| = r = \sqrt{x^2 + y^2}$, $\arg(z) = \theta$ (more precise later)



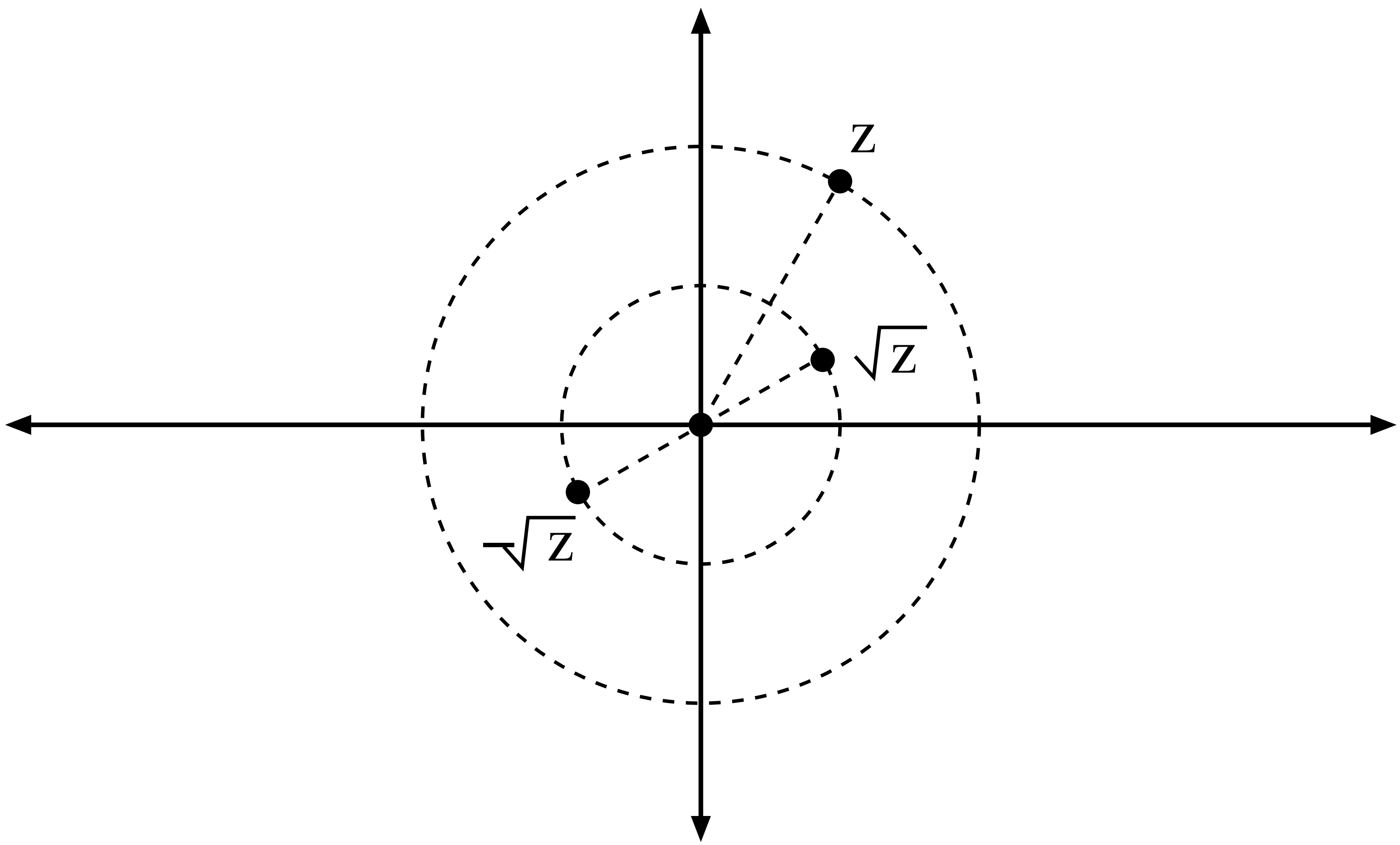
Addition: add coordinates



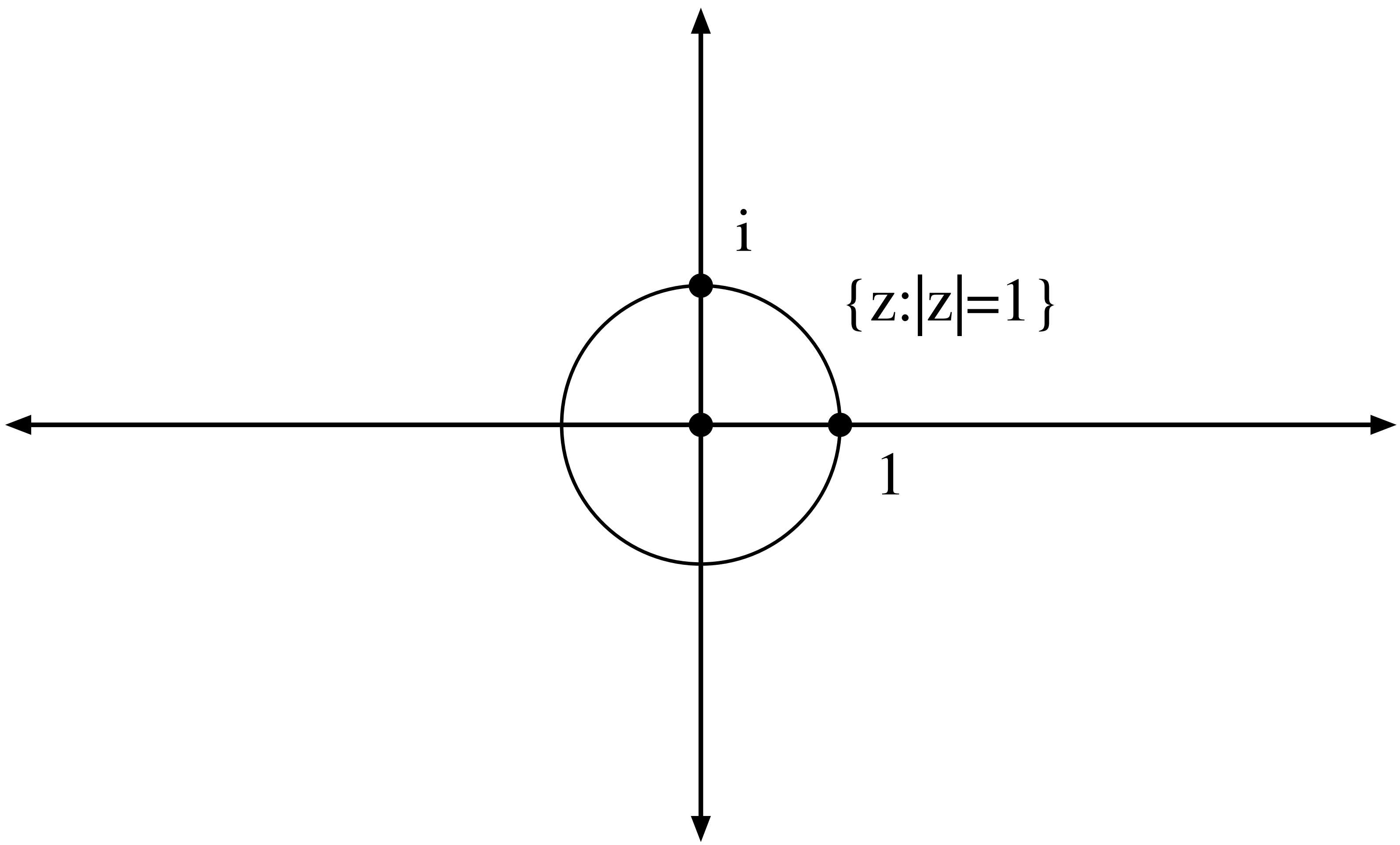
Multiplication: multiply out and use $i^2 = -1$

$$(x+iy)(u+iv) = xu + ixv + iyu + i^2yv = (xu - yv) + i(xv + yu)$$

Equivalent: multiply absolute values, add arguments: $(re^{i\theta})(se^{i\tau}) = (rs)e^{i(\theta+\tau)}$



Square roots: square root of modulus, halve angle, take negative



Unit circle $\mathbb{T} = \{z : |z| = 1\}$, Unit disk $\mathbb{D} = \{z : |z| < 1\}$

Complex number are a field. Division given by

$$\frac{x+iy}{u+iv} = \frac{x+iy}{u+iv}\frac{u-iv}{u-iv} = \frac{(xu+yv)+i(xu-yv)}{u^2+v^2}$$

Properties to check:

$$|zw| = |z| \cdot |w|$$

$$z\overline{z} = |z|^2$$

$$\mathrm{Re(z) = (z + \overline{z})/2}$$

$$\mathrm{Im(z) = (z - \overline{z})/2}$$

$$\overline{z + w} = \overline{z} + \overline{w}$$

$$\overline{zw} = \overline{z} \cdot \overline{w}$$

$$\arg(zw) = \arg z \arg(w) \text{ modulo } 2\pi$$

The triangle inequality: $|z + w| \leq |z| + |w|$

The triangle inequality: $|z+w| \leq |z| + |w|$

Proof.

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) \\ &= z\overline{z} + w\overline{z} + z\overline{w} + w\overline{w} \\ &= |z|^2 + 2\mathrm{Re}(\mathrm{w\overline{z}}) + |\mathrm{w}|^2 \\ &\leq |z|^2 + 2|w||\overline{z}| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

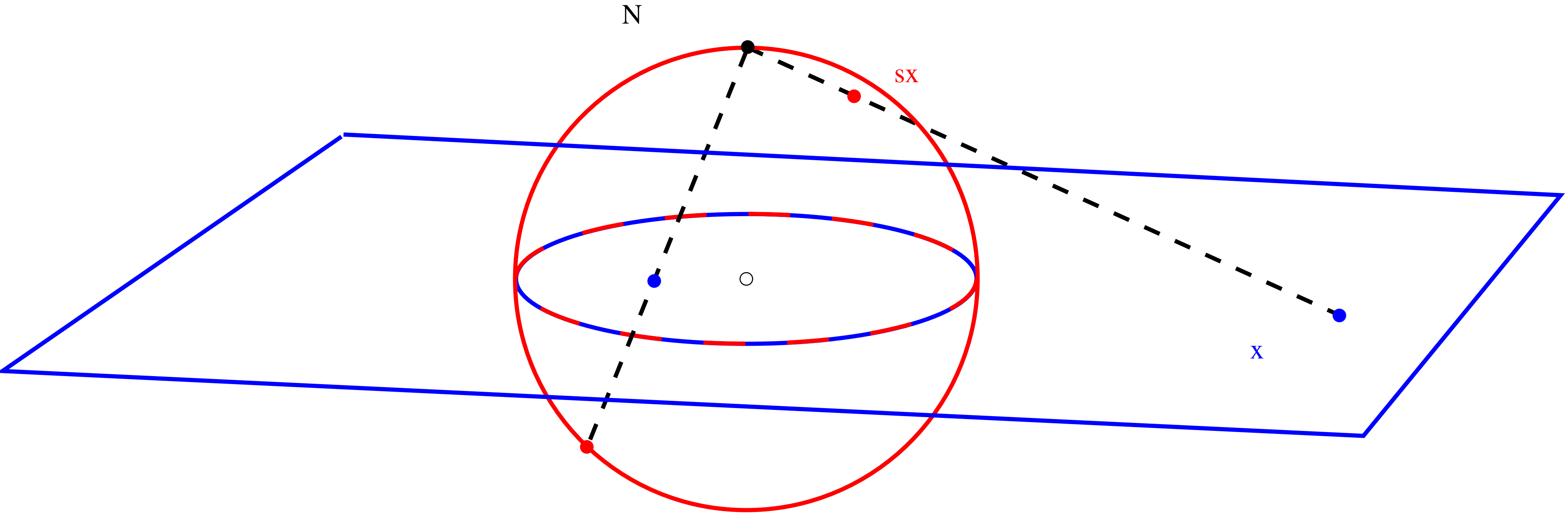
$\square$

The Cauchy-Schwarz inequality:

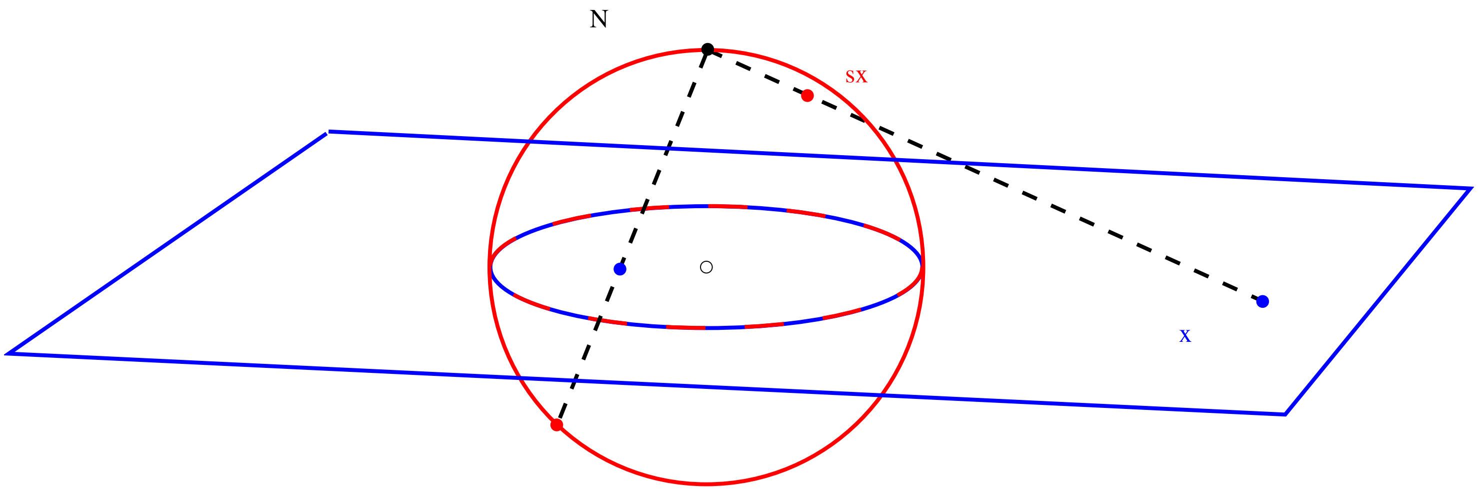
$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right| \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |b_j|^2\right)^{\frac{1}{2}}.$$

Integral version:

$$\left|\int_a^b f(t)\overline{g(t)}dt\right| \leq \left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2 dt\right)^{\frac{1}{2}}.$$

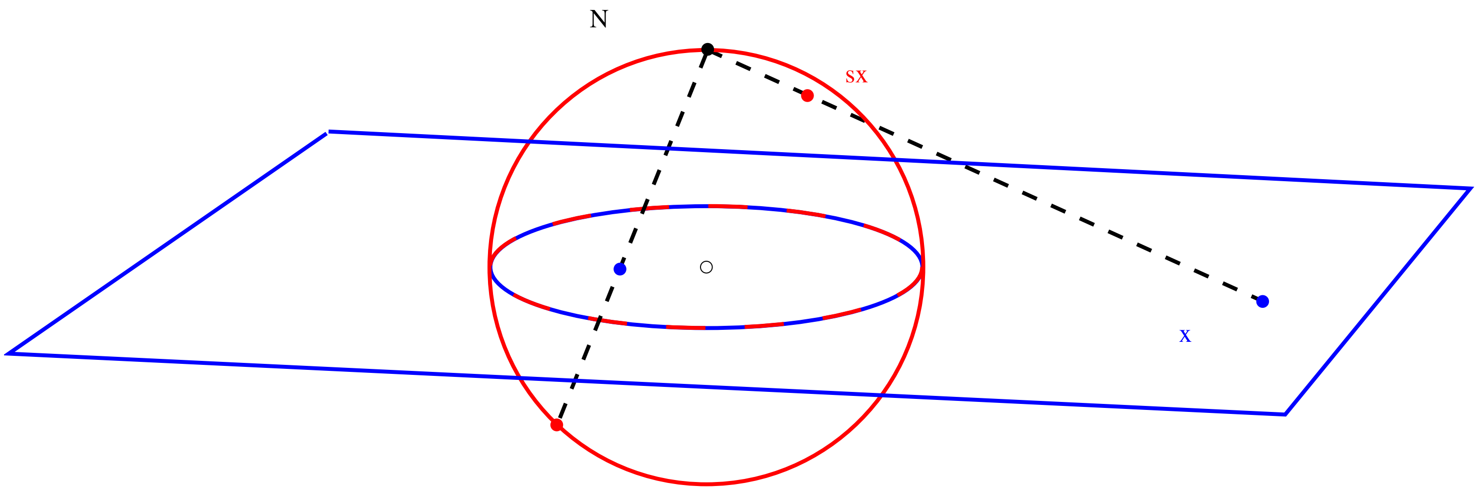


Stereographic projection from 2-sphere to plane



Suppose $z = x + iy$, and let z^* be the unique point on the unit sphere on the line from $(0,0,1)$ to $(x,y,0)$. Then z^* is a convex combination of the North pole $(1,0,0)$ and a point in the plane $(x,y,0)$:

$$\begin{aligned} z^* &= (x_1, x_2, x_3) = (0,0,1) + t[(x,y,0) - (0,0,1)] \\ &= = (1-t)(0,0,1) + t(x,y,0) \end{aligned}$$



Therefore

$$|z^*| = \sqrt{(tx)^2 + (ty)^2 + (1-t)^2} = 1,$$

which gives

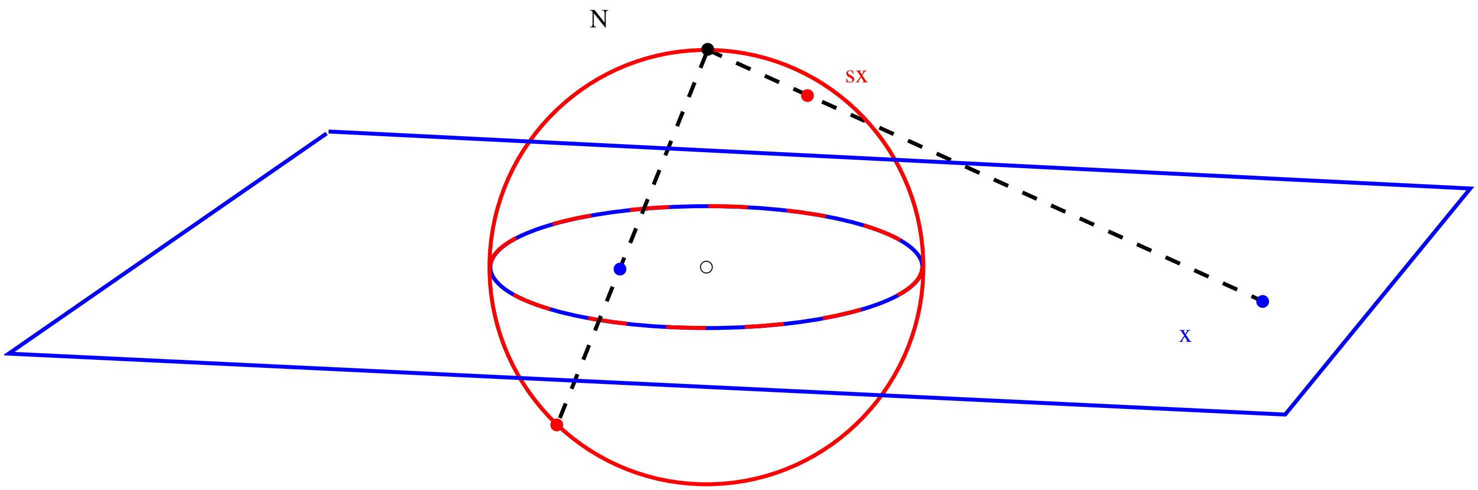
$$t = \frac{2}{x^2 + y^2 + 1},$$

where $0 < t \leq 2$, and

$$z^* = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

Theorem 1.4: *Under stereographic projection, circles and straight lines in* $\mathbb{C}$ *correspond precisely to circles on* $\mathbb{S}^2$.

Proof. Every circle on the sphere is given by the intersection of a plane with the sphere and conversely the intersection of a plane with a sphere is a circle or a point. (See Exercise I.6 of text).



If a plane is given by

$$Ax_1 + Bx_2 + Cx_3 = D$$

and if (x_1, x_2, x_3) corresponds to $(x, y, 0)$ under stereographic projection then

$$A\left(\frac{2x}{x^2+y^2+1}\right)+B\left(\frac{2y}{x^2+y^2+1}\right)+C\left(\frac{x^2+y^2-1}{x^2+y^2+1}\right)= D.$$

Equivalently,

$$(C-D)(x^2+y^2) + 2Ax + 2By = C + D.$$

If $C = D$, then this is the equation of a line, and all lines can be written this way. If $C \neq D$, then by completing the square, we get the equation of a circle, and all circles can be put in this form.

The chordal distance between two points on the sphere induces a metric, called the **chordal metric**, on $\mathbb{C}$ which is given by

$$\chi(z,w) = |z^* - w^*| = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}.$$