MAT 536, Spring 2024
PROBLEM SET 9, Due Monday, April 22

Problems are due in class on Mondays (if you can't attend class, email solutions to the grader before class begins). Please restate problem at each beginning of your solution, or attach this sheet to the front of your solution set. Solutions should be legible and written in complete, correct sentences. Handwritten or a PDF from LaTeX is acceptable. Cite any sources that you use, e.g., a Wikipedia article or another textbook (most problems will not require outside sources).

(1) Prove that every planar domain with finitely many boundary components can be conformally mapped to a domain where every boundary component is either a point or an analytic closed curve (the image of a circle under conformal map defined in some annular neighborhood of that circle).

(2) Let $S = [0,1]^2$ be a unit square. Prove there is a unique bounded harmonic function u on the interior of S that is continuous at all boundary points except the top two corners, and equals 1 along the top edge and zero along the other three edges. What is the value of u at the center of the square? Prove your answer.

(3) Find an explicit formula for the Green's function $G(z, w)$ for $\Omega = \mathbb{C} \setminus [-1, 1]$ with $w \in \Omega$.

(4) Suppose Ω is a bounded planar domain. Prove that a boundary point $x \in \partial\Omega$ is regular iff the Green's function $G(z, p)$ for Ω tends to zero as $z \to x$ through Ω (p can be any fixed point of Ω).

(5) The Dirichlet principle says that if f is continuous on the unit circle, then the harmonic extension of f to the unit disk minimizes $\iint |\nabla u|^2 dxdy$ among all differentiable extensions. But for some continuous functions f this integral is infinite even for the harmonic extension. Give an analytic example (with proof) of the form $f(z) = \sum_0^\infty a_n z^n$ where $\sum |a_n| < \infty$ but $\sum n|a_n|^2 = \infty$.

The Dirichlet principle was used by Riemann to prove the Riemann mapping theorem in his thesis, but Weierstrass pointed out it is not always valid, as above. The first valid proof of the mapping theorem was given by Osgood in 1900 using potential theory. Hilbert showed the Dirichlet principle approach is valid for Jordan domains. The now standard proof of the mapping theorem via normal families is due to Carathéodory in 1912.