MAT 536, Spring 2024
PROBLEM SET 8, Due Monday, April 15

Problems are due in class on Mondays (if you can't attend class, email solutions to the grader before class begins). Please restate problem at each beginning of your solution, or attach this sheet to the front of your solution set. Solutions should be legible and written in complete, correct sentences. Handwritten or a PDF from LaTeX is acceptable. Cite any sources that you use, e.g., a Wikipedia article or another textbook (most problems will not require outside sources).

(1) If $f : \mathbb{D} \to \Omega$ is a conformal map (1-1, holomorphic) onto a bounded simply connected domain, prove that $\lim_{r\to 1} f(re^{i\theta})$ exists for almost every $\theta \in [0, 2\pi)$. (Hint: Show $\int_{\mathbb{D}} |f'|^2 dxdy < \infty$ and deduce that $\int_0^1 |f'(re^{i\theta})|dr < \infty$ for almost every θ.)

(2) A compact set K is **locally connected** if for each $z \in K$ and $\epsilon > 0$ there is an open set $V \subset B(z, \epsilon)$ such that $V \cap K$ is connected and $z \in V$. Let Ω be a simply-connected region such that $\partial\Omega$ contains at least two points. Prove $\partial\Omega$ is locally connected if and only if a conformal map $\varphi : \mathbb{D} \to \Omega$ extends continuously to $\overline{\mathbb{D}}$. (Hint: first show that $\partial\Omega$ is locally connected iff it is the continuous image of $\mathbb{T}$. Then check that the proof in the textbook of continuity in Carathéodory's theorem still holds if we assume $\partial\Omega$ is locally connected.)

(3) Suppose Ω is bounded and simply connected and that the conformal map $f : \mathbb{D} \to \Omega$ extends continuously everywhere on the boundary. If $x \in \partial\Omega$, show that $f^{-1}(x) \subset \mathbb{T}$ contains only one or two points, except for at most countably many points $x \in \partial\Omega$. (Hint: look up and use the Moore triod theorem, a result in planar point set topology from 1928.)

(4) Give an example of a bounded harmonic function on $\mathbb{D}$ whose harmonic conjugate is not bounded. (Hint: consider conformal maps into a strip.)

(5) If γ is a closed Jordan curve in the plane, there is a conformal map f from $\mathbb{D}$ to the bounded complementary component of γ and a conformal map g of $\mathbb{D}^* = \{|z| > 1\}$ to the unbounded component. Show that $h = g^{-1} \circ f$ defines a homeomorphism of the unit circle to itself.

Such a homeomorphism is called a conformal welding. Every diffeomorphism of the circle to itself is a conformal welding, and it determines a curve γ that is unique up to Möbius transformations. Many "rougher" homeomorphisms are also conformal weldings, but there exist homeomorphisms that are not. Characterizing which circle homeomorphisms arise as conformal welding is a difficult open problem, and there are many results characterizing classes of closed curves in terms of the corresponding conformal weldings.